The Space $bv^0_k$ and Matrix Transformations

G. Canan Hazar Güleç and M. Ali Sarıgöl

1 Department of Mathematics, Faculty of Science and Arts, Pamukkale University, Denizli, Turkey, ORCID: 0000-0002-9825-5555
2 Department of Mathematics, Faculty of Science and Arts, Pamukkale University, Denizli, Turkey, ORCID: 0000-0002-9820-1024

* Corresponding Author E-mail: gchazar@pau.edu.tr

Abstract: In this study, we introduce the space $bv^0_k$, give its some algebraic and topological properties, and also characterize some matrix operators defined on that space. Also we extend some well known results.

Keywords: BK spaces, Matrix transformations, Sequence spaces.

1 Introduction

Let $\omega$ be the set of all complex sequences, $\ell_k$ and $c$ be the sets of $k$-absolutely convergent series and convergent sequences, respectively. By $bv$ we denote the space of all sequences of bounded variation, i.e.,

$$bv = \{x \in w : \Delta x \in \ell_k\}.$$ 

Let $U$ and $V$ be subspaces of $w$ and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the $A$-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series is convergent for $n \geq 0$. Then, we say that $A$ defines a matrix transformation from $U$ into $V$, and denote it by $A \in (U,V)$ if the sequence $A(x) = (A_n(x)) \in V$ for every sequence $x \in U$, also the sets $U^\beta = \{\varepsilon = (\varepsilon_v) : \Sigma e_v x_v \text{ converges for all } x \in U\}$ and

$$U_A = \{x \in \omega : A(x) \in U\}$$

are called the $\beta$ dual of $U$ and the domain of a matrix $A$ in $U$. Further, $U \subset w$ is said to be a $BK$-space if it is a Banach space with continuous coordinates $p_n : U \rightarrow C$ defined by $p_n(x) = x_n$ for $n \geq 0$. The sequence $(\varepsilon_v)$ is called a Schauder base (or briefly base) for a normed sequence space $U$ if for each $x \in U$ there exist unique scalar coefficients $(x_v)$ such that

$$\lim_{m \rightarrow \infty} \left\| x - \sum_{v=0}^{m} x_v \varepsilon_v \right\| = 0,$$

and we write

$$x = \sum_{v=0}^{\infty} x_v \varepsilon_v.$$ 

An infinite matrix $A = (a_{nv})$ is called a triangle if $a_{nn} \neq 0$ and $a_{nv} = 0$ for all $v > n$ for all $n, v [1]$.

We define the notations $\Gamma_\varepsilon$, $\Gamma_\infty$ and $\Gamma_\alpha$ for $v = 1, 2, \ldots$, as follows:

$$\Gamma_\varepsilon = \left\{ \varepsilon = (\varepsilon_v) : \lim_{m \rightarrow \infty} \sum_{v=0}^{m} \varepsilon_v \text{ exists for } r = 1, 2, \ldots \right\},$$

$$\Gamma_\infty = \left\{ \varepsilon = (\varepsilon_v) : \sup_{m, r} \left| \sum_{v=0}^{m} \varepsilon_v \right| < \infty, \quad r = 1, 2, \ldots \right\},$$

and

$$\Gamma_\alpha = \left\{ \varepsilon = (\varepsilon_v) : \sup_{m} \left\{ \frac{1}{\theta_r} \sum_{v=0}^{m} \varepsilon_v \right\}^{k^*} < \infty \right\},$$

where $k^*$ is the conjugate of $k$, that is, $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$. 

© CPOST 2019 169
More recently some new sequence spaces by means of the matrix domain of a particular limitation method or absolute summability methods have been defined and studied by several authors in many research papers (see, for instance [2–8]). In this study, we introduce the space $\text{bv}_k^\theta$, give its some algebraic and topological properties and characterize some matrix operators defined on that space. Also we extend some well known results.

The following lemmas are needed in proving our theorems.

**Lemma 1.** Let $1 \leq k < \infty$. Then, $A \in (\ell, \ell_k)$ if and only if

$$\sup_n \sum_{v=0}^{\infty} |a_{nv}|^k < \infty,$$

[9].

**Lemma 2.**

a-) For every $A \in (\ell, c)$ if and only if (i) $\lim_{n} a_{nv}$ exists for each $v$, and (ii) $\sup_{n,v} |a_{nv}| < \infty$.

b-) Let $1 < k < \infty$. Then $A \in (\ell_k, c)$ if and only if

$$\sup_{n,v} \sum_{n=0}^{\infty} |a_{nv}|^{k^*} < \infty$$

[10].

