SS-SUPPLEMENTED MODULES

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Abstract. A module $M$ is called \textit{ss-supplemented} if every submodule $U$ of $M$ has a supplement $V$ in $M$ such that $U \cap V$ is semisimple. It is shown that a finitely generated module $M$ is ss-supplemented if and only if it is supplemented and $\text{Rad}(M) \subseteq \text{Soc}(M)$. A module $M$ is called \textit{strongly local} if it is local and $\text{Rad}(M)$ is semisimple. Any direct sum of strongly local modules is ss-supplemented and coatomic. A ring $R$ is semiperfect if $\text{Rad}(R) \subseteq \text{Soc}(R)$ if every left $R$-module is (amply) ss-supplemented if $\mu R$ is a finite sum of strongly local submodules.

1. Introduction

Throughout this study, all rings are associative with identity and all modules are unitary left modules. Let $R$ be a ring and $M$ be an $R$-module. $U \subseteq M$ will mean that $U$ is a submodule of $M$. $\text{Rad}(M)$ and $\text{Soc}(M)$ will indicate radical and socle of $M$. A submodule $N$ of $M$ is called \textit{small} in $M$, denoted $N \ll M$, if $M \neq N + K$ for every proper submodule $K$ of $M$. Let $U$ and $V$ be submodules of $M$. $V$ is called a \textit{supplement} of $U$ in $M$ if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V \ll V$. The module $M$ is called \textit{supplemented} if every submodule of $M$ has a supplement in $M$. A submodule $U$ of $M$ has \textit{ample supplements} in $M$ if every submodule $L$ of $M$ such that $M = U + L$ contains a supplement of $U$ in $M$. The module $M$ is called \textit{amply supplemented} if every submodule of $M$ has ample supplements in $M$. For characterizations of supplemented and amply supplemented modules we refer to [7].

A non-zero module $M$ is called \textit{hollow} if every proper submodule of $M$ is small in $M$ and is called \textit{local} if the sum of all proper submodules of $M$ is also a proper submodule of $M$. Note that local modules are hollow and hollow modules are clearly amply supplemented. A ring $R$ is called \textit{local ring} if $\mu R$ is a local module.

In [3], Zhou and Zhang generalized the concept of socle of a module $M$ to that of $\text{Soc}_s(M)$ by considering the class of all simple submodules of $M$ that are small in $M$. 

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in place of the class of all simple submodules of $M$, that is, $\text{Soc}_s(M) = \sum \{ N \ll M \mid N \text{ is simple} \}$. It is clear that $\text{Soc}_s(M) \subseteq \text{Rad}(M)$ and $\text{Soc}_s(M) \subseteq \text{Soc}(M)$.

We call a module $M$ strongly local if it is local and $\text{Rad}(M)$ is semisimple. We call a ring $R$ left strongly local ring if $R$ is a strongly local module. Then we have that the following implications on modules:

$$\text{simple} \implies \text{strongly local} \implies \text{local}$$

Next we mention two examples which show that the above implications are proper. For the local left $\mathbb{Z}$-module $M = \mathbb{Z}_4$, we have $\text{Rad}(M) = \text{Soc}(M)$. Hence, $M$ is strongly local but not simple. On the other hand, for the local left $\mathbb{Z}$-module $M = \mathbb{Z}_8$, $\text{Soc}(M)$ is a proper submodule of $\text{Rad}(M)$. Thus $M$ is not a strongly local module.

In section 2 we study on strongly local modules and rings. We show that every left strongly local ring is left perfect and right perfect. A strongly local commutative domain is field.

Let $U$ and $V$ be submodules of a module $M$. $V$ is called a Rad-supplement of $U$ in $M$ if $M = U + V$ and $U \cap V \subseteq \text{Rad}(V)$. Since $\text{Soc}_s(V) \subseteq \text{Rad}(V)$, it is of interest to investigate the analogue of this notion by replacing “$\text{Rad}(V)$” with “$\text{Soc}_s(V)$”. Now, we give the following result playing a key role in our work as a proper generalization of direct summands. Firstly, we need the following well known facts that we include here for completeness.

**Lemma 1.** Let $M$ be a module and $N$ be a semisimple submodule of $M$ which is contained in $\text{Rad}(M)$. Then $N \ll M$.

