MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
8 (2) 170-177 (2020) ©MSAEN
HTTPS://DOI.ORG/10.36753/MATHENOT.672621

Some Properties of Generalized Frank Matrices

Efruz Özlem Mersin, Mustafa Bahşi* and Ayşe Dilek Maden

Abstract
In this paper, we first introduce a new generalization of Frank matrix which is a lower Hessenberg matrix. Then, we examine its algebraic structure, determinant, inverse, LU decomposition and characteristic polynomial.

Keywords: *Frank matrix; determinant; inverse; LU decomposition; characteristic polynomial.*

AMS Subject Classification (2020): *Primary: 15B99 ; Secondary: 15A09; 15A15; 15A23.*

****Corresponding author***

1. Introduction

In 1958, Frank [3] defined an $n \times n$ matrix F as

$$F = \begin{bmatrix} n & n-1 & 0 & \cdots & 0 & 0 \\ n-1 & n-1 & n-2 & \cdots & 0 & 0 \\ n-2 & n-2 & n-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}, \tag{1}$$

which is called Frank matrix [4,7]. One can easily generate the elements of matrix $F = [f_{ij}]$ by the rule:

$$f_{ij} = \begin{cases} n+1-\max(i,j), & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Frank matrix is one of the popular test matrices for eigenvalue routines, because the matrix F has well-conditioned and poorly conditioned eigenvalues [2,7]. Some properties of the Frank matrix F are reported [2,4] as: The eigenvalues of the matrix F are real and positive and also come in reciprocal pair, $\det(F) = 1$, the inverse of F is an upper Hessenberg matrix, LU decomposition of F exists, the characteristic polynomial of F has the recurrence relation

$$\chi_n(\lambda) = (1-\lambda)\chi_{n-1}(\lambda) - (n-1)\lambda\chi_{n-2}(\lambda),$$
$$\chi_1(\lambda) = 1-\lambda \text{ and } \chi_2(\lambda) = 1-3\lambda+\lambda^2.$$

Also, Varah [7] gave a generalization of the Frank matrix and showed how to compute its eigensystem accurately.

Frank matrix is also a Max matrix. There are many matrices defined on maximum and minimum concepts. Some of them have been mentioned by Kılıç and Arıkan [5]. Also, they have introduced new generalizations of the classical Max and Min matrices and have derived their inverses, LU and Cholesky decompositions and their inverse matrices.

In this paper, we define a new generalization of the Frank matrix and examine its some properties such as determinant and LU decomposition.

Received : *09–01–2020,* ***Accepted :*** *01–05–2020*
This article is a part of doctoral thesis being prepared by Efruz Özlem Mersin.

2. Structure of Generalized Frank Matrices

Consider the real n-tuple $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. The n-tuple $a = (a_1, a_2, \ldots, a_n)$ corresponds to well-known circulant, Min and Max matrices

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix}, \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_2 & a_3 & \cdots & a_n \\ a_3 & a_3 & a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & a_n \end{bmatrix},$$

respectively. For more information about the above matrices, please refer to [1,6].

Let us define a new matrix which corresponds to the real n-tuple $a = (a_1, a_2, \ldots, a_n)$ as follows:

$$F_a = \begin{bmatrix} a_n & a_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ a_{n-1} & a_{n-1} & a_{n-2} & 0 & \cdots & 0 & 0 \\ a_{n-2} & a_{n-2} & a_{n-2} & a_{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_2 & a_2 & a_2 & \cdots & a_2 & a_1 \\ a_1 & a_1 & a_1 & a_1 & \cdots & a_1 & a_1 \end{bmatrix}. \tag{3}$$

The elements of the matrix $F_a = [(f_a)_{ij}]$ is generated by the rule:

$$(f_a)_{ij} = \begin{cases} a_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

When we take $a_i = i \ (i = 1, 2, ..., n)$, the matrix F_a is reduced to the classical Frank matrix F in (1). So, we call F_a as the generalized Frank matrix. From matrix multiplication, it is easily seen that the matrix F_a is factored as

$$F_a = \widetilde{I}\Lambda P \widetilde{I} \tag{5}$$

where

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 \\ a_1 & a_2 - a_1 & \cdots & 0 & 0 \\ 0 & a_2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1} - a_{n-2} & 0 \\ 0 & 0 & \cdots & a_{n-1} & a_n - a_{n-1} \end{bmatrix}$$

and

$$\tilde{I} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}.$$

Hence, we have

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & n-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix},$$

for the classical Frank matrix.