2 The space $\text{bv}_k^\theta$ and matrix operators

In this section we introduce the space $\text{bv}_k^\theta$ as

$$\text{bv}_k^\theta = \left\{ x = (x_k) \in w : \left( \theta_n^{1/k^*} \Delta x_n \right) \in \ell_k \right\},$$

where $(\theta_n)$ is a sequence of nonnegative terms, $1 \leq k < \infty$ and $\Delta x_n = x_n - x_{n-1}$ for all $n$. Note that it includes some known spaces. For example, it is reduced to $\text{bv}^k$ for $\theta_n = 1$ for all $n$ and $\text{bv}_1^\theta = \text{bv}$, which have been studied by Malkowsky et al [11] and Jarrah and Malkowsky [6]. Moreover, recently Başar et al [3] have defined the sequence space $\text{bv}(u, p)$ and proved that this space is linearly isomorphic to the space $\ell(p)$ of Maddox [12] as generalized to paranormed space.

It is redefined as $\text{bv}_k^\theta = (\ell_k)_A$ with the notation (1), where the matrix $A$ is defined by

$$a_{nv} = \begin{cases} -\theta_n^{1/k^*}, & v = n - 1, \\
\theta_n^{1/k^*}, & v = n, \\
0, & v \neq n, n - 1. \end{cases}$$

Further, $\left| N_p^\theta \right|_k = \left( \text{bv}_k^\theta \right)_A$ and $\left| C_{\alpha} \right|_k = \left( \text{bv}_k^\theta \right)_B$, where $A$ and $B$ are Cesàro and Nörlund means of series $\Sigma x_n$ (see [8],[5,13]).

Now we begin with topological properties of $\text{bv}_k^\theta$, which also can be deduced from [3].

**Lemma 3.** Let $1 \leq k < \infty$ and $(\theta_n)$ be a sequence of nonnegative numbers. Then,

a-) The space $\text{bv}_k^\theta$ is a BK-space and norm isomorphic to the space $\ell_k$, i.e., $\text{bv}_k^\theta \approx \ell_k$.

b-) $(\text{bv}_k^\theta)^\beta = \Gamma_c \cap \Gamma_k$ for $1 < k < \infty$ and $(\text{bv}^\beta)^\beta = \Gamma_c \cap \Gamma_{\infty}$ for $k = 1$.

c-) Define the sequence $b^{(j)} = \left( b_n^{(j)} \right)$ such that, for $j, n \geq 0$,

$$b_n^{(j)} = \begin{cases} \theta_n^{-1/k^*}, & n \geq j, \\
0, & n < j. \end{cases}$$

Then, the sequence $b^{(j)} = \left( b_n^{(j)} \right)$ is the base of $\text{bv}_k^\theta$.

**Proof:** a-) Since $\ell_k$ is a BK-space with respect to its usual norm and $A$ is a triangle matrix, Theorem 4.3.2 of Wilansky [1, p. 61] gives the fact that $\text{bv}_k^\theta$ is a BK-space for $1 \leq k < \infty$. Now, consider $T : \text{bv}_k^\theta \rightarrow \ell_k$ defined by $y = T(x) = \left( \theta_n^{1/k^*} \Delta x_n \right)$ for all $x \in \text{bv}_k^\theta$. Then, it is clear that $T$ is a linear operator, and surjective since, if $y = (y_n) \in \ell_k$, then $x = (x_n) = \left( \sum_{j=0}^{n} \theta_j^{1/k^*} y_j \right) \in \text{bv}_k^\theta$, and also one to one. Further, it preserves the norm, since

$$\|T(x)\|_{\ell_k} = \left( \sum_{n=0}^{\infty} \theta_n^{k-1} |\Delta x_n|^k \right)^{1/k} = \|x\|_{\text{bv}_k^\theta},$$

which completes the proof.
b) This part can be proved together with Lemma 2.

c) Since the sequence \( e^{(j)} \) is a base of \( \ell_k \), where \( e^{(j)} = (e_n^{(j)})_{n=0}^{\infty} \) is the sequence whose only non-zero term is 1 in the \( n \)th place for each \( n \in \mathbb{N} \), it is clear that the sequence \( b^{(j)} \) is the base of \( b_k^0 \). In fact, we first note that \( T^{-1}(e^{(j)}) = b^{(j)} \). Now, if \( x \in b_k^0 \), then there exists \( y \in \ell_k \) such that \( y = T(x) \), and so it follows from (a) that

\[
\| x - \sum_{j=0}^{\infty} x_j b^{(j)} \|_{b_k^0} = \| y - \sum_{j=0}^{\infty} y_j e^{(j)} \|_{\ell_k} \to 0 \text{ as } m \to \infty,
\]

and it is easy to see that the representation \( x = \sum_{j=0}^{\infty} x_j b^{(j)} \) is unique. \( \square \)