**Proof.** Let $N + K = M$ for some submodule $K$ of $M$. Since $N$ is semisimple, there exists a submodule $N'$ of $N$ such that $N = (N \cap K) \oplus N'$. Hence $M = N + K = [(N \cap K) \oplus N'] + K = N' + K$. Since $N' \cap K = (N' \cap N) \cap K = N' \cap (N \cap K) = 0$, we have $M = N' \oplus K$. It follows from [7, 21.6 (5)] that $\text{Rad}(M) = \text{Rad}(N') \oplus \text{Rad}(K) = \text{Rad}(K)$ since $\text{Rad}(N') \subseteq \text{Rad}(N) = 0$. Then $M = N + K \subseteq \text{Rad}(M) + K \subseteq K$. It means that $N \ll M$.  

**Lemma 2.** Let $M$ be a module. Then $\text{Soc}_s(M) = \text{Rad}(M) \cap \text{Soc}(M)$.

**Proof.** Let $a \in \text{Rad}(M) \cap \text{Soc}(M)$. Then $Ra$ is semisimple and so there exist $n \in \mathbb{Z}^+$ and simple submodules $S_i$ of $M$ (1 ≤ $i$ ≤ $n$) such that $Ra = S_1 \oplus S_2 \oplus \ldots \oplus S_n$ by [7, Proposition 3.3]. Since $Ra$ is small in $M$, it follows from [7, 19.3 (2)] that each $S_i$ is small in $M$. Thus $a \in Ra \subseteq \text{Soc}_s(M)$.  

**Lemma 3.** Let $M$ be a module and $U, V$ be submodules of $M$. Then the following statements are equivalent:

1. $M = U + V$ and $U \cap V \subseteq \text{Soc}_s(V)$,
2. $M = U + V$, $U \cap V \subseteq \text{Rad}(V)$ and $U \cap V$ is semisimple,
3. $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is semisimple.
Proof. (1) $\implies$ (2) It follows that $U \cap V \subseteq \text{Soc}_s(V) \subseteq \text{Rad}(V) \cap \text{Soc}(V)$. Hence, we deduce that $U \cap V \subseteq \text{Rad}(V)$ and $U \cap V$ is semisimple.

(2) $\implies$ (3) It is clear by Lemma 1.

(3) $\implies$ (1) It is clear by Lemma 2.

We say that $V$ an ss-supplement of $U$ in $M$ if the equal conditions in the above lemma are satisfied. It is clear that the following implications on submodules of a module hold:

Direct summand $\implies$ ss-supplement $\implies$ supplement $\implies$ Rad-supplement

We call a module $M$ ss-supplemented if every submodule of $M$ has an ss-supplement in $M$. A submodule $U$ of a module $M$ has ample ss-supplements in $M$ if every submodule $V$ of $M$ such that $M = U + V$ contains an ss-supplement of $U$ in $M$. We call a module $M$ amply ss-supplemented if every submodule of $M$ has ample ss-supplements in $M$. It is clear that every ss-supplemented module is supplemented. Of course there exists the same relationship between amply ss-supplemented modules and amply supplemented modules. Later we shall give examples of (amply) supplemented modules which are not (amply) ss-supplemented (see Example 17 and Example 18).

In section 3 we characterize ss-supplemented and amply ss-supplemented modules. For modules with small radical, we give some conditions which are equivalent to being an ss-supplemented module in Theorem 20. It follows that a finitely generated module $M$ is ss-supplemented if and only if it is supplemented and $\text{Rad}(M) \subseteq \text{Soc}(M)$. Any direct sum of strongly local modules is ss-supplemented and coatomic. A module $M$ is amply ss-supplemented if and only if every submodule of the module $M$ is ss-supplemented. We show that a ring $R$ is semiperfect and $\text{Rad}(R) \subseteq \text{Soc}(R)$ if and only if every left $R$-module is (amply) ss-supplemented.

2. Strongly Local Modules and Rings

As we mentioned at introduction, we denote by $\text{Soc}_s(M)$ the sum of all simple submodules of a module $M$ that are small in $M$. Then we have:

Let $M$ be a non-zero module. $M$ is called indecomposable if the only direct summands of $M$ are $0$ and $M$.

**Lemma 4.** Let $M$ be an indecomposable module. Then $M$ is simple or $\text{Soc}(M) \subseteq \text{Rad}(M)$.

**Proof.** Suppose that $M$ is not simple. Let $M = \text{Soc}(M) + X$ for some submodule $X$ of $M$. Since $\text{Soc}(M)$ is semisimple, there exists a submodule $Y$ of $\text{Soc}(M)$ such that $\text{Soc}(M) = (\text{Soc}(M) \cap X) \oplus Y$. Therefore, $M = \text{Soc}(M) + X = [(\text{Soc}(M) \cap X) \oplus Y] + X = X \oplus Y$. Since $M$ is indecomposable and not simple, it follows that $Y = 0$. It means that $X = M$. Hence $\text{Soc}(M) << M$, that is, $\text{Soc}(M) \subseteq \text{Rad}(M)$.

