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A New Type Generalized Difference Sequence Space $m(\phi,p)(\Delta_m^n)$

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Abstract: Let (ϕ_n) be a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$. The class of all sequences (ϕ_n) is denoted by Φ. The sequence space $m(\phi)$ was introduced by Sargent [1] and he studied some of its properties and obtained some relations with the space ℓ_p. Later on it was investigated by Tripathy and Sen [2] and Tripathy and Mahanta [3]. In this work, using the generalized difference operator Δ_m^n, we generalize the sequence space $m(\phi)$ to sequence space $m(\phi,p)(\Delta_m^n)$, give some topological properties about this space and show that the space $m(\phi,p)(\Delta_m^n)$ is a $BK-$space by a suitable norm. The results obtained are generalizes some known results.

Keywords: Difference sequence, BK-space, Symmetric space, Normal space.

1 Introduction

By w, we denote the space of all complex (or real) sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$. We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Also by ℓ_1 and ℓ_p; we denote the spaces of all absolutely summable and $p-$absolutely summable sequences, respectively.

Let $x \in w$ and $S(x)$ denotes the set of all permutation of the elements x_n, i.e. $S(x) = \{(x_{\pi_{(n)}}) : \pi(n) \text{ is a permutation on } \mathbb{N}\}$. A sequence space E is said to be symmetric if $S(x) \subset E$ for all $x \in E$.

A sequence space E is said to be solid (normal) if $(y_n) \in E$, whenever $(x_n) \in E$ and $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$.

A sequence space E is said to be sequence algebra if $x.y \in E$, whenever $x, y \in E$.
A sequence space E is said to be perfect if $E = E^{\alpha\alpha}$.
It is well known that if E is perfect then E is normal.

A sequence space E with a linear topology is called a $K-$space provided each of the maps $p_i : E \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for each $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A $K-$space E is called an $FK-$space provided E is a complete linear metric space. An $FK-$space whose topology is normable is called a $BK-$space.

The notion of difference sequence spaces was introduced by Kızmaz [4] and it was generalized by Et and Çolak [5] for $X = \ell_\infty, c, c_0$ as follows:

Let n be a non-negative integer, then

$$\Delta^n(X) = \{x = (x_k) : (\Delta^n x_k) \in X\},$$

where $\Delta^n x_k = \Delta^{n-1}x_k - \Delta^{n-1}x_{k+1}$ for all $k \in \mathbb{N}$ and so $\Delta^n x_k = \sum_{v=0}^{n}(-1)^v\binom{n}{v}x_{k+v}$. Et and Çolak [5] showed that the sequence spaces $\Delta^n(c_0)$, $\Delta^n(c)$ and $\Delta^n(\ell_\infty)$ are $BK-$spaces with the norm

$$\|x\|_{\Delta 1} = \sum_{i=1}^{n}|x_i| + \|\Delta^n x\|_\infty .$$

After then, using a new difference operator Δ_m^n, Tripathy et al. ([6], [7], [8]) have defined a new type difference sequence space $\Delta_m^n(X)$ such as

$$\Delta_m^n(X) = \{x = (x_k) : (\Delta_m^n x_k) \in X\},$$

where $m, n \in \mathbb{N}$, $\Delta_m^0 x = x$, $\Delta_m^1 x = (x_k - x_{k+m})$, $\Delta_m^n x = (\Delta_m^n x_k) = \left(\Delta_m^{n-1}x_k - \Delta_m^{n-1}x_{k+m}\right)$ and so $\Delta_m^n x_k = \sum_{v=0}^{n}(-1)^n\binom{n}{v}x_{k+mv}$, and give some topological properties about this space and show that the spaces $\Delta_m^n(X)$ are $BK-$spaces by the norm

$$\|x\|_{\Delta 2} = \sum_{i=1}^{mn}|x_i| + \|\Delta_m^n x\|_\infty$$

for $X=\ell_\infty, c$ and c_0. Recently, difference sequences have been studied in ([9],[10],[11],[12],[13],[14],[15],[16],[17],[18]) and many others.

2 Main results

In this section, we introduce a new class $m(\phi,p)(\Delta_m^n)$ of sequences , establish some inclusion relations and some topological properties. The obtained results are more general than those of Çolak and Et [19], Sargent [1] and Tripathy and Sen [2] .

The notation φ_s denotes the class of all subsets of $\mathbb{N}$, those do not contain more than s elements. Let (ϕ_n) be a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \le (n+1)\phi_n$ for all $n \in \mathbb{N}$. The class of all sequences (ϕ_n) is denoted by Φ.

