Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (6) (2021), 1745 – 1755
DOI : 10.15672/hujms.678660

Research Article

On generalized weakly symmetric α-cosymplectic manifolds

Selahattin Beyendi[1], Mustafa Yıldırım*[2]

[1] Department of Mathematics, Faculty of Education, İnönü University, 44000, Malatya, Turkey
[2] Department of Mathematics, Faculty of Art and Science, Aksaray University, 68100, Aksaray, Turkey

Abstract

This study is concerned with some results on generalized weakly symmetric and generalized weakly Ricci-symmetric α-cosymplectic manifolds. We prove the necessary and sufficient conditions for an α-cosymplectic manifold to be generalized weakly symmetric and generalized weakly Ricci-symmetric. On the basis of these results, we give one proper example of generalized weakly symmetric α-cosymplectic manifolds.

Mathematics Subject Classification (2020). Primary 53C15, 53C25; Secondary 53D10, 53D15

Keywords. weakly symmetric manifold, weakly Ricci-symmetric manifold, generalized weakly symmetric manifold, generalized weakly Ricci-symmetric manifold, α-cosymplectic manifold

1. Introduction

In 1989, L. Tamassy and T. Q. Binh introduced the notions of weakly symmetric and weakly Ricci-symmetric manifolds [18]. Later on, many researchers have studied this topic. For details, we refer the reader to [3, 5, 8, 11, 12, 14, 16, 17, 21] and the references there in. In the view of [18], a $(2n+1)$-dimensional α- cosymplectic manifold is said to be weakly symmetric manifold, if its curvature tensor $\tilde{R}$ of type $(0,4)$ is not identically zero and admits the following identity:

$$\begin{aligned}(\nabla_W\tilde{R})(X_1,X_2,X_3,X_4) &= \mathcal{A}_1(W)\tilde{R}(X_1,X_2,X_3,X_4)+\mathcal{B}_1(X_1)\tilde{R}(W,X_2,X_3,X_4)\\ &\quad+\mathcal{B}_1(X_2)\tilde{R}(X_1,W,X_3,X_4)+\mathfrak{D}_1(X_3)\tilde{R}(X_1,X_2,W,X_4)\\ &\quad+\mathfrak{D}_1(X_4)\tilde{R}(X_1,X_2,X_3,W),\end{aligned} \tag{1.1}$$

where ∇ denotes the Levi-Civita connection with respect to metric g on M, also $\mathcal{A}_1,\mathcal{B}_1,\mathfrak{D}_1$ are non-zero 1-forms defined by $\mathcal{A}_1(W)=g(W,\sigma_1)$, $\mathcal{B}_1(W)=g(W,\varrho_1)$ and $\mathfrak{D}_1(W)=g(W,\pi_1)$, for all W and $\tilde{R}(X_1,X_2,X_3,X_4)=g(R(X_1,X_2)X_3,X_4)$. A $(2n+1)$-dimensional α-cosymplectic manifold of this kind is denoted by a $(WS)_{2n+1}$-manifold. Dubey [9] presented generalized recurrent manifold. In keeping with this work, we shall describe

*Corresponding Author.
Email addresses: selahattin.beyendi@inonu.edu.tr (S. Beyendi), mustafayldrm24@gmail.com (M. Yıldırım)
Received: 22.01.2020; Accepted: 27.07.2021

a $(2n+1)$-dimensional α-cosymplectic manifold generalized weakly symmetric (briefly $(GWS)_{2n+1}$-manifold) if it admits the following equation:

$$\begin{aligned}(\nabla_W \tilde{R})(X_1,X_2,X_3,X_4) &= \mathcal{A}_1(W)\tilde{R}(X_1,X_2,X_3,X_4)+\mathcal{B}_1(X_1)\tilde{R}(W,X_2,X_3,X_4)\\ &+\mathcal{B}_1(X_2)\tilde{R}(X_1,W,X_3,X_4)+\mathcal{D}_1(X_3)\tilde{R}(X_1,X_2,W,X_4)\\ &+\mathcal{D}_1(X_4)\tilde{R}(X_1,X_2,X_3,W)+\mathcal{A}_2(W)\tilde{G}(X_1,X_2,X_3,X_4)\\ &+\mathcal{B}_2(X_1)\tilde{G}(W,X_2,X_3,X_4)+\mathcal{B}_2(X_2)\tilde{G}(X_1,W,X_3,X_4)\\ &+\mathcal{D}_2(X_3)\tilde{G}(X_1,X_2,W,X_4)+\mathcal{D}_2(X_4)\tilde{G}(X_1,X_2,X_3,W),\end{aligned} \tag{1.2}$$

where

$$\tilde{G}(X_1,X_2,X_3,X_4)=[g(X_2,X_3)g(X_1,X_4)-g(X_1,X_3)g(X_2,X_4)] \tag{1.3}$$

and $\mathcal{A}_i,\mathcal{B}_i,\mathcal{D}_i$ are non-zero 1-forms defined by $\mathcal{A}_i(W)=g(W,\sigma_i)$, $\mathcal{B}_i(W)=g(W,\varrho_i)$ and $\mathcal{D}_i(W)=g(W,\pi_i)$, for $i=1,2$. There are interesting results of such $(GWS)_{2n+1}$-manifolds in that it exhibits

