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Obtaining New Fixed Point Theorems By Using Generalized Banach-Contraction Principle

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Abstract: In this study, a new three step iterative algorithm was introduced with the help of Jungck-contraction principle which is one of the remerkable generalizations of Banach-contraction principle. Also, the convergence and stability results were obtained for the pair of nonself mappings which satisfy a certain contractive condition by using this iterative algorithm in any Banach space. In addition, it was shown that the new iterative algorithm has a better convergence speed when compared the other Jungck-type iterative algorithms in the current literature, and to support this result, numerical examples were given.

Genelleştirilmiş Banach-Büzülme Prensibi Kullanılarak Yeni Sabit Nokta Teoremlerinin Elde Edilmesi

Anahtar Kelimeler
Jungck-Büzülme Prensibi,
Yeni İterasyon Algoritması,
Yakınsaklık,
Kararlılık

Öz: Bu çalışmada, Banach-büzülme prensibinin dikkate değer genellemelerinden biri olan Jungck-büzülme prensibi yardımıyla yeni üç adımlı iterasyon algoritması tanımlanmıştır. Ayrıca bu iterasyon algoritması kullanılarak, kendi üzerine olmayan ve belirli bir büzülme şartını sağlayan dönüşüm çifti için herhangi bir Banach uzayında yakınsaklık ve kararlılık sonuçları elde edilmiştir. Ek olarak, tanımlanan yeni algoritmanın literatürde bulunan diğer Jungck-tipindeki algoritmalarla kıyaslandığında yakınsama anlamında daha hızlı olduğu gösterilmiş ve bu sonucu destekleyen nümerik örnekler verilmiştir.

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1. Introduction

Fixed point theory has become an interesting and fundamental subject of nonlinear functional analysis with its wide range of applications in various fields of mathematics (differential and integral equations, linear algebra, approximation theory, control theory, game theory, etc.). This theory also has very fruitful applications in almost all branches of science such as chemistry, biology, statistics, computer science, engineering and economics. Particularly in the last fifty years, fixed point theory has been one of the most active areas of research that has rised on the basis of analysis and topology and today in this sense continues to attract the attention of many researchers as a dynamic field of study.

Let X is nonempty set. The point that provides the equation $Tx = x$ for the T mapping is called the fixed point of T. Geometrically, this means the points on the $y = x$ line. The basic idea in fixed point theory is to find the x point that provides the equation given above. Theorems constructed for the existence and uniqueness of the fixed point are called fixed point theorems.

The Banach contraction principle, which is one of the most famous theorems in fixed point theory, formulated and proved by Banach [1] guarantees the existence and uniqueness of a fixed point of a mapping defined on a complete metric space under appropriate conditions.

Also, it says that the sequence obtained from the Picard iteration will converge to this fixed point. Because of its simplicity and usefulness, this theorem has become a popular tool in the search for fixed points. Later, this theorem was extended and generalized in many ways by many authors (see [2]-[5]).However, the Banach contraction principle cannot guarantee the convergence of the sequence obtained from Picard iteration for non-expansive mappings. Therefore, new iterative algorithms are defined and fixed point theorems are obtained for different mapping classes in many spaces from Hilbert spaces to metric spaces (see [6]- [11]).

When defining an iterative algorithm, it is important that it must be faster and simpler in terms of convergence. Moving from this point, a new iterative algorithm of Jungck-type is defined in this study and convergence is obtained in Banach spaces. In addition, stability has been proved by using the newly defined iterative algorithm for a more general mapping class than the Jungck-contraction mapping. Finally, it has been shown that the new algorithm is faster than the other Jungck type algorithms in the literature.

2. Material and Method

One of the most important generalizations of the Banach contraction principle is obtained by Jungck [4] using commutative mappings as follows: In this paper we assume that X is a Banach space Y an arbitrary set and $S, T: Y \to X$ such that $T(Y) \subseteq S(Y)$.

Theorem 2.1. Suppose (X, d) is a complete metric space. Let f and g be two functions such that $f, g: X \to X$ satisfy the following conditions for all $x, y \in X$:

i. (f, g) are commutative mapping pair
ii. g is continuous
iii. $f(X) \subset g(X)$
iv. $d(fx, fy) \leq k.d(gx, gy)$ such that $k \in [0,1)$.

Then f and g have a unique common fixed point such as $p \in X$.

