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Perturbation of the Non-Resonance Eigenvalue of a Polyharmonic Matrix Operator

Polyharmonik Bir Matris Operatörün Rezonans Olmayan Özdeğerinin Pertürbasyonu

Sedef Karakılıç [1*]

[1] **Dokuz Eylül Üniversitesi Fen Fkültesi, Matematik Bölümü, İzmir, TÜRKİYE**
*Sorumlu Yazar / Corresponding Author *: sedef.erim@deu.edu.tr*

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Abstract

In this paper, we consider a matrix operator

$$H(l,V)u = (-\Delta)^l u + V(x)u,$$

where $(-\Delta)^l$ is a diagonal $s\times$s matrix, whose diagonal elements are the scalar polyharmonic operators, V is the operator of multiplication by a symmetric $s\times s$ matrix, $V(x)$ is periodic with respect to an arbitrary lattice and $s\geq 2$, $x=(x_1,x_2,\dots,x_d)\in R^d$, $d\geq 2$, $\frac{1}{2}<l<1$. We obtain asymptotic formulae of arbitrary order for the non- resonance eigenvalues of this operator.

Keywords: *system of polyharmonic operators, periodic, eigenvalue, asymptotic.*

Öz

Bu çalışmada, $x=(x_1,x_2,\dots,x_d)\in R^d$, $d\geq 2$, $s\geq 2$, $\frac{1}{2}<l<1$ olmak üzere,

$$H(l,q)u = (-\Delta)^l u + V(x)u$$

matris operatörünün resonans olmayan özdeğerleri için keyfi dereceden asimptotik formülleri elde edilmiştir. Bu gösterimde; $(-\Delta)^l$ dioganal elemanları skaler poliharmonik operatör olan diagonal $s\times$s matris, potansiyel $V(x)$ keyfi bir lattise göre periodik ve simetrik bir $s\times$s matristir.

Anahtar Kelimeler: *poliharmonik operatör sistemi, periodik, özdeğer, asimptotik.*

1. Introduction

For $\frac{1}{2}<l<1$, we consider the operator

$$H(l,q)u = (-\Delta)^l u + V(x)u \qquad (1)$$

in $L_2{}^s(R^d)$, where $(-\Delta)^l$ is a diagonal $s\times$s matrix, its diagonal elements being the scalar polyharmonic operators; $V(x)=(v_{ij}(x))$, $i,j=1,2,\dots,s$, is a symmetric $s\times s$ matrix, $V=V^T$ and $s\geq 2$, $x=(x_1,x_2,\dots,x_d)\in R^d$, $d\geq 2$.

We suppose that each entry $v_{ij}(x)$ is a real valued function of $W_2^m(K)$ and is periodic with respect to the same arbitrary lattice Ω , $K \equiv R^d \backslash \Omega$ is a fundamental domain of Ω and

$m > \frac{(4d-1)}{2}(d+20)3^{d+1} + \frac{d}{4}3^d + d + 1.$

Let $\Gamma = \{\gamma \in R^d : (\gamma, w) \in 2\pi Z, \forall w \in \Omega\}$ be the dual lattice of Ω and $K^* \equiv R^d/\Gamma$ be its fundamental domain. It is well known that the spectral analysis of $H(l,q)$ can be reduced to studying the operators $H_t(l,q)$ defined by the differential expression (1) in ${L_2}^s(K)$ and the quasiperiodic condition

$u(x+w) = e^{iw\cdot t}u(x), \qquad w \in \Omega, t \in K^*,$

$u(x) = \left(u_1(x), u_2(x), \dots, u_s(x)\right), \; x \in K . \qquad (2)$

Here, $\cdot$ denotes the innerproduct in R^d.

The spectrum of the operator $H_t(l,q)$ consists of the eigenvalues $\Lambda_1(t) \le \Lambda_2(t) \le \dots$ and $spec(H(l,q)) = \cup_{n=1}^{\infty} \{\Lambda_n(t) : t \in K^*\}$. Let $\Psi_{n,t}(x)$ denote the eigenfunction of $H_t(l,q)$ corresponding to the eigenvalue $\Lambda_n(t)$. The eigenvalues of the unpertubed operator $H_t(l,0)$ are $|\gamma+t|^{2l}$ and the corresponding eigenspaces are

$E_{\gamma,t} = span\{\Phi_{\gamma,t,1}(x), \Phi_{\gamma,t,2}(x), \dots, \Phi_{\gamma,t,m}(x)\},$

$\Phi_{\gamma,t,j}(x) = (0, \dots, 0, e^{i(\gamma+t)\cdot x}, 0, \dots, 0),$

$j = 1,2, \dots, s,$

for $\gamma \in \Gamma, t \in K^*$. We note that the non-zero component $e^{i(\gamma+t)\cdot x}$ of $\Phi_{\gamma,t,j}(x)$ stands in the jth component.

It is convenient to define a periodic function $v_{ij}(x)$ in $W_2^m(K)$ as a function satisfying the relation

$\sum_{\gamma\in\Gamma} |v_{ij_\gamma}|^2(1+|\gamma+t|^{2m}) < \infty, \qquad (3)$

where

$v_{ij_\gamma} = \left(v_{ij}(x), e^{i\gamma\cdot x}\right) = \int_K v_{ij}(x)e^{-i\gamma\cdot x}\,dx,$

$(.,.)$ is the inner product in $L_2(K)$. Moreover, for a big parameter ρ, we can write

$v_{ij}(x) = \sum_{\gamma\in\Gamma(\rho^\alpha)} v_{ij_\gamma}e^{i\gamma\cdot x} + O(\rho^{-p\alpha}) \qquad (4)$

and define

$M_{ij} = \sum_{\gamma\in\Gamma} |v_{ij_\gamma}| < \infty, \qquad (5)$

for all $i,j = 1,2, \dots, s$, where $p = m - d$, $\alpha > 0$ and

$\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 < |\gamma+t| < \rho^\alpha\}.$

If $\gamma = 0$, $v_{ij_0} = \int_K v_{ij}(x)\,dx$ and $V_0 = (v_{ij_0}) = \int_K V(x)dx$ is a symmetric $s \times s$ matrix.

