ASYMPTOTIC BEHAVIOUR OF RESONANCE EIGENVALUES
OF THE SCHRÖDINGER OPERATOR WITH A MATRIX
POTENTIAL

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Abstract. We will discuss the asymptotic behaviour of the eigenvalues of a Schrödinger operator with a matrix potential defined by the Neumann boundary condition in $L^2_m(F)$, where $F$ is a $d$-dimensional rectangle and the potential is an $m \times m$ matrix with $m \geq 2$, $d \geq 2$, when the eigenvalues belong to the resonance domain, roughly speaking they lie near the planes of diffraction.

1. Introduction

In this paper, we consider the Schrödinger operator with a matrix potential $V(x)$ defined by the differential expression

$$L\phi = -\Delta \phi + V\phi$$

and the Neumann boundary condition

$$\frac{\partial \phi}{\partial n}|_{\partial F} = 0,$$

in $L^2_m(F)$ where $F$ is the $d$ dimensional rectangle $F = [0,a_1] \times [0,a_2] \times \ldots \times [0,a_d]$, $\partial F$ is the boundary of $F$, $m \geq 2$, $d \geq 2$, $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal of the boundary $\partial F$, $\Delta$ is a diagonal $m \times m$ matrix whose diagonal elements are the scalar Laplace operators $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, $V$ is a real valued symmetric matrix $V(x) = (v_{ij}(x))$, $i, j = 1, 2, \ldots, m$, $v_{ij}(x) \in L^2(F)$, that is, $V^T(x) = V(x)$.

We denote the operator defined by (1)-(2) by $L(V)$, the eigenvalues and the corresponding eigenfunctions of $L(V)$ by $\Lambda_N$ and $\Psi_N$, respectively.

The eigenvalues of the operator $L(0)$ which is defined by the differential expression (1) when $V(x) = 0$ and the boundary condition (2) are $|\gamma|^2$, and the
corresponding eigenspaces are \( E_\gamma = \text{span}\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\} \), where

\[
\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \Gamma = \frac{\Gamma^0}{2},
\]

\[
\frac{\Gamma^0}{2} = \{\left(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \ldots, \frac{n_d\pi}{a_d}\right) : n_k \in \mathbb{Z}^+ \cup \{0\}, k = 1, 2, \ldots, d\},
\]

\[
\Phi_{\gamma,j}(x) = (0, \ldots, 0, u_\gamma(x), 0, \ldots, 0), j = 1, 2, \ldots, m,
\]

and the non-zero component of \( \Phi_{\gamma,j}(x) \) is \( u_\gamma(x) = \cos \frac{n_1\pi}{a_1} x_1 \cos \frac{n_2\pi}{a_2} x_2 \cdots \cos \frac{n_d\pi}{a_d} x_d \), which stands in the \( j \)th component. In particular, \( u_0(x) = 1 \) when \( \gamma = (0, 0, \ldots, 0) \).

It can be easily calculated that the norm of \( u_\gamma(x) \), \( \gamma \in \frac{\Gamma^0}{2} \), in \( L_2(F) \) is \( \sqrt{\frac{\mu(F)}{|A_\gamma|}} \), where \( \mu(F) \) is the measure of the \( d \)-dimensional parallelepiped \( F \), \( |A_\gamma| \) is the number of vectors in \( A_\gamma = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \frac{\Gamma}{2} : |\alpha_k| = |\gamma^k|, k = 1, 2, \ldots, d\} \), \( \frac{\Gamma}{2} = \{\left(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \ldots, \frac{n_d\pi}{a_d}\right) : n_k \in \mathbb{Z}, k = 1, 2, \ldots, d\} \).

From now on, \( \langle ., . \rangle \) and \( (., .) \) will denote the inner products in \( L_2^n(F) \) and \( L_2(F) \), respectively.

Since \( \{u_\gamma(x)\}_{\gamma \in \frac{\Gamma^0}{2}} \) is a complete system in \( L_2(F) \), for any \( q(x) \) in \( L_2(F) \) we have

\[
q(x) = \sum_{\gamma \in \frac{\Gamma^0}{2}} \frac{|A_\gamma|}{\mu(F)} (q, u_\gamma) u_\gamma(x).
\]  

In our study, it is convenient to use the equivalent decomposition (see [9])

\[
q(x) = \sum_{\gamma \in \frac{\Gamma^0}{2}} q_\gamma u_\gamma(x),
\]  

(4)

where \( q_\gamma = \frac{1}{\mu(F)} (q(x), u_\gamma(x)) \) for the sake of simplicity. That is, the decomposition (3) and (4) are equivalent for any \( d \geq 2 \). Thus, according to (4), each matrix element \( v_{ij}(x) \in L_2(F) \) of the matrix \( V(x) \) can be written in its Fourier series expansion

\[
v_{ij}(x) = \sum_{\gamma \in \frac{\Gamma}{2}} v_{ij\gamma} u_\gamma(x),
\]  

(5)

\[
v_{ij\gamma} = \frac{(v_{ij}, u_\gamma)}{\mu(F)}, \quad (v_{ij}, u_\gamma) = \frac{1}{\mu(F)} \int_F v_{ij}(x) u_\gamma(x) dx \quad \text{and} \quad v_{ij0} = \frac{1}{\mu(F)} \int_F v_{ij}(x) dx \quad i, j = 1, 2, \ldots, m.
\]

We assume that \( l > \frac{(d+20)(d-1)}{2} + d + 3 \) and the Fourier coefficients \( v_{ij\gamma} \) of \( v_{ij}(x) \) satisfy

\[
\sum_{\gamma \in \frac{\Gamma}{2}} |v_{ij\gamma}|^2 (1 + |\gamma|^2) < \infty,
\]  

(6)

for each \( i, j = 1, 2, \ldots, m \). Let \( \rho \) be a large parameter, \( \rho \gg 1 \) and \( \alpha \) be a positive number with \( 0 < \alpha < \frac{1}{\rho^2 + 25} \) then for \( \Gamma(\rho^2) = \{\gamma \in \frac{\Gamma}{2} : 0 \leq |\gamma| < \rho^2\} \) and \( p = l - d \)}
the condition (6) implies that

\[ v_{ij}(x) = \sum_{\gamma \in \Gamma(p^{\alpha})} v_{i\gamma} u_{\gamma}(x) + O(p^{-\rho(n)}). \]  

(7)

Here \( O(p^{-\rho(n)}) \) is a function in \( L_2(F) \) with norm of order \( p^{-\rho(n)} \). Furthermore, by (6), we have

\[ M_{ij} \equiv \sum_{\gamma \in \frac{1}{p} \mathbb{Z}^d} |v_{i\gamma}| < \infty, \]

(8)

for all \( i, j = 1, 2, \ldots, m \).

Notice that, if a function \( q(x) \) is sufficiently smooth \( q(x) \in W^2_2(F) \) and the support of \( \nabla q(x) = (\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, \ldots, \frac{\partial q}{\partial x_d}) \) is contained in the interior of the domain \( F \), then \( q(x) \) satisfies condition (6) (See [7]). There is also another class of functions \( q(x) \), such that \( q(x) \in W^2_2(F) \),

\[ q(x) = \sum_{\gamma \in \Gamma} q_{\gamma} u_{\gamma}(x), \]

which is periodic with respect to a lattice

\[ \Omega = \{(m_1 a_1, m_2 a_2, \ldots, m_d a_d) : m_k \in \mathbb{Z}, k = 1, 2, \ldots, d\} \]

and thus it also satisfies condition (6).

As in [17]-[22], we divide \( \mathbb{R}^d \) into two domains: Resonance and Non-resonance domains. In order to define these domains, let us introduce the following sets:

Let \( 0 < \alpha < \frac{1}{d+2\alpha}, \alpha_k = 3^k \alpha, k = 1, 2, \ldots, d-1 \) and

\[ V_b(p^{\alpha_1}) = \{ x \in \mathbb{R}^d : ||x|^2 - |x + b|^2 < p^{\alpha_1} \} \]

\[ E_1(p^{\alpha_1}, p) = \bigcup_{b \in \Gamma(p^{p\alpha})} V_b(p^{\alpha_1}) \]

\[ U(p^{\alpha_1}, p) = \mathbb{R}^d \setminus E_1(p^{\alpha_1}, p) \]

\[ E_k(p^{\alpha_k}, p) = \bigcup_{\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p^{p\alpha})} \left( \bigcap_{i=1}^{k} V_{\gamma_i}(p^{\alpha_k}) \right) \]

where \( b \neq 0, \gamma_i \neq 0, i = 1, 2, \ldots, k \) and the intersection \( \bigcap_{i=1}^{k} V_{\gamma_i}(p^{\alpha_k}) \) in \( E_k \) is taken over \( \gamma_1, \gamma_2, \ldots, \gamma_k \) which are linearly independent vectors and the length of \( \gamma_i \) is not greater than the length of the other vector in \( \Gamma \bigcap \gamma_i \). The set \( U(p^{\alpha_1}, p) \) is said to be a non-resonance domain, and the eigenvalue \( |\gamma|^2 \) is called a non-resonance eigenvalue if \( \gamma \in U(p^{\alpha_1}, p) \). The domains \( V_b(p^{\alpha_1}) \), for \( b \in \Gamma(p^{p\alpha}) \) are called resonance domains and the eigenvalue \( |\gamma|^2 \) is a resonance eigenvalue if \( \gamma \in V_b(p^{\alpha_1}) \).
As noted in [20]-[21], the domain $V_b(\rho^{a_1}) \setminus E_2$, called a single resonance domain, has asymptotically full measure on $V_b(\rho^{a_1})$, that is,
\[
\frac{\mu((V_b(\rho^{a_1}) \setminus E_2) \cap B(q))}{\mu(V_b(\rho^{a_1}) \cap B(q))} \to 1, \quad \text{as } \rho \to \infty,
\]
where $B(\rho) = \{x \in \mathbb{R}^d : |x| = \rho\}$, if
\[
2\alpha_2 - \alpha_1 + (d + 3)\alpha < 1, \quad \alpha_2 > 2\alpha_1, \tag{9}
\]
hold. Since $0 < \alpha < \frac{1}{\pi - 20}$, the conditions in (9) hold.

