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The Characterizations of the Spherical Images of Timelike W-Curves in Minkowski Space-Time

Minkowski Uzay-Zamanda Timelike W-Eğrilerin Küresel Göstergelerinin Karakterizasyonları

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Abstract

We know that W −curve is a curve which has constant Frenet curvatures. In this study, firstly, we have investigated the principal normal and binormal spherical images of a timelike W −curve on pseudohyperbolic space $\mathbb{H}_0^3$ in Minkowski space-time E_1^4. Besides, the binormal spherical image of the timelike W −curve is a spacelike curve which lies on pseudohyperbolic space $\mathbb{H}_0^3$. Hence, we have obtained the Frenet-Serret invariants of the mentioned image curve in terms of the invariants of the timelike W-curve in the same space. Finally, we have given some characterizations of the spherical image in the case of being helix for the timelike W −curve in Minkowski space-time E_1^4.

Keywords: *Spherical Images, Timelike W-Curve, General Helix, CCR-Curves*

Özet

W −eğrisinin sabit Frenet eğriliklerine sahip bir eğri olduğunu biliyoruz. Bu çalışmada, öncelikle, E_1^4 Minkowski uzay-zamanında, $\mathbb{H}_0^3$ pseudohiperbolik uzay üzerinde bir timelike W −eğrisinin asli normal ve binormal küresel göstergelerini araştırdık. Yanısıra, $\mathbb{H}_0^3$ pseudohiperbolik uzay üzerinde yatan timelike W −eğrisinin binormal küresel göstergesi spacelike bir eğridir. Bu nedenle, aynı uzayda, söz konusu görüntü eğrisinin Frenet-Serret değişmezlerini timelike W −eğrisinin değişmezleri cinsinden elde ettik. Son olarak, E_1^4 Minkowski uzay-zamanındaki timelike W −eğrisi için helis olması durumunda küresel göstergenin bazı karakterizasyonlarını verdik.

Anahtar Kelimeler: *Küresel Göstergeler, Timelike W −Eğrileri, Genel Helis, CCR-Eğriler*

1. Introduction

Lorentzian geometry helps to bridge the gap between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. Nearly a century ago, Einstein's formulation of general relativity expressed in terms of Lorentzian geometry was attractive for geometricians who could penetrate suprisingly into cosmology (redshift, expanding universe, big bang)[1].

Despite its long history, the theory of curve is still one of the most interesting topics in differential geometry and it is still being studied by many mathematicians until now. A tetrad of mutually orthogonal unit vectors (called tangent, normal, binormal, trinormal) was defined and constructed at each point of a differentiable curve. The rates of change of these vectors along the curve define the curvatures of the curve in the four dimensional space. Spherical images of a regular curve in the Euclidean space are obtained by means of Frenet-Serret frame vector fields, so the mentioned topic is a well-known concept in differential geometry of the curves [2]. Also, these kind of curves were studied in four dimensional Euclidean and Lorentzian space [3,4,5,6,7].

W-curve is another curve among the prominent curves which have the constant Frenet curvature. All $W-$curves in Minkowski 3-space are completely classified by Walrave in [3]. Besides, in Minkowski space-time, the spacelike, timelike, null W-curves are studied [8,9].

In this study, we have investigated the principal normal and binormal spherical images of a timelike $W-$curve on pseudohyperbolic space $\mathbb{H}_0^3$ in Minkowski space-time E_1^4. The binormal spherical image of the timelike W-curve is a spacelike curve which lies on pseudohyperbolic space $\mathbb{H}_0^3$. Hence, we have obtained the Frenet-Serret invariants of the mentioned image curve in terms of the invariants of the timelike $W-$curve. Finally, we have given some characterizations of the spherical image in the case of being helix for the timelike $W-$curve in Minkowski space-time E_1^4.