The aim of this paper is to obtain the high energy asymptotics of the non-resonance eigenvalues (roughly, the ones far away from the diffraction planes $\{x \in R^d : ||x|^{2l} - |x+b|^{2l}| < \rho\}$) of the operator (1) for arbitrary $\frac{1}{2} < l < 1$ and arbitrary $d \ge 2$, where the potential $V(x)$ satisfies (3).

Due to its physical importance, the most significant progress has been achieved in the case of the Schrödinger operator; i.e., the case $l = 1$ in (1). For the first time asymptotic formulae for the eigenvalues of the periodic (with respect to an arbitrary lattice) Schrödinger operator are obtained in the papers [1-4] by O.A. Veliev. Another proof of asymptotic formulae for quasiperiodic boundary conditions in two and three dimensional cases are obtained in [5, 6, 7, 8]. The asymptotic formulae for the eigenvalues of the Schrödinger operator with periodic boundary conditions are obtained in [9]. When this operator is considered with Dirichlet boundary conditions on 2-dimensional rectangle, the high energy asymptotics of the eigenvalues are obtained in [10]. In papers [11, 12, 13], we obtained the formulae for the eigenvalues of the Schrödinger operator considered with Dirichlet and Neumann boundary conditions on a d-dimensional parallelepiped, for arbitrary $d \ge 2$.

The high energy asymptotics of eigenvalues of $H(l,q)$ for $4l > d+1$ $(d \ge 2)$ are obtained by Yu. Karpeshina in [14] and for arbitrary $l \ge 1$ $(d \ge 2)$ by O.A. Veliev in [15], where he claimed that the asumption $l \ge 1$ can be replaced by $l > n_{m,d}$ for some number $n_{m,d} < 1$ that depends on m (the smoothness of $q(x)$) and d (the dimension) without giving any technical details.

For the matrix case, $s \ge 1$, d$\ge$ 2, $l \ge 1$ and $4l >$ d $+ 1$, asymptotic formulae for the eigenvalues of the operator (1) are obtained in [16].

In this paper, we obtain the asymptotic formulae of non-resonance eigenvalues of (1) when $\frac{1}{2}< l < 1$, $(n_{m,d} = \frac{1}{2})$, $s \geq 2$.

2. Material and Method

We use the same method introduced by O.A.Veliev in his papers [3,4,15] and define the following parameters:

$$\alpha(l) = \frac{a}{(d+20)3^{d+1}},$$

$$\alpha_1(l) = 3\alpha(l), \qquad (6)$$

where $l = \frac{1}{2} + a$, $0 < a < \frac{1}{2}$. By these notations (4) becomes

$$v_{ij}(x) = \sum_{\gamma' \in \Gamma(\rho^{\alpha(l)})} v_{ij_{\gamma'}} e^{i\gamma' \cdot x} + O(\rho^{-p\alpha(l)}), \quad (7)$$

where $\Gamma(\rho^{\alpha(l)}) = \{\gamma \in \frac{\Gamma}{2} : 0 < |\gamma + t| < \rho^{\alpha(l)}\}$, $p = m - d$ and ρ is a large parameter.

In the sequal, c_1, $c_2, c_3, \ldots$ denote the positive constants whose exact values are inessential (they do not dependent on ρ). Additionally, by $|a| \sim \rho$, we mean that there exist c_1, c_2 such that

$c_1\rho < |a| < c_2\rho$.

We divide the eigenvalues $|\gamma + t|^{2l}$ of the unperturbed operator into two groups. In order to define these groups, we introduce the following sets:

$$V_b^l(\rho^{\alpha_1(l)}) = \{x \in R^d : ||x|^{2l} - |x + b|^{2l}| < \rho^{\alpha_1(l)}\},$$

$$E_1^l(\rho^{\alpha_1(l)}, p) = \bigcup_{b \in \Gamma(p\rho^{\alpha(l)})} V_b^l(\rho^{\alpha_1(l)}),$$

$$U^l(\rho^{\alpha_1(l)}, p) = R^d \backslash E_1^l(\rho^{\alpha_1(l)}, p),$$

$$E_k^l(\rho^{\alpha_k(l)}, p) = \bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_k \in \Gamma(p\rho^{\alpha(l)})} (\bigcap_{i=1}^{k} V_{\gamma_i}^l(\rho^{\alpha_k(l)})),$$

where the intersection $\bigcap_{i=1}^{k} V_{\gamma_i}^l(\rho^{\alpha_k(l)})$ in E_k^l is taken over $\gamma_1, \gamma_2, \ldots, \gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vectors in

$\Gamma \cap \gamma_i R$. The set $U^l(\rho^{\alpha_1(l)}, p)$ is said to be a non-resonance domain and the eigenvalue $|\gamma + t|^{2l}$ is called a non-resonance eigenvalue if $\gamma \in U^l(\rho^{\alpha_1(l)}, p)$. The domains $V_b^l(\rho^{\alpha_1(l)})$, for all $b \in \Gamma(p\rho^{\alpha(l)})$, are called resonance domains and the eigenvalue $|\gamma + t|^{2l}$ is a resonance eigenvalue if $\gamma \in V_b^l(\rho^{\alpha_1(l)})$.