In this theorem, the condition given by (iv) is called the Jungck-contraction mapping. Furthermore, if $g(x) = x$ this theorem reduces to classical Banach contraction principle. As a result of this theorem, the following iterative algorithm is defined by Jungck [4]:

$$Sx_{n+1} = Tx_n \tag{1}$$

for all $n \in \mathbb{N}$. This equation is called Jungck iterative algorithm. If $S = I$ (unit mapping) and $Y = X$ in the above equation, it is clear that classical Picard iterative algorithm is obtained.

This approach, introduced by Jungck, has paved the way for many researchers to rewrite classical iterative algorithms in Jungck-type to obtain various types of fixed point theorems. Some of the studies carried out with this approach are as follows:

In 2005, Singh et al. [13] defined Jungck-Mann (JM) iterative algorithm as follows:

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n \tag{2}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$.

Jungck-Ishikawa (JI) and Jungck-Noor (JN) iterative algorithms were defined as follow respectively:

$$\begin{cases} Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n = (1 - \beta_n)Sx_n + \beta_n Tx_n \end{cases} \tag{3}$$

and

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)Sx_n + \beta_n Tz_n \\ Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ (see [14], [15]).

In 2011, Jungck-SP (JSP) iterative algorithm was defined as follows:

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sy_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)Sz_n + \beta_n Tz_n \\ Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ (see [16]).

Jungck-CR (JCR) iterative algorithm were defined by Hussain et al. as follow:

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sy_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)Tx_n + \beta_n Tz_n \\ Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{6}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ (see [17]).

In 2014, Khan et al. defined Jungck-Khan (JK) iterative algorithm as follow:

$$\begin{cases} Su_{n+1} = (1-\alpha_n-\beta_n)Su_n + \alpha_n Tv_n + \beta_n Tu_n \\ Sv_n = (1-b_n-c_n)Su_n + b_n Tw_n + c_n Tu_n \\ Sw_n = (1-a_n)Su_n + a_n Tu_n \end{cases} \tag{7}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty} \subset [0,1]$ (see [18]).

Remark 2.2.

- Putting $S = I$ and $Y = X$ in the JM iterative algorithm (2), we get classical Mann iterative algorithm [6].
- Putting $S = I$ and $Y = X$ in the JI iterative algorithm (3) and JN iterative algorithm (4), we get classical Ishikawa and classical Noor iterative algorithms, respectively [7], [8].
- Putting $S = I$ and $Y = X$ in the JSP iterative algorithm (5), we get classical SP iterative algorithm [19].
- Putting $S = I$ and $Y = X$ in the JCR iterative algorithm (6), we get classical CR iterative algorithm [20].
- Putting $\beta_n = c_n = 0$ in the JK iterative algorithm (7), we get JN iterative algorithm (4).

Also putting $\alpha_n = 0$ and $\alpha_n = 0, \beta_n = 1$ in the JCR iterative algorithm (6) respectively, we obtain the following Jungck-Agarwal (JA) iterative algorithm [21] and Jungck-Sahu (JS) iterative algorithm [17]:

$$\begin{cases} Sx_{n+1} = (1-\beta_n)Tx_n + \beta_n Ty_n \\ Sy_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{8}$$

and

$$\begin{cases} Sx_{n+1} = Ty_n \\ Sy_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{9}$$

It is shown that the sequence obtained from the iterative algorithm given by (1) converges to the common fixed point of S and T mappings which satisfies the Jungck-contraction condition by Jungck [4]. The following mapping class, which is more general than the Jungck-contraction mapping condition, was described by Olatinwo [14] and some convergence and stability results were obtained for the Jungck-Ishikawa iteration algorithm:

Definition 2.3. The pair $S,T\colon Y \to X$ is called contractive if there exist a real number $\delta \in [0,1)$ and a continuous function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$ and for all $x, y \in Y$, we have

$$\|Tx - Ty\| \leq \phi(\|Sx - Tx\|) + \delta\|Sx - Sy\|. \tag{10}$$

Hussain et al. [17] obtained some convergence and stability results by using this mapping for the JCR iterative algorithm (6) in any Banach space and showed that the convergence rate of this algorithm is better compared to other iterative algorithms.