The sequence spaces $m(\phi)$ and $m(\phi,p)$ were introduced by Sargent [1], Tripathy and Sen [2] as follows, respectively

$$m(\phi)=\left\{x=(x_k)\in w : \|x\|_{m(\phi)}=\sup_{s\ge 1,\ \sigma\in\varphi_s}\frac{1}{\phi_s}\sum_{k\in\sigma}|x_k|<\infty\right\},$$

$$m(\phi,p)=\left\{x=(x_k)\in w : \|x\|_{m(\phi,p)}=\sup_{s\ge 1,\ \sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}|x_k|^p\right)^{\frac{1}{p}}<\infty\right\}.$$

Let $m,n\in\mathbb{N}$ and $1\le p<\infty$. Now we define the sequence space $m(\phi,p)(\Delta_m^n)$ as

$$m(\phi,p)(\Delta_m^n)=\left\{x=(x_k)\in w : \sup_{s\ge 1,\ \sigma\in\varphi_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p<\infty\right\}.$$

From this definition it is clear that $m(\phi,p)\left(\Delta_m^0\right)=m(\phi,p)$ and $m(\phi,1)\left(\Delta_m^0\right)=m(\phi)$. In case of $m=1$, we shall write $m(\phi,p)(\Delta^n)$ instead of $m(\phi,p)(\Delta_m^n)$ and in case of $p=1$, we shall write $m(\phi)(\Delta_m^n)$ instead of $m(\phi,p)(\Delta_m^n)$. The sequence space $m(\phi,p)(\Delta_m^n)$ contains some unbounded sequences for $m,n\ge 1$. For example, the sequence $(x_k)=(k^n)$ is an element of $m(\phi,p)(\Delta_m^n)$ for $m=1$, but is not an element of ℓ_∞.

Theorem 1. The space $m(\phi,p)(\Delta_m^n)$ is a Banach space with the norm

$$\|x\|_{\Delta_m^n}=\sum_{i=1}^{r}|x_i|+\sup_{s\ge 1,\ \sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}},\quad 1\le p<\infty, \tag{1}$$

where $r=mn$ for $m\ge 1, n\ge 1$.

Proof. It is a routine verification that $m(\phi,p)(\Delta_m^n)$ is a normed linear space normed by (1) for $1\le p<\infty$. Let $\left(x^l\right)$ be a Cauchy sequence in $m(\phi,p)(\Delta_m^n)$, where $x^l=(x_k^l)_{k=1}^\infty=\left(x_1^l,x_2^l,...\right)\in m(\phi,p)(\Delta_m^n)$, for each $l\in\mathbb{N}$. Then given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\left\|x^l-x^t\right\|_{\Delta_m^n}=\sum_{i=1}^{r}\left|x_i^l-x_i^t\right|+\sup_{s\ge 1,\ \sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l-x_k^t\right)\right|^p\right)^{\frac{1}{p}}<\varepsilon \tag{2}$$

for all $l,t>n_0$. Hence we obtain

$$\left|x_k^l-x_k^t\right|\to 0 \text{ as } l,t\to\infty, \text{ for each } k\in\mathbb{N}.$$

Therefore $(x_k^l)_{l=1}^\infty=\left(x_k^1,x_k^2,...\right)$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, it is convergent, that is,

$$\lim_l x_k^l=x_k$$

for each $k\in\mathbb{N}$. Using these infinite limits $x_1,x_2,x_3,...$ let us define the sequence $x=(x_k)$. We should show that $x\in m(\phi,p)(\Delta_m^n)$ and $\left(x^l\right)\to x$. Taking limit as $t\to\infty$ in (2) , we get

$$\left\|x^l-x\right\|_{\Delta_m^n}=\sum_{i=1}^{r}\left|x_i^l-x_i\right|+\sup_{s\ge 1,\ \sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l-x_k\right)\right|^p\right)^{\frac{1}{p}}<\varepsilon \tag{3}$$

for all $l \geq n_0$. This shows that $\left(x^l\right) \to x$ as $l \to \infty$. From (3) we also have

$$\sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l - x_k\right)\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

for all $l \geq n_0$. Hence $x^l - x = \left(x_k^l - x_k\right)_k \in m(\phi,p)(\Delta_m^n)$. Since $x^l - x, x^l \in m(\phi,p)(\Delta_m^n)$ and $m(\phi,p)(\Delta_m^n)$ is a linear space, we have $x = x^l - \left(x^l - x\right) \in m(\phi,p)(\Delta_m^n)$. Therefore $m(\phi,p)(\Delta_m^n)$ is complete.

Theorem 2. The space $m(\phi,p)(\Delta_m^n)$ is a $BK-$space.

Proof. Omitted.