In most cases, it is important to know the asymptotic behavior of the eigenvalues of the Schrödinger operator $L(V)$. In this paper, [3] and [8], we construct the asymptotic formulas in the high energy region for eigenvalues of the operator $L(V)$. In [3], we obtain the asymptotic formulas of arbitrary order for the eigenvalue of $L(V)$ corresponding to the non-resonance eigenvalues $|\gamma|^2$ of $L(0)$ in arbitrary dimension $d \geq 2$.

In [8], we constructed the high energy asymptotics of arbitrary order for the eigenvalue of $L(V)$ corresponding to resonance eigenvalue $|\gamma|^2$ when $\gamma$ belongs to the special single resonance domains $V_b(\rho^{a_2}) \setminus E_2$, where $\delta$ is from $\{e_1, e_2, \ldots, e_d\}$ and $e_1 = \left(\frac{\pi}{a_1}, 0, \ldots, 0\right), \ldots, e_d = \left(0, \ldots, \frac{\pi}{a_d}\right)$, $d \geq 2$.

In this paper, we study the case for which $|\gamma|^2$ is a resonance eigenvalue. More precisely, in Theorem (1) and (2) of Section (2), we assume that $\gamma \in \bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{a_k}) \setminus E_{k+1}$, $k = 1, 2, \ldots, d - 1$ and $\gamma \notin V_{\gamma_i}(\rho^{a_k})$ for $k = 1, 2, \ldots, d$ and prove that the corresponding eigenvalue of $L(V)$ is close to the sum of the eigenvalue of the matrix $V_0$ and the eigenvalue of the matrix $C = C(\gamma, \gamma_1, \ldots, \gamma_k)$ (See (14)).

In Section (3), this time we assume that $\gamma \in V_b(\rho^{a_2}) \setminus E_2$, $\delta \in \left[\frac{\pi}{2}\right] \setminus \{e_1, e_2, \ldots, e_d\}$, that is, $\gamma$ is in a single resonance domain and we prove the main result Theorem (7) which gives a connection between the eigenvalues of $L(V)$ corresponding to a single resonance domain and the eigenvalues of the Sturm-Liouville operators.

Note that, the case $\delta = e_i, \ i = 1, 2, \ldots, d$, was considered in [8], by a different but simpler method and better formulas were obtained.

2. Asymptotic Formulas for the Eigenvalues in the Resonance Domain

We assume that $\gamma \notin V_{\gamma_i}(\rho^{a_k})$ for $k = 1, 2, \ldots, d$, and $|\gamma|^2$ is a resonance eigenvalue of the operator $L(0)$, that is, $\gamma \in \bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{a_k}) \setminus E_{k+1}$, $k = 1, 2, \ldots, d - 1$, such that $|\gamma| \sim \rho$ where $|\gamma| \sim \rho$ means that $|\gamma|$ and $\rho$ are asymptotically equal, that is, there exist $c_1, c_2$ satisfying the inequality $c_1\rho \leq |\gamma| \leq c_2\rho$, $c_i, \ i = 1, 2, 3, \ldots$
are positive real constants which do not depend on $\rho$. To obtain the asymptotic formulas for the eigenvalues of $L(V)$ corresponding to $|\gamma|^2$ we use the binding formula (see (9) in [3])

$$(\Lambda_N - |\gamma|^2) \langle \Psi_N, \Phi_{\gamma,j} \rangle = \langle \Psi_N, V \Phi_{\gamma,j} \rangle. \tag{10}$$

Now, we decompose $V(x)\Phi_{\gamma,j}(x)$ with respect to the basis $\{\Phi_{\gamma,i}(x)\}_{\gamma \in \mathbb{F}, j = 1, 2, \ldots, m}$. By definition of $\Phi_{\gamma,j}(x)$, it is obvious that

$$V(x)\Phi_{\gamma,j}(x) = (v_{1j}(x)u_{\gamma}(x), \ldots, v_{mj}(x)u_{\gamma}(x)). \tag{11}$$

Substituting the decomposition (7) of $v_{ij}(x)$ in (11), we get

$$V(x)\Phi_{\gamma,j}(x) = \left( \sum_{\gamma \in \Gamma(\rho^s)} v_{1j,\gamma}u_{\gamma}(x)u_{\gamma}(x), \ldots, \sum_{\gamma \in \Gamma(\rho^s)} v_{mj,\gamma}u_{\gamma}(x)u_{\gamma}(x) + O(\rho^{-p\alpha}). \right)$$

Since $\gamma$ does not belong to the domains $V_{ek}(\rho^{p_1})$, for each $k = 1, 2, \ldots, d$, we may use the following equation

$$\sum_{\gamma \in \Gamma(\rho^s)} v_{ij,\gamma}u_{\gamma}(x)u_{\gamma}(x) = \sum_{\gamma \in \Gamma(\rho^s)} v_{ij,\gamma}u_{\gamma-\gamma}(x)$$

which is proved in [9] (see equation (18) in [9]), and obtain

$$V(x)\Phi_{\gamma,j}(x) = \left( \sum_{\gamma \in \Gamma(\rho^s)} v_{1j,\gamma}u_{\gamma-\gamma}(x), \ldots, \sum_{\gamma \in \Gamma(\rho^s)} v_{mj,\gamma}u_{\gamma-\gamma}(x) + O(\rho^{-p\alpha}) \right)$$

Substituting (12) into (10), we obtain

$$\langle \Psi_N, \Phi_{\gamma,j} \rangle = \frac{\langle \Psi_N, V \Phi_{\gamma,j} \rangle}{(\Lambda_N - |\gamma|^2)} = \sum_{i=1}^{m} \sum_{\gamma \in \Gamma(\rho^s)} v_{ij,\gamma} \frac{\langle \Psi_N, \Phi_{\gamma-\gamma,i} \rangle}{(\Lambda_N - |\gamma|^2)} + O(\rho^{-p\alpha}) \tag{13}$$

for every vector $\gamma \in \mathbb{F}$, satisfying the condition

$$|\Lambda_N - |\gamma|^2| > \frac{1}{2} \rho^{p_1}.$$

Letting $p_1 = \lfloor \frac{p+1}{2} \rfloor$, that is, $p_1$ is the integer part of $\frac{p+1}{2}$, we define the following sets

$$B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) = \{ b : b = \sum_{i=1}^{k} n_i \gamma_i, n_i \in \mathbb{Z}, |b| < \frac{1}{2} \rho^{\frac{1}{2} n_k+1} \},$$

$$B_k(\gamma) = \gamma + B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) = \{ \gamma + b : b \in B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) \},$$

$$B_k(\gamma, p_1) = B_k(\gamma) + \Gamma(p_1 \rho^s).$$
Let $h, \tau = 1, 2, \ldots, b_k$ denote the vectors of $B_k(\gamma, p_1)$, $b_k$ the number of the vectors in $B_k(\gamma, p_1)$. By its definition, it can easily be obtained that $b_k = O(\epsilon^{2\alpha})$, since $\alpha_k = 3^k \alpha$, $2 \leq k \leq d$. We define the $mb_k \times mb_k$ matrix $C = C(\gamma, \gamma_1, \ldots, \gamma_k)$ by

$$C = \begin{bmatrix} |h|^2 I - V_0 & V_{h_1 - h_2} & \cdots & V_{h_1 - h_{b_k}} \\ V_{h_2 - h_1} & |h|^2 I - V_0 & \cdots & V_{h_2 - h_{b_k}} \\ \vdots & \vdots & \ddots & \vdots \\ V_{h_{b_k} - h_1} & V_{h_{b_k} - h_2} & \cdots & |h_{b_k}|^2 I - V_0 \end{bmatrix}, \quad (14)$$

where $V_{h_{r-\xi}}, \tau, \xi = 1, 2, \ldots, b_k$ are the $m \times m$ matrices defined by

$$V_{h_{r-\xi}} = \begin{bmatrix} v_{11_{h_{r-\xi}}} & v_{12_{h_{r-\xi}}} & \cdots & v_{1m_{h_{r-\xi}}} \\ v_{21_{h_{r-\xi}}} & v_{22_{h_{r-\xi}}} & \cdots & v_{2m_{h_{r-\xi}}} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1_{h_{r-\xi}}} & v_{m2_{h_{r-\xi}}} & \cdots & v_{mm_{h_{r-\xi}}} \end{bmatrix}. \quad (15)$$

Writing equation (13) for all $h_{r} \in B_k(\gamma, p_1), \tau = 1, 2, \ldots, b_k$ and $j = 1, 2, \ldots, m$, we get

$$(\Lambda_N - |h|^2) < \Psi_N, \Phi_{h_{r}, j} = \sum_{i=1}^{m} \sum_{\gamma \in \Gamma(\rho^s)} v_{ij \gamma} < \Psi_N, \Phi_{h_{r-\gamma}, i} > + O(\rho^{-\alpha}). \quad (16)$$

Similar system of equations for quasi-periodic boundary condition was investigated in [19], [21] and [22]. More recently, in [22], Lemma 2.2.1. states that for $\gamma \in \left( \bigcap_{i=1}^{k} V_{\gamma_i}(\rho^s) \right) \setminus E_{k+1}, h_{r} \in B_k(\gamma, p_1)$ and $\gamma', \gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma(\rho^s)$, if $h_{r} - \gamma' \notin B_k(\gamma, p_1)$ then

$$||\gamma|^2 - |h_{r} - \gamma' - \gamma_1 - \ldots - \gamma_s|^2| > \frac{1}{5} \rho^{\alpha_{k+1}}, \quad (17)$$

for $s = 0, 1, 2, \ldots, p_1 - 1$.