In the light of the studies mentioned above, we have defined a new Jungck type iterative algorithm (JY) as follows:

$$\begin{cases} Sx_{n+1} = Ty_n \\ Sy_n = (1-\alpha_n)Sz_n + \alpha_n Tz_n \\ Sz_n = Tx_n \end{cases} \tag{11}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$.

Definition 2.4. Let X be a nonempty set and $S,T\colon X \to X$ be mappings.

i. If $Tx = Sx$, then $x \in X$ is called coincidence point of T and S.
ii. If $x = Tx = Sx$, then $x \in X$ is called common fixed point of T and S.
iii. If $p = Tx = Sx$ for some $x \in X$, then p is called the point of coincidence of T and S.
iv. If $TSx = STx$ for all $x \in X$, then a pair (S,T) is called commuting.
v. If $TSx = STx$ whenever $Tx = Sx$ for some $x \in X$, then a pair (S,T) is called weakly compatible [22].

Lemma 2.5. Let $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality

$$\sigma_{n+1} \leq (1-\lambda_n)\sigma_n + \mu_n$$

where $\lambda_n \in (0,1)$ for all $n \geq n_0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\frac{\mu_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \sigma_n = 0$ [23].

Definiton 2.6. Let $\{c_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ be two sequences of real numbers with limits c and d, respectively. Suppose that there exists

$$\lim_{n\to\infty} \frac{|c_n - c|}{|d_n - d|} = l$$

i. If $l = 0$, then we say that $\{c_n\}_{n=0}^{\infty}$ converges faster to c than $\{d_n\}_{n=0}^{\infty}$ to d.
ii. If $0 < l < \infty$, then we say that $\{c_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ have the same rate of convergence [24].

Definition 2.7. Let $S,T\colon Y \to X$, $T(Y) \subseteq S(Y)$ and $p = Tx = Sx$. For any $x_0 \in Y$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the iterative algorithm $Sx_{n+1} = f(T,x_n)$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set $\epsilon_n = d\big(Sy_{n+1}, f(T,y_n)\big), n = 0,1,2,\ldots.$ Then the iterative algorithm $f(T,x_n)$ will be called (S,T)-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} Sy_n = p$ [13].

3. Results

For the sake of simplicity in the rest of this paper, we assume that $S, T: Y \to X$ satisfy contractive condition (10) where $T(Y) \subseteq S(Y)$ such that $S(Y)$ is complete subset of X and $C(S,T)$ denotes the set of coincidence points of S and T.

Theorem 3.1. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (11) with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Assume that there exist a $z \in C(S,T)$ such that $p = Tz = Sz$. Then $\{Sx_n\}_{n=0}^{\infty}$ converges to p. In adittion, p is a unique common fixed point of S and T if $Y = X$ and S and T are weakly compatible.
Proof. By using iterative algorithm (11) and contractive condition (10), we obtain

$$\begin{aligned}\|Sx_{n+1} - p\| = \|Ty_n - p\| &\leq \phi(\|Sz - Tz\|) + \delta\|Sz - Sy_n\| \\ &= \delta\|Sy_n - p\|\end{aligned} \tag{12}$$

Also

$$\begin{aligned}\|Sy_n - p\| &= \|(1-\alpha_n)Sz_n + \alpha_n Tz_n - p\| \\ &\leq (1-\alpha_n)\|Sz_n - p\| + \alpha_n\|Tz_n - p\| \\ &\leq (1-\alpha_n)\|Sz_n - p\| + \alpha_n\{\phi(\|Sz - Tz\|) + \delta\|Sz - Sz_n\|\} \\ &= [1 - \alpha_n(1-\delta)]\|Sz_n - p\|\end{aligned} \tag{13}$$

and

$$\begin{aligned}\|Sz_n - p\| = \|Tx_n - p\| &\leq \phi(\|Sz - Tz\|) + \delta\|Sz - Sx_n\| \\ &= \delta\|Sx_n - p\|.\end{aligned} \tag{14}$$

Substituting (14) in (13) and (13) in (12) respectively, we have

$$\|Sx_{n+1} - p\| \leq \delta^2[1 - \alpha_n(1-\delta)]\|Sx_n - p\| \tag{15}$$

Then

$$\begin{aligned}\|Sx_{n+1} - p\| &\leq \prod_{k=0}^{n} \delta^2[1 - \alpha_k(1-\delta)]\|Sx_0 - p\| \\ &\leq \frac{\delta^{2n}}{e^{(1-\delta)\sum_{k=0}^{n}\alpha_k}}\|Sx_0 - p\|\end{aligned} \tag{16}$$

Taking the limit in both sides of the above inequality, it can be seen that $\lim_{n\to\infty}\|Sx_n - p\| = 0$.