Using Lemma 2 and Lemma 4, we have the following result.
Corollary 5. Let $M$ be a local module which is not simple. Then $\text{Soc}_s(M) = \text{Soc}(M)$.

Recall that a module $M$ is called radical if $M$ has no maximal submodules, that is, $M = \text{Rad}(M)$. Let $P(M)$ be the sum of all radical submodules of $M$. It is easy to see that $P(M)$ is the largest radical submodule of $M$. If $P(M) = 0$, $M$ is called reduced.

**Proposition 6.** Let $M$ be a strongly local module. Then $M$ is reduced.

**Proof.** Since $M$ is strongly local, we get $P(M) \subseteq \text{Rad}(M) \subseteq \text{Soc}(M)$. This implies that $P(M)$ is semisimple and so $P(M) = \text{Rad}(P(M)) = 0$. This completes the proof. $\square$

Note that the condition “strongly” in the above proposition is necessary. The following example shows that in general a local module need not be reduced.

**Example 7.** Let $K$ be a field. In the polynomial ring $K[x_1, x_2, \ldots]$ with countably many indeterminates $x_n$, $n \in \mathbb{Z}^+$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ generated by $x_1^2$ and $x_{n+1}^2 - x_n$ for each $n \in \mathbb{Z}^+$. Then as shown in [7, Example 6.2], the quotient ring $R = K[x_1^2, x_2, \ldots]/I$ is a local ring with the unique maximal ideal $J = (x_1, x_2, \ldots) = J^2$. Let $M$ be the left $R$-module $\_R$. Then $M$ is a local module. On the other hand, $M$ is not reduced because $P(M) = \text{Rad}(J) = J \neq 0$.

**Proposition 8.** Every factor module of a strongly local module is strongly local.

**Proof.** Let $M$ be a strongly local module and $N$ be a submodule of $M$. Then the factor module $\frac{M}{N}$ is local. Since $\text{Rad}(M)$ is the unique maximal submodule of $M$, it follows from [7] 21.2 (1) that $\text{Rad}(\frac{M}{N}) = \frac{\text{Rad}(M)}{N} \subseteq \frac{\text{Soc}(M)}{N} = \pi(\text{Soc}(M)) \subseteq \text{Soc}(\frac{M}{N})$, where $\pi : M \to \frac{M}{N}$ is the canonical projection. Hence $\frac{M}{N}$ is strongly local. $\square$

**Proposition 9.** Let $R$ be a left strongly local ring. Then $(\text{Rad}(R))^2 = 0$. In particular, $\text{Rad}(R)$ is nilpotent.

**Proof.** Since $\text{Rad}(R) \subseteq \text{Soc}(\_R)$, it follows from [7] 21.12 (4) that $(\text{Rad}(R))^2 = 0$. It means that $\text{Rad}(R)$ is nilpotent. $\square$

Recall from [7] that an ideal $I$ of a ring $R$ is right $t$-nilpotent if for every sequence $a_1, a_2, \ldots, a_k$ of elements in $I$, there is a $k \in \mathbb{Z}^+$ with $a_1a_2\ldots a_k = 0$. Similarly left $t$-nilpotent is defined. Following [7] 43.9, $R$ is called left perfect (respectively, right perfect) if $R$ is semilocal and $\text{Rad}(R)$ is right $t$-nilpotent (respectively, left $t$-nilpotent). Here a ring $R$ is semilocal if $\frac{R}{\text{Rad}(R)}$ is an artinian semisimple ring (see [4]). Note that nilpotent ideals are left and right $t$-nilpotent. Using this fact, we have the following:

**Corollary 10.** Every left strongly local ring is left perfect and right perfect.
Proof. Let $R$ be a left strongly local ring. Since local rings are semilocal, it follows from Proposition 9 that $R$ is left perfect and right perfect.

It is well known that an artinian commutative domain is field. We have:

**Proposition 11.** A strongly local commutative domain is field.

Proof. Let $R$ be a strongly local commutative domain and $a$ be any element of $R$. If $a \in R \backslash \text{Rad}(R)$, we can write $Ra = R$ because $R$ is local. Therefore, $a$ is an invertible element of $R$. Suppose that $a \in \text{Rad}(R)$. It follows from Proposition 9 that $a^2 \in (\text{Rad}(R))^2 = 0$. By the hypothesis, we get $a = 0$. Hence, $R$ is field.

