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Research Article

A Study On Hyperbolic Lifted Developable Surfaces

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Abstract

In this paper, we build an envelope ruled surface by means of hyperbolic lifting transformation which is performed to a plane curve,. We demonstrate that it is a developable surface in three dimensional Euclidean space and we impute the statements to be a minimal surface of this surface and constant mean curvature surface. Finally, we construct some examples and graph the curves and surfaces given in examples.

Keywords: Ruled surface, developable surface, envelope surface, curve lifting

Hiperbolik Yükseltme İle Açılabilir Yüzeyler Üzerine Bir Çalışma

Öz

Bu makalede, bir düzlemsel eğriye hiperbolik yükseltme dönüşümü uygulayarak, regle zarf yüzeyi inşa ettik. Daha sonra bu yüzeyin üç boyutlu Öklid uzayında açılabilir yüzey olduğunu gösterdik ve bu yüzeyin minimal yüzey ve sabit ortalama eğrilikli yüzey olabilmesi için koşulları verdik. Son olarak, bazı örnekler verdik ve örneklerdeki eğrilerin ve yüzeylerin grafiklerini çizdik.

Anahtar Kelimeler: Regle yüzey, açılabilir yüzey, zarf yüzeyi, eğri yükseltme

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1. Introduction

Surfaces are called developable, when they can be created through ordinary bending of a planar surface without stretching, cutting or wrinkling the material. These surfaces are characterized by only bending in one direction at a time, like the cylinder or cone. Developable surfaces are useful because they allow round forms to be made out of .at materials like plywood, sheet metal or cloth. This is why they are used for shiping building, tent sewing and fabrication of ventiliation ducts ect.

Developable surfaces have natural applications in many areas of engineering and manufacturing. For example; they are used to design the airplane wings by aircraft designer or to connect two tubes of different shapes with planar segments of metal sheets by tinsmith.

So developable surfaces are more efficient then the general surfaces. There are many works about developable surfaces in literature. Choi, Kim and Elber constructed a developable surface as the envelope of a one parameter family of tangent planes along the lifted curve on the paraboloid $z = x^2 + y^2$, by lifting a rational planar curve in (Choi et al., 1997). Izumiya and Takeuchi gave a classification of special developable surfaces under the condition of the existence of such a special curve as a geodesic in (Izumiya and Takeuchi, 2004).

In the surface theory of geometry, ruled surfaces were found by French mathematician Gaspard Monge who was a founder of constructive geometry. Recently, many mathematicians have studied the ruled surfaces on Euclidean space and Minkowski space for a long time. For the information about these topic, see, e.g., Ali et al. (2013); Çöken et al. (2008); Izumiya and Takeuchi (2003); Yu et al. (2014) for a systematic work. Also Kühnel studied a type of ruled surfaces which was called ruled W-surface and the properties of being a ruled W-surface were given in (Kühnel, 1994). Ravani and Ku showed that, predominantly, entire ruled surface could have a double infinity of Bertrand offsets, yet a linear equation should be provided among the torsion and curvature of a developable ruled surface's edge of regression for it to possess a developable Bertrand offset, in (Ravani and Ku, 1991).

A ruled surface in R^3 is surface which can be described as the set of points swept out by moving a straight line in surface. It therefore has a parametrization of the form

$$\Phi(s, v) = \alpha(s) + v\delta(s), \tag{1}$$

where α and δ are curves lying on the surface called base curve and ruling, respectively. By using the equation of ruled surface we assume that α' is never zero and δ is not identically zero. The ruled surface's rulings are asymptotic curves. In addition to this, the ruled surface's Gaussian curvature is everywhere non-positive. A ruled surface is doubly ruled if through every one of its points there are two distinct lines that lie on the surface. Cylinder, cone, helicoid, Mobius strip, right conoid are some examples of ruled surfaces and hyperbolic paraboloid and hyperboloid of one sheet are doubly ruled surfaces (Gray, 1993).

Minimal surface is a shape whose surface has the least amount of area needed to occupy space. Since the material which is needed to build shapes is least, minimal surfaces are preferable in architecture, industrial design and mathematics. So mathematicians was investigated this kind of surfaces for a long time, see (Osserman, 1986). Some well known minimal surfaces are plane, helicoid and catenoid.

In this study, we construct a developable surface as the envelope of one parameter family of tangent planes along the lifted curve which is lifted from the planar curve to the hyperbolic paraboloid $z = x^2 - y^2$. Then we give the conditions of being minimal surface and constant mean curvature surface of this developable surface. Also we illustrate some examples.

2. Preliminaries

In Choi et al. (1997), the envelope surface of the one parameter family of tangent planes along a curve of the paraboloid was given by the following procedure.