Now we show that p is a unique common fixed point of S and T when $Y = X$. Assume that there exist another point of coincide p_* of the pair (S,T). Then there exist $z_* \in C(S,T)$ such that $Sz_* = Tz_* = p_*$. By using inequality (10), we get

$$0 \leq \|p - p_*\| = \|Tz - Tz_*\| \leq \phi(\|Sz - Tz\|) + \delta\|Sz - Sz_*\| = \delta\|p - p_*\|$$

which implies that $p = p_*$. Also S and T are weakly compatible and $Sz = Tz = p$, then $Tp = TTz = TSz = STz$ implies $Tp = Sp$. Hence, Tp is a point of coincidence of the pair (S,T) and because point of coincidence is unique, then $Tp = p$. So, $Sp = Tp = p$ implies that p is a unique common fixed point of S and T.

Theorem 3.2. Assume that $z \in C(S,T)$ such that $p = Tz = Sz$. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (11) with $0 < \alpha_1 < \alpha_n$ converges to p. Also let $\{Su_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and let $\epsilon_n = d\big(Su_{n+1}, f(T,u_n)\big), n = 0,1,2, \dots$. Then iterative algorithm (11) will be called (S,T)-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} Su_n = p$.

Proof. Let $\{Su_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and let $Sv_n = (1-\alpha_n)Sw_n + \alpha_n Tw_n, Sw_n = Tu_n$ and $\lim_{n\to\infty} \epsilon_n = 0$ such that $\epsilon_n = \|Su_{n+1} - Tv_n\|$. It will be shown that $\lim_{n\to\infty} Su_n = p$.

$$\begin{aligned} \|Su_{n+1} - p\| &\le \|Su_{n+1} - Tv_n\| + \|Tv_n - p\| \\ &\le \epsilon_n + \|Tz - Tv_n\| \\ &\le \epsilon_n + \phi(\|Sz - Tz\|) + \delta\|Sz - Sv_n\| \\ &= \epsilon_n + \delta\|Sv_n - p\| \end{aligned} \tag{17}$$

and

$$\begin{aligned} \|Sv_n - p\| &\le (1-\alpha_n)\|Sw_n - p\| + \alpha_n\|Tw_n - p\| \\ &\le (1-\alpha_n)\|Sw_n - p\| \\ &\quad +\alpha_n\{\phi(\|Sz - Tz\|) + \delta\|Sz - Sw_n\|\} \\ &= [1 - \alpha_n(1-\delta)]\|Sw_n - p\| \end{aligned} \tag{18}$$

Moreover

$$\|Sw_n - p\| \le \phi(\|Sz - Tz\|) + \delta\|Sz - Su_n\| = \delta\|Su_n - p\| \tag{19}$$

Substituting (19) in (18) and (18) in (17) respectively, we have

$$\|Su_{n+1} - p\| \le \epsilon_n + \delta^2[1 - \alpha_n(1-\delta)]\|Su_n - p\| \tag{20}$$

Hence $0 < \alpha_1 < \alpha_n$ and $\delta \in [0,1)$, we get $[1 - \alpha_n(1-\delta)] \le [1 - \alpha_1(1-\delta)] < 1$. Then we obtain $\lim_{n\to\infty} Su_n = p$.
Now, we suppose that $\lim_{n\to\infty} Su_n = p$. It will be shown that $\lim_{n\to\infty} \epsilon_n = 0$:

$$\begin{aligned} \epsilon_n = \|Su_{n+1} - Tv_n\| &\le \delta\|Su_{n+1} - p\| + \|Tv_n - p\| \\ &\le \|Su_{n+1} - p\| + \phi(\|Sz - Tz\|) + \delta\|Sz - Sv_n\| \\ &= \|Su_{n+1} - p\| + \delta\|Sv_n - p\| \end{aligned} \tag{21}$$

From (18) it can be seen easily $\|Sv_n - p\| \le [1 - \alpha_n(1-\delta)]\|Sw_n - p\|$. Then we get

$$\epsilon_n \le \|Su_{n+1} - p\| + \delta[1 - \alpha_n(1-\delta)]\|Sw_n - p\|$$

Also $\|Sw_n - p\| \le \delta\|Su_n - p\|$. Hence we obtain

$$\epsilon_n \leq \|Su_{n+1} - p\| + \delta^2[1 - \alpha_n(1-\delta)]\|Su_n - p\|$$

Taking the limit in both sides of the above inequality, it can be seen that $\lim_{n\to\infty} \epsilon_n = 0$.

