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ON THE BOUNDS FOR THE NORMS OF TOEPLITZ MATRICES WITH THE JACOBSTHAL AND JACOBSTHAL LUCAS NUMBERS

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Abstract

In this study, we compute the value of the various norms of the Toeplitz matrices whose elements are Jacobsthal numbers, Jacobsthal Lucas numbers and upper and lower bounds for the spectral norms of these matrices. Also, the Euclidean norm of Kronecker product of Toeplitz matrices with Jacobsthal and the Jacobsthal Lucas numbers are denoted. Finally, the upper bound for the spectral norm of Hadamard product mentioned above matrices are demonstrated.

Keywords: Jacobsthal, Jacobsthal Lucas sequences, Toeplitz matrices, Hadamard product, Kronecker product

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1. Introduction and preliminaries

Jacobsthal and Jacobsthal Lucas numbers are famous special integer sequences. Therefore, there have been many studies about their generalizations in recent years. For example, in [2], by using two parameters, a new generalization of these numbers were studied. In [1], the Jacobsthal sequence $\{j_n\}_{n\in N}$, and the Jacobsthal Lucas sequence $\{c_n\}_{n\in N}$ are defined, respectively, by

$$j_n = j_{n-1} + 2j_{n-2}, \qquad j_0 = 0,\ \ j_1 = 1, \qquad n \geq 2,$$

$$c_n = c_{n-1} + 2c_{n-2}. \qquad c_0 = 2,\ \ c_1 = 1, \qquad n \geq 2.$$

The first some Jacobsthal numbers are 0, 1, 1, 3, 5, 11 … and the first some Jacobsthal Lucas numbers are 2, 1, 5, 7, 17, 31…

These recurrences involve the characteristic equation

$$x^2 - x - 2 = 0$$

with roots

$$\alpha = 2, \quad \beta = -1.$$

There have been several papers on the norms of special matrices [6-11]. In [6], Solak and Bozkurt have studied the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices. Solak [7] has given some bounds for the circulant matrices whose elements are Fibonacci and Lucas numbers concerned with the spectral and Euclidean norms. In [11], Akbulak and Bozkurt have studied the norms of Toeplitz matrices involving Fibonacci and Lucas numbers. Shen [10] has given upper and lower bounds for the spectral norms of Toeplitz matrices involving k-Fibonacci and k-Lucas numbers. In [9], Daşdemir demonstrated the norms of Toeplitz matrices with the Pell, Pell-Lucas and modified Pell numbers. In this paper, different norms of Toeplitz matrices with the Jacobsthal and the Jacobsthal Lucas numbers are obtained. Then the lower and upper bounds for the spectral norms of Toeplitz matrices $A = T(j_0, j_1, \dots, j_{n-1})$ and $B = T(c_0, c_1, \dots, c_{n-1})$ are investigated.

Binet's formulas for Jacobsthal and Jacobsthal Lucas numbers are denoted, respectively, by

$$j_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

and

$$c_n = \alpha^n + \beta^n. \tag{2}$$

Jacobsthal and Jacobsthal Lucas numbers with negative indices are computed as

$$j_{-n} = \frac{(-1)^{n+1}}{2^n} j_n, \qquad c_{-n} = \frac{(-1)^n}{2^n} c_n. \tag{3}$$

A matrix $T = [t_{ij}] \epsilon M_n(C)$ is called a Toeplitz matrix if it is of the form $t_{ij} = t_{i-j}$ for i, j=1,2,…n

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{bmatrix}. \tag{4}$$

For any $A = [a_{ij}] \epsilon M_{m,n}(C)$, the maximum absolute column sum (1-norm) and the maximum absolute row sum (∞-norm) are given, respectively, by

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|,$$

$$\|A\|_{\infty} = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

For any $A = [a_{ij}] \epsilon M_{m,n}(C)$, the Frobenious (or Euclidean) norm of matrix A is

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}, \tag{5}$$

and the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1\le i\le m} |\lambda_i(A^H A)|} \tag{6}$$

where A^H is defined as the conjugate transpose of matrix A and $\lambda_i(A^H A)$ is eigenvalue of $A^H A$.