### 3. SS-Supplemented Modules

It is known that a ring $R$ is semiperfect if and only if every finitely generated $R$-module is (amply) supplemented (see [7, 42.6]). In this section we obtain new characterizations of semiperfect rings via their ss-supplemented modules.

Recall that for a maximal submodule $U$ of a module $M$, a submodule $V$ of $M$ is a supplement of $U$ in $M$ if and only if $M = U + V$ and $V$ is local (see [7, 41.1 (3)]). Analogous to that we have:

**Proposition 12.** Let $M$ be a module and $U$ be a maximal submodule of $M$. A submodule $V$ of $M$ is an ss-supplement of $U$ in $M$ if and only if $M = U + V$ and $V$ is strongly local.

Proof. Let $V$ be an ss-supplement of $U$ in $M$. By [7, 41.1.(3)], $V$ is local and $U \cap V = \text{Rad}(V)$ is the unique maximal submodule of $V$. Since $U \cap V$ is semisimple, we have $\text{Rad}(V) \subseteq \text{Soc}(V)$. Thus $V$ is strongly local.

Conversely, since $V$ is local and $M = U + V$, we can write $U \cap V \subseteq \text{Rad}(V)$. It follows from assumption that $U \cap V$ is semisimple. Hence, $V$ is an ss-supplement of $U$ in $M$.

Now, we give examples of (amply) supplemented modules which are not (amply) ss-supplemented. We first need the following facts.

**Lemma 13.** Let $M$ be an ss-supplemented module and $N$ be a small submodule of $M$. Then $N \subseteq \text{Soc}_s(M)$.

Proof. By the assumption, $M$ is the unique ss-supplement of $N$ in $M$ and so $N \cap M = N$ is semisimple. Hence, $N \subseteq \text{Soc}_s(M)$ by Lemma 2.

The following result is a direct consequence of Lemma 13.

**Corollary 14.** Let $M$ be an ss-supplemented module and $\text{Rad}(M) \ll M$. Then $\text{Rad}(M) \subseteq \text{Soc}(M)$.

It is well known that every local module is amply supplemented. Now we give an analogous characterization of this fact for amply ss-supplemented modules.

**Proposition 15.** Every strongly local module is amply ss-supplemented.
Proof. Let $M$ be a strongly local module. Then, $M$ is local and so it is amply supplemented. Note that $M$ has no supplement submodule except for 0 and $M$. Since $\text{Rad}(M) \subseteq \text{Soc}(M)$, $M$ is amply ss-supplemented.

**Proposition 16.** Let $R$ be a ring and $M$ be a hollow $R$-module. $M$ is (amply) ss-supplemented if and only if it is strongly local.

Proof. Suppose that $M$ is ss-supplemented. Let $m \in \text{Rad}(M)$. Then we get $Rm \ll M$. Since $M$ is ss-supplemented, it follows from Lemma 13 that $Rm \subseteq \text{Soc}(M)$. It means that $m \in \text{Soc}(M)$ and so $\text{Rad}(M) \subseteq \text{Soc}(M)$. Suppose that $M = \text{Rad}(M)$. Since $M = \text{Rad}(M) = \text{Soc}(M)$ and the radical of a semisimple module is zero, we have that $M = 0$. This is a contradiction because $M$ is hollow. It means that $M \neq \text{Rad}(M)$, that is, $M$ is local by [7, 41.4]. Therefore $M$ is strongly local. The converse follows from Proposition 15.

**Example 17.** For any prime integer $p$, consider the left $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$. Note that $M$ is a hollow module which is not local. Since hollow modules are (amply) supplemented, $M$ is (amply) supplemented. However, $M$ is not (amply) ss-supplemented module by Proposition 16.

Every artinian module is supplemented. The next example shows that in general artinian modules need not to be ss-supplemented.

**Example 18.** Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z}_{p^k}$, for $p$ is any prime integer and $k \geq 3$. Note that $M$ is artinian. Since $\text{Soc}(\mathbb{Z}_{p^k}) = \text{Soc}(\mathbb{Z}_p) = \mathbb{Z}_p$ and $\text{Rad}(M) = p\mathbb{Z}_{p^k}$, $M$ is not strongly local and so it is not ss-supplemented by Proposition 16.