Theorem 3.3. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (11) with $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$. Let $\{Su_n\}_{n=0}^{\infty}$ be iterative sequence (7) with $\lim_{n\to\infty} \beta_n = 0$ and $0 \leq \alpha_n + \beta_n \leq \frac{1}{1+\delta}$. Assume that p is the unique common fixed point of the pair (S,T). Then $\{Sx_n\}_{n=0}^{\infty}$ converges to p faster than $\{Su_n\}_{n=0}^{\infty}$ for $x_0 = u_0 \in Y$.

Proof. From inequality (16) in Theorem 3.1., we have

$$\|Sx_{n+1} - p\| \leq \delta^{2(n+1)} \prod_{i=0}^{n}[1 - \alpha_i(1-\delta)]\|Sx_0 - p\| \tag{22}$$

Also from JK iterative algorithm (7), we have

$$\|Sw_n - p\| \leq (1 - a_n)\|Su_n - p\| + a_n\|Tu_n - p\| \tag{23}$$

$$\leq [1 - a_n(1-\delta)]\|Su_n - p\|$$

and

$$\|Sv_n - p\| \leq (1 - b_n - c_n)\|Su_n - p\| + b_n\|Tw_n - p\| + c_n\|Tu_n - p\| \tag{24}$$

$$\leq (1 - b_n - c_n)\|Su_n - p\| + b_n\delta\|Sw_n - p\| + c_n\delta\|Su_n - p\|.$$

Substituting (23) in (24), we obtain

$$\|Sv_n - p\| \leq \{1 - b_n - c_n + b_n\delta[1 - a_n(1-\delta)] + c_n\delta\}\|Su_n - p\| \tag{25}$$

Also

$$\|Su_{n+1} - p\| \geq (1 - \alpha_n - \beta_n)\|Su_n - p\| - \alpha_n\|Tv_n - p\| - \beta_n\|Tu_n - p\| \tag{26}$$

and because $\|Tv_n - p\| \leq \delta\|Sv_n - p\|$, we have

$$\|Su_{n+1} - p\| \geq (1 - \alpha_n - \beta_n)\|Su_n - p\|$$

$$-\alpha_n\delta\{1 - b_n - c_n + b_n\delta[1 - a_n(1-\delta)] + c_n\delta\}\|Su_n - p\|$$

$$-\beta_n\delta\|Su_n - p\| \tag{27}$$

$$= \left\{\begin{matrix} 1 - \alpha_n - \beta_n \\ -\alpha_n\delta\{1 - b_n - c_n + b_n\delta[1 - a_n(1-\delta)] + c_n\delta\} - \beta_n\delta \end{matrix}\right\}\|Su_n - p\|$$

Because $c_n\delta < c_n$, $b_n\delta < b_n$ and $[1 - a_n(1-\delta)] \leq 1$, we obtain

$$\|Su_{n+1} - p\| \geq (1 - \alpha_n - \beta_n - \alpha_n\delta - \beta_n\delta)\|Su_n - p\|$$

$$= [1 - (\alpha_n + \beta_n)(1+\delta)]\|Su_n - p\|$$

From the above inequality we obtain

$$\|Su_{n+1} - p\| \geq \prod_{i=0}^{n}[1 - (\alpha_i + \beta_i)(1 + \delta)]\|Su_0 - p\| \tag{28}$$

By using (22) and (28), we get

$$\left\|\frac{Sx_{n+1} - p}{Su_{n+1} - p}\right\| \leq \frac{\delta^{2(n+1)} \prod_{i=0}^{n}[1 - \alpha_i(1 - \delta)]\|Sx_0 - p\|}{\prod_{i=0}^{n}[1 - (\alpha_i + \beta_i)(1 + \delta)]\|Su_0 - p\|}$$