Let V_n be a set of all $n \times n$ generalized Frank matrices as in (3). The following theorem gives us the algebraic structure of V_n.

Theorem 2.1. *V_n is an n-dimensional vector space.*

Proof. Let $F_a = [(f_a)_{ij}]$, $F_b = [(f_b)_{ij}] \in V_n$ for $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ and α, β be real numbers. If $D = \alpha F_a + \beta F_b = [d_{ij}]$, then by equation (4) we have,

$$d_{ij} = \alpha (f_a)_{ij} + \beta (f_b)_{ij} = \begin{cases} \alpha a_{n+1-\max(i,j)} + \beta b_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Also,

$$\alpha a + \beta b = (\alpha a_1 + \beta b_1, \ldots, \alpha a_n + \beta b_n) = (c_1, c_2, \ldots, c_n) = c \in \mathbb{R}^n, \tag{7}$$

where $c_i = \alpha a_i + \beta b_i$ $(i = 1, ..., n)$. If $F_c = [(f_c)_{ij}] \in V_n$ corresponds to the n-tuple $c = (c_1, c_2, \ldots, c_n)$, then

$$\begin{aligned} (f_c)_{ij} &= \begin{cases} c_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \alpha a_{n+1-\max(i,j)} + \beta b_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, considering the equations (6), (7) we get

$$F_c = F_{\alpha a + \beta b} = \alpha F_a + \beta F_b \in V_n.$$

Thus, V_n is a subspace of the vector space of all $n \times n$ matrices.

Let 1_s be an n-tuple such that its sth element is one and the others are zero. For example, $1_3 = (0, 0, 1, 0, ..., 0)$. Then for every $F_a \in V_n$, we have

$$F_a = \sum_{s=1}^{n} a_s F_{1_s}$$

and the matrices $F_{1_1}, F_{1_2}, \ldots, F_{1_n}$ are linear independent. Then the n matrices $F_{1_1}, F_{1_2}, \ldots, F_{1_n}$ form a basis for V_n. That is, the dimension of V_n is n.

□

The question is while the matrix F is generalized to the matrix F_a, how we can generalize the determinant, inverse, LU decomposition and characteristic polynomial formulas given in the study of Hake [4]? In the present paper, we seek answer to this question.

To characterize our results, we use the term a_0 in this paper. Our readers should know that $a_0 = 0$ throughout this paper.

Next section presents our results.

3. Main Results

Theorem 3.1. *The determinant of the $n \times n$ matrix F_a is*

$$\det(F_a) = \prod_{i=1}^{n} (a_i - a_{i-1}).$$

Proof. According to equation (5), we obtain

$$\det(F_a) = \det(\widetilde{I}) \det(\Lambda) \det(P) \det(\widetilde{I}).$$

Since $\det(\widetilde{I}) = \mp 1$ and $\det(P) = 1$, we get

$$\begin{aligned} \det(F_a) &= \det(\Lambda) = a_1(a_2 - a_1)(a_3 - a_2)...(a_n - a_{n-1}) \\ &= \prod_{i=1}^{n} (a_i - a_{i-1}). \end{aligned}$$

□

Corollary 3.1. *F_a is invertible if and only if $a_i \neq a_{i-1}$ for $i = 1, 2, \ldots, n$.*

Theorem 3.2. *Let the matrix $(B_a)_n = (\beta_{ij})_{i,j=1}^{n}$ be the inverse of the $n \times n$ matrix F_a. Then, we have*

$$\beta_{ij} = \begin{cases} \frac{1}{a_n - a_{n-1}}, & i = j = 1 \\ \frac{a_{n+2-i}}{(a_{n+2-i} - a_{n+1-i})\,(a_{n+1-i} - a_{n-i})}, & i = j \neq 1 \\ -\frac{1}{a_{n+2-i} - a_{n+1-i}}, & i = j + 1 \\ 0, & i > j + 1 \\ (-1)^{j-i} \prod_{k=1}^{j-i} \beta_{ii} \frac{a_{n+1-i-k}}{(a_{n+1-i-k} - a_{n-i-k})}, & i < j \end{cases}$$

where $a_i \neq a_{i-1}$ $(i = 1, 2, \ldots, n)$.