The maximum column length norm $c_1(A)$ and the maximum row length norm $r_1(A)$ of matrix of order $n\ x\ n$ are defined as

$$c_1(A) = \max_j \sqrt{\sum_{i=1}^{n} |a_{ij}|^2},$$
$$r_1(A) = \max_i \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}.$$

For any $A, B \epsilon M_{m,n}(C)$, the Hadamard product of A, B is defined by [4,5]

$$A \circ B = (a_{ij} b_{ij}),$$

and have the following properties

$$\|AoB\| \le r_1(A) c_1(B), \qquad \|AoB\| \le \|A\| \|B\|. \tag{7}$$

Let $A \epsilon M_{m,n}(C)$, $B \epsilon M_{p,q}(C)$, be given, then the Kronecker product of A, B is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

and have the following property [12]

$$\|A \otimes B\|_E = \|A\|_E \|B\|_E . \tag{8}$$

For any $A \epsilon M_{m,n}(C)$, then the inequality holds [3]

$$\frac{1}{\sqrt{n}} \|A\|_E \le \|A\|_2 \le \|A\|_E. \tag{9}$$

2. Some sum formulas for Jacobsthal numbers and Jacobsthal Lucas numbers

The summation formulas for the Jacobsthal and Jacobsthal Lucas sequences are

$$\sum_{k=0}^{n} j_k = \frac{j_{n+1} + 2j_n - 1}{2}, \tag{10}$$

$$\sum_{k=0}^{n} c_k = \frac{c_{n+2} - 1}{2} \ . \tag{11}$$

The summation formulas for the Jacobsthal sequence with different indices are

$$\sum_{k=1}^{n-1} j_{2k} = \frac{4j_{2n-2} - n + 1}{3}, \tag{12}$$

$$\sum_{k=1}^{n-1} j_{2k+2} = \frac{16j_{2n-2} - n + 1}{3}. \tag{13}$$

The summation of the squares of Jacobsthal sequence and Jacobsthal sequence with negative indices are denoted by using Jacobsthal numbers as the following:

$$\sum_{k=1}^{n} j_k^2 = \frac{1}{9}[j_{2n+2} + 2(-1)^{n+1} j_{n+1} + n + 1] \,, \tag{14}$$

$$\sum_{k=1}^{n} (j_{-k})^2 = \sum_{k=1}^{n} \left(\frac{j_k}{2^k}\right)^2 = \frac{1}{9}\left(\frac{j_{2n}}{2^{2n}} + 2\frac{j_n}{2^n} + n\right). \tag{15}$$

The summation of the squares of Jacobsthal Lucas sequence and Jacobsthal Lucas sequence with negative indices are demonstrated by using Jacobsthal numbers as the following:

$$\sum_{k=0}^{n-1} c_k^2 = j_{2n} - 2(-1)^n j_n + n, \tag{16}$$

$$\sum_{k=1}^{n} c_{-k}^2 = \frac{j_{2n}}{2^{2n}} - 2\frac{j_n}{2^n} + n. \tag{17}$$

For the alternating sums of the Jacobsthal sequence, the following equalities hold:

$$\sum_{k=1}^{n-1} (-1)^{k+1} j_{k+1} = \frac{4(-1)^n j_{n-1} - n + 1}{3}, \tag{18}$$

$$\sum_{k=1}^{n-1}(-1)^k j_k = -\frac{2(-1)^n j_{n-1} - n + 1}{3}. \tag{19}$$

The different summation formulas of the Jacobsthal sequence are demonstrated by using Jacobsthal numbers as the following:

$$\sum_{k=1}^{n-1}\frac{j_{2k}}{2^{2k}} = \frac{1}{3}\left(n - 1 - \frac{j_{2n-2}}{2^{2n-2}}\right) \tag{20}$$

$$\sum_{k=1}^{n-1}\frac{j_k}{2^k} = \frac{1}{3}\left(n - 1 + \frac{j_{n-1}}{2^{n-1}}\right). \tag{21}$$