**Lemma 19.** Let $M$ be a supplemented module and $\text{Rad}(M) \subseteq \text{Soc}(M)$. Then $M$ is ss-supplemented.

Proof. Let $U \subseteq M$. Since $M$ is supplemented, there exists a submodule $V$ of $M$ such that $M = U + V$ and $U \cap V \ll V$. Then $U \cap V \subseteq \text{Rad}(V) \subseteq \text{Rad}(M)$ and so $U \cap V$ is semisimple by the assumption. Hence $V$ is an ss-supplement of $U$ in $M$. It means that $M$ is ss-supplemented.

**Theorem 20.** Let $M$ be a module with $\text{Rad}(M) \ll M$. Then the following statements are equivalent:

1. $M$ is ss-supplemented,
2. $M$ is supplemented and $\text{Rad}(M)$ has an ss-supplement in $M$,
3. $M$ is supplemented and $\text{Rad}(M) \subseteq \text{Soc}(M)$.

Proof. (1) $\implies$ (2) It is clear.

(2) $\implies$ (3) It follows from Lemma 13.

(3) $\implies$ (1) By Lemma 19.

Since finitely generated modules have small radical, we have the following result.
Corollary 21. Let \( M \) be a finitely generated module. Then \( M \) is ss-supplemented if and only if it is supplemented and \( \text{Rad}(M) \subseteq \text{Soc}(M) \).

Next, in order to prove that every finite sum of ss-supplemented modules is ss-supplemented, we use the following standard lemma (see, [7, 41.2]).

Lemma 22. Let \( M \) be a module and \( M_1, U \) be submodules of \( M \) with \( M_1 \) ss-supplemented. If \( M_1 + U \) has an ss-supplement in \( M \), \( U \) also has an ss-supplement in \( M \).

Proof. Suppose that \( X \) is an ss-supplement of \( M_1 + U \) in \( M \) and \( Y \) is an ss-supplement of \( (X+U) \cap M_1 \) in \( M_1 \). Then \( M = X + Y + U \) and \( (X+Y) \cap U \ll X+Y \). Moreover, \( X \cap (Y + U) \) is semisimple as a submodule of the semisimple module \( X \cap (M_1 + U) \). Note that \( Y \cap [(X + U) \cap M_1] = Y \cap (X + U) \) is semisimple. It follows from [3, 8.1.5] that \( (X + Y) \cap U \) is semisimple. Hence \( X + Y \) is an ss-supplement of \( U \) in \( M \).

Proposition 23. Let \( M_1, M_2 \) be any submodules of a module \( M \) such that \( M = M_1 + M_2 \). Then if \( M_1 \) and \( M_2 \) are ss-supplemented, \( M \) is ss-supplemented.

Proof. Let \( U \) be any submodule of \( M \). The trivial submodule \( 0 \) is ss-supplement of \( M = M_1 + M_2 + U \) in \( M \). Since \( M_1 \) is ss-supplemented, \( M_2 + U \) has an ss-supplement in \( M \) by Lemma 22. Again applying Lemma 22, we also have that \( U \) has an ss-supplement in \( M \). This shows that \( M \) is ss-supplemented.

Using this fact we obtain the following corollary.

Corollary 24. Every finite sum of ss-supplemented modules is ss-supplemented.

Now we give an example of an ss-supplemented module which is not strongly local.

Example 25. The \( \mathbb{Z} \)-module \( M = \mathbb{Z}_4 \oplus \mathbb{Z}_4 \) is ss-supplemented as a sum of strongly local modules. However, \( M \) is not (strongly) local.

Then we have the following proper implications on modules hold:

\[
\text{strongly local} \quad \text{local} \quad \text{ss-supplemented} \quad \text{supplemented}
\]

Proposition 26. If \( M \) is a (amply) ss-supplemented module, then every factor module of \( M \) is (amply) ss-supplemented.
Proof. Let $M$ be an ss-supplemented module and $\frac{M}{L}$ be a factor module of $M$. By the assumption, for any submodule $U$ of $M$ which contains $L$, there exists a submodule $V$ of $M$ such that $M = U + V$, $U \cap V << V$ and $U \cap V$ is semisimple. Let $\pi : M \rightarrow \frac{M}{L}$ be the canonical projection. Then we have that $\frac{M}{L} = \frac{U}{L} + \frac{V}{L}$ and $\frac{U}{L} \cap \frac{V}{L} = \frac{(U \cap V) + L}{L} = \pi(U \cap V) << \pi(V) = \frac{V}{L}$ by [7, 19.3(4)]. Since $U \cap V$ is semisimple, it follows from [3, 8.1.5] that $\pi(U \cap V) = \frac{(U \cap V) + L}{L} = \frac{U}{L} \cap \frac{V}{L}$ is semisimple. That is, $\frac{V}{L}$ is an ss-supplement of $\frac{U}{L}$ in $\frac{M}{L}$, as required.