Thus, if an eigenvalue $\Lambda_N$ of $L(V)$ satisfies

$$|\Lambda_N - |\gamma|^2| < \frac{1}{2} \rho^{\alpha_1}, \quad (18)$$

then by (17) and (18), we have

$$|\Lambda_N - |h_{r} - \gamma' - \gamma_1 - \ldots - \gamma_s|^2| > \frac{1}{6} \rho^{\alpha_{k+1}}. \quad (19)$$

Now, we prove that if (18) holds then

$$O(\rho^{-\alpha}) = \sum_{i=1}^{m} \sum_{\gamma \in \Gamma(\rho^s)} v_{ij \gamma} < \Psi_N, \Phi_{h_{r-\gamma}, i} > \quad (20)$$
for any $j = 1, 2, \ldots, m$. Here we remark that $\gamma^j \neq 0$. If it were the case, then we would have from $h_r - \gamma^j \notin B_k(\gamma, p_1)$ that $h_r \notin B_k(\gamma, p_1)$ which is a contradiction. So, to prove (20), we argue as Theorem 2.2.2 (a) of [22]: Since $\Lambda_N$ satisfies the inequality (18), by (19) (for $s = 0$) we have $\| \Lambda_N - | h_r - \gamma^j |^2 \| > \frac{1}{6} \rho^{\alpha_{k+1}}$. Using this, in the equation (13) instead of $\gamma$ we write $h_r - \gamma^j$ to get

$$< \Psi_N, \Phi_{h_r - \gamma^j} > = \sum_{i=1}^{m} \sum_{\gamma \in \Gamma(\rho^\alpha)} v_{ij} \gamma_1 < \Psi_N, \Phi_{h_r - \gamma - \gamma_1} > \frac{\Lambda_N - | h_r - \gamma^j |^2}{(\Lambda_N - | h_r - \gamma^j |^2)} + O(\rho^{-\alpha}).$$

(21)

Substituting this equation (21) into the right hand side of (20), we obtain

$$\sum_{\gamma \notin \Gamma(\rho^\alpha)} v_{ij} \gamma < \Psi_N, \Phi_{h_r - \gamma^j} > = \sum_{\gamma \notin \Gamma(\rho^\alpha)} v_{ij} \gamma \frac{\Lambda_N - | h_r - \gamma^j |^2}{(\Lambda_N - | h_r - \gamma^j |^2)} \sum_{i=1}^{m} \sum_{\gamma \notin \Gamma(\rho^\alpha)} v_{ij} \gamma_1 < \Psi_N, \Phi_{h_r - \gamma - \gamma_1} > + O(\rho^{-\alpha}).$$

In this manner, iterating $p_1$ times, we get

$$\sum_{\gamma \notin \Gamma(\rho^\alpha)} v_{ij} \gamma < \Psi_N, \Phi_{h_r - \gamma^j} > = \sum_{i_1, i_2, \ldots, i_{p_1}} \sum_{\gamma \notin \Gamma(\rho^\alpha)} v_{ij} \gamma \frac{\Lambda_N - | h_r - \gamma^j |^2}{(\Lambda_N - | h_r - \gamma^j |^2)} \sum_{i_{p_1}} < \Psi_N, \Phi_{h_r - \gamma - \gamma_1 - \cdots - \gamma_{p_1}} > + O(\rho^{-\alpha}).$$

Taking norm of both sides of the last equality, using (19), the relation (8) and the fact that $p_1 \alpha_{k+1} \geq p_1 \alpha_2 > p \alpha$, we obtain

$$| \sum_{\gamma \notin \Gamma(\rho^\alpha)} v_{ij} \gamma < \Psi_N, \Phi_{h_r - \gamma^j} > | = O(\rho^{-\alpha}),$$

which implies (20). Therefore, the equation (16) becomes

$$(\Lambda_N - | h_r |^2) < \Psi_N, \Phi_{h_r, j} > = \sum_{i=1}^{m} \sum_{\gamma \notin \Gamma(\rho^\alpha)} v_{ij} \gamma < \Psi_N, \Phi_{h_r - \gamma^j} > + O(\rho^{-\alpha}).$$

(22)

Since $h_r - \gamma^j \in B_k(\gamma, p_1)$, using the notation $h_{\xi} = h_r - \gamma^j$, the decomposition (22) can be written as

$$(\Lambda_N - | h_r |^2) < \Psi_N, \Phi_{h_r, j} > = \sum_{i=1}^{m} \sum_{h_r - h_{\xi} \notin \Gamma(\rho^\alpha)} v_{ij} h_r - h_{\xi} < \Psi_N, \Phi_{h_{\xi}, i} > + O(\rho^{-\alpha}).$$

(23)
Isolating the terms where \( h_{\tau} - h_{\xi} = 0 \) in (23), we get

\[
(\Lambda_N - |h_{\tau}|^2) < \Phi_N, \Phi_{h_{\tau}, i} > = \sum_{i=1}^{m} v_{i0} < \Phi_N, \Phi_{h_{\tau}, i} >
+ \sum_{i=1}^{m} \sum_{h_{\tau} - h_{\xi} \in V(\alpha)} v_{ij} h_{\tau} - h_{\xi} < \Phi_N, \Phi_{h_{\xi}, i} >
+ O(\rho^{-p\alpha}).
\]

Writing the equation (24) for all \( j = 1, 2, \ldots, m \) and for any \( \tau = 1, 2, \ldots, b_k \), we get the system of equations

\[
[(\Lambda_N - |h_{\tau}|^2) I - V_0] A(N, h_{\tau}) = \sum_{i=1}^{b_k} V_{h_{\tau} - h_{\xi}} A(N, h_{\xi}) + O(\rho^{-p\alpha}),
\]

where \( I \) is an \( m \times m \) identity matrix, \( V_{h_{\tau} - h_{\xi}} \) is given by (15), \( O(\rho^{-p\alpha}) = (O(\rho^{-p\alpha}), \ldots, O(\rho^{-p\alpha})) \) is an \( m \times 1 \) vector and \( A(N, h_{\xi}) \) is the \( m \times 1 \) vector

\[
A(N, h_{\xi}) = ( < \Phi_N, \Phi_{h_{\xi}, 1} >, < \Phi_N, \Phi_{h_{\xi}, 2} >, \ldots, < \Phi_N, \Phi_{h_{\xi}, m} >)
\]

for any \( \xi = 1, 2, \ldots, b_k \). Letting \( \lambda_{N, \tau} = \Lambda_N - |h_{\tau}|^2 \), we have

\[
\begin{bmatrix}
\lambda_{N, 1} I - V_0 & -V_{h_1 - h_2} & \cdots & -V_{h_1 - h_{b_k}} \\
-V_{h_2 - h_1} & \lambda_{N, 2} I - V_0 & \cdots & -V_{h_2 - h_{b_k}} \\
\vdots & \vdots & \ddots & \vdots \\
-V_{h_{b_k} - h_1} & -V_{h_{b_k} - h_2} & \cdots & \lambda_{N, b_k} I - V_0
\end{bmatrix}
\begin{bmatrix}
A(N, h_1) \\
A(N, h_2) \\
\vdots \\
A(N, h_{b_k})
\end{bmatrix}
= \begin{bmatrix}
O(\rho^{-p\alpha}) \\
O(\rho^{-p\alpha}) \\
\vdots \\
O(\rho^{-p\alpha})
\end{bmatrix}.
\]

We may write the system (27) as

\[
[\Lambda_N I - C] A(N, h_1, h_2, \ldots, h_{b_k}) = O(\rho^{-p\alpha}),
\]

where \( I \) is an \( mb_k \times mb_k \) identity matrix, \( C \) is given by (14), \( A(N, h_1, h_2, \ldots, h_{b_k}) \) is the \( mb_k \times 1 \) vector

\[
A(N, h_1, h_2, \ldots, h_{b_k}) = (A(N, h_1), A(N, h_2), \ldots, A(N, h_{b_k}))
\]

and the right side of the system (28) is the \( mb_k \times 1 \) vector whose norm is

\[
|O(\rho^{-p\alpha})| = O(\sqrt{b_k\rho^{-p\alpha}}).
\]

**Theorem 1.** Let \( |\gamma|^2 \) be a resonance eigenvalue of the operator \( L(0) \), that is, \( \gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}, \ k = 1, 2, \ldots, d - 1 \) where \( |\gamma| \sim \rho \), and \( \Lambda_N \) an eigenvalue.
of the operator \( L(V) \) for which (18) holds and its corresponding eigenfunction \( \Psi_N \) satisfies
\[
|\Phi_{\gamma,j}, \Psi_N| > c_4\rho^{-c_{\alpha}}.
\] (31)

Then there exists an eigenvalue \( \eta_s(\gamma) \), \( 1 \leq s \leq mb_k \) of the matrix \( C \) such that
\[
\Lambda_N = \eta_s(\gamma) + O(\rho^{-(p-c-\frac{d}{4})\alpha}).
\]

Proof. Since (18) is satisfied, (28) holds. Then multiplying both sides of the equation (28) by \([\Lambda_N I - C]^{-1}\), then taking norm of both sides and by (30), we get
\[
|\mathcal{A}(N, h_1, h_2, \ldots, h_{mb})| \leq \| [\Lambda_N I - C]^{-1} \| O(\sqrt{b_k}\rho^{-p\alpha}).
\] (32)

Using the fact that \( \gamma \) is one of \( h_1, h_2, \ldots, h_r \) (See definition of \( B_k(\gamma, p_1) \)) and hence by (31) and (32), we obtain
\[
c_5\rho^{-c_{\alpha}} < |\mathcal{A}(N, h_1, h_2, \ldots, h_{mb})| \leq \| [\Lambda_N I - C]^{-1} \| \sqrt{b_k}c_5\rho^{-p\alpha}.
\]