3. Lower and Upper Bounds of Toeplitz Matrices involving Jacobsthal numbers and Jacobsthal Lucas numbers

Theorem 3.1 Let $A = T(j_0, j_1, \dots, j_{n-1})$ be a Toeplitz matrix, then the maximum absolute column sum norm (1-norm) and the maximum absolute row sum norm (∞-norm) of matrix A are

$$\|A\|_1 = \|A\|_\infty = \frac{j_n + 2j_{n-1} - 1}{2}.$$

Proof. Clearly, the explicit form of Toeplitz matrix is as follows:

$$A = \begin{bmatrix} j_0 & j_{-1} & j_{-2} & \cdots & j_{1-n} \\ j_1 & j_0 & j_{-1} & \cdots & j_{2-n} \\ j_2 & j_1 & j_0 & \cdots & j_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_{n-1} & j_{n-2} & j_{n-3} & \cdots & j_0 \end{bmatrix}. \tag{22}$$

By the definitions of 1-norm and ∞-norm, and (10), it is easily seen:

$$\|A\|_1 = \max_j \sum_{i=1}^{n}\left|a_{ij}\right| = \sum_{i=1}^{n}\left|a_{i1}\right| = \sum_{k=0}^{n-1} j_k = \frac{j_n + 2j_{n-1} - 1}{2},$$

$$\|A\|_\infty = \max_i \sum_{i=1}^{n}\left|a_{ij}\right| = \sum_{j=1}^{n}\left|a_{nj}\right| = \sum_{k=0}^{n-1} j_k = \frac{j_n + 2j_{n-1} - 1}{2}.$$

Theorem 3.2 Let $A = T(j_0, j_1, \dots, j_{n-1})$ be a Toeplitz matrix, then the Frobenious (or Euclidean) norm of matrix A is

$$\|A\|_E = \frac{1}{3}\sqrt{\frac{16j_{2n-2} + 8(-1)^n j_{n-1}}{3} - \frac{j_{2n-2}}{3.\,2^{2n-2}} + \frac{j_{n-1}}{3.2^{n-2}} + n^2 - 1}. \tag{23}$$

Proof. Let A be a $n\ x\ n$ matrix as in (22). Then by (5), (14), (15)

$$\|A\|_E{}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = nj_0{}^2 + \sum_{k=1}^{n-1}(n-k)j_k{}^2 + \sum_{k=1}^{n-1}(n-k)j_{-k}{}^2$$
$$= \sum_{k=1}^{n-1}\sum_{j=1}^{k} j_i{}^2 + \sum_{k=1}^{n-1}\sum_{j=1}^{k}\left(\frac{j_i}{2^i}\right)^2$$
$$= \frac{1}{9}\sum_{k=1}^{n-1}[j_{2k+2} + 2(-1)^{k+1}j_{k+1} + k + 1] + \frac{1}{9}\sum_{k=1}^{n-1}\left[\frac{j_{2k}}{2^{2k}} + 2\frac{j_k}{2^k} + k\right]$$

and then by using the sum formulas (13), (18), (20), (21) the following result is obtained:

$$\|A\|_E{}^2 = \frac{1}{9}\left[\frac{16j_{2n-2} - n + 1}{3} + 2\frac{4(-1)^n j_{n-1} - n + 1}{3} + \frac{(n-1)n}{2} + n - 1\right]$$
$$+ \frac{1}{9}\left[\frac{1}{3}\left(n - 1 - \frac{j_{2n-2}}{2^{2n-2}}\right) + \frac{2}{3}\left(n - 1 + \frac{j_{n-1}}{2^{n-1}}\right) + \frac{(n-1)n}{2}\right]$$
$$= \frac{1}{9}\left[\frac{16j_{2n-2} + 8(-1)^n j_{n-1}}{3} - \frac{j_{2n-2}}{3.\,2^{2n-2}} + \frac{j_{n-1}}{3.2^{n-2}} + n^2 - 1\right].$$