Remark *If $x \in R^d$, $|x| \sim \rho$ and $\gamma_1 \in \Gamma$ then $|x + \gamma_1| \sim \rho$ and by the Mean Value Theorem*

$$|x|^{2l} - |x + \gamma_1|^{2l} = \xi^{2(l-1)}(|x|^2 - |x + \gamma_1|^2) \quad (8)$$

where $\xi \sim \rho$. Therefore for $\frac{1}{2} < l < 1$, $V_{\gamma_1}^l(\rho^{\alpha_1(l)}) \subset V_{\gamma_1}^1(\rho^{\alpha_1(l)-2l+2})$ from which we have

$$(\cap_{i=1}^{k} V_{\gamma_i}^l(\rho^{\alpha_k(l)})) \subset \cap_{i=1}^{k} V_{\gamma_i}^1(\rho^{\alpha_k(l)-2l+2})),$$

$$U^1(\rho^{\alpha_1(l)-2l+2}, p) \subset U^l(\rho^{\alpha_1(l)}, p), \qquad (9)$$

for $k = 1,2,\ldots$.

As noted in the Remark1 of the paper [15], the expression (9) implies that the non-resonance domain $U^l(\rho^{\alpha_1(l)}, p)$ has asymptotically full measure in R^d in the sense that $\frac{\mu(U^l(\rho^{\alpha_1(l)},p) \cap B(\rho))}{\mu(B(\rho))} \to 1$ as $\rho \to \infty$, where $B(\rho) = \{x \in R^d : |x| = \rho\}$, if

$$\alpha_1(l) - 2l + 2 + d\alpha(l) < 1 - \alpha(l) \qquad (10)$$

holds. By the definitions (6) of $\alpha(l)$ and $\alpha_k(l)$ the condition (10) holds.

From now on, we assume that $\gamma \in U^l(\rho^{\alpha_1(l)}, p)$ with $|\gamma + t| \sim \rho$. To prove the asymptotic formulae for eigenvalue $\Lambda_N(t)$ of the operator $H(l, q)$, we use the following well-known formula:

$$(\Lambda_N(t) - |\gamma + t|^{2l}) < \Psi_{N,t}, \Phi_{\gamma,t,j} > \; = \; < \Psi_{N,t}, V(x)\Phi_{\gamma,t,j} >, \qquad (11)$$

where $<\cdot,\cdot>$ denotes the inner product in $L_2{}^s(K)$. We substitute the decomposition (7) of $v_{ij}(x)$ into the formula (11) to obtain

$$(\Lambda_N(t) - |\gamma + t|^{2l})c(N, j, \gamma) = \sum_{i=1}^{s} \sum_{\gamma_1 \in \Gamma(\rho^{\alpha(l)})} v_{ij_{\gamma_1}} c(N, i, \gamma + \gamma_1) + O(\rho^{-p\alpha(l)}),$$

where $c(N, i, \gamma) = < \Psi_{N,t}, \Phi_{\gamma,t,i} >$. If we isolate the terms with the coefficient $c(N, i, \gamma)$; that is, the terms with $\gamma_1 = 0$ for each $i = 1,2,\ldots, s$, then we get

$$(\Lambda_N-|\gamma+t|^{2l})c(N,j,\gamma)=\sum_{i=1}^{s}v_{ij_0}c(N,i,\gamma)+\sum_{i_1=1}^{s}\sum_{\gamma_1\in\Gamma(\rho^{\alpha(l)})}v_{i_1j_{\gamma_1}}c(N,j,\gamma+\gamma_1)+O(\rho^{-p\alpha}). \quad (12)$$

Also, (11) together with (7) imply

$$c(N,j,\tilde{\gamma})=\frac{<\Psi_{N,t},V\Phi_{\tilde{\gamma},t,j}>}{\Lambda_N(t)-|\tilde{\gamma}+t|^{2l}}=\sum_{i=1}^{s}\sum_{\gamma_1\in\Gamma(\rho^{\alpha(l)})}v_{ij_{\gamma_1}}\frac{c(N,i,\tilde{\gamma}+\gamma_1)}{\Lambda_N(t)-|\tilde{\gamma}+t|^{2l}}+O(\rho^{-p\alpha(l)}), \quad (13)$$

for every vector $\tilde{\gamma}\in\frac{\Gamma}{2}$ satisfying the condition

$$|\Lambda_N(t)-|\tilde{\gamma}+t|^{2l}|>\frac{1}{2}\rho^{\alpha_1(l)} \quad (14)$$

which is called the iterability condition. Note that, if $\gamma\in U^l(\rho^{\alpha_1(l)},p)$ and

$$|\Lambda_N(t)-|\gamma+t|^{2l}|<\frac{1}{2}\rho^{\alpha_1(l)}, \quad (15)$$

then (14) holds for $\tilde{\gamma}=\gamma+b,\ \forall b\in\Gamma(p\rho^{\alpha(l)})$. Hence, when $\gamma_1\in\Gamma(\rho^{\alpha(l)})$, we may substitute $\gamma+\gamma_1$ for $\tilde{\gamma}$ in (13) and then the equation (12) becomes

$$(\Lambda_N(t)-|\gamma+t|^{2l})c(N,j,\gamma)=\sum_{i=1}^{s}v_{ij_0}c(N,i,\gamma)+\sum_{i_1,i_2=1}^{s}\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha(l)})}v_{i_1j_{\gamma_1}}v_{i_2i_{1\gamma_2}}\frac{c(N,i_2,\gamma+\gamma_1+\gamma_2)}{\Lambda_N(t)-|\gamma+\gamma_1+t|^{2l}}+O(\rho^{-p\alpha(l)}).$$

