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Solutions of Singular Differential Equations by means of Discrete Fractional Analysis

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Abstract: Recently, many researchers demonstrated the usefulness of fractional calculus in the derivation of particular solutions of linear ordinary and partial differential equation of the second order. In this study, we acquire new discrete fractional solutions of singular differential equations (homogeneous and nonhomogeneous) by using discrete fractional nabla operator $\nabla^{\upsilon}(0<\upsilon<1)$.

Keywords: Discrete fractional analysis, Nabla operator, Singular differential equations.

1 Introduction

The remarkably widely investigated subject of fractional and discrete fractional calculus has gained importance and popularity during the past three decades or so, due chiefly to its demonstrated applications in numerous seemingly diverse fields of science and engineering [1]-[4]. The analogous theory for discrete fractional analysis was initiated and properties of the theory of fractional differences and sums were established. Recently, many articles related to discrete fractional analysis have been published [5]-[9]. The fractional nabla operator have been applied to various singular ordinary and partial differential equations such as the second-order linear ordinary differential equation of hypergeometric type [10], the Bessel equation [11], the Hermite equation [12], the non- fuchsian differential equation [13], the hydrogen atom equation [14].

The aim of this article is to obtain new dfs of the singular differential equation by means of fractional calculus operator.

2 Preliminary and properties

Here we only give a very short introduction to the basic definitions in discrete fractional calculus. For more on the subject we refer the reader to [5, 13].

Let $\zeta\in\mathbb{R}^{+}$, $n\in\mathbb{Z}$, such that $n-1\leq\zeta<n$. The $\zeta^{th}-$ order fractional sum of F is defined as

$$\nabla_c^{-\zeta}F(t)=\frac{1}{\Gamma(\zeta)}\sum_{\tau=c}^{t}(t-\rho(\tau))^{\overline{\zeta-1}}F(\tau), \tag{1}$$

where $t\in\mathbb{N}_{\alpha}=\{\alpha,\alpha+1,\alpha+2,...\}$, $\alpha\in\mathbb{R},\ \rho(t)=t-1$ is the backward jump operator.

The rising factorial power and rising function is given by

$$t^{\overline{n}}=t(t+1)(t+2)...(t+n-1),\ n\in\mathbb{N},\ t^{\overline{0}}=1,$$

$$t^{\overline{\zeta}}=\frac{\Gamma(t+\zeta)}{\Gamma(t)},\ \zeta\in\mathbb{R},\ t\in\mathbb{R}\backslash\{...,-2,-1,0\},\ 0^{\overline{\zeta}}=0. \tag{2}$$

Note that

$$\nabla\left(t^{\overline{\zeta}}\right)=\zeta t^{\overline{\zeta-1}}, \tag{3}$$

where $\nabla\phi(t)=\phi(t)-\phi(\sigma(t))=\phi(t)-\phi(t-1)$.

The $\zeta^{th}-$ order fractional difference of F is defined by

$$\begin{aligned}\nabla_c^{\zeta}F(t)&=\nabla^n\left[\nabla_c^{\zeta-(n-\zeta)}F(t)\right]\\&=\nabla^n\left[\frac{1}{\Gamma(n-\zeta)}\sum_{\tau=c}^{t}(t-\sigma(\tau))^{\overline{n-\zeta-1}}F(\tau)\right],\end{aligned} \tag{4}$$

where F is defined on $\mathbb{N}_\alpha$.
Lemma 1. (Linearity). Let F and G be analytic and single-valued functions. Then

$$[c_1 F(t) + c_2 G(t)]_\zeta = c_1 F_\zeta(t) + c_2 G_\zeta(t), \tag{5}$$

where c_1 and c_2 are constants, $\zeta \in \mathbb{R};\ t \in \mathbb{C}$.
Lemma 2. (Index law). Let ϕ be an analytic and single-valued function. The following equality holds

$$\left(F_\zeta(t)\right)_\eta = F_{\zeta+\eta}(t) = \left(F_\eta(t)\right)_\zeta \ \left(F_\zeta(t) \neq 0;\ F_\eta(t) \neq 0;\ \zeta, \eta \in \mathbb{R};\ t \in \mathbb{C}\right). \tag{6}$$

Lemma 3. (Leibniz Rule). Suppose that F and G are analytic and single-valued functions. Then

$$\nabla_0^\zeta (FG)(t) = \sum_{n=0}^{t} \binom{\zeta}{n} \left[\nabla_0^{\zeta-n} F(t-n)\right] \left[\nabla^n G(t)\right], \ \zeta \in \mathbb{R};\ t \in \mathbb{C}, \tag{7}$$

where $\nabla^n G(t) = G_n(t)$ is the ordinary derivative of G of order $n \in \mathbb{N}_0$.
Definition 4. μ shift operator is given by

$$\mu^n F(t) = F(t-n) \tag{8}$$

where $n \in \mathbb{N}$.