2. Material and Method

Minkowski spacetime E_1^4 is a real vector space R^4 furnished with the standard indefinite flat metric g defined by

$$g = -dx_1 + dx_2 + dx_3 + dx_{4,}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4 [1]. Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characters; it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0, v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), recpectively. Also, the norm of the vector v is given by

$$\|v\| = \sqrt{|g(v,v)|}.$$

The vector v is a unit vector if $g(v,v) = \mp 1$. Vectors v, w in E_1^4 are said to be orthogonal if $g(v,w) = 0$ [10]. Let u and v be two spacelike vectors in E_1^4, then there is a unique real number $0 \leq \delta \leq \pi$, called the angle between u and v, such that $g(u,v) = \|u\|\|v\| cos\delta$ [11].

The pseudohyperbolic space with the center $m = (m_1, m_2, m_3, m_4) \in E_1^4$ and radius $r \in \mathbb{R}^+$ in the spacetime E_1^4 is the hyperquadric

$$\mathbb{H}_0^3 = \{a = (a_1, a_2, a_3, a_4) \in E_1^4 \\ \mid g(a-m, a-m) = -r^2\},$$

with dimension 3 and index 0 [1].

Let $\varphi = \varphi(s)$ be a curve in E_1^4. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix. Recall that, if a regular curve has constant Frenet-Serret curvatures ratios in E_1^4, then it is called a ccr-curve [12,13,14]. Also, if these curvatures are non-zero constants, the curve is said to be $W-$curve (or helix) [15,16,17].

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet-Serret frame along the curve $\varphi(s)$ in E_1^4. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the binormal (the first binormal) and the trinormal (the second binormal) vector fields. A spacelike or timelike curve $\varphi(s)$ is said to be parametrized by arc-length function s, if $g(\varphi'(s), \varphi'(s)) = \pm 1$.

Let $\varphi(s)$ be a timelike curve in E_1^4, parametrized by arc-length function s. Then the following Frenet-Serret equations are given in [3]:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \qquad (1)$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying equations

$g(T,T) = -1,$

$g(N,N) = g(B_1,B_1) = g(B_2,B_2) = 1,$

and where κ, τ, σ are the first, second, and third curvatures of the curve φ, respectively.

In the same space, the authors expressed a characterization of spacelike curves lying on $\mathbb{H}_0^3$ by the following theorem:

Theorem 2.1. Let φ(s) be an unit speed spacelike curve in E_1^4, with the spacelike vectors N,B_1 and the curvatures $\kappa \neq 0, \tau \neq 0$, $\sigma \neq 0$ for each $s \in I \subset \mathbb{R}$. Then, the curve φ lies on pseudohyperbolic space if and only if

$$\frac{\sigma}{\tau}\frac{d\rho}{ds} = \frac{d}{ds}\left[\frac{1}{\sigma}\left(\rho\tau + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d\rho}{ds}\right)\right)\right], \qquad (2)$$

where

$$\left\{\frac{1}{\sigma}\left(\rho\tau + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d\rho}{ds}\right)\right)\right\}^2 > \rho^2 + \left(\frac{1}{\tau}\frac{d\rho}{ds}\right)^2$$

and $\rho = \frac{1}{\kappa}$ [15].

Definition 2.2. Let $a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in E_1^4. The vector product is defined by

$$a \times b \times c = -\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

where e_1, e_2, e_3, e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$e_1 \times e_2 \times e_3 = e_4, \qquad e_2 \times e_3 \times e_4 = e_1,$

$e_3 \times e_4 \times e_1 = e_2, \qquad e_4 \times e_1 \times e_2 = -e_3,$

[5].