By isolating the terms with coefficient $c(N,i_2,\gamma)$ in the last equation, we obtain

$$(\Lambda_N(t)-|\gamma+t|^{2l})c(N,j,\gamma)=\sum_{i=1}^{s}v_{ij_0}c(N,i,\gamma)+\sum_{i_1,i_2=1}^{s}\sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha(l)})\\ \gamma_1+\gamma_2=0}}v_{i_1j_{\gamma_1}}v_{i_2i_{1\gamma_2}}\frac{c(N,i_2,\gamma)}{\Lambda_N(t)-|\gamma+\gamma_1+t|^{2l}}+\sum_{i_1,i_2=1}^{s}\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha(l)})}v_{i_1j_{\gamma_1}}v_{i_2i_{1\gamma_2}}\frac{c(N,i_2,\gamma+\gamma_1+\gamma_2)}{\Lambda_N(t)-|\gamma+\gamma_1+t|^{2l}}+O(\rho^{-p\alpha(l)}).$$

If we write this equation for $j=1,2,\ldots,s$ and $i=1,2,\ldots,s$, after the first step of the iteration, we obtain the following system:

$$[(\Lambda_N(t)-|\gamma+t|^{2t})I-V_0]A(N,\gamma)=S^1A(N,\gamma)+R^1+O(\rho^{-p\alpha(l)}),$$

where I is the $s\times s$ identity matrix,

$A(N,\gamma)=(c(N,j,\gamma))$,

$S^1=(s_{ji}^1)$ is the $s\times s$ matrix whose entries are

$$s_{ji}^1=\sum_{i_1=1}^{s}\sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha(l)})\\ \gamma_1+\gamma_2=0}}\frac{v_{i_1j\gamma_1}v_{ii_1\gamma_2}}{(\Lambda_N(t)-|\gamma+\gamma_1+t|^{2l})},$$

and $R^1=(r_j^1)$ is the vector whose components are

$$r_j^1=\sum_{i_1,i_2=1}^{s}\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha(l)})}\frac{v_{i_1j\gamma_1}v_{i_2i_1\gamma_2}c(N,i_2,\gamma+\gamma_1+\gamma_2)}{(\Lambda_N(t)-|\gamma+\gamma_1+t|^{2l})},$$

$j,i=1,2,\ldots,s$.

In this way, if we repeate the iteration $p_1=[\frac{p+1}{3}]$ times and each time we isolate the terms with coefficient $c(N,i_k,\gamma)$, we have

$$[(\Lambda_N(t)-|\gamma+t|^{2})I-V_0]A(N,\gamma)=\left(\sum_{k=1}^{p_1}S^k\right)A(N,\gamma)+R^{p_1}+O(\rho^{-p\alpha(l)}), \quad (16)$$

where

$S^k(\Lambda_N(\mathrm{t}))=\left(s_{ji}^k(\Lambda_N(t))\right),\quad k=1,\ldots,p_1,\quad j,i=1,,\ldots,s,$

$$s_{ji}^k(\Lambda_N(t))=\sum_{i_1,i_2,\ldots,i_k=1}^{s}\sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{k+1}\in\Gamma(\rho^{\alpha(l)})\\ \gamma_1+\gamma_2+\cdots+\gamma_{k+1}=0}}\frac{v_{i_1j\gamma_1}v_{i_2i_1\gamma_2}\cdots v_{ii_k\gamma_{k+1}}}{(\Lambda_N(t)-|\gamma+\gamma_1+\mathrm{t}|^{2l})\ldots(\Lambda_N(t)-|\gamma+\gamma_1+\cdots+\gamma_k+t|^{2l})}, \quad (17)$$

$R^{p_1}=(r_j^{p_1})_j$ and (18)

$$r_j^{p_1}=\sum_{i_1,i_2,\ldots,i_{p_1+1}=1}^{s}\sum_{\gamma_1,\gamma_2,\ldots,\gamma_{p_1+1}\in\Gamma(\rho^{\alpha(l)})}\frac{v_{i_1j\gamma_1}\ldots v_{i_{p_1+1}i_{p_1}\gamma_{p_1+1}}c(N,i_{p_1+1},\gamma+\gamma_1+\cdots+\gamma_{p_1+1})}{(\Lambda_N(t)-|\gamma+\gamma_1+\mathrm{t}|^{2l})\ldots(\Lambda_N(t)-|\gamma+\gamma_1+\cdots+\gamma_{p_1}+t|^{2l})}.$$

Since the vectors $\gamma_i \in \Gamma(\rho^{\alpha(l)})$, we have $|b| = |\gamma_1 + \gamma_2 + \ldots + \gamma_i| < p_1 \rho^{\alpha(l)}$, for all $i = 1,2,\ldots,p_1$, in (17) and (18). Therefore, (14) together with (5) imply

$$S^k(\Lambda_N(t)) = O(\rho^{-k\alpha_1(l)}), R^{p_1} = O\left(\rho^{-p_1\alpha_1(l)}\right) \quad (19)$$

for $k = 1,2,\ldots,p_1$. To obtain (19), we have only used the iterability condition in (14); that is, $\Lambda_N(t) \in I = [|\gamma + t|^{2l} - \frac{1}{2}\rho^{\alpha_1(l)}, |\gamma + t|^{2l} + \frac{1}{2}\rho^{\alpha_1(l)}]$. Hence, we may conclude that

$$S^k(a) = O\left(\rho^{-k\alpha_1(l)}\right),$$

$$\sum_{i=1}^{p_1} S^i(a) = O\left(\rho^{-\alpha_1(l)}\right), \quad \forall a \in I \quad (20)$$

and

$$[D(\Lambda_N, \gamma) - S(a, p_1)]A(N, \gamma) = O(\rho^{-p\alpha(l)}), \quad (21)$$

where $D(\Lambda_N, \gamma) \equiv (\Lambda_N(t) - |\gamma + t|^{2l})I - V_0$ and $S(a, p_1) \equiv \sum_{k=1}^{p_1} S^k(a)$. We note that since V is symmetric, V_0 and $S(a, p_1)$ are symmetric real valued matrices; hence $D(\Lambda_N, \gamma) - S(a, p_1)$ is a symmetric real valued matrix.