Theorem 3. [2] i) The space $m(\phi,p)$ is a symmetric space,
ii) The space $m(\phi,p)$ is a normal space.

Theorem 4. The sequence space $m(\phi,p)(\Delta_m^n)$ is not sequence algebra, is not solid and is not symmetric, for $m,n,p \geq 1$.

Proof. For the proof of the Theorem, consider the following examples:

Example 1. It is obvious that, if $x = \left(k^{n-2}\right)$, $y = \left(k^{n-2}\right)$ and $m = 1$, then $x, y \in m(\phi,p)(\Delta_m^n)$, but $x.y \notin m(\phi,p)(\Delta_m^n)$. Hence $m(\phi,p)(\Delta_m^n)$ is not a sequence algebra.

Example 2. It is obvious that, if $x = \left(k^{n-1}\right)$ and $m = 1$, then $x \in m(\phi,p)(\Delta_m^n)$, but $(\alpha_k x_k) \notin m(\phi,p)(\Delta_m^n)$ for $(\alpha_k) = \left((-1)^k\right)$. Hence $m(\phi,p)(\Delta_m^n)$ is not solid.

Example 3. Let us consider the sequence $x = \left(k^{n-1}\right)$. Then $x \in m(\phi,p)(\Delta_m^n)$ for $m = 1$. Let (y_k) be a rearrangement of (x_k) which is defined as follows:

$$y_k = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...\}.$$

Then $y \notin m(\phi,p)(\Delta_m^n)$. Hence $m(\phi,p)(\Delta_m^n)$ is not symmetric.

The following result is a consequence of Theorem 4.

Corollary 1. The sequence space $m(\phi,p)(\Delta_m^n)$ is not perfect, for $m,n,p \geq 1$.

Theorem 5. $m(\phi)(\Delta_m^n) \subset m(\phi,p)(\Delta_m^n)$ for each $m,n,p \geq 1$.

Proof. Omitted.

Theorem 6. $m(\phi,p)(\Delta_m^n) \subset m(\psi,p)(\Delta_m^n)$ if and only if $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$.

Proof. Suppose that $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$. Then $\phi_s \leq K\psi_s$ for every s and for some positive number K. If $x \in m(\phi,p)(\Delta_m^n)$, then,

$$\sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} < \infty.$$

Now, we have

$$\sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\psi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}\%} < \sup_{s\geq 1}(K) \sup_{s\geq 1\quad k\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} < \infty.$$

Hence $x \in m(\psi,p)(\Delta_m^n)$.

Conversely let $m(\phi,p)(\Delta_m^n) \subset m(\psi,p)(\Delta_m^n)$ and suppose that $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = \infty$. Then, there exists a sequence (s_i) of natural numbers such that $\lim\limits_i\left(\frac{\phi_{s_i}}{\psi_{s_i}}\right) = \infty$. Then, for $x \in m(\phi,p)(\Delta_m^n)$ we have

$$\sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\psi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} \geq \sup_{i\geq 1}\left(\frac{\phi_{s_i}}{\psi_{s_i}}\right) \sup_{i\geq 1,\ \sigma\in\varphi_{s_i}} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}} = \infty.$$

Therefore $x \notin m(\psi,p)(\Delta_m^n)$. This contradict to $m(\phi,p)(\Delta_m^n) \subset m(\psi,p)(\Delta_m^n)$. Hence $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$.

From Theorem 6, we get the following result.

Corollary 2. $m(\phi,p)(\Delta_m^n)=m(\psi,p)(\Delta_m^n)$ if and only if $0<\inf_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right)\leq \sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right)<\infty$.

Theorem 7. $m(\phi,p)\left(\Delta_m^{n-1}\right)\subset m(\phi,p)(\Delta_m^n)$ and the inclusion is strict.

*Proof.*Let $x\in m(\phi,p)\left(\Delta_m^{n-1}\right)$. It is well known that, for $1\leq p<\infty$, $|a+b|^p\leq 2^p\left(|a|^p+|b|^p\right)$. Hence, for $1\leq p<\infty$, we have

$$\frac{1}{\phi_s}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\leq 2^p\left(\frac{1}{\phi_s}\sum_{k\in\sigma}\left|\Delta_m^{n-1}x_k\right|^p+\frac{1}{\phi_s}\sum_{k\in\sigma}\left|\Delta_m^{n-1}x_{k+1}\right|^p\right)$$

Hence $x\in m(\phi,p)(\Delta_m^n)$.

To show the inclusion is strict consider the following example.

Example 4. Let $\phi_n=1$, for all $n\in\mathbb{N}$, $m=1$ and $x=\left(k^{n-1}\right)$, then $x\in\ell_p(\Delta_m^n)\backslash\ell_p\left(\Delta_m^{n-1}\right)$.