Since \([\Lambda_N I - C]^{-1}\) is symmetric matrix with the eigenvalues \( \frac{1}{\Lambda_N - \eta_s(\gamma)} \), \( s = 1, \ldots, mb_k \), we have
\[
\max_{s=1,\ldots,mb_k} |\Lambda_N - \eta_s(\gamma)|^{-1} = \| [\Lambda_N I - C]^{-1} \| > c_7c_8^{-1}b_k^{-\frac{1}{2}}\rho^{-c_{\alpha}+p\alpha},
\]

where \( b_k = O(\rho^{\frac{d}{4}3^{d\alpha}}) \), thus
\[
\min_{s=1,2,\ldots,mb_k} |\Lambda_N - \eta_s(\gamma, \lambda_i)| \leq c_9\rho^{-(p-c-\frac{d}{4})\alpha},
\]

and
\[
\Lambda_N = \eta_s(\gamma, \lambda_i) + O(\rho^{-(p-c-\frac{d}{4})\alpha}).
\]

\( \square \)

**Theorem 2.** Let \(|\gamma|^2\) be a resonance eigenvalue of the operator \( L(0) \), that is, \( \gamma \in \bigcap_{i=1}^k V_i(\rho_{\alpha_i}) \setminus E_{k+1} \), \( k = 1, 2, \ldots, d - 1 \) where \(|\gamma| \sim \rho, \eta_s(\gamma) \) an eigenvalue of the matrix \( C \) such that \( |\eta_s(\gamma) - |\gamma|^2| < \frac{3}{5}\rho^{\alpha_1} \). Then there is an eigenvalue \( \Lambda_N \) of the operator \( L(V) \) satisfying
\[
\Lambda_N = \eta_s(\gamma) + O(\rho^{-(p-c-\frac{d}{4})\alpha} + \frac{d}{4}).
\] (33)

Proof. By the general perturbation theory, there is an eigenvalue \( \Lambda_N \) of the operator \( L(V) \) such that \( |\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1} \) holds. Thus one can use the system (28) and we prove the theorem for this eigenvalue \( \Lambda_N \):

Let \( \eta_s, s = 1, 2, \ldots, mb_k \) be an eigenvalue of the matrix \( C \) and \( \theta_s = (\theta_s^1, \theta_s^2, \ldots, \theta_s^{mb_k})_{mb_k \times 1} \) the corresponding normalized eigenvector, where \( \theta_s^\tau = (\theta_s^{\tau^1}, \theta_s^{\tau^2}, \ldots, \theta_s^{\tau^m})_{m \times 1}, \tau = 1, 2, \ldots, b_k \). Multiplying the equation (28) by \( \theta_s \), since \( C \) is symmetric (see (14) and (15)), we get
\[
|\Lambda_N - \eta_s| |\mathcal{A}(N, h_1, h_2, \ldots, h_{mb}) \cdot \theta_s| = |\mathcal{O}(\rho^{-p\alpha}) \cdot \theta_s|.
\] (34)
By using \( b_k = O(d^{\frac{d}{2}}) \), (30) and the Cauchy Schwartz Inequality for the right hand side of (34), we have

\[
|A_N - \eta_s| |A_N(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s| = O(\rho^{-\alpha + \frac{d}{2}}).
\]

(35)

So we need to prove that

\[
|A_N(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s| > \gamma_{10} \rho^{-\frac{d}{2}} - \frac{d}{4},
\]

(36)

from which the theorem follows.

For this purpose, we first consider the decomposition of the matrix \( C \) as

\[
C = A + B,
\]

where

\[
A = \begin{bmatrix}
|h_1|^2I & 0 \\
0 & |h_{b_k}|^2I
\end{bmatrix}, \quad B = \begin{bmatrix}
V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\
V_{h_2-h_1} & V_0 & \cdots & V_{h_2-h_{b_k}} \\
\vdots & \ddots & \ddots & \vdots \\
V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & V_0
\end{bmatrix}.
\]

(37)

The eigenvalues and the corresponding eigenspaces of the matrix \( A \) are \(|h_{\tau}|^2 \) and \( E_\tau = \text{span}\{e_j : (\tau - 1)m + 1 \leq j \leq \tau m\} \), respectively, where

\[
\{e_j = (0, \ldots, 0, 1, 0, \ldots, 0)\}_{j=1}^{mb_k}
\]

is the standard basis of \( R^{mb_k} \). Now, we use the following notation

\[
\theta_s(h_{\tau,j}) = \theta_s \cdot e_j = \theta_s^{(j)} \quad \text{if} \quad (\tau - 1)m + 1 \leq j \leq \tau m,
\]

(38)

for \( \tau = 1, 2, \cdots, b_k \).

Multiplying \((A + B)\theta_s = \eta_s \theta_s \) by \( e_j \), since \( A \) and \( B \) are symmetric, we get

\[
(\eta_s - |h_{\tau}|^2)\theta_s(h_{\tau,j}) = \theta_s \cdot B e_j
\]

(39)

and \((\tau - 1)m + 1 \leq j \leq \tau m\), and \( \tau = 1, 2, \cdots, b_k \).

On the other hand, if we consider the sum of the elements in the i-th row of the matrix \( B \), by (8)

\[
\sum_{j=1}^{mb_k} \sum_{\tau=1}^m v_{ij}h_{j-h_{\tau}} < \sum_{j=1}^m M_{ij},
\]

(40)

for all \( i = 1, 2, \ldots, m \). Since \( B \) is a symmetric matrix and by (40), the sum of elements in each row of \( B \) is less then \( M = \max_{i=1,2,\ldots,m} \{ \sum_{j=1}^m M_{ij} \} \), the eigenvalues of \( B \) are also less then \( M \) from which we have \( \| B \| \leq M \).

Thus, by (26), (36), (38), we have

\[
|A(N, h_1, \ldots, h_{b_k}) \cdot \theta_s| = |(\psi_N, \sum_{\tau=1}^{b_k} \sum_{j=1}^m \theta_s(h_{\tau,j}) \phi_{h_{\tau,j}})|,
\]

(41)
which, together with Parseval’s relation, imply

\[ \begin{align*}
1 &= \left\| \sum_{\tau=1}^{b_k} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) \Phi_{h_{\tau,i}} \right\|^2 \\
&= \sum_{N:|\Lambda_N-\gamma|^2 \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau=1}^{b_k} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
&\quad + \sum_{N:|\Lambda_N-\gamma|^2 < \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau=1}^{b_k} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2. 
\end{align*} \tag{42} \]

Now we estimate the first summation in the expression (42):

\[ \begin{align*}
&\sum_{N:|\Lambda_N-\gamma|^2 \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau=1}^{b_k} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
&= \sum_{N:|\Lambda_N-\gamma|^2 \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s-|h_{\tau}|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
&\quad + \sum_{N:|\Lambda_N-\gamma|^2 \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s-|h_{\tau}|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
&< 2 \sum_{N:|\Lambda_N-\gamma|^2 \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s-|h_{\tau}|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
&\quad + 2 \sum_{N:|\Lambda_N-\gamma|^2 \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s-|h_{\tau}|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2. \tag{43} 
\end{align*} \]

Using Bessel’s inequality, Parseval’s relation, orthogonality of the functions \( \Phi_{h_{\tau,i}}(x) \), \( \tau = 1, 2, \ldots, b_k \), \( i = 1, 2, \ldots, m \), the binding formula (39) and \( \| B \| \leq M \), we have

\[ \begin{align*}
&\sum_{N:|\Lambda_N-\gamma|^2 \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s-|h_{\tau}|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
&\leq \left\| \sum_{\tau:|\eta_s-|h_{\tau}|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) \Phi_{h_{\tau,i}} \right\|^2 \\
&= \sum_{\tau:|\eta_s-|h_{\tau}|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \left| \theta_s(h_{\tau,i}) \right|^2 \| \Phi_{h_{\tau,i}} \|^2 \\
&= \sum_{\tau:|\eta_s-|h_{\tau}|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \left| \theta_s \cdot B e_i \right|^2 \| \eta_s - |h_{\tau}|^2 \|^2 = O(\rho^{-2\alpha_1}). \tag{44} 
\end{align*} \]
The assumption $|\eta_s - |\gamma|^2| < \frac{3}{8}\rho^{2\alpha_1}$ of the theorem and $|\eta_s - |h^*|^2| < \frac{1}{4}\rho^{2\alpha_1}$ imply that $||\gamma|^2 - |h^*|^2| < \frac{1}{8}\rho^{2\alpha_1}$. So by the well-known formula

$$\frac{1}{\Lambda_N - |h^*|^2} = \frac{1}{\Lambda_N - |\gamma|^2} \left\{ \sum_{n=0}^{k} \left( \frac{|h^*|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2} \right)^n + O(\rho^{-(k+1)\alpha_1}) \right\},$$

for $|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}$, and $||\gamma|^2 - |h^*|^2| < \frac{1}{8}\rho^{2\alpha_1}$, using (39), we have

$$\sum_{N:|\Lambda_N-|\gamma|^2| \geq \frac{1}{8}\rho^{2\alpha_1}} \frac{1}{\tau:|\eta_s-|h^*|^2| < \frac{1}{4}\rho^{\alpha_1}} \sum_{j=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_{\tau,i}} > |^2$$

$$= \sum_{N:|\Lambda_N-|\gamma|^2| \geq \frac{1}{8}\rho^{2\alpha_1}} \frac{1}{\tau:|\eta_s-|h^*|^2| < \frac{1}{4}\rho^{\alpha_1}} \sum_{j=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_{\tau,i}} > \Lambda_N - |h^*|^2$$

$$\leq \sum_{N:|\Lambda_N-|\gamma|^2| \geq \frac{1}{8}\rho^{2\alpha_1}} (k + 1) \sum_{\tau:|\eta_s-|h^*|^2| < \frac{1}{4}\rho^{\alpha_1}} \sum_{j=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_{\tau,i}} > \Lambda_N - |\gamma|^2$$

$$+ \sum_{N:|\Lambda_N-|\gamma|^2| \geq \frac{1}{8}\rho^{2\alpha_1}} (k + 1) \sum_{\tau:|\eta_s-|h^*|^2| < \frac{1}{4}\rho^{\alpha_1}} \sum_{j=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_{\tau,i}} > \left( \frac{|h^*|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2} \right)^2$$

$$+ \sum_{N:|\Lambda_N-|\gamma|^2| \geq \frac{1}{8}\rho^{2\alpha_1}} (k + 1) \sum_{\tau:|\eta_s-|h^*|^2| < \frac{1}{4}\rho^{\alpha_1}} \sum_{j=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_{\tau,i}} > O(\rho^{-(k+1)\alpha_1}) |^2.$$ (45)