Let $A(t) = (x(t), y(t))$ be a rational planar curve which is regular (i.e $A'(t) \neq 0$). Then a curve is constructed as

$$\hat{A}(t) = \left(x(t),\ y(t),\ x(t)^2 + y(t)^2\right), \tag{2}$$

by lifting the curve A(t) to the paraboloid Q; $z = x^2 + y^2$. Let the unnormalized normal vector field of the surface Q be n(t). Then n(t) can be count up by the gradient function of the surface as;

$$n(t) = (2x(t),\ 2y(t),\ -1). \tag{3}$$

The derivative of the unnormalized normal vector field and vector product of n(t) and its derivative are easily attained as;

$$n'(t) = (2x'(t),\ 2y'(t),\ 0),$$

$$n(t) \times n'(t) = \begin{pmatrix} 2y'(t), -2x'(t), \\ 4x(t)y'(t) - 4x'(t)y(t) \end{pmatrix}.$$

By bearing in mind one parameter family of tangent planes $\{T_{\hat{A}(t)}(Q)\}$ of the surface Q; $z = x^2 + y^2$ along the curve $\hat{A}(t)$, the envelope surface is constructed by taking $n(t) \times n'(t)$ as a ruling direction and $\hat{A}(t)$ as a base curve. Therefore the envelope surface of one-parameter family of tangent planes $\{T_{\hat{A}(t)}(Q)\}$ is defined by;

$$D_{A(t)}(t, v) = \hat{A}(t) + v(n(t) \times n'(t)). \tag{4}$$

This surface is rational when the curve A(t) is rational and it is developable because its Gaussian curvature $K(t, v)$ is zero. Since the envelope surface $D_{A(t)}(t, v)$ in equation (4) is tangent to paraboloid Q; $z = x^2 + y^2$ along the curve $\hat{A}(t)$, the unnormalized normal vector field of the paraboloid Q; $z = x^2 + y^2$ and the envelope surface $D_{A(t)}(t, v)$ in (4) are same (Choi et al., 1997).

Now lets give some basic conception which can be encountered in (Do Carmo, 1976; Gray, 1993; Kühnel, 2006).

- An envelope surface is a surface that is tangent to each member of a family of surfaces.
- A developable surface possesses negative Gaussian curvature every place.
- A surface's mean curvature vanishes necessary and sufficient condition it is minimal.
- A surface's mean curvature is constant necessary and sufficient condition it is constant mean curvature surfaces. This comprises minimal surfaces as a subset , but they are dealed by special case.

For a surface given with the parametrization $\Phi(u, v)$ and the surface's normal vector field n, the Gaussian and mean curvatures are attained respectively by

$$K = \frac{eg - f^2}{EG - F^2}, \tag{5}$$

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} \tag{6}$$

where

$$\begin{aligned} E &= \langle \Phi_u, \Phi_u \rangle, & e &= \langle \Phi_{uu}, n \rangle, \\ F &= \langle \Phi_u, \Phi_v \rangle, & f &= \langle \Phi_{uv}, n \rangle, \\ G &= \langle \Phi_v, \Phi_v \rangle, & g &= \langle \Phi_{vv}, n \rangle. \end{aligned} \tag{7}$$

3. Envelope Surface Of The Hyperbolic Paraboloid Surface

Before giving the envelope surface of the hyperbolic paraboloid $z = x^2 - y^2$, lets express the following theorems and their proofs about the envelope surface $D_{A(t)}(t, v)$ of the paraboloid given above.

Remark 1. The planar curve $A(t) = (x(t), y(t))$ can be lifted to all of the quadratics and the curve $\hat{A}(t)$ can be created by the equation of the quadratic surface. Also the envelope surface of the one parameter family of tangent planes of the quadratic surface is given with the equation (4).

Theorem 1. The envelope surface $D_{A(t)}(t, v)$ given with the equation (4) of the one parameter family of tangent planes of any quadratic surface, is minimal if and only if the sequent equation is provided;

$$\langle \hat{A}'', n\rangle\langle n \times n', n \times n'\rangle + v\langle n, n' \times n''\rangle\langle n \times n', n \times n'\rangle = 0,$$

where n is the unnormalized normal vector field of the quadratic surface.