Define

$$\psi_n = \frac{\delta^{2(n+1)} \prod_{i=0}^{n}[1 - \alpha_i(1 - \delta)]\|Sx_0 - p\|}{\prod_{i=0}^{n}[1 - (\alpha_i + \beta_i)(1 + \delta)]\|Su_0 - p\|}$$

Then we have

$$\frac{\psi_{n+1}}{\psi_n} = \frac{\delta^2[1 - \alpha_n(1 - \delta)]}{[1 - (\alpha_{n+1} + \beta_{n+1})(1 + \delta)]}$$

By assumption $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, we get $\lim_{n\to\infty} \frac{\psi_{n+1}}{\psi_n} = \delta^2 < 1$. That is $\lim_{n\to\infty} \psi_n = 0$ which implies that $\{Sx_n\}_{n=0}^{\infty}$ converges to p faster than $\{Su_n\}_{n=0}^{\infty}$.

The following examples show that our newly defined iterative algorithm is faster than other Jungck-type algorithms which mentioned in this paper:

Example 3.4. Let $Y = [-1,1] \subset \mathbb{R}$ be endowed with usual metric. Define $T, S: [-1,1] \to [-1,1]$ with a common fixed point $p = 0$ by $Tx = \frac{x}{8}$ and $Sx = \frac{x}{2}$. It is clear that $T([-1,1]) \subseteq S([-1,1])$ and $S([-1,1])$ is a complete subset of $[-1,1]$. The pair (S, T) satisfies condition (10) with $\delta \in [\frac{2}{5}, 1)$ and $\phi(t) = \frac{t}{8}$. Let $x_0 = 0.79$ and $\alpha_n = \beta_n = \gamma_n = a_n = b_n = c_n = \frac{1}{4}$.

The following table shows that the new iterative algorithm (11) converges to $p = 0$ faster than all of Jungck-type iterative algorithm:

Table 1. Comparison of rate of convergence of some Jungck-type iterative algorithms for $x_0 = 0.79$ initial point

Iter. No.	JY	JCR	JSP	JN	JA	JI	JS	JM	JK
1	0,790000	0,790000	0,790000	0,790000	0,790000	0,790000	0,790000	0,790000	0,790000
2	0,040117	0,152947	0,423738	0,632039	0,188242	0,632617	0,160469	0,641875	0,474656
3	0,002037	0,029611	0,227283	0,505662	0,044855	0,506588	0,032595	0,521523	0,285187
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	0,000000	0,000215	0,035073	0,258946	0,000607	0,260134	0,000273	0,279733	0,061857
7	⋮	0,000042	0,018813	0,207170	0,000145	0,208310	0,000055	0,227283	0,037165
8		0,000008	0,010091	0,165746	0,000034	0,166811	0,000011	0,184668	0,022330
9		0,000002	0,005412	0,132605	0,000008	0,133579	0,000002	0,150042	0,013417
10		0,000000	0,002903	0,106090	0,000002	0,106968	0,000000	0,121909	0,008061
11		⋮	0,001557	0,084877	0,000000	0,085658	0,000000	0,099051	0,004843
⋮			⋮	⋮	⋮	⋮	⋮	⋮	⋮
23			0,000001	0,005837		0,005956		0,008199	0,000011
24			0,000000	0,004670		0,004769		0,006661	0,000006
⋮			⋮	⋮		⋮		⋮	⋮
29				0,001531		0,001570		0,002359	0,000001
30				0,001225		0,001258		0,001916	0,000000
⋮				⋮		⋮		⋮	⋮
64				0,000001		0,000001		0,000002	
65				0,000000		0,000001		0,000001	
66				⋮		0,000000		0,000001	
⋮						⋮			
69								0,000001	
70								0,000000	

Table 1 shows that while the newly defined iterative algorithm reaches the fixed point at the 6^{th} step;

- Jungck-CR iterative algorithm at the 10^{th} step,
- Jungck-SP iterative algorithm at the 24^{th} step,
- Jungck-Noor iterative algorithm at the 65^{th} step,
- Jungck-Agarwal iterative algorithm at the 11^{th} step,
- Jungck-Ishikawa iterative algorithm at the 66^{th} step,
- Jungck-Sahu iterative algorithm at the 10^{th} step,
- Jungck-Mann iterative algorithm at the 70^{th} step,
- Jungck-Khan iterative algorithm reaches to the fixed point at the 30^{th} step.