To calculate the order of each term in (45), we use Bessel’s inequality and the orthogonality of $\Phi_{h_{\tau,i}}$. So we have

$$2 \sum_{N:|\Lambda_N-|\gamma|^2| \geq \frac{1}{8}\rho^{2\alpha_1}} (k + 1)$$

$$\times \left( \sum_{\tau:|\eta_s-|h^*|^2| < \frac{1}{4}\rho^{\alpha_1}} \sum_{j=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_{\tau,i}} > \left( \frac{|h^*|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2} \right)^r \right)^2$$

$$= 2 \sum_{N:|\Lambda_N-|\gamma|^2| \geq \frac{1}{8}\rho^{2\alpha_1}} (k + 1) \left( \frac{|h^*|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2} \right)^{2(r+1)}.$$
\[
\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_{\tau,i}} > (|h_\tau|^2 - |\gamma|^2)^r \geq c_{11}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \\
\times \sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2 - |\gamma|^2)^r V \Phi_{h_{\tau,i}} \geq c_{12}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \\
\times \sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2 - |\gamma|^2)^r V \Phi_{h_{\tau,i}} \leq c_{13}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \\
\times \sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2 - |\gamma|^2)^r V \Phi_{h_{\tau,i}} \geq c_{14}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \\
\times \sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2 - |\gamma|^2)^r V \Phi_{h_{\tau,i}} \leq c_{15}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2 - |\gamma|^2)^r V \Phi_{h_{\tau,i}} \geq O(\rho^{-2(r+1)\alpha_1}),
\]

for \( r = 0, 1, 2, \ldots, k \). Now let \( K \) be the number of \( h_\tau \) satisfying \( |\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1} \), then the order of the last summation in (46) is:

\[
\sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} (k+1) \\
\times \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_{\tau,i}} > O(\rho^{-(k+1)\alpha_1}) \geq K \sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} (k+1) \\
\times \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} |O(\rho^{-(k+1)\alpha_1})|^2 \cdot |\theta_s(h_{\tau,i})|^2 \cdot | < \Psi_N, V \Phi_{h_{\tau,i}} > \geq c_{16} \cdot K \cdot \rho^{-2(k+1)\alpha_1} \cdot \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1}} \| V(x) \Phi_{h_{\tau,i}} \|^2 \leq c_{17} \cdot K^2 \cdot M^2 \cdot \rho^{-2(k+1)\alpha_1} = K^2 \cdot 0(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}),
\]
since $K = O(\rho^d\alpha_d)$ and we can always choose $k$ in $O(\rho^{-2(k+1)\alpha_1})$ such that

$$K^2 \cdot O(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}),$$

which together with the estimations (44), (45), and (46) imply

$$O(\rho^{-\alpha_1}) = \sum_{N:|\Lambda_N-\gamma|^2 \geq \frac{1}{2}\rho^{2\alpha_1}} \left| \frac{b_k}{\rho} \sum_{\tau=1}^{m} \sum_{i=1}^{m} \theta_s(h_{\tau, i}) < \Psi_N, \Phi_{h_{\tau, i}} > \right|^2.$$

Therefore, from the decomposition (42) we have

$$1 - O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N-\gamma|^2 < \frac{1}{2}\rho^{2\alpha_1}} \left| \frac{b_k}{\rho} \sum_{\tau=1}^{m} \sum_{i=1}^{m} \theta_s(h_{\tau, i}) < \Psi_N, \Phi_{h_{\tau, i}} > \right|^2.$$

Since the number of indexes $N$ satisfying $|\Lambda_N-\gamma|^2 < \frac{1}{2}\rho^{2\alpha_1}$ is less than $\rho^{d-1}$, we have

$$1 - O(\rho^{-2\alpha_1}) \leq \rho^{d-1} \max_{N:|\Lambda_N-\gamma|^2 < \frac{1}{2}\rho^{2\alpha_1}} \left\{ \frac{b_k}{\rho} \sum_{\tau=1}^{m} \sum_{i=1}^{m} \theta_s(h_{\tau, i}) < \Psi_N, \Phi_{h_{\tau, i}} > \right\}^2$$

which implies together with the relation (41) that

$$|A(N, h_1, h_2, \ldots, h_b) : \theta_s|^2 \geq \frac{1 - O(\rho^{-2\alpha_1})}{\rho^{d-1}}.$$

It follows from the equation (35) and the estimation (48) that

$$\Lambda_N = \eta_s + \frac{O(\rho^{-\alpha_1/2})}{O(\rho^{-\alpha_1})},$$

that is, (36) holds.

3. ASYMPTOTIC FORMULAS FOR THE EIGENVALUES IN A SINGLE RESONANCE DOMAIN

Now, we investigate in detail the eigenvalues of $L(V)$ in a single resonance domain. In order the inequalities

$$0 < \alpha < \frac{1}{d+20}, \quad 2\alpha_2 - \alpha_1 + (d+3)\alpha < 1 \quad (49)$$

and

$$\alpha_2 > 2\alpha_1, \quad (50)$$

to be satisfied, we can choose $\alpha, \alpha_1$ and $\alpha_2$ as follows

$$\alpha = \frac{1}{d+p}, \quad \alpha_1 = \frac{p_2}{d+p}, \quad \alpha_2 = \frac{2p_2 + 1}{d+p}.$$
where \( p_2 = \left\lfloor \frac{\alpha}{3} \right\rfloor - 1 \). Let \( \gamma \in V_\delta(\rho^\alpha) \setminus E_2, \delta \in \pi \setminus \{ e_i \} \), where \( \delta \) is minimal in its direction. Consider the following sets:

\[
B_1(\delta) = \{ b : b = n\delta, n \in Z, |b| < \frac{1}{2}\rho^{2\alpha_2} \},
\]

\[
B_1(\gamma) = \gamma + B_1(\delta) = \{ \gamma + b : b \in B_1(\delta) \},
\]

\[
B_1(\gamma, p_1) = B_1(\gamma) + \Gamma(p_1\rho^\alpha).
\]

As before, denote by \( h, \tau, \sigma = 1, 2, \ldots, b_1 \) the vectors of \( B_1(\gamma, p_1) \), where \( b_1 \) is the number of vectors in \( B_1(\gamma, p_1) \). Then the matrix \( C(\gamma, \delta) = (c_{ij}), i, j = 1, 2, \ldots, mb_1 \) is defined by

\[
C(\gamma, \delta) = \begin{bmatrix}
|h_1|^2 I - V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_b} \\
V_{h_2-h_1} & |h_2|^2 I - V_0 & \cdots & V_{h_2-h_b} \\
\vdots & \vdots & \ddots & \vdots \\
V_{h_b-h_1} & V_{h_b-h_2} & \cdots & |h_b|^2 I - V_0
\end{bmatrix},
\]

(51)

where \( V_{h_i-h_{i+1}}, \tau, \sigma = 1, 2, \ldots, b_1 \) are the \( m \times m \) matrices defined by (15). Also we define the matrix \( D(\gamma, \delta) = (c_{ij}) \) for \( i, j = 1, 2, \ldots, ma_1 \), where \( h_1, h_2, \ldots, h_{a_1} \) are the vectors of \( B_1(\gamma, p_1) \cap \{ \gamma + n\delta : n \in Z \} \), and \( a_1 \) is the number of vectors in \( B_1(\gamma, p_1) \cap \{ \gamma + n\delta : n \in Z \} \). Clearly \( a_1 = O(\rho^{2\alpha_2}) \).

**Lemma 3.**

a) If \( \eta_{j_s} \) is an eigenvalue of the matrix \( C(\gamma, \delta) \) such that \( |\eta_{j_s} - |h_s|^2| < M \) for \( s = 1, 2, \ldots, a_1, 1 + (s-1)m \leq j_s \leq ms \), then

\[
|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4} \rho^{\alpha_2}, \ \forall \tau = a_1 + 1, a_1 + 2, \ldots, b_1.
\]

b) If \( \eta_{j_s} \) is an eigenvalue of the matrix \( C(\gamma, \delta) \) such that \( |\eta_{j_s} - |h_s|^2| < M \) for \( s = a_1 + 1, a_1 + 2, \ldots, b_1 \) and \( 1 + (s-1)m \leq j_s \leq ms \), then

\[
|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4} \rho^{\alpha_2}, \ \forall \tau = 1, 2, \ldots, a_1.
\]

**Proof.** First we prove

\[
|\eta_{j_s} - |h_s|^2| \geq \frac{1}{3} \rho^{\alpha_2}, \ \forall s \leq a_1, \ \forall \tau > a_1.
\]

(52)

By definition, if \( s \leq a_1 \) then \( h_s = \gamma + n\delta \), where \( |n\delta| < \frac{1}{2}\rho^{2\alpha_2} + p_1\rho^\alpha \). If \( \tau > a_1 \) then \( h_\tau = \gamma + s'\delta + a \), where \( |s'\delta| < \frac{1}{2}\rho^{2\alpha_2}, a \in \Gamma(p_1\rho^\alpha) \setminus \delta R \). Therefore

\[
|h_\tau|^2 - |h_s|^2 = 2(\gamma \cdot a + 2s' \delta \cdot a + 2s' \gamma \cdot \delta + |s'\delta|^2 + |a|^2 - 2n\gamma \cdot \delta - |n\delta|^2.
\]

Since \( \gamma \notin V_\delta(\rho^\alpha) \), \( |a| < p_1\rho^\alpha \), we have

\[
|2\gamma \cdot a| > \rho^{\alpha_2} - c_0\rho^{2\alpha}.
\]

The relation \( \gamma \in V_\delta(\rho^{\alpha_1}) \) and the inequalities for \( s' \) and \( n \) imply that

\[
2s' \gamma \cdot \delta + 2s' \gamma \cdot a + |a|^2 - 2n\gamma \cdot \delta = O(\rho^{\frac{\alpha_2}{2} + \alpha_1}),
\]

\[
|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4} \rho^{\alpha_2}, \ \forall \tau = 1, 2, \ldots, a_1.
\]
Thus (52) follows from these relations, since $\frac{1}{2}\alpha_2 + \alpha_1 < \alpha_2$ and $\frac{1}{2}\alpha_2 + \alpha < \alpha_2$.