**Theorem 1.** Let \( A = (a_{nv}) \) be an infinite matrix of complex numbers for all \( n, v \geq 0 \), \( (\theta_n) \) be a sequence of nonnegative numbers and \( 1 \leq k < \infty \). Then, \( A \in (b, b_k^0) \) if and only if

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} a_{nj} \text{ exists for each } v, \tag{2}
\]

\[
\sup_{n, v} \left| \sum_{j=0}^{\infty} a_{nj} \right| < \infty \tag{3}
\]

and

\[
\sup_{\nu} \sum_{n=0}^{\infty} \left| \theta_n^{1/k} \sum_{j=0}^{\infty} (a_{nj} - a_{n-1,j}) \right|^k < \infty. \tag{4}
\]

**Proof:** \( A \in (b, b_k^0) \) iff \( (a_{nj})_{j=0}^{\infty} \in b\theta^0 \) and \( A(x) \in b_k^0 \) for every \( x \in b \), and also, by Lemma 3, \( (a_{nj})_{j=0}^{\infty} \in b\theta^0 \) iff (2) and (3) hold.

Now, to prove necessity and sufficiency of the condition (4), consider the operators \( B : b \to \ell \) and \( B' : b_k^0 \to \ell_k \) defined by

\[
B_n(x) = \Delta x_n, \quad B'_n(x) = \theta_n^{1/k} \Delta x_n,
\]

respectively. As in Lemma 3, these operators are bijection and the matrices corresponding to these operators are triangles. Further, let \( x \in b \) be given. Then, \( B(x) = y \in \ell \) if \( x = S(y) \), where \( S \) is the inverse of \( B \) and it is given by

\[
s_{nv} = \begin{cases} 
1, & 0 \leq \nu \leq n, \\
0, & \nu > n.
\end{cases}
\]

On the other hand, if any matrix \( R = (r_{nv}) \in (\ell, c) \), then, the series \( R_n(x) = \sum_{v=0}^{\infty} x_v \) is convergent uniformly in \( n \), since, by Lemma 2, the remaining term tends to zero uniformly in \( n \), that is,

\[
\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \left( \sup_{n, v} |r_{nv}| \right) \sum_{v=m}^{\infty} |x_v| \to 0 \text{ as } m \to \infty,
\]

and so

\[
\lim_{n} R_n(x) = \lim_{n} \sum_{v=0}^{\infty} r_{nv} x_v. \tag{5}
\]

Now, it is easily seen from (2) and (3) that \( H = \left( h_{nv}(n) \right) \in (\ell, c) \), which gives us, by (5), that

\[
A_n(x) = \lim_{n} a_{nv} y_r = \sum_{r=0}^{m} \sum_{v=r}^{\infty} a_{nv} y_r,
\]

converges for all \( n \geq 0 \), where, for \( r, m = 0, 1, \ldots \),

\[
h_{nv}(n) = \begin{cases} 
s_{uv}, & 0 \leq r \leq m, \\
0, & r > m.
\end{cases}
\]

This shows that the mapping sequence \( A(x) = (A_n(x)) \) exists. On the other hand, since \( S \) is the infinite triangle matrix, it is clear that \( A(x) = A(S(y)) \in b_k^0 \) for every \( x \in b \) iff \( B'(A(S(y))) \in \ell_k \), i.e., \( (B' o A o S)(y) \in \ell_k \), which implies that \( D = B' o A o S : \ell \to \ell_k \).
Therefore, it can be written that $A : \ell \to \ell_k$ and also $D = B' \circ \hat{A}$, where $\hat{A} = \text{AoS}$. Now, a few calculations reveal that

$$\hat{a}_{n,v} = \sum_{j=v}^{\infty} a_{nj}s_{j,v} = \sum_{j=v}^{\infty} a_{nj}$$

and so

$$d_{n,v} = \sum_{j=0}^{n} b_{n,j} \hat{a}_{j,v} = \theta_n^{1/k}\sum_{j=\nu}^{\infty} (a_{nj} - a_{n-1,j})$$

Now, let us apply Lemma 1 with the matrix $D$. Then, it can be easily obtained from the definition of the matrix $D$ that $D : \ell \to \ell_k$ if condition (4) holds. This completes the proof.

If $A$ is an infinite triangle matrix in Theorem 1, then (2) and (3) hold, and so it reduces to the following result.

**Corollary 1.** If $A$ is an infinite triangle matrix of complex numbers for all $n, v \geq 0$ and $1 \leq k < \infty$, then $A \in \left( \ell_k, \ell_k \right)$ if and only if

$$\sup_{\nu} \sum_{n=0}^{\infty} \left| \sum_{j=\nu}^{\infty} \left( a_{nj} - a_{n-1,j} \right) \right| ^k < \infty.$$

**Acknowledgement**

This study is supported by Pamukkale University Scientific Research Projects Coordinatorship (Grant No. 2019KRM004-029).

3 References