Theorem 3.3 Let $A = T(j_0, j_1, \dots, j_{n-1})$ be a Toeplitz matrix, then the lower and upper bounds for the spectral norm of A are obtained as

$$\frac{1}{3}\sqrt{\frac{16j_{2n-2} + 8(-1)^n j_{n-1}}{3n} - \frac{j_{2n-2}}{3n.\,2^{2n-2}} + \frac{j_{n-1}}{3n.\,2^{n-2}} + \frac{n^2-1}{n}} \le \|A\|_2$$

$$\|A\|_2 \le \frac{1}{9}\sqrt{(j_{2n-2} + 2(-1)^{n-1}j_{n-1} + n)(j_{2n} + 2(-1)^n j_n + n)}.$$

Proof. From (23), we get

$$\|A\|_E{}^2 = \frac{1}{9}\left[\frac{16j_{2n-2} + 8(-1)^n j_{n-1}}{3} - \frac{j_{2n-2}}{3.\,2^{2n-2}} + \frac{j_{n-1}}{3.2^{n-2}} + n^2 - 1\right].$$

Then by using the property (9), the left-hand side of the inequality is completed. The explicit form of this matrix is as follows:

$$\frac{1}{3}\sqrt{\frac{16j_{2n-2} + 8(-1)^n j_{n-1}}{3n} - \frac{j_{2n-2}}{3.\,2^{2n-2}} + \frac{j_{n-1}}{3n.\,2^{n-2}} + \frac{n^2-1}{n}} = \frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2.$$

On the other hand, let $A = B \circ C$ whereas

$$B = b_{ij} = \begin{cases} b_{ij} = 1, & j = 1, \\ b_{ij} = j_{i-j}, & j \neq 1, \end{cases}$$

$$C = c_{ij} = \begin{cases} c_{ij} = j_{i-j}, & j = 1, \\ c_{ij} = 1, & j \neq 1. \end{cases}$$

The explicit form of this matrix is as follows:

$$B = \begin{bmatrix} 1 & j_{-1} & j_{-2} & \cdots & j_{1-n} \\ 1 & j_0 & j_{-1} & \cdots & j_{2-n} \\ 1 & j_1 & j_0 & \cdots & j_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & j_{n-2} & j_{n-3} & \cdots & j_0 \end{bmatrix}, \qquad C = \begin{bmatrix} j_0 & 1 & 1 & \cdots & 1 \\ j_1 & 1 & 1 & \cdots & 1 \\ j_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_{n-1} & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then, by using the sum formula (14) and the definition of the maximum column length norm and the maximum row length norm, the following equalities hold:

$$r_1(B) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} \left|b_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n} \left|b_{nj}\right|^2} = \sqrt{\sum_{k=0}^{n-2} {j_k}^2 + 1} = \frac{1}{3}\sqrt{j_{2n-2} + 2(-1)^{n-1} j_{n-1} + n}$$

$$c_1(C) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} \left|c_{ij}\right|^2} = \sqrt{\sum_{i=1}^{n} \left|c_{j1}\right|^2} = \sqrt{\sum_{k=0}^{n-1} {j_k}^2} = \frac{1}{3}\sqrt{j_{2n} + 2(-1)^{n} j_{n} + n}.$$

By using the property (7), the right-hand side of the inequality is completed as

$$r_1(B)c_1(C) = \frac{1}{9}\sqrt{(j_{2n-2} + 2(-1)^{n-1} j_{n-1} + n)(j_{2n} + 2(-1)^{n} j_{n} + n)}.$$

Theorem 3.4 Let the elements of the Toeplitz matrix be Jacobsthal Lucas numbers, $A = T(c_0, c_1, \dots, c_{n-1})$ then 1-norm, ∞-norm of A are

$$\|A\|_1 = \|A\|_\infty = \frac{c_{n+2}-1}{2}.$$

Proof. Clearly, the explicit form of this matrix as follows:

$$A = \begin{bmatrix} c_0 & c_{-1} & c_2 & \cdots & c_{1-n} \\ c_1 & c_0 & c_{-1} & \cdots & c_{2-n} \\ c_2 & c_1 & c_0 & \cdots & c_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix}. \tag{26}$$