The eigenvalues of $D(\gamma, \delta)$ and $C(\gamma, \delta)$ lay in $M$-neighborhood of the numbers $|h_k|^2$ for $k = 1, 2, ... , a_1$ and for $k = 1, 2, ... , b_1$, respectively. The inequality (52) shows that one can enumerate the eigenvalues $\eta_j$ ($j = 1, 2, \ldots , mb_1$) of $C$ in the following way:

$$\eta_j \equiv \eta_{j_1}, \quad j_s \leq ma_1, \quad 1 + (s - 1)m \leq j_s \leq sm$$

when for $s \leq a_1$, $\eta_j$ lay in $M$-neighborhood of $|h_s|^2$ and

$$\eta_j \equiv \eta_{j_m}, \quad j_t \geq ma_1, \quad 1 + (\tau - 1)m \leq j_t \leq \tau m$$

when for $\tau > a_1$, $\eta_j$ lay in $M$-neighborhood $|h_{\tau}|^2$. Then by (52), we get

$$|\eta_{j_	au} - |h_{\tau}|^2| > \frac{1}{4} \rho^\alpha,$$

for $s \leq a_1$, $\tau > a_1$ and $s > a_1$, $\tau \leq a_1$.

Now, using the notation $h_s = \gamma - (\frac{s}{2})\delta$ if $s$ is even, $h_s = \gamma + (\frac{s-1}{2})\delta$ if $s$ is odd, for $s = 1, 2, \ldots , a_1$, (without loss of generality assume that $a_1$ is even) and using the orthogonal decomposition of $\gamma \in \mathbb{R}_+$, $\gamma = \beta + (l + v(\beta))\delta$, where $\beta \in H_\delta \equiv \{x \in \mathbb{R}^d : x \cdot \delta = 0\}$, $l \in \mathbb{Z}$, $v \in [0, 1)$ we can write the matrix $D(\gamma, \delta)$ as

$$D(\gamma, \delta) = |\beta|^2 I + E(\gamma, \delta),$$

where $I$ is a maximal identity matrix and $E(\gamma, \delta)$ is

$$E(\gamma, \delta) = \begin{bmatrix}
(v_\delta) & v_{\delta - \delta} & \ldots & v_{\delta - b_1}\delta \\
v_{-\delta} & (v_{-\delta - 2\delta}) + v_0 & \ldots & v_{-\delta - b_1}\delta \\
v_\delta & v_\delta & \ldots & v_\delta + v_0 \\
\ldots & \ldots & \ldots & \ldots \\
v_{\delta - \delta} & \ldots & \ldots & (v_{\delta - \delta - 2\delta}) + v_0
\end{bmatrix}$$

Denote $n_k = \frac{k}{2}$ if $k$ is even, $n_k = \frac{k-1}{2}$ if $k$ is odd. The system \( \{e^{i(nk + 1 + v)t} : k = 1, 2, \ldots \} \) is a basis in $L^2_\gamma[0, 2\pi]$. Let $T(\gamma, \delta) \equiv T(P(t), \beta)$ be the operator in $\ell_2$ corresponding to the Sturm-Liouville operator $T$, generated by

$$-|\beta|^2 Y''(t) + P(t)Y(t) = \mu Y(t), \quad \mu = \frac{1}{4} \rho^\alpha,$$

$$Y(t + 2\pi) = e^{i2\pi v(\beta)}Y(t),$$

where $P(t) = \langle p_{ij}(t) \rangle, p_{ij}(t) = \sum_{k=1}^\infty v_{ijnk}\delta e^{inkt}, v_{ijnk}\delta = (\langle v_{ij}(x) \rangle, \frac{1}{|A_{nk}\delta|} \sum_{\alpha \in A_{nk}\delta} e^{i(\alpha \cdot x)}), t = x \cdot \delta$. It means that $T(\gamma, \delta)$ is the infinite matrix $(Te^{i((n_k + 1 + v)t)}e^{i((l+n_m + v)t)})$, $k, m = 1, 2, \ldots$.
To find the relation between the eigenvalues of $L(V)$ in a single resonance domain and the eigenvalues of the Sturm-Liouville operators defined by (55), we need the following theorems.

**Theorem 4.** Let $2V(1) \subset E$ and $|\gamma| \sim \rho$. Then, for any eigenvalue $\eta_{js}(\gamma)$ of the matrix $C(\gamma, \delta)$ satisfying

$$|\eta_{js} - |h_s|^2| < M, \quad 1 + (s - 1)m \leq j_s \leq sm, \quad s = 1, 2, \ldots, a_1$$

there exists an eigenvalue $\tilde{\eta}_{k(js)}$ of the matrix $D(\gamma, \delta)$ such that

$$\eta_{js} = \tilde{\eta}_{k(js)} + O(\rho^{-\frac{3}{2}a_2}).$$

**Proof.** Let $\eta_{js}$ be an eigenvalue of the matrix $C(\gamma, \delta)$ satisfying (56) and $\eta_{js} = (\theta_{js}^1, \theta_{js}^2, \ldots, \theta_{js}^n)_{mb \times 1}$ be the corresponding normalized eigenvector, $|\theta_{js}| = 1$. Now, we consider the decomposition $C = A + B$ and the matrices $A, B$ which are defined in (37). Writing the binding formula (39) for $\eta_{js}$ and using (38), we get

$$\eta_{js} = \tilde{\eta}_{k(js)} + O(\rho^{-\frac{3}{2}a_2}).$$

For simplicity, we use the following notation in the sequel:

$$e_{\zeta} = e \quad \text{if} \quad 1 + (\zeta - 1)m \leq k \leq \zeta m, \quad \zeta = 1, \ldots, b_1; \quad Be_i \cdot e_{k_1} = Be_{r,i} \cdot e_{\xi,k_1} = b(\tau, i, \xi, k_1).$$

Thus, substituting the orthogonal decomposition

$$Be_i = Be_{r,i} = \sum_{\xi = 1, 2, \ldots, b_1}^{1 + (m - 1)\xi \leq k_1 \leq m\xi} b(\tau, i, \xi, k_1)e_{\xi,k_1}$$

into the formula (57), we get

$$\eta_{js} = \tilde{\eta}_{k(js)} + O(\rho^{-\frac{3}{2}a_2}).$$

It is clear that

$$b(\tau, i, \xi, k_1) = \begin{cases} 0 & \text{if } \xi = \tau, \\ \psi_{k_1} & \text{if } \xi \neq \tau, \end{cases}$$

which implies

$$\sum_{\xi = 1, 2, \ldots, b_1}^{1 + (m - 1)\xi \leq k_1 \leq m\xi} b(\tau, i, \xi, k_1) = \sum_{\xi = 1, 2, \ldots, b_1}^{1 + (m - 1)\xi \leq k_1 \leq m\xi} \psi_{k_1}.$$
Thus one has

\[
(\eta_{j_*} - |h|_r^2)\theta_{j_*}(h, i) = \sum_{\xi=1,2,...,b_1} \sum_{v} \theta_{j_*}(h, k_1)
\]

\[
= \sum_{\xi=1,2,...,a_1} \sum_{v} \theta_{j_*}(h, k_1)
\]

\[
+ \sum_{\xi=a_1+1,2,...,b_1} \sum_{v} \theta_{j_*}(h, k_1).
\]

(58)

Now, writing the equation (58) for all \( h, \tau = 1, 2, ..., a_1 \), we get the system of linear algebraic equations:

\[
(\eta_{j_*} - |h|_1^2)\theta_{j_*}(h, i) = \sum_{\xi=1,2,...,a_1} \sum_{v} \theta_{j_*}(h, k_1)
\]

\[
= \sum_{\xi=a_1+1,2,...,b_1} \sum_{v} \theta_{j_*}(h, k_1)
\]

(59)

Using the binding formula (57), the relation (53), and \( \| B \| \leq M \), for any \( \tau = 1, 2, \ldots, a_1 \), we find

\[
\sum_{\xi=a_1+1,2,...,b_1}^{\xi \neq \tau} v_{k_1 h_\xi - h_r} \theta_{j_*}(h, k_1) = \sum_{\xi=a_1+1,2,...,b_1}^{\xi \neq \tau} v_{k_1 h_\xi - h_r} \theta_{j_*}(h, k_1)
\]

\[
\leq \sum_{\xi=a_1+1,2,...,b_1}^{\xi \neq \tau} |v_{k_1 h_\xi - h_r}| \theta_{j_*}(h, k_1)
\]

\[
\leq 4\rho^{-a_2} M \sum_{\xi=a_1+1,2,...,b_1}^{\xi \neq \tau} |v_{k_1 h_\xi - h_r}|
\]

\[
\leq 4\rho^{-a_2} M^2
\]

= \mathcal{O}(\rho^{-a_2})

(60)
and
\begin{align*}
\sum_{\tau=a_1+1, \ldots, b_1 \atop i=1, 2, \ldots, m} |\theta_{j_s}(h_\tau, i)|^2 &= \sum_{\tau=a_1+1, \ldots, b_1 \atop i=1, 2, \ldots, m} \left| \frac{\theta_{j_s} \cdot B e_{\tau, i}}{(\eta_{j_s} - |h_\tau|^2)} \right|^2 \\
&= \sum_{\tau=a_1+1, \ldots, b_1 \atop i=1, 2, \ldots, m} \frac{|B \theta_{j_s} \cdot e_{\tau, i}|^2}{(\eta_{j_s} - |h_\tau|^2)^2} \\
&\leq 16M^2\rho^{-2\alpha_2} \\
&= O(\rho^{-2\alpha_2}).
\end{align*}