We denote the eigenvalues of V_0, counted with multiplicity, and the corresponding orthonormal eigenvectors by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s$ and $\omega_1, \omega_2, \ldots, \omega_s$, respectively. Thus

$$V_0\omega_i = \lambda_i\omega_i, \quad \omega_i \cdot \omega_j = \delta_{ij}.$$

We let $\beta_i \equiv \beta_i(\Lambda_N, \gamma, a)$ denote an eigenvalue of the matrix $D(\Lambda_N, \gamma) - S(a, p_1)$ and $f_i \equiv f_i(\Lambda_N, \gamma, a)$ its corresponding normalized eigenvector. That is,

$$[D(\Lambda_N, \gamma) - S(a, p_1)]f_i = \beta_i f_i, \quad (22)$$

where $f_i \cdot f_j = \delta_{ij}, i, j = 1,2,\ldots,s$.

3. Results

Lemma 1 *Suppose* $\frac{1}{2} < l < 1$, $\gamma \in U^l(\rho^{\alpha(l)}, p)$ *and* $|\gamma + t| \sim \rho$.

(a) Let β_i be an eigenvalue of the matrix $D(\Lambda_N, \gamma) - S(a, p_1)$ and $f_i = (f_{i_1}, f_{i_2}, \ldots, f_{i_s})$ its corresponding normalized eigenvector. Then there exists an integer $N \equiv N_i$ such that $\Lambda_N(t)$ satisfies (15) and

$$|A(N, \gamma) \cdot f_i| > c_3\rho^{\frac{-(d-1)}{2}}. \quad (23)$$

(b) Let $\Lambda_N(t)$ be an eigenvalue of the operator $H_t(l, V)$ satisfying the inequality (15). Then there exists an eigenfunction $\Phi_{\gamma,t,i}(x)$ of the operator $H_t(l, 0)$ such that

$$|c(N, i, \gamma)| > c_4\rho^{\frac{-(d-1)}{2}}. \quad (24)$$

Proof. **(a)** By a well-known result from perturbation theory, the Nth eigenvalue of the operator $H_t(l, V)$ lies in M-neighborhood of the Nth eigenvalue of the operator $H_t(l, 0)$; that is, there is an integer N such that

$$|\Lambda_N(t) - |\gamma + t|^{2l}| < \frac{1}{2}\rho^{\alpha_1(l)}.$$

On the other hand, since $H_t(l, V)$ is a self adjoint operator, the eigenfunctions $\{\Psi_{N,t}(x)\}_{N=1}^{\infty}$ of $H_t(l, V)$ form an orthonormal basis for $L_2^s(K)$. By using Parseval's relation, we have

$$\| \sum_{j=1}^{s} f_{ij}\Phi_{\gamma,t,j} \|^2 = \sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|<\frac{1}{2}\rho^{\alpha_1(l)}} | < \sum_{j=1}^{s} f_{ij}\Phi_{\gamma,t,j}, \Psi_{N,t} > |^2 + \sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} | < \sum_{j=1}^{s} f_{ij}\Phi_{\gamma,t,j}, \Psi_{N,t} > |^2. \quad (25)$$

Now, we estimate the last expression in (25). By using the Cauchy-Schwartz inequality and (11), we get

$$\sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} | < \sum_{j=1}^{s} f_{ij}\Phi_{\gamma,t,j}, \Psi_{N,t} > |^2$$

$$= \sum_{\substack{N: \\ |\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}}} |\sum_{j=1}^{s} f_{ij} < \Phi_{\gamma,t,j}, \Psi_{N,t} > |^2$$

$$\leq \sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} [\sum_{j=1}^{s} |f_{ij}|^2 \sum_{j=1}^{s} |c(N, j, \gamma)|^2]$$

$$= \sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{j=1}^{s} \frac{| < \Psi_{N,t}, V\Phi_{\gamma,t,j} > |^2}{|\Lambda_N(t) - |\gamma+t|^{2l}|^2}$$

$$\leq (\frac{\rho^{\alpha_1(l)}}{2})^{-2} \sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{\rho^{\alpha_1(l)}}{2}}$$

$$\sum_{j=1}^{s} |<\Psi_{N,t}, V\Phi_{\gamma,t,j}>|^2$$

$$\leq (\frac{1}{2}\rho^{\alpha_1(l)})^{-2}\sum_{j=1}^{s} \| V\Phi_{\gamma,t,j} \|^2$$

from which, together with (5), we obtain $\sum_{N:|\Lambda_{N(t)}-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} |<\sum_{j=1}^{s} f_{ij}\Phi_{\gamma,j}, \Psi_{N,t}>|^2 = O(\rho^{-2\alpha_1(l)})$.