3 Main results

Theorem 1. Let $F \in \{F : 0 \neq |F_\upsilon| < \infty;\ \upsilon \in \mathrm{R}\}$. Then the following homogeneous ordinary differential equation:

$$s(1-s)\ F_2 + [(\alpha - 2\gamma)\ s + \gamma + \sigma]\ F_1 + \gamma(\alpha - \gamma + 1)\ F = 0, \quad (s \in \mathrm{C}\backslash\{0, 1\}), \tag{9}$$

has particular solutions of the forms:

$$F = k\left\{s^{-(\upsilon\tau+\gamma+\sigma)}(1-s)^{-(\upsilon\tau+\gamma-\alpha-\sigma)}\right\}_{-(1+\upsilon)}, \tag{10}$$

and

$$F = ks^{1-(\gamma+\sigma)}\left\{s^{-(\upsilon\tau-\gamma-\sigma+2)}(1-s)^{-(\upsilon\tau+\gamma-\alpha-\sigma)}\right\}_{-(1+\upsilon)} \tag{11}$$

where $F_n = d^n F/ds^n\ (n = 0, 1, 2)$, $F_0 = F = F(s)$, $\alpha \neq 0$, γ, σ are given constants, k is an arbitrary constant and τ is a shift operator [15].
Proof. ***(i)*** When we operate ∇^υ to the both sides of (9), we readily obtain;

$$\nabla^\upsilon [F_2 s(1-s)] + \nabla^\upsilon \{F_1 [(\alpha - 2\gamma)\ s + \gamma + \sigma]\} + \nabla^\upsilon [F\gamma(\alpha - \gamma + 1)] = 0. \tag{12}$$

Using $(5) - (7)$ we have

$$\nabla^\upsilon [F_2 s(1-s)] = F_{2+\upsilon} s(1-s) + F_{1+\upsilon}\upsilon\tau(1-2s) - F_\upsilon \upsilon(\upsilon-1)\ \tau^2 \tag{13}$$

and

$$\nabla^\upsilon \{F_1 [(\alpha - 2\gamma)\ s + \gamma + \sigma]\} = F_{1+\upsilon} [(\alpha - 2\gamma)\ s + \gamma + \sigma] + F_\upsilon \upsilon\tau(\alpha - 2\gamma) \tag{14}$$

where τ is a shift operatÃűr. By substituting (13), (14) into the (12), we obtain

$$F_{2+\upsilon} s(1-s) + F_{1+\upsilon}[\upsilon\tau(1-2s) + (\alpha - 2\gamma)\ s + \gamma + \sigma]$$

$$+F_\upsilon\left[\upsilon(1-\upsilon)\ \tau^2 + \upsilon\tau(\alpha - 2\gamma) + \gamma(\alpha - \gamma + 1)\right] = 0. \tag{15}$$

Choose υ such that

$$\upsilon(1-\upsilon)\ \tau^2 + \upsilon\tau(\alpha - 2\gamma) + \gamma(\alpha - \gamma + 1) = 0,$$

$$\upsilon = \left[(\tau + \alpha - 2\gamma) \pm \sqrt{(\tau + \alpha - 2\gamma)^2 + 4\gamma(\alpha - \gamma + 1)}\right]/2\tau. \tag{16}$$

From Eq. (16), one can easily see that

$$\left[(\tau + \alpha - 2\gamma)^2 \geq 4\gamma(-\alpha + \gamma - 1)\right],$$

we have then

$$F_{2+\upsilon} s(1-s) + F_{1+\upsilon}[\upsilon\tau(1-2s) + (\alpha - 2\gamma)\ s + \gamma + \sigma] = 0, \tag{17}$$

from (15) and (16) .
Next, writing:

$$F_{1+\upsilon} = f(s) \quad \left[F = f_{-(1+\upsilon)}\right], \tag{18}$$

we have

$$f_1 + f\left[\frac{\upsilon\tau(1-2s) + (\alpha - 2\gamma)\ s + \gamma + \sigma}{s(1-s)}\right] = 0, \tag{19}$$

from eqs. (17) and (18) . A particular solution of linear ordinary differential equation (19) :

$$f = ks^{-(\upsilon\tau+\gamma+\sigma)}(1-s)^{-(\upsilon\tau+\gamma-\alpha-\sigma)}. \tag{20}$$