Proof. We use principle of mathematical induction on n. It is clear that the result is true for $n = 2$, that is,

$$F_a = \begin{bmatrix} a_2 & a_1 \\ a_1 & a_1 \end{bmatrix}$$

and

$$\begin{aligned} (B_a)_2 &= \frac{1}{a_2 a_1 - a_1^2} \begin{bmatrix} a_1 & -a_1 \\ -a_1 & a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a_2 - a_1} & \frac{-a_1}{(a_2 - a_1)a_1} \\ \frac{-1}{a_2 - a_1} & \frac{a_2}{(a_2 - a_1)a_1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{a_2 - a_1} & \frac{-a_1}{(a_2 - a_1)(a_1 - a_0)} \\ \frac{-1}{a_2 - a_1} & \frac{a_2}{(a_2 - a_1)(a_1 - a_0)} \end{bmatrix} \end{aligned}$$

Assume that the result is true for $n - 1$, then

$$(B_a)_{n-1} = (\beta_{ij})_{i,j=1}^{n-1} = \begin{cases} \frac{1}{a_{n-1} - a_{n-2}}, & i = j = 1 \\ \frac{a_{n+1-i}}{(a_{n+1-i} - a_{n-i})\,(a_{n-i} - a_{n-1-i})}, & i = j \neq 1 \\ -\frac{1}{a_{n+1-i} - a_{n-i}}, & i = j + 1 \\ 0, & i > j + 1 \\ (-1)^{j-i} \prod_{k=1}^{j-i} \beta_{ii} \frac{a_{n-i-k}}{a_{n-i-k} - a_{n-1-i-k}}, & i < j. \end{cases}$$

Now, we must show that the result is true for n. Let the matrices F_a and $(B_a)_n$ be partitioned as

$$F_a = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } (B_a)_n = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$A_{11} = [a_n],$$
$$A_{12} = [a_{n-1} \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0],$$
$$A_{21} = \begin{bmatrix} a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 \end{bmatrix}^T$$

and

$$A_{22} = \begin{bmatrix} a_{n-1} & a_{n-2} & 0 & 0 & \cdots & 0 & 0 \\ a_{n-2} & a_{n-2} & a_{n-3} & 0 & \cdots & 0 & 0 \\ a_{n-3} & a_{n-3} & a_{n-3} & a_{n-4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_2 & a_2 & a_2 & \cdots & a_2 & a_1 \\ a_1 & a_1 & a_1 & a_1 & \cdots & a_1 & a_1 \end{bmatrix}.$$

Using the assumption, we have $A_{22}^{-1} = (B_a)_{n-1}$. Then, the equation

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

yields

$$B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = \frac{1}{a_n - a_{n-1}},$$

$$\begin{aligned} B_{12} &= -B_{11}A_{12}A_{22}^{-1} \\ &= \begin{bmatrix} -\frac{x_1 a_{n-1}}{a_{n-1}-a_{n-2}} & \frac{x_1 a_{n-1}a_{n-2}}{(a_{n-1}-a_{n-2})(a_{n-2}-a_{n-3})} & \cdots & (-1)^{n-1}x_1\prod_{i=1}^{n-1}\frac{a_i}{a_i-a_{i-1}} \end{bmatrix} \end{aligned}$$

where $x_1 = \frac{1}{a_n - a_{n-1}}$,

$$B_{21} = -A_{22}^{-1}A_{21}B_{11} = \begin{bmatrix} -\frac{1}{a_n-a_{n-1}} & 0 & 0 & \cdots & 0 \end{bmatrix}^T$$