Theorem 2.3. Let φ(s) be an arbitrary spacelike curve in E_1^4. The Frenet-Serret apparatus of the curve φ can be written as follows:

$$\begin{aligned} T &= \frac{\varphi'}{\|\varphi'\|}, \\ \mathrm{N} &= \frac{\|\varphi'\|^2\varphi'' - g(\varphi',\varphi'')\varphi'}{\|\|\varphi'\|^2\varphi'' - g(\varphi',\varphi'')\varphi'\|}, \\ B_1 &= \mu N \times T \times B_2, \\ B_2 &= \mu\frac{T \times N \times \varphi'''}{\|T \times N \times \varphi'''\|}, \end{aligned} \qquad (3)$$

and

$$\begin{aligned} \kappa &= \frac{\|\|\varphi'\|^2\varphi'' - g(\varphi',\varphi'')\varphi'\|}{\|\varphi'\|^4}, \\ \tau &= \frac{\|T \times N \times \varphi'''\|\|\varphi'\|}{\|\|\varphi'\|^2\varphi'' - g(\varphi',\varphi'')\varphi'\|}, \\ \sigma &= \frac{g(\varphi^{IV}, B_2)}{\|T \times N \times \varphi'''\|\|\varphi'\|}, \end{aligned} \qquad (4)$$

where μ is taken -1 or 1 to make 1 the determinant of $\{T, N, B_1, B_2\}$ matrix [5].

3. Results

3.1. The principal normal spherical image of a timelike W −curve in E_1^4

In this section, we give the definition of the principal normal spherical image for the timelike W −curves in Minkowsk space-time E_1^4.

Definition 3.1. Let $\beta = \beta(s)$ be a unit speed timelike W −curve in Minkowski space-time E_1^4. If we translate the principal normal vector to the center O of the pseudohyperbolic space $\mathbb{H}_0^3$, we obtain a curve $\delta = \delta(s_\delta)$. This curve is called the principal normal spherical indicatrix or image of the curve β in E_1^4.

Theorem 3.2. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\delta = \delta(s_\delta)$ be its principal normal spherical image. Then,

i) $\delta = \delta(s_\delta)$ is a spacelike curve if the first and second curvatures of $\beta(s)$ satisfy the following

$$\tau < \kappa < 0, \qquad 0 < \kappa < \tau.$$

ii) Frenet-Serret apparatus of the curve $\delta, \{T_\delta, N_\delta, B_{1\delta}, B_{2\delta}, \kappa_\delta, \tau_\delta, \sigma_\delta\}$ can be formed by the apparatus of the curve β.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W-curve and $\delta = \delta(s_\delta)$ be its principal normal spherical indicatrix. It can be written as

$$\delta = N(s). \tag{5}$$

Differentiating the equation (5) with respect to s, then we obtain

$$\delta' = \dot{\delta}\frac{ds_\delta}{ds} = \kappa T + \tau B_1. \tag{6}$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_δ by a dot. Thus, we obtain the unit tangent vector of the principal normal spherical image curve δ as

$$T_\delta = \frac{\kappa T + \tau B_1}{\sqrt{\tau^2 - \kappa^2}}, \tag{7}$$

and

$$\|\delta'\| = \frac{ds_\delta}{ds} = \sqrt{\tau^2 - \kappa^2}. \tag{8}$$

The causal character of the principal normal spherical image curve δ is determined by the following inner product:

$$g(\delta', \delta') = \tau^2 - \kappa^2. \tag{9}$$

From the expression (9), we will take the spherical image curve as spacelike one by assuming the conditions

$$\tau < \kappa < 0, \qquad 0 < \kappa < \tau. \tag{10}$$

Considering the previous method and using the property of the curve to be W −curve, we form the following differentiations with respect to s:

$$\begin{aligned} \delta'' &= (\kappa^2 - \tau^2)N + \tau\sigma B_2, \\ \delta''' &= \kappa(\kappa^2 - \tau^2)T + \tau(\kappa^2 - \tau^2 - \sigma^2)B_1, \\ \delta^{(IV)} &= ((\kappa^2 - \tau^2)^2 + \tau^2\sigma^2)N + \tau\sigma(\kappa^2 - \tau^2 - \sigma^2)B_2. \end{aligned} \tag{11}$$