By the definitions of 1-norm and ∞-norm, and (11), it is easily seen:

$$\|A\|_1 = \max_j \sum_{i=1}^{n} \left|a_{ij}\right| = \sum_{i=1}^{n} \left|a_{i1}\right| = \sum_{k=0}^{n-1} c_k = \frac{c_{n+1}-1}{2},$$

$$\|A\|_\infty = \max_i \sum_{i=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{nj}| = \sum_{k=0}^{n-1} c_k = \frac{c_{n+1}-1}{2}.$$

Theorem 3.5 Let $A = T(c_0, c_1, \dots, c_{n-1})$ be a Toeplitz matrix with Jacobsthal Lucas numbers, then the Frobenious (or Euclidean) norm of matrix A is

$$\|A\|_E = \sqrt{\frac{1}{3}\left(16j_{2n-2} + 8(-1)^n j_{n-1} - \frac{j_{2n-2}}{2^{2n-2}} - \frac{j_{n-1}}{2^{n-2}} + 13 - 7n + 3n^2\right)}. \qquad (27)$$

Proof. Let A be an *n x n* matrix as in (26). Then by (5), (16), (17)

$$\begin{aligned}
\|A\|_E{}^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = nc_0{}^2 + \sum_{k=1}^{n-1}(n-k)c_k{}^2 + \sum_{k=1}^{n-1}(n-k)c_{-k}{}^2 \\
&= 4n + \sum_{k=1}^{n-1}\sum_{i=1}^{k} c_i{}^2 + \sum_{k=1}^{n-1}\sum_{i=1}^{k}\left(\frac{c_i}{2^i}\right)^2 \\
&= 4n + \sum_{k=1}^{n-1}\left([j_{2k+2} + 2(-1)^{k+1}j_{k+1} + k - 3] + \left[\frac{j_{2k}}{2^{2k}} - 2\frac{j_k}{2^k} + k\right]\right).
\end{aligned}$$

Then by using the sum formulas (13), (19), (20), (21) the following result is obtained:

$$\begin{aligned}
\|A\|_E{}^2 &= 4n + \frac{1}{3}(16j_{2n-2} - n + 1) + \frac{2}{3}(4(-1)^n j_{n-1} - n + 1) + \frac{n(n-1)}{2} - 3n \\
&\quad + 3 + \frac{1}{3}\left(n - 1 - \frac{j_{2n-2}}{2^{2n-2}}\right) - \frac{2}{3}\left(n - 1 + \frac{j_{n-1}}{2^{n-1}}\right) + \frac{n(n-1)}{2} \\
&= \frac{1}{3}\left(16j_{2n-2} + 8(-1)^n j_{n-1} - \frac{j_{2n-2}}{2^{2n-2}} - \frac{j_{n-1}}{2^{n-2}} + 13 - 7n + 3n^2\right).
\end{aligned}$$

Theorem 3.6 Let $A = T(c_0, c_1, \dots, c_{n-1})$ be a Toeplitz matrix, then the lower and upper bounds for the spectral norm of A are obtained as

$$\sqrt{\frac{1}{3n}\left(16j_{2n-2} + 8(-1)^n j_{n-1} - \frac{j_{2n-2}}{2^{2n-2}} - \frac{j_{n-1}}{2^{n-2}} + 13 - 7n + 3n^2\right)} \le \|A\|_2$$

$$\|A\|_2 \le \sqrt{(j_{2n-2} + 4(-1)^{n-1}j_{n-1} + n)(j_{2n} - 2(-1)^n j_n + n)}.$$

Proof. From (27), we get

$$\|A\|_E{}^2 = \frac{1}{3}\left(16j_{2n-2} + 8(-1)^n j_{n-1} - \frac{j_{2n-2}}{2^{2n-2}} - \frac{j_{n-1}}{2^{n-2}} + 13 - 7n + 3n^2\right).$$