and

$$B_{22} = A_{22}^{-1} - A_{22}^{-1}A_{21}B_{11}A_{12}A_{22}^{-1}$$

$$= \begin{bmatrix} x_2 & -\frac{x_2 a_{n-2}}{a_{n-2}-a_{n-3}} & \frac{x_2 a_{n-2}a_{n-3}}{(a_{n-2}-a_{n-3})(a_{n-3}-a_{n-4})} & \cdots & (-1)^{n-2}x_2\prod_{i=1}^{n-2}\frac{a_i}{a_i-a_{i-1}} \\ -\frac{1}{a_{n-1}-a_{n-2}} & x_3 & -\frac{x_3 a_{n-3}}{a_{n-3}-a_{n-4}} & \cdots & (-1)^{n-3}x_3\prod_{i=1}^{n-3}\frac{a_i}{a_i-a_{i-1}} \\ 0 & -\frac{1}{a_{n-2}-a_{n-3}} & x_4 & \cdots & (-1)^{n-4}x_4\prod_{i=1}^{n-4}\frac{a_i}{a_i-a_{i-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{bmatrix}$$

where $\underset{2\le s\le n}{x_s} = \frac{a_{n+2-s}}{(a_{n+2-s}-a_{n+1-s})(a_{n+1-s}-a_{n-s})}$. Thus,

$$(B_a)_n = \begin{bmatrix} x_1 & -\frac{x_1 a_{n-1}}{a_{n-1}-a_{n-2}} & \frac{x_1 a_{n-1}a_{n-2}}{(a_{n-1}-a_{n-2})(a_{n-2}-a_{n-3})} & \cdots & (-1)^{n-1}x_1\prod_{i=1}^{n-1}\frac{a_i}{a_i-a_{i-1}} \\ -\frac{1}{a_n-a_{n-1}} & x_2 & -\frac{x_2 a_{n-2}}{a_{n-2}-a_{n-3}} & \cdots & (-1)^{n-2}x_2\prod_{i=1}^{n-2}\frac{a_i}{a_i-a_{i-1}} \\ 0 & -\frac{1}{a_{n-1}-a_{n-2}} & x_3 & \cdots & (-1)^{n-3}x_3\prod_{i=1}^{n-3}\frac{a_i}{a_i-a_{i-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{bmatrix}.$$

□

Theorem 3.3. *The LU decomposition of F_a exists for all n. Its factors $L = (l_{ij})$ and $U = (u_{ij})$ are given by*

$$l_{ij} = \begin{cases} 0, & i < j \\ 1, & i = j \\ \frac{a_{n+1-i}}{a_{n+1-j}}, & \text{otherwise} \end{cases} \quad \text{and } u_{ij} = \begin{cases} a_n, & i = j = 1 \\ \frac{(a_{n+1-i})(a_{n+2-i}-a_{n+1-i})}{a_{n+2-i}}, & i = j \neq 1 \\ a_{n-i}, & i = j-1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Matrix multiplication yields the result. □

Theorem 3.4. *The characteristic polynomial of F_a satisfies the recurrence relation*

$$P_n(\lambda) = (\lambda - a_n + a_{n-1})\,P_{n-1}(\lambda) - a_{n-1}\lambda P_{n-2}(\lambda),$$

$$P_1(\lambda) = \lambda - a_1 \text{ and } P_2(\lambda) = \lambda^2 - (a_1 + a_2)\,\lambda + a_1a_2 - a_1^2.$$