By the expressions (2), we arrive at

$$\|\delta'\|^2\delta'' - g(\delta', \delta'')\delta' = -(\kappa^2 - \tau^2)^2 N + \tau\sigma(\tau^2 - \kappa^2)B_2. \tag{12}$$

Then, we can write the principal normal vector of the spherical image curve δ

$$N_\delta = \frac{\kappa^2 - \tau^2}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}N + \frac{\tau\sigma}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}.B_2, \tag{13}$$

and the first curvature of the spherical image curve δ is obtained by

$$\kappa_\delta = \frac{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}{\tau^2 - \kappa^2}. \tag{14}$$

Now, calculate the vector product $U = T_\delta \times N_\delta \times \delta'''$, that is, we have the vector U as

$$U = \frac{-\kappa\tau\sigma^2}{\sqrt{\tau^2 - \kappa^2}}\left(\frac{-\tau\sigma}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}N + \frac{\kappa^2 - \tau^2}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}B_2\right). \tag{15}$$

Hence, we obtain the trinormal (second binormal) vector field of the principal normal spherical image curve δ as follows:

$$B_{2\delta} = \mu\left(\frac{\tau\sigma}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}N + \frac{\tau^2 - \kappa^2}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}B_2\right). \tag{16}$$

Taking the norm of both sides of the expressions (15) and (12) then the second curvature of the principal normal spherical image curve δ is

$$\tau_\delta = \frac{-\kappa\tau\sigma^2}{(\tau^2 - \kappa^2)\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}. \tag{17}$$

To obtain the binormal vector field of the principal normal spherical image curve δ,we express $V = N_\delta \times T_\delta \times B_{2\delta}$ as follows:

$$V = -\frac{\tau}{\sqrt{\tau^2 - \kappa^2}}T - \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}B_1. \tag{18}$$

From the expression (18), then we get the binormal vector of the principal normal spherical image curve δ

$$B_{1\delta} = \mu\left(-\frac{\tau}{\sqrt{\tau^2-\kappa^2}}T - \frac{\kappa}{\sqrt{\tau^2-\kappa^2}}B_1\right) \tag{19}$$

Using the equation (16), the third curvature is given by

$$\sigma_\delta = \mu\frac{\kappa\sigma}{\sqrt{(\tau\sigma)^2+(\tau^2-\kappa^2)^2}}. \tag{20}$$

Corollary 3.3. Frenet-Serret apparatus of the principal normal spherical image curve δ is an orthonormal frame of Minkowski space-time E_1^4.

Proof. It can be straightforwardly seen by using the equations (7), (13), (16), (19).

Corollary 3.4. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\delta = \delta(s_\delta)$ be its principal normal spherical image. Then, the curve δ is also a helix.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W −curve. We know that the curvature functions are constants. Therefore, we know that the curvature functions of the principal normal spherical image $\delta(s_\delta)$ are constants by means of the equations (14), (17) and (20). Hence, the curve $\delta(s_\delta)$ becomes W −curve which is the special case of helix.

Theorem 3.5. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\delta = \delta(s_\delta)$ be its principal normal spherical image. If δ is a general helix, then its fixed direction Φ is composed

$$\begin{aligned}\Phi = &\left(-\frac{1}{2}x_1\kappa s^2 - x_2\kappa s + x_3\right)T \\ &+(x_1 s + x_2)N \\ &+\begin{pmatrix}-\frac{1}{2\tau}x_1\kappa^2 s^2 - \frac{1}{\tau}x_2\kappa^2 s \\ +\ \frac{1}{\tau}x_3\kappa + \frac{x_1}{\tau}\end{pmatrix}B_1 \\ &+\begin{pmatrix}\frac{1}{6\tau}x_1\kappa^2\sigma s^3 + \frac{1}{2\tau}x_2\kappa^2\sigma s^2 \\ -\frac{1}{\tau}x_3\kappa\sigma - \frac{x_1\sigma}{\tau}s + x_4\end{pmatrix}B_2,\end{aligned} \tag{21}$$

