On Some Generalized Deferred Cesàro Means-II

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Abstract: In this study, using the generalized difference operator \( \Delta^m \), we introduce some new sequence spaces and investigate some topological properties of these sequence spaces

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1 Introduction

Let \( w \) be the set of all sequences of real or complex numbers and \( \ell_\infty \), \( c \) and \( c_0 \) be respectively the Banach spaces of bounded, convergent and null sequences \( x = (x_k) \) with the usual norm \( \| x \|_\infty = \sup |x_k| \), where \( k \in \mathbb{N} = \{1, 2, \ldots\} \), the set of positive integers. Also by \( \ell_b \), \( \ell_s \) and \( \ell_p \); we denote the spaces of all bounded, convergent, absolutely summable and \( p \)-absolutely summable sequences, respectively.

A sequence space \( X \) with a linear topology is called a \( K \)-space provided each of the maps \( p_k : X \rightarrow \mathbb{C} \) defined by \( p_k (x) = x_i \) is continuous for each \( i \in \mathbb{N} \), where \( \mathbb{C} \) denotes the complex field. A \( K \)-space \( X \) is called an \( FK \)-space provided \( X \) is a complete linear metric space. An \( FK \)-space whose topology is normable is called a \( BK \)-space. We say that an \( FK \)-space \( X \) has \( AK \) (or has the \( AK \) property), if \( (e_k) \) (the sequence of unit vectors) is a Schauder bases for \( X \).

The notion of difference sequence spaces was introduced by Kizmaz [1] and the notion was generalized by Et and Çolak [2]. Later on Et and Nuray [3] generalized these sequence spaces to the following sequence spaces:

Let \( X \) be any sequence space and let \( m \) be a non-negative integer. Then,

\[
\Delta^m (X) = \{ x = (x_k) : (\Delta^m x_k) \in X \}
\]

\[
\Delta^0 x = (x_k), \Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}) \quad \text{and so} \quad \Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}, \text{ is a Banach space normed by}
\]

\[
\| x \|_\Delta = \sum_{i=1}^{\infty} |x_i| + \| \Delta^m x_k \|_\infty.
\]

If \( x \in X (\Delta^m) \) then there exists one and only one \( y = (y_k) \in X \) such that

\[
x_k = \sum_{i=1}^{k-m} (-1)^m \binom{k-i-1}{m-1} y_i = \sum_{i=1}^{k} (-1)^m \binom{k+m-i-1}{m-1} y_i-m, \quad y_{m+1} = y_{2m} = \cdots = y_0 = 0
\]

for sufficiently large \( k \), for instance \( k > 2m \). Recently, a large amount of work has been carried out by many mathematicians regarding various generalizations of sequence spaces. For a detailed account of sequence spaces one may refer to ([2-13]).

In 1932, Agnew [4] introduced the concept of deferred Cesaro mean of real (or complex) valued sequences \( x = (x_k) \) defined by

\[
(D_{p,q} x)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \quad n = 1, 2, 3, \ldots,
\]

where \( p = \{ p(n) \} \) and \( q = \{ q(n) \} \) are the sequences of non-negative integers satisfying

\[
p(n) < q(n) \quad \text{and} \quad \lim_{n \to \infty} q(n) = \infty.
\]

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2 Topological Properties of $X(\Delta^m)$

In this section we prove some results involving the sequence spaces $C_0^d(\Delta^m), C_1^d(\Delta^m)$ and $C_\infty^d(\Delta^m)$.

**Definition 1.** Let $m$ be a fixed non-negative integer and let $\{p(n)\}$ and $\{q(n)\}$ be two sequences of non-negative integers satisfying the condition (1). We define the following sequence spaces:

$$C_0^d(\Delta^m) = \left\{ x = (x_k) : \lim_{n} \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k = 0 \right\},$$

$$C_1^d(\Delta^m) = \left\{ x = (x_k) : \lim_{n} \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} (\Delta^m x_k - L) = 0 \right\},$$

$$C_\infty^d(\Delta^m) = \left\{ x = (x_k) : \sup_{n} \left( \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k \right) < \infty \right\}.$$

The above sequence spaces contain some unbounded sequences for $m \geq 1$, for example let $x = (k^m)$, then $x \in C_\infty^d(\Delta^m)$, but $x \notin \ell_\infty$.

**Theorem 1.** The sequence spaces $C_0^d(\Delta^m), C_1^d(\Delta^m)$ and $C_\infty^d(\Delta^m)$ are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \sup_{n} \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k.$$  

**Proof:** Proof follows from Theorem 1 of Et and Nuray [3].

**Theorem 2.** $X(\Delta^{m-1}) \subset X(\Delta^m)$ and the inclusion is strict for $X = C_0^d, C_1^d$ and $C_\infty^d$.

**Proof:** The inclusions part of the proof are easy. To see that the inclusions are strict, let $m = 2$ and $q(n) = n, p(n) = 0$ and consider a sequence defined by $x = (k^2)$, then $x \in C_1^d(\Delta^2)$, but $x \notin C_1^d(\Delta)$ ( if $x = (k^2)$, then $(\Delta^2 x_k) = (2, 2, 2, ...)$).

**Theorem 3.** The inclusions $C_0^d(\Delta^m) \subset C_1^d(\Delta^m) \subset C_\infty^d(\Delta^m)$ are strict.

**Proof:** First inclusion is easy. Second inclusion follows from the following inequality

$$\frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k \leq \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k - L + \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} L \leq \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k - L + L.$$

For strict the inclusion, observe that $x = (1, 0, 1, 0, ...) \in C_\infty^d(\Delta^m)$, but $x \notin C_1^d(\Delta^m)$ ( if $x = (1, 0, 1, 0, ...)$, then $(\Delta^m x_k) = (-1)^{m+1} 2^{m+1}$).

**Theorem 4.** $C_1^d(\Delta^m)$ is a closed subspace of $C_\infty^d(\Delta^m)$.
Proof: Proof follows from Theorem 4 of Et and Nuray [3].

Theorem 5. $C^d_{\infty}(\Delta^m)$ is a nowhere dense subset of $C^d_{\infty}(\Delta^m)$.

Proof: Proof follows from the fact that $C^d_{\infty}(\Delta^m)$ is a proper and complete subspace of $C^d_{\infty}(\Delta^m)$.

Theorem 6. $C^d_{\infty}(\Delta^m)$ is not separable, in general.

Proof: Suppose that $C^d_{\infty}(\Delta^m)$ is separable for some $m \geq 1$, for example let $m = 2$ and $q(n) = n, p(n) = 0$. In this case $C^d_{\infty}(\Delta^2)$ is separable. In Theorem 5, Bhardwaj et al. [5] show that $C^d_{\infty}(\Delta^2)$ is not separable. So $C^d_{\infty}(\Delta^m)$ is not separable, in general.

Theorem 7. $C^d_{\infty}(\Delta^m)$ does not have Schauder basis. separable, in general.

Proof: Proof follows from the fact that if a normed space has a Schauder basis, then it is separable.

Theorem 8. $C^d_{\infty}(\Delta^m)$ is separable.

Proof: Proof follows from Theorem 5 of Et and Nuray [3].

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4 References