By adapting this argument we can prove similarly that if $M$ is amply ss-supplemented, then so is every factor module of $M$. \hfill \Box

Recall that a module $M$ is said to be coatomic if every proper submodule of $M$ is contained in a maximal submodule of $M$. It is easy to see that every coatomic module has small radical.

Let $p$ be a prime integer and consider the localization ring $R = \mathbb{Z}(p) = \{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } p \nmid b \}$. Note that $R$ is a local ring. Let $M$ be the left $R$-module $R(\mathbb{N})$. Then $M$ is the direct sum of local submodules but it is not supplemented. Since $R$ is not perfect, $\text{Rad}(M)$ is not small in $M$ and so $M$ is not also coatomic. However, any arbitrary direct sum of strongly local modules is ss-supplemented and coatomic, as the next result shows.

**Theorem 27.** Let $M = \bigoplus_{i \in I} M_i$, where each $M_i$ is a strongly local module. Then, $M$ is ss-supplemented and coatomic.

**Proof.** Since $M_i$ is strongly local for every $i \in I$, it is local and $\text{Rad}(M_i) \subseteq \text{Soc}(M_i)$ and so $\text{Rad}(M) = \bigoplus_{i \in I} \text{Rad}(M_i) \subseteq \bigoplus_{i \in I} \text{Soc}(M_i) = \text{Soc}(M)$ by [7, 21.6 (5) and 21.2 (5)]. Applying Lemma 1, we get that $\text{Rad}(M)$ is a small submodule of $M$. Since strongly local modules are local, it follows from [10, Theorem 1.4 (A)] that $M$ is supplemented. Hence, $M$ is ss-supplemented by Theorem 20.

Let $U$ be a proper submodule of $M$. It follows from [7, 41.1 (6)] that $U$ is contained in a maximal submodule of $M$, that is, $M$ is coatomic. \hfill \Box

Let $M$ be a module. A module $N$ is called $M$-generated if there exists an epimorphism $f : M(I) \rightarrow N$ for some index set $I$.

**Corollary 28.** Let $M$ be a strongly local module. Then every $M$-generated module is ss-supplemented and coatomic.

**Proof.** Suppose that $N$ is $M$-generated. Then, there exists an epimorphism $f : M(I) \rightarrow N$ for some index set $I$. By Theorem 27, $M(I)$ is ss-supplemented and coatomic. Hence $N$ is ss-supplemented by Proposition 26 and it is coatomic by [10, Lemma 1.5 (a)]. \hfill \Box

**Corollary 29.** Let $R$ be a left strongly local ring. Then every left $R$-module is ss-supplemented.
Proof. Since all left $R$-modules are $R$-generated, the proof follows from Corollary 28. □

A submodule $U$ of a module $M$ is said to be cofinite if $M/U$ is finitely generated (see [1]). Note that maximal submodules of $M$ are cofinite.

**Theorem 30.** The following statements are equivalent for a module $M$:

1. $M$ is the sum of all strongly local submodules,
2. $M$ is ss-supplemented and coatomic,
3. $M$ is coatomic and every cofinite submodule of $M$ has an ss-supplement in $M$,
4. $M$ is coatomic and every maximal submodule of $M$ has an ss-supplement in $M$.

**Proof.** (1) $\implies$ (2) Let $M = \sum_{i \in I} M_i$, where each $M_i$ is strongly local submodules. Put $N = \bigoplus_{i \in I} M_i$. Then, by Theorem 27, $N$ is ss-supplemented and coatomic. Now we consider the epimorphism $f : N \to M$ via $f((m_i)_{i \in I}) = \sum_{i \in I} m_i$ for all $(m_i)_{i \in I} \in N$. It follows from Proposition 26 and [10, Lemma 1.5 (a)] that $M$ is ss-supplemented and coatomic.

(2) $\implies$ (3) $\implies$ (4) are clear.

(4) $\implies$ (1) Let $S$ be the sum of all strongly local submodules of $M$. Assume that $S \neq M$. Since $M$ is coatomic, there exists a maximal submodule $K$ of $M$ with $S \subseteq K$. By (4), $K$ has an ss-supplement, say $V$, in $M$. It follows from Proposition 12 that $V$ is strongly local. Therefore, $V \subseteq S \subseteq K$, a contradiction. □

The following fact is a direct consequence of Theorem 30.