Proof: For the envelope surface $D_{A(t)}(t,v)$, the following derivatives are obtained as;

$$
\begin{aligned}
&D_t = \hat{A}' + v(n \times n''),\\
&D_v = n \times n',\\
&D_{tt} = \hat{A}'' + v(n' \times n'' + n \times n'''),\\
&D_{tv} = n \times n'',\\
&D_{vv} = 0.
\end{aligned} \tag{8}
$$

By using the equations (7), we have;

$$
\begin{aligned}
&E = \langle \hat{A}', \hat{A}' \rangle + 2v\langle \hat{A}', n \times n'' \rangle\\
&+v^2\langle n \times n'', n \times n'' \rangle,\\
&F = \langle \hat{A}', n \times n' \rangle + v\langle n \times n'', n \times n' \rangle,\\
&G = \langle n \times n', n \times n' \rangle,
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
&e = \langle \hat{A}'', n \rangle + v\langle n, n' \times n'' \rangle,\\
&f = \langle n, n \times n'' \rangle = 0,\\
&g = 0.
\end{aligned} \tag{10}
$$

From the equations (6), (9) and (10) , the mean curvature of the envelope surface $D_{A(t)}(t,v)$ is obtained as;

$$H = \frac{eG}{2(EG - F^2)},$$

which yields to

$$
\begin{aligned}
&eG = \langle \hat{A}'', n \rangle \langle n \times n', n \times n' \rangle\\
&+v\langle n' \times n'', n \rangle \langle n \times n', n \times n' \rangle.
\end{aligned}
$$

Thus the result is clear.

Theorem 2. The envelope surface $D_{A(t)}(t,v)$ given with the equation (4) of the one parameter family of tangent planes of any quadratic surface is developable.

Proof: By the equations in (9), (10) and using the Gaussian curvature in (5), we have directly;

$$K = 0$$

which means the surface is developable.

Now lets construct the envelope surface by using the planar curve $A(t) = (x(t), y(t))$ and lifting it to the hyperbolic paraboloid S; $z = x^2 - y^2$. We call this lifting as hyperbolic lifting and the curve on the surface S is imputed by

$$\hat{A}(t) = \left(x(t),\ y(t),\ x(t)^2 - y(t)^2\right). \tag{11}$$

The envelope surface of the one parameter family of tangent planes $\{T_{\hat{A}(t)}(S)\}$ of the hyperbolic paraboloid surface S; $z = x^2 - y^2$ along the curve $\hat{A}(t)$ which we call $\Phi_{A(t)}(t,v)$ same as the equation (1) is imputed by

$$\Phi_{A(t)}(t,v) = \hat{A}(t) + v(n(t) \times n'(t)). \tag{12}$$

Here $\hat{A}(t)$ is the base curve and $n(t) \times n'(t)$ is direction of ruling.

The unnormalized normal vector field n(t) of the surface S; $z = x^2 - y^2$, its derivative and the vectoral product of n(t) and its derivative can be calculated simply as;

$$
\begin{aligned}
&n(t) = (2x(t),\ -2y(t),\ -1),\\
&n'(t) = (2x'(t),\ -2y'(t),\ 0),\\
&n(t) \times n'(t) = \begin{pmatrix} -2y'(t), -2x'(t), \\ 4x'(t)y(t) - 4x(t)y'(t) \end{pmatrix}.
\end{aligned} \tag{13}
$$

Notation 1: The envelope surface of the one parameter family of tangent planes $\{T_{\hat{A}(t)}(S)\}$ of the hyperbolic paraboloid surface S; $z = x^2 - y^2$ can be also denoted with the parametric equation

$$\Phi_{A(t)}(t,v) = \begin{pmatrix} x(t) - 2vy'(t),\ y(t) - 2vx'(t), \\ x(t)^2 - y(t)^2 + 4v\begin{pmatrix} x'(t)y(t) \\ -x(t)y'(t) \end{pmatrix} \end{pmatrix}.$$

Corollary 1: The envelope surface of the one parameter family of tangent planes $\{T_{\hat{A}(t)}(S)\}$ of the hyperbolic paraboloid surface S; $z = x^2 - y^2$ is developable.

Proof: By means of the Theorem 3, it could be simply revealed.

Theorem 3. The envelope surface of the one parameter family of tangent planes $\{T_{\hat{A}(t)}(S)\}$ of the hyperbolic paraboloid surface S; $z = x^2 - y^2$ is minimal if the curve in plane is a line parametrized by

$$A(t) = (x(t), \pm x(t) + k),$$

where k is a constant.

Proof: For the envelope surface given in (12), if we use the equations (7) and (13), we get;

$$
\begin{aligned}
G &= \langle n\times n', n\times n'\rangle \\
&= 4x'(t)^2 + 4y'(t)^2 + 16\big(y(t)x'(t) - y'(t)x(t)\big)^2 \\
&= 4\big(x'(t)^2 + y'(t)^2\big) + 16\left(\frac{x(t)}{y(t)}\right)'^2 y(t)^4.
\end{aligned}
$$