It follows from the last equation and (25) that $\sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|<\frac{1}{2}\rho^{\alpha_1(l)}} |<\sum_{j=1}^{s} f_{ij}\Phi_{\gamma,t,j}, \Psi_{N,t}>|^2 = \sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|<\frac{1}{2}\rho^{\alpha_1(l)}} |A(N,\gamma)\cdot f_i|^2 =$

$$1 - O(\rho^{-2\alpha_1(l)}). \quad (26)$$

On the other hand, if $a \sim \rho$, then the number of $\gamma \in \frac{\Gamma}{2}$ satisfying $||\gamma|^2 - a^2| < 1$ is less than $c_5\rho^{d-1}$. Therefore, the number of eigenvalues of $H_t(l,0)$ lying in (a^2-1, a^2+1) is less than $c_6\rho^{d-1}$. By this result and the result of perturbation theory, the number of eigenvalues $\Lambda_N(t)$ of $H_t(l,V)$ in the interval $[|\gamma+t|^{2l} - \frac{1}{2}\rho^{\alpha_1(l)}, |\gamma+t|^{2l} + \frac{1}{2}\rho^{\alpha_1(l)}]$ is less than $c_7\rho^{d-1}$. Thus

$$1 - O(\rho^{-2\alpha_1(l)}) = \sum_{N:|\Lambda_N(t)-|\gamma+t|^{2l}|<\frac{1}{2}\rho^{\alpha_1(l)}} |A(N,\gamma)\cdot f_i|^2 < c_7\rho^{d-1}|A(N,\gamma)\cdot f_i|^2 \quad (27)$$

from which we get (23).

(b) Since $H_t(l,0)$ is a self adjoint operator the set of eigenfunctions $\{\Phi_{\gamma,t,i}(x)\}_{\gamma\in\Gamma, i=1,2,\dots,m}$ of $H_t(l,0)$ forms an orthonormal basis for ${L_2}^s(K)$. By Parseval's relation, we have

$$\| \Psi_{N,t} \|^2 = \sum_{\gamma:|\Lambda_N(t)-|\gamma+t|^{2l}|<\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{i=1}^{s} |c(N,i,\gamma)>|^2 + \sum_{\gamma:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{i=1}^{s} |c(N,i,\gamma)|^2. \quad (28)$$

We estimate the last expression in (28). Hence for a fixed $i = 1,2,\dots,s$, using (11) together with (5) we get

$$\sum_{\gamma:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{i=1}^{s} |c(N,i,\gamma)|^2$$

$$= \sum_{\gamma:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{i=1}^{s} \frac{|<\Psi_{N,t}, V\Phi_{\gamma,t,i}>|^2}{|\Lambda_N(t)-|\gamma+t|^{2l}|^2}$$

$$\leq (\frac{1}{2}\rho^{\alpha_1(l)})^{-2} \sum_{\gamma:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{i=1}^{s} |<V\Psi_{N,t}, \Phi_{\gamma,t,i}>|^2$$

$$\leq (\frac{1}{2}\rho^{\alpha_1(l)})^{-2} \| V\Psi_{N,t} \|^2; \quad (29)$$

that is,

$$\sum_{\gamma:|\Lambda_N(t)-|\gamma+t|^{2l}|\geq\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{i=1}^{s} |c(N,i,\gamma)|^2 = O(\rho^{-2\alpha_1(l)}).$$

From the last equality and (28), we obtain

$$\sum_{\gamma:|\Lambda_N(t)-|\gamma+t|^{2l}|<\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{i=1}^{s} |c(N,i,\gamma)|^2 = 1 - O(\rho^{-2\alpha_1(l)}).$$

Arguing as in the proof of part(a), we get

$$1 - O(\rho^{-2\alpha_1(l)}) = \sum_{\gamma:|\Lambda_N(t)-|\gamma+t|^{2l}|<\frac{1}{2}\rho^{\alpha_1(l)}} \sum_{i=1}^{s} |c(N,i,\gamma)|^2 \leq c_8\rho^{d-1}|c(N,i,\gamma)|^2$$

from which (24) follows.

Theorem 2 *Suppose* $\frac{1}{2} < l < 1$, $\gamma \in U^l(\rho^{\alpha(l)}, p)$ *and* $|\gamma + t| \sim \rho$.

(a) For each eigenvalue λ_i of the matrix V_0, there exists an eigenvalue $\Lambda_N(t)$ of the operator $H_t(l,V)$ satisfying

$$\Lambda_N(t) = |\gamma+t|^{2l} + \lambda_i + O(\rho^{-\alpha_1(l)}). \quad (30)$$

(b) For each eigenvalue $\Lambda_N(t)$ of the operator $H_t(l,V)$ satisfying (15), there exists an eigenvalue λ_i of the matrix V_0 satisfying (30).

Proof. **(a)** By Lemma(1a), there exists an eigenvalue $\Lambda_N(t)$ of the operator $H_t(l,V)$ satisfying (15); that is, $\Lambda_N(t) \in I$ and (23) holds. Thus, we consider the equation (21) for $a = \Lambda_N(t)$; that is,

$$[D(\Lambda_N,\gamma) - S(\Lambda_N,p_1)]A(N,\gamma) = O(\rho^{-p\alpha(l)}).$$

Multiplying both sides of the above equation by f_i gives

$$\beta_i[A(N,\gamma)\cdot f_i] = O(\rho^{-p\alpha(l)}).$$

By using the inequality (23) in the above equation, we get

$$\beta_i = O(\rho^{-(p-\frac{d-1}{2\alpha})\alpha(l)}).$$

(31)

Since $D(\Lambda_N,\gamma)$ and $S(\Lambda_N,p_1)$ are symmetric real valued matrices, by a well known result in matrix theory (see [13]),

$$|\beta_i - (\Lambda_N(t) - |\gamma+t|^{2l} - \lambda_i)| \leq \| S(\Lambda_N,p_1) \|$$

which together with (18) imply that

$$\beta_i = \Lambda_N(t) - |\gamma+\mathrm{t}|^{2l} - \lambda_i + O(\rho^{-\alpha_1(l)}).$$

(32)

Hence, by choosing $p > \frac{d-1}{2\alpha(l)} + 1$ and using (32) and (31), we get the result.