**Corollary 31.** For a coatomic module $M$, the following statements are equivalent:

1. $M$ is the sum of all strongly local submodules,
2. $M$ is ss-supplemented,
3. Every cofinite (maximal) submodule of $M$ has an ss-supplement in $M$.

A ring $R$ is called left max if every non-zero left $R$-module has a maximal submodule. Note that if $R$ is a left max ring, then every left $R$-module is coatomic. Using this fact and Corollary 31, we obtain the following result.

**Corollary 32.** Let $R$ be a left max ring and $M$ be a non-zero left $R$-module. Then $M$ is the sum of all strongly local submodules of $M$ if and only if it is ss-supplemented.

**Proposition 33.** Let $M$ be a module. If every submodule of $M$ is ss-supplemented, then $M$ is amply ss-supplemented.

**Proof.** Let $U$ and $V$ be two submodules of $M$ such that $M = U + V$. Since $V$ is ss-supplemented, there exists a submodule $V'$ of $V$ such that $V = (U \cap V) + V'$, $U \cap V' \ll V'$ and $U \cap V'$ is semisimple. Note that $M = U + V = U + ((U \cap V') + V') =$
$U + V'$. It means that $U$ has ample ss-supplements in $M$. Hence $M$ is amply ss-supplemented.

Lemma 34. Let $M$ be amply ss-supplemented module and $V$ be an ss-supplement submodule in $M$. Then $V$ is amply ss-supplemented.

Proof. Let $V$ be an ss-supplement of a submodule $U$ of $M$. Let $X$ and $Y$ be submodules of $V$ such that $V = X + Y$. Then $M = (U + X) + Y$. Since $M$ is amply ss-supplemented, $U + X$ has an ss-supplement $Y' \subseteq Y$ in $M$. It follows that $X + Y' \subseteq V$. By the minimality of $V$, we have $V = X + Y'$. In addition, $X \cap Y' \subseteq (U + X) \cap Y' \ll Y'$, that is, $X \cap Y' \ll Y'$. Since $(U + X) \cap Y'$ is semisimple, $X \cap Y'$ is also semisimple by [3, 8.1.5]. It means that $Y'$ is an ss-supplement of $X$ in $V$. Finally, $V$ is amply ss-supplemented.

The next result gives a useful characterization of amply ss-supplemented modules.

Theorem 35. Let $M$ be a module. Then, $M$ is amply ss-supplemented if and only if every submodule $U$ of $M$ is of the form $U = X + Y$, where $X$ is ss-supplemented and $Y \subseteq \text{Soc}_s(M)$.

Proof. Let $U$ be a submodule of $M$. Since $M$ is ss-supplemented, $U$ has an ss-supplement $V$ in $M$. Then $M = U + V$. By the assumption, there exists a submodule $X$ of $U$ such that $X$ is an ss-supplement of $V$ in $M$. Put $Y = U \cap V$. Since $V$ is an ss-supplement of $U$ in $M$, we have that $Y \subseteq \text{Soc}_s(V) \subseteq \text{Soc}_s(M)$. Applying the modular law, we get $U = U \cap M = U \cap (X + V) = X + U \cap V = X + Y$. Note that $X$ is ss-supplemented by Lemma 34.

Conversely, let $U$ be a submodule of $M$. By the assumption, there exist submodules $X$ and $Y$ of $M$ such that $U = X + Y$, $X$ ss-supplemented and $Y \subseteq \text{Soc}_s(M)$. By Proposition 23, $U$ is ss-supplemented. Hence $M$ is amply ss-supplemented from Proposition 33.

The next result is crucial.

Corollary 36. For a module $M$, the following statements are equivalent:

1. $M$ is amply ss-supplemented,
2. Every submodule of $M$ is ss-supplemented,
3. Every submodule of $M$ is amply ss-supplemented.

Note that it is not in general true that any submodule of an amply supplemented module is (amply) supplemented. Let $R$ be a local Dedekind domain which is not field. Suppose that $M = R^{(n)}$. Then, $M$ is not (amply) supplemented. The group $F = R \times M$ can be converted to a ring by the following operation: $(x, y) \cdot (x', y') = (xx', xy' + x'y)$ where $x, x' \in R$ and $y, y' \in M$. Then $F$ is a commutative local ring and so $F$ is amply supplemented. Put $L = \{0\} \times M$. Therefore, $L$ is an ideal of $F$. Hence the submodule $L$ of $F$ is not a (amply) supplemented $F$-module.
A module $M$ is said to be $\pi$-projective if whenever $U$ and $V$ are submodules of $M$ such that $M = U + V$, there exists an endomorphism $f$ of $M$ such that $f(M) \subseteq U$ and $(1 - f)(M) \subseteq V$. Hollow (local) modules and self-projective modules are $\pi$-projective and $\pi$-projective supplemented modules are amply supplemented. Similarly, we show that $\pi$-projective ss-supplemented modules are amply ss-supplemented. The proof is virtually the same that of [7, 41.15], but we give it for completeness.