where x_1 is a non-zero constant and x_2, x_3, x_4 are constants.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\delta = \delta(s_\delta)$ be its principal normal spherical image. If δ is a general helix, then for a spacelike vector Φ, we may express

$$g(T_\delta, \Phi) = \cos\theta, \tag{22}$$

where θ is a constant angle. The equation (22) is also congruent to

$$g\left(\frac{\kappa T + \tau B_1}{\sqrt{\tau^2-\kappa^2}}, \Phi\right) = \cos\theta. \tag{23}$$

The constant vector Φ according to $\{T, N, B_1, B_2\}$ is formed as

$$\Phi = \varepsilon_1 T + \varepsilon_2 N + \varepsilon_3 B_1 + \varepsilon_4 B_2 \tag{24}$$

Differentiating the expression (24) with respect to s, then we have the following system of ordinary differential equations

$$\begin{cases}\varepsilon_1' + \varepsilon_2\kappa = 0\\ \varepsilon_1\kappa + \varepsilon_2' - \varepsilon_3\tau = 0\\ \varepsilon_2\tau - \varepsilon_4\sigma + \varepsilon_3' = 0\\ \varepsilon_4' + \varepsilon_3\sigma = 0\end{cases} \tag{25}$$

We know that $-\varepsilon_1\kappa + \varepsilon_3\tau = x_1 \neq 0$ is a non-zero constant. Since the curve $\beta = \beta(s)$ is a W −curve, its curvature functions are constants. Then the solution of the system (25) can be obtained as

$$\begin{aligned}\varepsilon_1 &= -\frac{1}{2}x_1\kappa s^2 - x_2\kappa s + x_3,\\ \varepsilon_2 &= x_1 s + x_2,\\ \varepsilon_3 &= -\frac{1}{2\tau}x_1\kappa^2 s^2 - \frac{1}{\tau}x_2\kappa^2 s\\ &\quad +\ \frac{1}{\tau}x_3\kappa + \frac{x_1}{\tau},\\ \varepsilon_4 &= \frac{1}{6\tau}x_1\kappa^2\sigma s^3 + \frac{1}{2\tau}x_2\kappa^2\sigma s^2\\ &\quad -\frac{1}{\tau}x_3\kappa\sigma s - \frac{x_1\sigma}{\tau}s + x_4.\end{aligned} \tag{26}$$

3.2. The binormal spherical image of a timelike W −curve in E_1^4

In this section, we give the definition of the binormal spherical image for timelike W −curves in Minkowski space-time E_1^4.

Definition 3.6. Let $\beta = \beta(s)$ be a unit speed timelike W −curve in Minkowski space-time E_1^4. If we translate the binormal vector to the center O of the pseudohyperbolic space $\mathbb{H}_0^3$, we obtain a curve $\varphi = \varphi(s_\varphi)$. This curve is called the binormal spherical indicatrix or image of the curve β in E_1^4.

Theorem 3.7. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\varphi = \varphi(s_\varphi)$ be its binormal spherical image. Then,

i) $\varphi = \varphi(s_\varphi)$ is a spacelike curve.

ii) Frenet-Serret apparatus of the curve $\varphi, \{T_\varphi, N_\varphi, B_{1\varphi}, B_{2\varphi}, \kappa_\varphi, \tau_\varphi, \sigma_\varphi\}$ can be formed by the apparatus of the curve β.