By (60) and (54), (59) becomes
\begin{align}
[\theta^1_{j_s}, \theta^2_{j_s}, \ldots, \theta^{a_1}_{j_s}]^t &= (D(\gamma, \delta) - \eta_{j_s} I)^{-1}[O(\rho^{-\alpha_2}), O(\rho^{-\alpha_2}), \ldots, O(\rho^{-\alpha_2})]^t. 
\end{align}

By the Parseval’s identity and (61), we get
\begin{align*}
\sum_{\tau=1, 2, \ldots, a_1 \atop i=1, 2, \ldots, m} |\theta_{j_s}(h_\tau, i)|^2 &= \sum_{\tau=1, 2, \ldots, b_1 \atop i=1, 2, \ldots, m} |\theta_{j_s}(h_\tau, i)|^2 - \sum_{\tau=a_1+1, \ldots, b_1 \atop i=1, 2, \ldots, m} |\theta_{j_s}(h_\tau, i)|^2 \\
&\geq 1 - O(\rho^{-2\alpha_2}).
\end{align*}

Now, taking norm of both sides in (62) and using the above inequality we have
\[\sqrt{1 - O(\rho^{-2\alpha_2})} < \left( \sum_{\tau=1, 2, \ldots, a_1 \atop i=1, 2, \ldots, m} |\theta_{j_s}(h_\tau, i)|^2 \right)^{\frac{1}{2}} \leq \| (D(\gamma, \delta) - \eta_{j_s} I)^{-1} \| O(\sqrt{\alpha_1}\rho^{-\alpha_2}).
\]

Thus
\[\max |\eta_{j_s} - \tilde{\eta}_{k(j_s)}|^{-1} > \frac{\sqrt{1 - O(\rho^{-2\alpha_2})}}{\sqrt{\alpha_1}\rho^{-\alpha_2}},
\]
or
\[\min |\eta_{j_s} - \tilde{\eta}_{k(j_s)}| = O(\sqrt{\alpha_1}\rho^{-\alpha_2}) = O(\rho^{-\frac{3}{4}\alpha_2}),
\]
where the maximum (minimum) is taken over all $\tilde{\eta}_{k(j_s)}$, $s = 1, 2, \ldots, a_1$. So the result follows.

**Theorem 5.** For any eigenvalue $\tilde{\eta}_r$ of the matrix $D(\gamma, \delta)$, there exists an eigenvalue $\eta_{j_s(\tau)}$ of the matrix $C(\gamma, \delta)$ such that
\[\eta_{j_s(\tau)} = \tilde{\eta}_r + O(\rho^{-\frac{3}{4}\alpha_2}).\]
Proof. Define the matrix \( D' = D'(\gamma, \delta) \) by

\[
D' = \begin{bmatrix}
    |h_{1}1^2I - V_0| & v_{h_{1} - h_{2}} & \cdots & v_{h_{1} - h_{a_1}} & 0 & 0 & \cdots & 0 \\
v_{h_{2} - h_{1}} & |h_{2}1^2I - V_0| & \cdots & v_{h_{2} - h_{a_1}} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{h_{a_1} - h_{1}} & v_{h_{a_1} - h_{2}} & \cdots & |h_{a_1}1^2I - V_0| & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & |h_{a_1 + 1}1^2I| & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & |h_{b_1}1^2I|
\end{bmatrix}
\] (61)

So that the spectrum of the matrix \( D' \) is

\[
\text{spec}(D') = \text{spec}(D(\gamma, \delta)) \bigcup \{ |h_{a_1 + 1}|^2, |h_{a_1 + 2}|^2, \ldots, |h_{b_1}|^2 \}
\]

\[
\equiv \{ \tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_{ma_1}, |h_{a_1 + 1}|^2, |h_{a_1 + 2}|^2, \ldots, |h_{b_1}|^2 \}.
\]

Let us denote by \( \Upsilon_\tau = (\Upsilon_1^\tau, \Upsilon_2^\tau, \ldots, \Upsilon_{ma_1}^\tau, 0, \ldots, 0)_{mb_1 \times 1} \), \( \Upsilon_1^\tau = (\Upsilon_1^t, \Upsilon_2^t, \ldots, \Upsilon_{ma_1}^t)_{m \times 1} \) the normalized eigenvector corresponding to the \( \tau \)-th eigenvalue of the matrix \( D' \), for \( \tau = 1, 2, \ldots, ma_1 \) and by \( \{ e_{k,i} \}_{i=1,2,\ldots,m} \) the eigenvector corresponding to the \( k \)-th eigenvalue \( |h_k|^2 \) of \( D' \), for \( k = a_1 + 1, a_1 + 2, \ldots, b_1 \).

Now, using (62) from the previous theorem, we have

\[
(D' - \eta_{j_\tau})[^{\theta_1}_{j_\tau} \theta_2^{j_\tau} \cdots \theta_{b_\tau}^{j_\tau}]^t = [(D(\gamma, \delta) - \eta_{j_\tau})[^{\theta_1}_{j_\tau} \theta_2^{j_\tau} \cdots \theta_{b_\tau}^{j_\tau}]^t, (|h_{a_1 + 1}|^2 - \eta_{j_\tau})^{\theta_1}_{j_\tau} + 1, \ldots, (|h_{b_1}|^2 - \eta_{j_\tau})^{\theta_{b_\tau}}_{j_\tau}]
\]

Taking inner product of both sides of the last equality by \( \Upsilon_\tau \) for \( \tau = 1, 2, \ldots, ma_1 \), using that \( D' \) is symmetric and \( D' \Upsilon_\tau = \tilde{\eta}_\tau \Upsilon_\tau \) we have

\[
(\eta_{j_\tau(\tau)} - \tilde{\eta}_\tau) \sum_{k=1}^{a_1} \theta_j^{\kappa}_{j_\tau} \cdot \Upsilon_\tau^k = \sum_{k=1}^{a_1} O(\rho^{-a_2}) \Upsilon_\tau^k,
\] (64)

For the right hand side of the equation (64) using the Cauchy-Schwarz inequality, we get

\[
|\sum_{k=1}^{a_1} O(\rho^{-a_2}) \Upsilon_\tau^k| \leq \sqrt{\sum_{k=1}^{a_1} O(\rho^{-a_2})^2} \sqrt{\sum_{k=1}^{a_1} |\Upsilon_\tau^k|^2} \leq \sqrt{a_1(\rho^{-a_2})^2} = O(\sqrt{a_1}\rho^{-a_2}),
\]

where \( a_1 = O(\rho^{-\frac{a_2}{2}}) \). Thus, the equation (64) can be written as

\[
(\eta_{j_\tau(\tau)} - \tilde{\eta}_\tau) \sum_{k=1}^{a_1} \theta_j^{\kappa}_{j_\tau} \cdot \Upsilon_\tau^k = O(\rho^{-\frac{a_2}{2}}).
\] (65)
In order to get the result, we need to show that for any \( m = 1, 2, \ldots, ma_1 \) there exists \( \theta_{j_s(\tau)} \) such that

\[
|\sum_{k=1}^{a_1} \theta_{j_s(\tau)}^k \cdot Y^k_\tau| = |\theta_{j_s(\tau)} \cdot Y_\tau| > \sqrt{1 - O(\rho^{-\frac{3}{2} \alpha_2})} > c_{18} \rho^{-\frac{1}{2} \alpha_2}.
\] (66)

For this, we consider the orthogonal decomposition \( Y_\tau = \sum_{s=1}^{mb_1} (Y_\tau \cdot \theta_{j_s}) \theta_{j_s} \) and the Parseval’s identity

\[
1 = \sum_{s=1}^{mb_1} |Y_\tau \cdot \theta_{j_s}|^2 = \sum_{s=1}^{ma_1} |Y_\tau \cdot \theta_{j_s}|^2 + \sum_{s=ma_1+1}^{mb_1} |Y_\tau \cdot \theta_{j_s}|^2.
\] (67)

First, let us show that

\[
\sum_{s=ma_1+1}^{mb_1} |Y_\tau \cdot \theta_{j_s}|^2 = O(\rho^{-\frac{3}{2} \alpha_2}).
\]

Using the decomposition \( Y_\tau = \sum_{k=1,2,\ldots,a_1} (Y_\tau \cdot e_{k,i}) e_{k,i} \), the binding formula (57) for \( C(\gamma, \delta) \) and \( A \), the relation (53), and the Bessel’s inequality we obtain the estimation

\[
\sum_{s=ma_1+1}^{mb_1} |Y_\tau \cdot \theta_{j_s}|^2 \\ \leq 16 \sum_{s=ma_1+1}^{mb_1} \rho^{-2 \alpha_2} ( \sum_{k=1,2,\ldots,a_1}^{a_1} |Y^k_\tau|^2 |\theta_{j_s} \cdot Be_{k,i}|)^2 \\ \leq 16 a_1 m \rho^{-2 \alpha_2} \left( \sum_{k=1,2,\ldots,a_1}^{a_1} |Y^k_\tau|^2 |\theta_{j_s} \cdot Be_{k,i}| \right)^2 \\ \leq 16 \rho^{-2 \alpha_2} a_1 m \sum_{s=ma_1+1}^{mb_1} |Y^k_\tau|^2 \sum_{s=ma_1+1}^{mb_1} |\theta_{j_s} Be_{k,i}|^2.
\]
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\[ \sum_{k=1,2,...,a_1}^{a_1} |\gamma^k| |B e_{k,i}|^2 \leq 16 \rho^{-2a_2} |a_1| m M^2 \sum_{k=1,2,...,a_1}^{a_1} |\gamma^k|^2 \]

\[ \leq 16 |a_1| m \rho^{-2a_2} M^2 = O(\rho^{-\frac{3}{2}a_2}). \]

Therefore one has

\[ \sum_{s=1}^{m a_1} |\gamma \cdot \theta_j|^2 = 1 - O(\rho^{-\frac{3}{2}a_2}) \]

from which it follows that there exists an eigenvector \( \theta_{j,(\tau)} \) such that (66) holds.