(b) By Lemma(1b), there exists $\Phi_{\gamma,t,i}(x)$ satisfying (24) from which we have

$$\| A(N,\gamma) \| > c_9\rho^{\frac{-(d-1)}{2}}. \quad (33)$$

Now, we consider the equation (16) for these (N,γ) pairs

$$[(\Lambda_N(t) - |\gamma+\mathrm{t}|^{2l})I - V_0]A(N,\gamma) = S(\Lambda_N,p_1)A(N,\gamma) + O(\rho^{-p\alpha(l)}).$$

First, we apply $\frac{1}{\|A(N,\gamma)\|}[(\Lambda_N(t) - |\gamma+\mathrm{t}|^{2l})I - V_0]^{-1}$ to both sides of the above equation. Next, we take the norm of both sides and use (33) to obtain the following inequality

$$1 \leq \| [(\Lambda_N(t) - |\gamma+\mathrm{t}|^{2l})I - V_0]^{-1} \| \| \sum_{k=1}^{p_1} S^k \| + \| [(\Lambda_N(t) - |\gamma+\mathrm{t}|^{2l})I - V_0]^{-1} \| O(\rho^{-(p\alpha(l) - \frac{(d-1)}{2})}).$$

By estimation (20) and choosing $p > \frac{d-1}{2\alpha(l)} + 1$, we get

$$1 \leq \max_{i=1,\dots,s} \frac{1}{|\Lambda_N(t) - |\gamma+\mathrm{t}|^{2l} - \lambda_i|} O(\rho^{-\alpha_1(l)}),$$

Hence,

$$\min_{i=1,2,\dots,s} |\Lambda_N(t) - |\gamma+t|^{2l} - \lambda_i| \leq c_{10}\rho^{-\alpha_1(l)},$$

where minimum (maximum) is taken over all eigenvalues of the matrix V_0, from which we obtain the result.

In the interest of saving space, we use the notation

$$a_{\gamma,k} = |\gamma+\mathrm{t}|^{2l} + \lambda_k + \| F_{j-1} \|,$$

where

$$F_0 = 0, \quad F_1 = S^1(|\gamma+\mathrm{t}|^{2l} + \lambda_k),$$

$$F_j = S(a_{\gamma,k}, j), \quad j \geq 2. \quad (34)$$

Then, we have

$$\| F_j \| = O(\rho^{-\alpha_1(l)}) \quad (35)$$

for all $j = 1,2,\dots,p-c$, $c = [\frac{d-1}{2\alpha(l)}] + 1$. Indeed, since $F_0 = 0$, $\| F_0 \| = 0$ and if we assume that $\| F_{j-1} \| = O(\rho^{-\alpha_1(l)})$, then since $a_{\gamma,k} \in I$, by (20), we have $\| F_j \| = O(\rho^{-\alpha_1(l)})$.

By (35), we have $a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}) \in I$. Thus, we let $a \equiv a_{\gamma,k} + O(\rho^{-j\alpha_1(l)})$ in (20), to get

$$[D(\Lambda_N,\gamma) - S(a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}), p_1)]A(N,\gamma) = O(\rho^{-p\alpha(l)}). \quad (36)$$

We add and subtract the term $F_jA(N,\gamma) = S(a_{\gamma,k}, j)A(N,\gamma)$ into the left hand side of the equation (36) to obtain

$$[D(\Lambda_N,\gamma) - F_j]A(N,\gamma) - E_jA(N,\gamma) = O(\rho^{-p\alpha(l)}), \quad (37)$$

where

$$E_j = [S(a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}), j) - S(a_{\gamma,k}, j)] + (\sum_{i=j+1}^{p_1} S^k(\mu_{\gamma,k} + \| F_{j-1} \| + O(\rho^{-j\alpha_1(l)}))).$$

By (20), we have

$$\sum_{i=j+1}^{p_1} S^k\left(a_{\gamma,k} + O(\rho^{-j\alpha_1(l)})\right) = O(\rho^{-(j+1)\alpha_1(l)}). \quad (38)$$

If we prove that

$$\| S(a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}), j) - S(a_{\gamma,k}, j) \| = O(\rho^{-(j+1)\alpha_1(l)}), \quad (39)$$

then it follows from (38) and (39) that

$$\| E_j \| = O(\rho^{-(j+1)\alpha_1(l)}). \qquad (40)$$

Since $a_{\gamma,k} \in I$, we have

$$|a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}) - |\gamma + \gamma_1 + \ldots + \gamma_t + t|^{2l}| > \frac{1}{2}\rho^{\alpha_1(l)},$$

$$|a_{\gamma,k} - |\gamma + \gamma_1 + \ldots + \gamma_t + t|^{2l}| > \frac{1}{2}\rho^{\alpha_1(l)}, \qquad (41)$$

for all $\gamma_t \in \Gamma(\rho^{\alpha(l)})$ and $t = 1,2,\ldots,p_1$. We first calculate the order of the first term of the summation in (39). To do this, we consider each entry of this term, and use (41) and (5):