Therefore, we obtain (10) from (18) and (20) .
(ii) Set

$$F = s^{\eta}\Phi, \quad \Phi = \Phi(s). \tag{21}$$

The first and second derivatives of (21) are acquired as follows:

$$F_1 = \eta s^{\eta-1}\Phi + s^{\eta}\Phi_1 \tag{22}$$

and

$$F_2 = \eta(\eta-1)\ s^{\eta-2}\Phi + 2\eta s^{\eta-1}\Phi_1 + s^{\eta}\Phi_2. \tag{23}$$

Substitute $(21) - (23)$ into (9) , we obtain

$$s(1-s)\ \Phi_2 + [(1-s)\ 2\eta + (\alpha - 2\gamma)\ s + \gamma + \sigma]\ \Phi_1$$

$$+\left[\left(\left(\eta^2-\eta\right) + (\gamma+\sigma)\ \eta\right)\ s^{-1} - \left(\eta^2-\eta\right) + (\alpha-2\gamma)\ \eta + \gamma(\alpha-\gamma+1)\right]\ \Phi{=}0. \tag{24}$$

Choose η such that

$$\left(\eta^2-\eta\right) + (\gamma+\sigma)\ \eta = 0,$$

that is

$$\eta = 0, \quad \eta = 1 - (\gamma+\sigma).$$

In the case $\eta = 0$, we have the same results as i.

Let $\eta = 1 - (\gamma+\sigma)$. From (21) and (24) , we have

$$F = s^{1-(\gamma+\sigma)}\Phi \tag{25}$$

and

$$s(1-s)\ \Phi_2 + [(2\sigma+\alpha-2)\ s - (\gamma+\sigma-2)]\ \Phi_1 + [(1-\sigma)\ (\sigma+\alpha)]\ \Phi = 0 \tag{26}$$

respectively.
Applying the discrete operator ∇^{υ} to both sides of (26) , we obtain

$$\Phi_{2+\upsilon}s(1-s) + \Phi_{1+\upsilon}\left[\upsilon\tau(1-2s) + (2\sigma+\alpha-2)\ s - (\gamma+\sigma-2)\right]$$

$$+\Phi_{\upsilon}\left[\upsilon(1-\upsilon)\ \tau^2 + \upsilon\tau(2\sigma+\alpha-2) + (1-\sigma)\ (\alpha+\sigma)\right] {=}0. \tag{27}$$

Choose υ such that

$$\upsilon(1-\upsilon)\ \tau^2 + \upsilon\tau(2\sigma+\alpha-2) + (1-\sigma)\ (\alpha+\sigma)\ {=}0,$$

$$\upsilon = \left[(\tau+2\sigma+\alpha-2) \pm \sqrt{(\tau+2\sigma+\alpha-2)^2 - 4(\sigma-1)\ (\alpha+\sigma)}\right]/2\tau. \tag{28}$$

From Eq. (28), one can get

$$\left[(\tau+2\sigma+\alpha-2)^2 \geq 4(\sigma-1)\ (\alpha+\sigma)\right],$$

then we have

$$\Phi_{2+\upsilon}s(1-s) + \Phi_{1+\upsilon}\left[\upsilon\tau(1-2s) + (2\sigma+\alpha-2)\ s - (\gamma+\sigma-2)\right] = 0, \tag{29}$$

from (27) and (28).

Next, by writing

$$\Phi_{1+\upsilon} = g\left(s\right), \quad \left[\Phi = g_{-(1+\upsilon)}\right], \tag{30}$$

we have

$$g_1 + g\left[\frac{\upsilon\tau\left(1-2s\right)+\left(2\sigma+\alpha-2\right)\, s-\left(\gamma+\sigma-2\right)}{s\left(1-s\right)}\right] = 0, \tag{31}$$

from (29) and (30) . A particular solution to this linear differential equation is given by

$$g = ks^{-(\upsilon\tau-\gamma-\sigma+2)}(1-s)^{-(\upsilon\tau+\gamma-\alpha-\sigma)}. \tag{32}$$

Thus we obtain the solution (11) from (25), (30) and (32).

4 Conclusion

In this article, we applied the nabla operator of discrete fractional analysis to the second order linear differential equations. We obtained the discrete fractional solutions of these equations via this new operator method.

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