iii) $\varphi = \varphi(s_\varphi)$ is also a helix lying on the pseudohyperbolic sphere $\mathbb{H}_0^3$ in E_1^4.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\varphi = \varphi(s_\varphi)$ be its binormal spherical image. It can be written as

$$\varphi = B_1(s). \tag{27}$$

Differentiating the equation (27) with respect to s, then we obtain

$$\varphi' = \dot{\varphi}\frac{ds_\varphi}{ds} = -\tau N + \sigma B_2. \tag{28}$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_φ by a dot. Thus, we obtain the unit tangent vector of the binormal spherical image curve φ as

$$T_\varphi = \frac{-\tau N + \sigma B_2}{\sqrt{\tau^2 + \sigma^2}}, \tag{29}$$

and

$$\|\varphi'\| = \frac{ds_\varphi}{ds} = \sqrt{\tau^2 + \sigma^2}. \tag{30}$$

The causal character of the binormal spherical image curve φ is determined by the following inner product:

$$g(\varphi', \varphi') = \tau^2 + \sigma^2. \tag{31}$$

According to the expression (31), the binormal spherical image is a spacelike curve.

Considering the previous method and using the property of the curve to be W −curve, the following differentiations with respect to s are formed:

$$\begin{aligned} \varphi'' &= -\tau\kappa T - (\tau^2 + \sigma^2)B_1, \\ \varphi''' &= \tau\begin{pmatrix}\tau^2 + \sigma^2 \\ -\kappa^2\end{pmatrix}N - \sigma(\tau^2 + \sigma^2)B_2, \\ \varphi^{(IV)} &= \tau(\kappa(\tau^2 + \sigma^2) - \kappa^3)T \\ &\quad + ((\tau^2 + \sigma^2)^2 - \tau^2\kappa^2)B_1. \end{aligned} \tag{32}$$

By the expressions (2), then we get

$$\|\varphi'\|^2\varphi'' - g(\varphi', \varphi'')\varphi' = -(\tau^2 + \sigma^2)\tau\kappa T - (\tau^2 + \sigma^2)^2 B_1. \tag{33}$$

Then, we can get the principal normal vector of the binormal spherical image curve φ

$$\begin{aligned} N_\varphi = &-\frac{\kappa\tau}{\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|}}T \\ &-\frac{\tau^2 + \sigma^2}{\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|}}B_1, \end{aligned} \tag{34}$$

and the first curvature of the binormal spherical image curve φ is as:

$$\kappa_\varphi = \frac{\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|}}{\tau^2 + \sigma^2}. \tag{35}$$

The vector product $X = T_\varphi \times N_\varphi \times \varphi'''$ is given by

$$X = -\frac{1}{\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|(\tau^2 + \sigma^2)}}\begin{pmatrix}(\tau^2 + \sigma^2)T \\ +(\kappa\tau)B_1\end{pmatrix}.$$

(36)

Using the expression (36), then the trinormal (second binormal) vector field of the binormal spherical image curve φ is obtained as

$$B_{2\varphi} = -\frac{\mu}{\kappa^2\tau\sigma\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|}}\begin{pmatrix}(\tau^2 + \sigma^2)T \\ +(\kappa\tau)B_1\end{pmatrix}. \tag{37}$$

Taking the norm of both sides of the equations (33) and (36), then we find the second curvature of the binormal spherical image curve φ

$$\tau_\varphi = \frac{\kappa^2\tau\sigma}{(\tau^2 + \sigma^2)\sqrt{(\tau^2 + \sigma^2)^2 - (\kappa\tau)^2}}. \tag{38}$$

The binormal vector field of the the binormal spherical image curve φ is expressed as

$$B_{1\varphi} = -\frac{\mu}{\sqrt{\tau^2-\kappa^2}}\begin{pmatrix}\sigma N\\ +\tau B_2\end{pmatrix}. \tag{39}$$

Finally, using the equation (39), then the third curvature of the the binormal spherical image curve φ is obtained by

$$\sigma_\varphi = 0. \tag{40}$$

Corollary 3.8. Frenet-Serret apparatus of the binormal spherical image φ is an orthonormal frame of Minkowski space-time E_1^4.

Proof. It can be straightforwardly seen by using the equations (29), (34), (37), (39).