$$|s^1_{ni}(a_{\gamma,k} + O(\rho^{-j\alpha_1(l)})) - s^1_{ni}(a_{\gamma,k})| \leq \sum_{i_1=1}^{s} \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma(\rho^{\alpha(l)}) \\ \gamma_1+\gamma_2=0}} \frac{|v_{i_1 n \gamma_1}||v_{i i_1 \gamma_2}| O(\rho^{-j\alpha_1(l)})}{|(a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}) - |\gamma + \gamma_1 + t|^{2l})||(a_{\gamma,k} - |\gamma + \gamma_1 + t|^{2l})|} \leq c_{11}\rho^{-(j+2)\alpha_1(l)},$$

for each $n, i = 1,2,\ldots,s$, which implies

$$\| S^1(a_{\gamma,k} + O(\rho^{-j\alpha_1(l)})) - S^1(a_{\gamma,k}) \| = O(\rho^{-(j+2)\alpha_1(l)}).$$

Therefore, by direct calculations, it can be easily seen that

$$\| S^k(a_{\gamma,k} + O(\rho^{-j\alpha_1(l)})) - S^k(a_{\gamma,k}) \| = O(\rho^{-(j+k+1)\alpha_1(l)})$$

from which we obtain (39).

Theorem 3 *Suppose* $\frac{1}{2} < l < 1$, $\gamma \in U^l(\rho^{\alpha(l)}, p)$ *and* $|\gamma + t| \sim \rho$.

(a) For any eigenvalue λ_i, $i = 1,2,\ldots,s$ of the matrix V_0, there exits an eigenvalue $\Lambda_N(t)$ of the operator $H_t(l,V)$ satisfying the following formula:

$$\Lambda_N(\mathrm{t}) = |\gamma + t|^{2l} + \lambda_i + \| F_{k-1} \| + O(\rho^{-k\alpha_1(l)}), \qquad (42)$$

where F_{k-1} is given by (34), $k = 1,2,\ldots,p-c$.

(b) For any eigenvalue $\Lambda_N(t)$ of the operator $H_t(l,V)$ satisfying (15), there is an eigenvalue λ_i of the matrix V_0 satisfying (42).

Proof. **(a)** By Lemma(1a), there exist $\Lambda_N(t)$ and $\Psi_{N,t}(x)$ satisfying (15) and (23), respectively. We prove the theorem by induction. For $k = 1$, we obtain the result by Theorem(2a).

Now, assume that for $k = j - 1$ the formula (42) is true; that is,

$$\Lambda_N(\mathrm{t}) = |\gamma + t|^{2l} + \lambda_i + \| F_{j-1} \| + O(\rho^{-j\alpha_1(l)}). \qquad (43)$$

Let β_i be an eigenvalue of the matrix $D(\Lambda_N, \gamma) - S((a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}), p_1)$. If we multiply both sides of the equation (36) by its corresponding normalized eigenvector f_i, and use (23), then we obtain

$$\beta_i = O(\rho^{-(p-c)\alpha(l)}). \qquad (44)$$

On the other hand, the matrix

$D(\Lambda_N, \gamma) - S((a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}), p_1)$

in (36) is decomposed as follows

$$D(\Lambda_N, \gamma) - S((a_{\gamma,k} + O(\rho^{-j\alpha_1(l)}), p_1) = D(\Lambda_N, \gamma) - F_j - E_j.$$

Thus, by (40), (44) and a well known result in matrix theory,

$$\left|\beta_i - \left(\Lambda_N(t) - (\,|\gamma + t|^{2l} + \lambda_i)\right)\right| \leq \| F_j \| + O(\rho^{-(j+1)\alpha_1(l)}),$$

where $1 \leq j + 1 \leq p - c$, we get the proof of (42).

(b) Again, we prove this part of the theorem by induction. For $j = 1$, we obtain the result by Theorem(2b).

Now, assume that for $k = j - 1$ the formula (42) is true. To prove (42) for $k = j$, we use the equation (37) and the definition of the matrix $D(\Lambda_N, \gamma)$ and get

$$[(\Lambda_N(t) - |\gamma + t|^{2l})I - D_j]A(N,\gamma) = E_j A(N,\gamma) + O(\rho^{-p\alpha(l)}),$$

where $D_j = V_0 + F_j$.

First, we apply $\frac{1}{\|A(N,\gamma)\|}[(\Lambda_N(t) - |\gamma + t|^{2l})I - D_j]^{-1}$ to both sides of the above equation and then, take the norm of both sides and use the estimations (33) and (40) to obtain

$1 \leq$

$$\| [(\Lambda_N(t) - |\gamma + t|^{2l})I - D_j]^{-1} \| [O(\rho^{-(j+1)\alpha_1(l)}]$$
$$+\| [(\Lambda_N(t) - |\gamma + t|^{2l})I - D_j]^{-1} \| [O(\rho^{-(p-c)\alpha(l)})]$$

$$\leq$$

$$\max_{i=1,2,\dots,s} \frac{1}{|\Lambda_N(t) - |\gamma + t|^{2l} - \tilde{\lambda}_i(j)|} [O(\rho^{-(j+1)\alpha_1(l)})],$$

or

$$\min_{i=1,2,\dots,s} |\Lambda_N(t) - |\gamma + t|^{2l} - \tilde{\lambda}_i(j)| \leq c_{12}\rho^{-(j+1)\alpha_1(l)},$$

where minimum is taken over all eigenvalues $\tilde{\lambda}_i(j)$ of the matrix D_j, $1 \leq j+1 \leq p-c$. By the last inequality and the well known result in matrix theory, $|\tilde{\lambda}_i(j) - \lambda_i| \leq \| F_j \|$ and the result follows.

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