For $e = \langle \hat{A}'', n\rangle + v\langle n, n'\times n''\rangle$, firstly we get;

$$\hat{A}(t) = \big(x(t),\ y(t),\ x(t)^2 - y(t)^2\big),$$

$$\hat{A}'(t) = \big(x'(t),\ y'(t),\ 2x'(t)x(t) - 2y'(t)y(t)\big),$$

$$\hat{A}''(t) = \left(x'(t),\ y'(t),\ 2\begin{pmatrix} x'(t)^2 - y'(t)^2 \\ +x(t)x''(t) \\ -y(t)y''(t)\end{pmatrix}\right),$$

and

$$\langle \hat{A}'', n\rangle = 2\big(y'(t)^2 - x'(t)^2\big),$$

$$n'\times n'' = \big(0, 0, 4\big(y'(t)x''(t) - x'(t)y''(t)\big)\big),$$

$$\langle n, n'\times n''\rangle = 4\left(\frac{y'(t)}{x'(t)}\right)' x'(t)^2.$$

Since f=0 and g=0, the mean curvature is obtained as;

$$H = \frac{eG}{2(EG - F^2)}.$$

Here by the equation of G given above, we are conscious of $G \neq 0$. Thereby if we have $\langle \hat{A}'', n\rangle = 0$ and $\langle n, n'\times n''\rangle = 0$. Then the surface is minimal. By solving the last equations, we acquire that;

$$
\begin{aligned}
&-x'(t)^2 + y'(t)^2 = 0 \\
&\Rightarrow y'(t) = |x'(t)| \\
&\Rightarrow y'(t) = \pm x'(t) \\
&\Rightarrow y(t) = \pm x(t) + k,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\frac{y'(t)}{x'(t)}\right)' = 0 \\
&\Rightarrow \left(\frac{y'(t)}{x'(t)}\right) = c \\
&\Rightarrow y'(t) = c x'(t) \\
&\Rightarrow y(t) = c\, x(t) + k,
\end{aligned}
$$

where k is a constant. By these solutions we have e=0 which means H=0.

Theorem 4. The envelope surface of the one parameter family of tangent planes $\{T_{\hat{A}(t)}(S)\}$ of the hyperbolic paraboloid surface S; $z = x^2 - y^2$ is constant mean curvature surface if the planar curve is a line parametrized by

$$A(t) = \big(x(t),\ c\,x(t) + k\big),$$

where c and k are constants.

Proof: It can be attained by a straightforward computation.

Example 1. Lets conceive a line given by

$$A(t) = (t,\ 3t),$$

then the lifted curve $\hat{A}(t)$ of $A(t)$ to the hyperbolic paraboloid is;

$$\hat{A}(t) = (t,\ 3t,\ -8t^2),$$

and the developable surface is parametrized by

$$\Phi_{A(t)}(t, v) = (t - 6v,\ 3t - 2v,\ -8t^2).$$

It follows that the Gaussian and mean curvates are calculated as;

$$K = 0, \quad H = -10240.$$

In Figure 1, we illustrate the generator curve $A(t)$, the lifted curve $\hat{A}(t)$ and the constant mean curvature surface $\Phi_{A(t)}$.

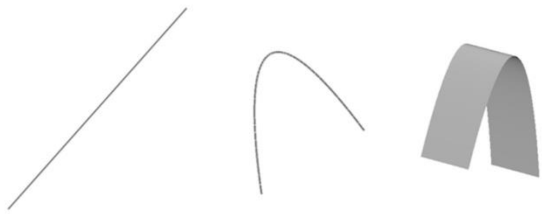

Figure 1. The generator curve $A(t)$, the lifted curve $\hat{A}(t)$, the constant mean curvature surface $\Phi_{A(t)}$, respectively.

Example 2. Let us consider the hyperbola parametrized by

$$A(t) = (\sinh t,\ \cosh t).$$

One can calculate the lifted curve of $A(t)$ to the hyperbolic paraboloid as

$$\hat{A}(t) = (\sinh t,\ \cosh t,\ -1).$$

Lastly the developable surface can be parametrized as (in Figure 2);

$$\Phi_{A(t)}(t, v) = \begin{pmatrix} \sinh t - 2v\sinh t, \\ \cosh t - 2v\cosh t,\ -1 + 4v \end{pmatrix}.$$

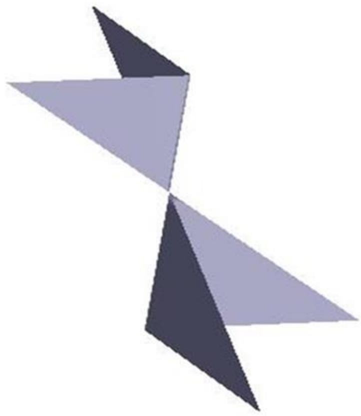

Figure 2. The developable surface $\Phi_{A(t)}$

A simple calculation implies that

$$K = 0, \quad H = -16(2v-1)^2(2\cosh t^2 + 3).$$

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