Proof. For the characteristic polynomial of F_a, we have

$$P_n(\lambda)=\begin{vmatrix} \lambda-a_n & -a_{n-1} & 0 & \cdots & 0 & 0\\ -a_{n-1} & \lambda-a_{n-1} & -a_{n-2} & \cdots & 0 & 0\\ -a_{n-2} & -a_{n-2} & \lambda-a_{n-2} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -a_2 & -a_2 & -a_2 & \cdots & \lambda-a_2 & -a_1\\ -a_1 & -a_1 & -a_1 & \cdots & -a_1 & \lambda-a_1 \end{vmatrix}$$

$$=(\lambda-a_n)\begin{vmatrix} \lambda-a_{n-1} & -a_{n-2} & 0 & \cdots & 0 & 0\\ -a_{n-2} & \lambda-a_{n-2} & -a_{n-3} & \cdots & 0 & 0\\ -a_{n-3} & -a_{n-3} & \lambda-a_{n-3} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -a_2 & -a_2 & -a_2 & \cdots & \lambda-a_2 & -a_1\\ -a_1 & -a_1 & -a_1 & \cdots & -a_1 & \lambda-a_1 \end{vmatrix}$$

$$+(a_{n-1})\begin{vmatrix} -a_{n-1} & -a_{n-2} & 0 & \cdots & 0 & 0\\ -a_{n-2} & \lambda-a_{n-2} & -a_{n-3} & \cdots & 0 & 0\\ -a_{n-3} & -a_{n-3} & \lambda-a_{n-3} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -a_2 & -a_2 & -a_2 & \cdots & \lambda-a_2 & -a_1\\ -a_1 & -a_1 & -a_1 & \cdots & -a_1 & \lambda-a_1 \end{vmatrix}$$

The first determinant of the right hand side of the last equality corresponds to the $P_{n-1}(\lambda)$. Let $q(\lambda)$ denotes the second determinant of the right hand side of the last equality. Then,

$$\begin{aligned} q(\lambda) &= \begin{vmatrix} \lambda-a_{n-1} & -a_{n-2} & 0 & \cdots & 0\\ -a_{n-2} & \lambda-a_{n-2} & -a_{n-3} & \cdots & 0\\ -a_{n-3} & -a_{n-3} & \lambda-a_{n-3} & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ -a_2 & -a_2 & -a_2 & \cdots & -a_1\\ -a_1 & -a_1 & -a_1 & \cdots & \lambda-a_1 \end{vmatrix} \\ &\quad - \begin{vmatrix} \lambda & -a_{n-2} & 0 & \cdots & 0 & 0\\ 0 & \lambda-a_{n-2} & -a_{n-3} & \cdots & 0 & 0\\ 0 & -a_{n-3} & \lambda-a_{n-3} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & -a_2 & -a_2 & \cdots & \lambda-a_2 & -a_1\\ 0 & -a_1 & -a_1 & \cdots & -a_1 & \lambda-a_1 \end{vmatrix} \\ &= P_{n-1}(\lambda)-\lambda P_{n-2}(\lambda). \end{aligned}$$

Thus, we have

$$\begin{aligned} P_n(\lambda) &= (\lambda-a_n)P_{n-1}(\lambda)+a_{n-1}(P_{n-1}(\lambda)-\lambda P_{n-2}(\lambda))\\ &= (\lambda-a_n+a_{n-1})\,P_{n-1}(\lambda)-a_{n-1}\lambda P_{n-2}(\lambda)\,. \end{aligned}$$

Also, it is clear that $P_1(\lambda)=\lambda-a_1$ and $P_2(\lambda)=\lambda^2-(a_1+a_2)\lambda+a_1a_2-a_1^2$. □

Theorem 3.5. *Let $P_n(\lambda)=\lambda^n+\gamma_{n-1}^{(n)}\lambda^{n-1}+\cdots+\gamma_1^{(n)}\lambda+\gamma_0^{(n)}$ be the characteristic polynomial of the $n\times n$ matrix F_a. Then,*

$$\gamma_0^{(n)}=(a_{n-1}-a_n)\gamma_0^{(n-1)}=(-1)^n\det(F_a),$$
$$\gamma_{n-1}^{(n)}=\gamma_{n-2}^{(n-1)}-a_n=-tr(F_a)$$

and

$$\gamma_i^{(n)}=\gamma_{i-1}^{(n-1)}+(a_{n-1}-a_n)\,\gamma_i^{(n-1)}-a_{n-1}\gamma_{i-1}^{(n-2)}$$

are valid for $1\le i\le n-2$.