Corollary 3.9. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\varphi = \varphi(s_\varphi)$ be its binormal spherical image. Then, the curve φ is also a helix.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W −curve. We know that the curvature functions are constants. We know that the curvature functions of the binormal spherical image $\varphi(s_\varphi)$ are constants. Hence, the curve $\varphi(s_\varphi)$ becomes W −curve which is the special case of helix.

Theorem 3.10. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\varphi = \varphi(s_\varphi)$ be its binormal spherical image. If φ is a general helix, then its fixed direction Φ is composed

$$\begin{aligned}\Phi = &\left(\frac{1}{6\tau}x_1\sigma^2\kappa s^3 + \frac{x_2\sigma^2\kappa s^2}{2\tau} - \frac{1}{\tau}x_3\kappa\sigma + \frac{1}{\tau}x_1\kappa\sigma + x_4\right)T\\ &+\left(-\frac{1}{2\tau}x_1\sigma^2 s^2 - \frac{1}{\tau}x_2\sigma^2 s + \frac{1}{\tau}x_3\sigma - \frac{x_1}{\tau}\right)N\\ &+(x_1 s + x_2)B_1\\ &+\left(-\frac{1}{2}x_1\sigma s^2 - x_2\sigma s + x_3\right)B_2,\end{aligned} \tag{41}$$

where x_1 is a non-zero constant and x_2, x_3, x_4 are constants.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\varphi = \varphi(s_\varphi)$ be its binormal spherical indicatrix. If φ is a general helix, then for a constant spacelike vector Φ, we may express

$$g(T_\varphi, \Phi) = \cos\theta, \tag{42}$$

where θ is a constant angle. The equation (28) is also congruent to

$$g\left(\frac{-\tau N + \sigma B_2}{\sqrt{\tau^2+\sigma^2}}, \Phi\right) = \cos\theta. \tag{43}$$

The constant vector Φ according to $\{T, N, B_1, B_2\}$ is formed as

$$\Phi = \varepsilon_1 T + \varepsilon_2 N + \varepsilon_3 B_1 + \varepsilon_4 B_2 \tag{44}$$

Differentiating the expression (43) with respect to s, then we have the following system of ordinary differential equations

$$\begin{cases}\varepsilon_1' + \varepsilon_2\kappa = 0\\ \varepsilon_1\kappa + \varepsilon_2' - \varepsilon_3\tau = 0\\ \varepsilon_2\tau - \varepsilon_4\sigma + \varepsilon_3' = 0\\ \varepsilon_4' + \varepsilon_3\sigma = 0\end{cases} \tag{45}$$

We know that $-\varepsilon_2\tau + \varepsilon_4\sigma = x_1 \neq 0$ is a non-zero constant. Since the curve $\beta = \beta(s)$ is a W −curve, its curvature functions are constants. Then the solution of the system (44) can be obtained as

$$\begin{aligned}\varepsilon_1 &= \frac{1}{6\tau}x_1\sigma^2\kappa s^3 + \frac{x_2\sigma^2\kappa s^2}{2\tau} - \frac{1}{\tau}x_3\kappa\sigma + \frac{1}{\tau}x_1\kappa\sigma + x_4,\\ \varepsilon_2 &= -\frac{1}{2\tau}x_1\sigma^2 s^2 - \frac{1}{\tau}x_2\sigma^2 s + \frac{1}{\tau}x_3\sigma - \frac{x_1}{\tau},\\ \varepsilon_3 &= x_1 s + x_2,\\ \varepsilon_4 &= -\frac{1}{2}x_1\sigma s^2 - x_2\sigma s + x_3.\end{aligned} \tag{46}$$

4. Discussion and Conclusion

In the present work, we extend spherical image concept to timelike W −curve in Minkowski space-time. We investigate principal normal and binormal spherical images of a timelike $W-$curve and observe that principal normal spherical curves are spacelike curves under certain conditions, and also binormal spherical images occur entirely as spacelike curves. Thereafter, we determine relations between Frenet-Serret invariants of the base curve and its spherical images.

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