Dividing both sides of (65) by (66) we get the result

\[ \eta_{j,(\tau)} = \bar{\eta}_\tau + O(\rho^{-\frac{3}{2}a_2}). \]

\[ \Box \]

**Theorem 6.** For every eigenvalue \( \zeta_s \) of the Sturm-Liouville operator \( T(\gamma, \delta) \), there exists an eigenvalue \( \widetilde{\zeta}_s \) of the matrix \( \widetilde{E}(\gamma, \delta) \) such that

\[ \zeta_s = \widetilde{\zeta}_s + O(\rho^{-\frac{3}{2}a_2}). \]

**Proof.** Decompose the infinite matrix \( T(\gamma, \delta) \) as \( T(\gamma, \delta) = \begin{bmatrix} A + B \end{bmatrix} \) where the matrix \( \tilde{A} \) is defined by

\[
\tilde{A} = \begin{bmatrix}
(l + v)^2|\delta|^2 & (l - 1 + v)^2|\delta|^2 & 0 \\
(l - 1 + v)^2|\delta|^2 & (l - v)^2|\delta|^2 & 0 \\
0 & 0 & (l - \frac{a_2}{2} + v)^2|\delta|^2 \end{bmatrix}
\]

(68)

and \( \tilde{B} = T(\gamma, \delta) - \tilde{A} \). Let \( \zeta_s \) be an eigenvalue of \( T(\gamma, \delta) \), and \( \Theta_s = (\Theta_s^1, \Theta_s^2, \Theta_s^3, \ldots) \), \( \Theta_s^* = (\Theta_s^1, \ldots, \Theta_s^m) \) be the corresponding normalized eigenvector, that is, \( T \Theta_s = \zeta_s \Theta_s \). span\{\epsilon_i : (\tau - 1)m + 1 \leq i \leq \tau m\} is the eigenspace of the matrix \( \tilde{A} \) which corresponds to the eigenvalue \( |(\tau' + v)|^2 \), where \( \tau' = l - \frac{\tau}{2} \) if \( \tau \) is even, \( \tau' = l + \frac{\tau - 1}{2} \) if \( \tau \) is odd, for \( \tau = 1, 2, \ldots \) and \( \{\epsilon_i\} \) is the standard basis for \( l_2 \).

One can easily verify that

\[ \left( \zeta_s - |(\tau' + v)|^2 \right) \Theta_s^* = \Theta_s \cdot \tilde{B} e_{\tau,i} \]

(69)

where \( e_{\tau,i} \equiv e_i \), if \( (m - 1)\tau + 1 \leq i \leq m\tau \).

Using the orthogonal decomposition \( \tilde{B} e_{\tau,i} = \sum_{j=1}^{\infty} (\tilde{B} e_{\tau,i} \cdot e_{k,j}) e_{k,j} \), (69) reduces to

\[ \left( \zeta_s - |(\tau' + v)|^2 - |v_{i0}|^2 \right) \Theta_s^{*i} = \sum_{j=1}^{\infty} (\tilde{B} e_{\tau,i} \cdot e_{k,j}) \Theta_s^{*j} \]
and since $\vec{B}e_{r,i} \cdot e_{k,j} = v_{ji(n_k-n_r)\delta}$ for $k \neq r$,

$$
(\zeta_s - (\tau' + v)\delta)^2 \Theta^r_s = \sum_{j=1}^{\infty} \sum_{k=1}^{a_1} v_{ji(n_k-n_r)\delta} \Theta^{k,j}_s = \sum_{j=1}^{m} \sum_{k=a_1+1}^{\infty} v_{ji(n_k-n_r)\delta} \Theta^{k,j}_s. \tag{70}
$$

Now take any eigenvalue $\zeta_s$ of $T(\gamma, \delta)$, satisfying $|\zeta_s - |(i' + v)\delta|^2| < \sup |P(t)|$ for $s = 1, 2, ..., \frac{m+1}{2}$, where $i' = l - \frac{v}{2}$ if $s$ is even, $i' = l + \frac{v+1}{2}$ if $s$ is odd. The relations $\gamma \in \mathcal{X}_\delta(\rho^{\alpha_1})$ ($\delta \neq e_i$) and $\gamma = \beta + (l + v)\delta$, $\beta \cdot \delta = 0$ imply

$$
|2\gamma \cdot \delta + |\delta|^2| = |(l + v)|^2 + |\delta|^2 < \rho^{\alpha_1}, \quad |l| < c_{19}\rho^{\alpha_1}.
$$

Therefore, using the definition of $i'$ and $\tau'$, we have

$$
|((i' + v)\delta| < \frac{|a_1\delta|}{4} + c_{20}\rho^{\alpha_1}
$$

for $s = 1, 2, ..., \frac{a_1}{2}$ and

$$
|((\tau' + v)\delta| > \frac{|a_1\delta|}{2} - c_{21}\rho^{\alpha_1}
$$

for $\tau > a_1$. Since $|a_1| > c_{22}\rho^{\frac{\alpha_2}{2}}$ and $\alpha_2 > 2\alpha_1$, we have

$$
|((i' + v)\delta|^2 - |(\tau' + v)\delta|^2| > c_{23}\rho^{\alpha_2} \tag{71}
$$

for $s \leq \frac{a_1}{2}$, $\tau > a_1$, which implies

$$
|\zeta_s - |(\tau' + v)|^2| = ||\zeta_s - |(i' + v)\delta|^2| - |(\tau' + v)|^2| - |(i' + v)\delta|^2| > c_{24}\rho^{2\alpha_2}, \tag{72}
$$

for $s = 1, 2, ..., \frac{a_1}{2}$, $\tau > a_1$.

Since $\vec{B}$ corresponds to the operator $P : Y \rightarrow P(t)Y$ in $L^0_\beta[0, 2\pi]$, which has norm $\sup |P(t)| \leq M$. Using this, equation (69) and (72), we have for the right hand side of (70) that

$$
\left|\sum_{j=1}^{m} \sum_{k=a_1+1}^{\infty} v_{ji(n_k-n_r)\delta} \Theta^{k,j}_s \right| \left|\Theta = \frac{\Theta_s \cdot \vec{B}e_{k,j}}{\zeta_s - (k' + v)\delta^2} \right| \leq M\rho^{-\alpha_2} \sum_{j=1}^{m} \sum_{k=a_1+1}^{\infty} |v_{ji(n_k-n_r)\delta}| \leq c_{25}\rho^{-\alpha_2}, \tag{73}
$$

Therefore writing the equation (70) for all $\tau = 1, 2, ..., a_1$, and using (73) we get the following system

$$
(E(\gamma, \delta) - \zeta_s I)[\Theta^1_s, \Theta^2_s, ..., \Theta^a_s] = [O(\rho^{-\alpha_2}), O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2})], \tag{74}
$$
where $I$ is an $ma_1 \times ma_1$ identity matrix. Using $\Theta_s = \sum_{\tau=1}^{\infty} \Theta_{\tau}^s e_{\tau,i}$, the formula (69) and the inequality (72), we have

$$\sum_{\tau=a_1+1}^{\infty} |\Theta_{\tau}^s|^2 = \sum_{\tau=a_1+1}^{\infty} \left| \frac{\Theta_s \cdot B e_{\tau,i}}{s_s - |(\tau + v)\delta|^2} \right|^2 = O(\rho^{-2\alpha_2})$$

and thus

$$\sum_{\tau=1}^{a_1} |\Theta_{\tau}^s|^2 = 1 - O(\rho^{-2\alpha_2}). \quad (75)$$

Multiplying both sides of (74) by $(E(\gamma, \delta) - \zeta_s I)^{-1}$,

$$[\Theta_1, \Theta_2^s, \ldots, \Theta_{a_1}^s] = (E(\gamma, \delta) - \zeta_s I)^{-1}[O(\rho^{-\alpha_2}), \ldots, O(\rho^{-\alpha_2})],$$

then taking norm of both sides and using (75), we get

$$\sqrt{1 - O(\rho^{-2\alpha_2})} = \|(E(\gamma, \delta) - \zeta_s I)^{-1}\|O(\sqrt{a_1}\rho^{-\alpha_2})$$

or

$$\min_\tau |\zeta_s - \zeta_\tau| = O(\sqrt{\frac{\sqrt{a_1}\rho^{-\alpha_2}}{\sqrt{1 - O(\rho^{-2\alpha_2})}}} \cdot \sqrt{m} = O(\rho^{-3\alpha_2}),$$

where the minimum is taken over all eigenvalues $\zeta_\tau$ of the matrix $E(\gamma, \delta)$. Thus, the result follows.

**Theorem 7.** *(Main result)* For every $\beta \in H_\delta$, $|\beta| \sim \rho$ and for every eigenvalue $\zeta_s(v(\beta))$ of the Sturm-Liouville operator $T(\gamma, \delta)$, there is an eigenvalue $\Lambda_N$ of the operator $L(V)$ satisfying

$$\Lambda_N = |\beta|^2 + \zeta_s + O(\rho^{-1}\alpha_2).$$

**Proof.** From Theorem 6 and the definition of $E(\gamma, \delta)$, there exists an eigenvalue $\tilde{\eta}_{\tau(s)}$ of the matrix $D(\gamma, \delta)$, where $\gamma$ has a decomposition $\gamma = \beta + (\tau + v(\beta))\delta$, satisfying $\tilde{\eta}_{\tau(s)} = |\beta|^2 + \zeta_s + O(\rho^{-1}\alpha_2)$. Therefore, the result follows from Theorem 5 and Theorem 2. 

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