Example 3.5. Let $Y = [4,5] \subset \mathbb{R}$ be endowed with usual metric. Define $T, S: [4,5] \to [16,25]$ with a coincidence point $p = 16$ by $Tx = 2x + 8$ and $Sx = x^2$. It is clear that $T([4,5]) \subseteq S([4,5])$ and $S([4,5])$ is a complete subset of $[16,25]$. The pair (S, T) satisfies condition (10) with $\delta \in [\frac{1}{5}, 1)$ and $\phi(t) = \frac{t}{8}$. Let $x_0 = 5$ and $\alpha_n = \beta_n = \gamma_n = a_n = b_n = c_n = \frac{1}{4}$.
The convergence result for various Jungck-type iterative algorithms to $p = 16 = S4 = T4$ are listed in the following table:

Table 2. Comparison of rate of convergence of some Jungck-type iterative algorithms for $x_0 = 5$ initial point

Iter. No	JY	JCR	JSP	JN	JA	JI	JS	JM	JK
1	5,000000	5,000000	5,000000	5,000000	5,000000	5,000000	5,000000	5,000000	5,000000
2	4,049142	4,189182	4,552299	4,812098	4,232129	4,812579	4,200435	4,821825	4,616665
3	4,002491	4,036464	4,301403	4,657939	4,054970	4,658752	4,040615	4,674229	4,376874
4	4,000127	4,007054	4,163259	4,531936	4,013079	4,532962	4,008246	4,552299	4,228900
5	4,000006	4,001365	4,088045	4,429293	4,003115	4,430436	4,001675	4,451822	4,138459
6	4,000000	4,000264	4,047366	4,345926	4,000742	4,347113	4,000340	4,369205	4,083535
7	⋮	4,000051	4,025447	4,278388	4,000177	4,279567	4,000069	4,301403	4,050317
8		4,000010	4,013661	4,223795	4,000042	4,224927	4,000014	4,245851	4,030278
9		4,000002	4,007331	4,179746	4,000010	4,180808	4,000003	4,200400	4,018209
10		4,000000	4,003933	4,144261	4,000002	4,145238	4,000001	4,163259	4,010947
11		⋮	4,002110	4,115711	4,000001	4,116596	4,000000	4,132937	4,006579
12			4,001132	4,092766	4,000000	4,093558	⋮	4,108205	4,003954
⋮			⋮	⋮	⋮	⋮		⋮	⋮
24			4,000001	4,006430		4,006555		4,009021	4,000009
25			4,000000	4,005145		4,005250		4,007331	4,000005
⋮			⋮	⋮		⋮		⋮	⋮
29				4,002108		4,002159		4,003196	4,000001
30				4,001687		4,001729		4,002597	4,000000
⋮				⋮		⋮		⋮	⋮
62				4,000001		4,000001		4,000003	
63				4,000001		4,000001		4,000003	
⋮				⋮		⋮		⋮	
66				4,000001		4,000001		4,000001	
67				4,000000		4,000000		4,000001	
⋮				⋮		⋮		⋮	
71								4,000001	
72								4,000000	

Table 2 shows that while the newly defined iterative algorithm reaches the fixed point at the 6^{th} step;

- Jungck-CR iterative algorithm at the 10^{th} step,
- Jungck-SP iterative algorithm at the 25^{th} step,
- Jungck-Noor iterative algorithm at the 67^{th} step,
- Jungck-Agarwal iterative algorithm at the 12^{th} step,
- Jungck-Ishikawa iterative algorithm at the 67^{th} step,
- Jungck-Sahu iterative algorithm at the 11^{th} step,
- Jungck-Mann iterative algorithm at the 72^{th} step,
- Jungck-Khan iterative algorithm reaches to the fixed point at the 30^{th} step.

4. Discussion and Conclusion

Considering the results obtained in Theorem 3.1, Theorem 3.2, Theorem 3.3 and Table 1 - Table 2;

- The fact that the newly defined Jungck-type iterative algorithm (11) has a higher convergence rate compared to other algorithms in the literature shows that this algorithm has good potential for future applications.
- In addition to the high convergence rate, it is observed that stability results can be obtained by using the new iterative algorithm (11) for more general mapping classes.
- It is also seen from Table 1 and Table 2 that the rate of convergence of an iterative algorithm can be changed depend on its control sequences.

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