Then by using the property (9), the left-hand side of the inequality is completed as

$$\sqrt{\frac{1}{3n}\left(16j_{2n-2} + 8(-1)^n j_{n-1} - \frac{j_{2n-2}}{2^{2n-2}} - \frac{j_{n-1}}{2^{n-2}} + 13 - 7n + 3n^2\right)} = \frac{1}{\sqrt{n}}\|A\|_E = \|A\|_2.$$

On the other hand, let $A = B \circ C$ where B, C are

$$B = (b_{ij}) = \begin{cases} b_{ij} = 1 & j = 1, \\ b_{ij} = c_{i-j} & j \neq 1, \end{cases}$$

$$C = (c_{ij}) = \begin{cases} c_{ij} = c_{i-j} & j = 1, \\ c_{ij} = 1 & j \neq 1. \end{cases}$$

The explicit form of this matrix is as follows:

$$B = \begin{bmatrix} 1 & c_{-1} & c_{-2} & \cdots & c_{1-n} \\ 1 & c_0 & c_{-1} & \cdots & c_{2-n} \\ 1 & c_1 & c_0 & \cdots & c_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 & 1 & 1 & \cdots & 1 \\ c_1 & 1 & 1 & \cdots & 1 \\ c_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then, by using the sum formula (16) and the definition of the maximum column length norm and the maximum row length norm, the following equalities hold:

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{m} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{nj}|^2} = \sqrt{\sum_{k=0}^{n-2} c_k^2 + 1} = \sqrt{j_{2n-2} - 2(-1)^{n-1} j_{n-1} + n},$$

$$c_1(C) = \max_{1 \leq i \leq n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |c_{j1}|^2} = \sqrt{\sum_{k=0}^{n} c_k{}^2} = \sqrt{j_{2n} - 2(-1)^n j_n + n}.$$

By using the property (7), the right-hand side of the inequality is completed as

$$r_1(B)\, c_1(C) = \sqrt{(4j_{2n-2} + 4(-1)^{n-1} j_{n-1} + n + 1)(4j_{2n} + 4(-1)^n j_n + n)}.$$

Corollary 3.7 Let $A = C(j_0, j_1, \dots, j_{n-1})$ and $B = C(c_0, c_1, \dots, c_{n-1})$ be Toeplitz matrices with Jacobsthal and the Jacobsthal Lucas numbers, then the Euclidean norm of Kronecker product of these matrices is

$$\|A \otimes B\|_E = \frac{1}{3} \sqrt{\frac{16 j_{2n-2} + 8(-1)^n j_{n-1}}{3} - \frac{j_{2n-2}}{3.\,2^{2n-2}} + \frac{j_{n-1}}{3.2^{n-2}} + n^2 - 1}$$
$$\times \sqrt{\frac{1}{3}\left(16 j_{2n-2} + 8(-1)^n j_{n-1} - \frac{j_{2n-2}}{2^{2n-2}} - \frac{j_{n-1}}{2^{n-2}} + 13 - 7n + 3n^2\right)}.$$

Proof. By using (8), (23), (27), the proof is easily seen.

Theorem 3.8 Let $A = C(j_0, j_1, \dots, j_{n-1})$ and $B = C(c_0, c_1, \dots, c_{n-1})$ be Toeplitz matrices with Jacobsthal and the Jacobsthal Lucas numbers, then the upper bound for any norm of Hadamard product of these matrices is

$$\|A \circ B\|_E \leq \frac{1}{3}\sqrt{\frac{16j_{2n-2}+8(-1)^n j_{n-1}}{3}-\frac{j_{2n-2}}{3.\,2^{2n-2}}+\frac{j_{n-1}}{3.2^{n-2}}+n^2-1}$$
$$\times\sqrt{\frac{1}{3}\left(16j_{2n-2}+8(-1)^n j_{n-1}-\frac{j_{2n-2}}{2^{2n-2}}-\frac{j_{n-1}}{2^{n-2}}+13-7n+3n^2\right)}.$$

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Proof. The proof is seen easily by using $\|A \circ B\| \leq \|A\|\|B\|$ and (23), (27).

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