Theorem 8. We have $\ell_p(\Delta_m^n)\subset m(\phi,p)(\Delta_m^n)\subset\ell_\infty(\Delta_m^n)$.

Proof. Since $m(\phi,p)(\Delta_m^n)=\ell_p(\Delta_m^n)$ for $\phi_n=1$, for all $n\in\mathbb{N}$, then $\ell_p(\Delta_m^n)\subset m(\phi,p)(\Delta_m^n)$. Now assume that $x\in m(\phi,p)(\Delta_m^n)$. Then we have

$$\sup_{s\geq 1,\ \sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}}<\infty \text{ and so } \left|\Delta_m^n x_k\right|<K\phi_1,$$

for all $k\in\mathbb{N}$ and for some positive number K. Thus, $x\in\ell_\infty(\Delta_m^n)$.

Theorem 9. If $0<p<q$, then $m(\phi,p)(\Delta_m^n)\subset m(\phi,q)(\Delta_m^n)$.

Proof. Proof follows from the following inequality

$$\left(\sum_{k=1}^n|x_k|^q\right)^{\frac{1}{q}}\leq\left(\sum_{k=1}^n|x_k|^p\right)^{\frac{1}{p}},\quad(0<p<q).$$

3 References

[1] W. L. C. Sargent, *Some sequence spacess related to ℓ_p spaces*, J. London Math. Soc. **35** (1960), 161-171.
[2] B. C. Tripathy, M. Sen, *On a new class of sequences related to the space ℓ_p*, Tamkang J. Math. **33**(2) (2002), 167-171.
[3] B. C. Tripathy, S. Mahanta, *On a class of sequences related to the ℓ_p* space defined by Orlicz functions, Soochow J. Math. **29**(4) (2003), 379–391.
[4] H. Kızmaz, *On certain Sequence spaces*, Canad. Math. Bull. **24**(2) (1981), 169-176.
[5] M. Et, R. Çolak, *On generalized difference sequence spaces*, Soochow J. Math. **21**(4) (1995), 377-386.
[6] A. Esi, B. C. Tripathy, B. Sarma, *On some new type generalized difference sequence spaces*, Math. Slovaca **57**(5) (2007), 475–482.
[7] A. Esi, B. C. Tripathy, *A New Type Of Difference Sequence Spaces*, International Journal of Science & Technology **1**(1) (2006), 11-14.
[8] B . C. Tripathy, A. Esi, B. K. Tripathy, *On a new type of generalized difference Cesaro Sequence spaces*, Soochow J. Math. **31**(3) (2005), 333-340.
[9] Y. Altin, *Properties of some sets of sequences defined by a modulus function*, Acta Math. Sci. Ser. B Engl. Ed. **29**(2) (2009), 427–434.
[10] H. Altinok, M. Et, R. Çolak, *Some remarks on generalized sequence space of bounded variation of sequences of fuzzy numbers*, Iran. J. Fuzzy Syst. **11**(5) (2014), 39–46, 109.
[11] S. Demiriz, C. Çakan, *Some topological and geometrical properties of a new difference sequence space*. Abstr. Appl. Anal. 2011, Art. ID 213878, 14 pp.
[12] S. Erdem, S. Demiriz, *On the new generalized block difference sequence spaces*, Appl. Appl. Math., Special Issue No. **5** (2019), 68–83.
[13] M. Et, A. Alotaibi, S. A. Mohiuddine, *On (Δ^m, I)-statistical convergence of order α*, The Scientific World Journal, 2014, 535419 DOI: 10.1155/2014/535419.
[14] M. Et, M. Mursaleen, M. Işık, *On a class of fuzzy sets defined by Orlicz functions*, Filomat **27**(5) (2013), 789–796.
[15] M. Et, V. Karakaya, *A new difference sequence set of order α and its geometrical properties*, Abstr. Appl. Anal. 2014, Art. ID 278907, 4 pp.
[16] M. Karakaş, M. Et, V. Karakaya, *Some geometric properties of a new difference sequence space involving lacunary sequences*, Acta Math. Sci. Ser. B (Engl. Ed.) **33**(6) (2013), 1711–1720.
[17] M. A. Sarıgöl, *On difference sequence spaces*, J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math.-Phys. **10** (1987), 63–71.
[18] E. Savaş, M. Et, *On $(\Delta_\lambda^m, I)-$statistical convergence of order α*, Period. Math. Hungar. **71**(2) (2015), 135–145.
[19] R. Çolak, M. Et, *On some difference sequence sets and their topological properties*, Bull. Malays. Math. Sci. Soc. **28**(2) (2005), 125–130.