**Proposition 37.** Let $M$ be a $\pi$-projective and ss-supplemented module. Then $M$ is amply ss-supplemented.

**Proof.** Let $U$ and $V$ be submodules of $M$ such that $M = U + V$. Since $M$ is $\pi$-projective, there exists an endomorphism $f$ of $M$ such that $f(M) \subseteq U$ and $(1 - f)(M) \subseteq V$. Note that $(1 - f)(U) \subseteq U$. Let $V'$ be an ss-supplement of $U$ in $M$. Then $M = f(M) + (1 - f)(M) = f(M) + (1 - f)(U + V') \subseteq U + (1 - f)(V')$, so that $M = U + (1 - f)(V')$. Note that $(1 - f)(V')$ is a submodule of $V$. Let $y \in U \cap (1 - f)(V')$. Then, $y \in U$ and $y = (1 - f)(x) = x - f(x)$ for some $x \in V'$. Next $x = y + f(x) \in U$ so that $y \in (1 - f)(U \cap V')$. Since $U \cap V' << V'$, $U \cap (1 - f)(V') = (1 - f)(U \cap V') << (1 - f)(V')$ by [7, 19.3(4)]. By [3, 8.1.5], $U \cap (1 - f)(V') = (1 - f)(U \cap V')$ is semisimple because $U \cap V'$ is semisimple. Thus $(1 - f)(V')$ is an ss-supplement of $U$ in $M$. Therefore $M$ is amply ss-supplemented module.

Since every projective module is $\pi$-projective, the following result follows from Proposition 37 and Corollary 36.

**Corollary 38.** Any submodule of a projective ss-supplemented module is ss-supplemented.

Now, we characterize the rings whose modules are ss-supplemented. Firstly, we need the following lemmas.

**Lemma 39.** Let $M$ be a projective module. Then $M$ is ss-supplemented if and only if it is supplemented and $\text{Rad}(M) \subseteq \text{Soc}(M)$.

**Proof.** Suppose that $M$ is projective supplemented module. Therefore we have $\text{Rad}(M) << M$ by [7, 42.5]. Then the proof is obvious from Theorem 20. □

**Lemma 40.** Let $R$ be a ring. Then every left $R$-module is ss-supplemented if and only if every left $R$-module is the sum of all strongly local submodules.

**Proof.** Assume that every left $R$-module $M$ is ss-supplemented. Then, by [7, 43.9], $R$ is left perfect. This implies that $R$ is a left max ring. Applying Corollary 32, $M$ is the sum of all strongly local submodules of $M$. The converse follows from Theorem 30. □
Theorem 41. The following statements are equivalent for a ring $R$:

1. $R$ is $ss$-supplemented,
2. $R$ is semiperfect and $\text{Rad}(R) \subseteq \text{Soc}(R)$,
3. $R$ is semilocal and $\text{Rad}(R) \subseteq \text{Soc}(R)$,
4. Every projective left $R$-module is (amply) $ss$-supplemented,
5. Every left $R$-module is (amply) $ss$-supplemented,
6. Every left $R$-module is the sum of all strongly local submodules,
7. $R$ is a finite sum of strongly local submodules,
8. Every maximal left ideal of $R$ has an $ss$-supplement in $R$.

Proof. (1) $\implies$ (2) $\implies$ (3) By Corollary 21 and [7, 42.6].
(3) $\implies$ (4) Let $M$ be a projective $R$-module. Then, by [7, 21.17 (2)], we can write $\text{Rad}(M) = \text{Rad}(R)M \subseteq \text{Soc}(R)M = \text{Soc}(M)$. From [7, 43.9] and Lemma 39, the proof is completed.
(4) $\implies$ (5) follows [7, 18.6] and Proposition 26.
(5) $\implies$ (6) By Lemma 40.
(6) $\implies$ (7) is obvious.
(7) $\implies$ (8) By Theorem 30.
(8) $\implies$ (1) By Corollary 31.

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