Proof. By using the recurrence relation in Theorem 3.4 and the coefficients of $P_n(\lambda)$, $P_{n-1}(\lambda)$ and $P_{n-2}(\lambda)$, we have

$$\lambda^n + \gamma_{n-1}^{(n)}\lambda^{n-1} + \cdots + \gamma_1^{(n)}\lambda + \gamma_0^{(n)} = (\lambda - a_n + a_{n-1})(\lambda^{n-1} + \gamma_{n-2}^{(n-1)}\lambda^{n-2} + \cdots + \gamma_1^{(n-1)}\lambda + \gamma_0^{(n-1)})$$
$$-a_{n-1}\lambda(\lambda^{n-2} + \gamma_{n-3}^{(n-2)}\lambda^{n-3} + \cdots + \gamma_1^{(n-2)}\lambda + \gamma_0^{(n-2)}).$$

Comparison of the coefficients yields the desired formulas. Also, we have

$$\begin{aligned} \gamma_0^{(n)} &= (a_{n-1} - a_n)\gamma_0^{(n-1)} = (a_{n-1} - a_n)(a_{n-2} - a_{n-1})\gamma_0^{(n-2)} \\ &= \cdots = (-1)^n \prod_{i=1}^{n} (a_i - a_{i-1}) = (-1)^n \det(F_a) \end{aligned}$$

and

$$\gamma_{n-1}^{(n)} = \gamma_{n-2}^{(n-1)} - a_n = \gamma_{n-3}^{(n-2)} - a_{n-1} - a_n = \cdots = -(a_1 + a_2 + \cdots + a_n) = -tr(F_a).$$

□

4. Conclusion

In this paper we introduced a new generalized Frank matrix. Then, we examined its algebraic structure, determinant, inverse, *LU* decomposition and characteristic polynomial. We showed that the determinant, inverse, *LU* decomposition and characteristic polynomial formulas of the generalized Frank matrix are the general forms those of classical Frank matrix. When we take $a_i = i$ $(i = 1, 2, ..., n)$, our results are reduced the results of Hake [4].

References

[1] Davis, P. J.: Circulant Matrices. Wiley, New York. Chichester, Brisbane (1979).

[2] Eberlein, P. J.: *A note on the matrices denoted* B_n^*. Siam J. Appl. Math. **20**, 87-92 (1971). https://doi.org/10.1137/0120012

[3] Frank, W.L.: *Computing eigenvalues of complex matrices by determinant evaluation and by methods of Danilewski and Wielandt*. J. Soc. Indust. Appl. Math. **6** (4), 378-392 (1958). https://doi.org/10.1137/0106026

[4] Hake, J-F.: *A remark on Frank matrices*. Computing. **35**, 375-379 (1985). https://doi.org/10.1007/BF02240202

[5] Kılıç, E. and Arıkan, T.: *Studying new generalizations of Max-Min matrices with a novel approach*. Turkish Journal of Mathematics. **43**, 2010-2024 (2019). https://doi:10.3906/mat-1811-95

[6] Mattila, M. and Haukkanen, P.: *Studying the various properties of Min and Max matrices - elementary vs. more advanced methods*. Spec. Matrices. **4**, 101-109 (2016). https://doi.org/10.1515/spma-2016-0010

[7] Varah, J.M.: *A generalization of the Frank matrix*. Siam J.Sci. Stat. Comput. **7**(3), 835-839 (1986). https://doi.org/10.1137/0907056

Affiliations

EFRUZ ÖZLEM MERSIN
ADDRESS: Aksaray University, Department of Mathematics, 68000, Aksaray-Turkey.
E-MAIL: efruzmersin@gmail.com
ORCID ID:0000-0001-6260-9063

MUSTAFA BAHŞI
ADDRESS: Aksaray University, Department of Mathematics and Science Education, 68000, Aksaray-Turkey.
E-MAIL: mhvbahsi@yahoo.com
ORCID ID:0000-0002-6356-6592

AYŞE DILEK MADEN
ADDRESS: Selçuk University, Department of Mathematics, 42130, Konya-Turkey.
E-MAIL: aysedilekmaden@selcuk.edu.tr.
ORCID ID:0000-0001-7717-0241