(i) (for $\mathcal{A}_i=\mathcal{B}_i=\mathcal{D}_i=0$), locally symmetric space [6],

(ii) (for $\mathcal{A}_1\neq 0,\mathcal{A}_2=\mathcal{B}_i=\mathcal{D}_i=0$), recurrent space [20],

(iii) (for $\mathcal{A}_i\neq 0,\mathcal{B}_i=\mathcal{D}_i=0$), generalized recurrent space [9],

(iv) (for $\frac{\mathcal{A}_1}{2}=\mathcal{B}_1=\mathcal{D}_1=H_1\neq 0,\mathcal{A}_2=\mathcal{B}_2=\mathcal{D}_2=0$), pseudo symmetric space [7],

(v) (for $\frac{\mathcal{A}_i}{2}=\mathcal{B}_i=\mathcal{D}_i=H_i\neq 0$), generalized pseudo symmetric space [3],

(vi)) (for $\mathcal{A}_i=\mathcal{B}_2=\mathcal{D}_2=0,\mathcal{B}_1=\mathcal{D}_1\neq 0$), semi-pseudo symmetric space [19],

(vii) (for $\mathcal{A}_i=0,\mathcal{B}_i=\mathcal{D}_i\neq 0$), generalized semi-pseudo symmetric space [3],

(viii) (for $\mathcal{A}_1=H_1+K_1,\mathcal{B}_1=\mathcal{D}_1=H_1\neq 0$ *and* $\mathcal{A}_2=\mathcal{B}_2=\mathcal{D}_2=0$), almost pseudo symmetric space [7],

(ix) (for $\mathcal{A}_i=H_i+K_i,\mathcal{B}_i=\mathcal{D}_i=H_i\neq 0$), almost generalized pseudo symmetric space [3],

(x) (for $\mathcal{A}_1,\mathcal{B}_1,\mathcal{D}_i\neq 0,\mathcal{A}_2=\mathcal{B}_2=\mathcal{D}_2=0$), weakly symmetric space [18] .

In the present paper, we have investigated some properties of the generalized weakly symmetric α-cosymplectic manifolds. In Section 2, we review basic formulas and definitions for α-cosymplectic manifolds. In Section 3, we have examined a generalized weakly symmetric α-cosymplectic manifold and it is observed that such a space is an η-Einstein manifold provided $\mathcal{D}_1(\xi)\neq -\alpha$. We also present tables of different types of curvature restrictions for which α-cosymplectics manifolds are sometimes Einstein and some other times remain η-Einstein. In Section 4, we have given an example of the existence of such manifolds. Finally, we have investigated a generalized weakly Ricci-symmetric α-cosymplectic manifold which is also found to be η-Einstein space.

2. Preliminaries

Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a $(2n+1)$-dimensional almost contact metric manifold, where φ is a $(1,1)$-tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well known that the (φ,ξ,η,g) structure satisfies the following conditions [5]:

$$\varphi(\xi)=0,\qquad \eta(\varphi)=0,\qquad \eta(\xi)=1, \tag{2.1}$$

$$\varphi^2 W = -W + \eta(W)\xi, \quad g(W,\xi) = \eta(W), \tag{2.2}$$

$$g(\varphi W, \varphi X_1) = g(W, X_1) - \eta(W)\eta(X_1), \tag{2.3}$$

for any vector fields W and X_1 on M^{2n+1}. If in addition,

$$\nabla_W \xi = -\alpha\varphi^2 W, \tag{2.4}$$

$$(\nabla_W \eta)X_1 = \alpha[g(W, X_1) - \eta(W)\eta(X_1)], \tag{2.5}$$

where ∇ denotes the Riemannian connection holds and α is a real number, then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called an α-cosymplectic manifold ([1, 2, 10, 13, 15]). In this case, it is well known that [4]

$$R(W, X_1)\xi = \alpha^2[\eta(W)X_1 - \eta(X_1)W], \tag{2.6}$$

$$S(W,\xi) = -2n\alpha^2\eta(W), \tag{2.7}$$

$$S(\xi,\xi) = -2n\alpha^2, \tag{2.8}$$

where S denotes the Ricci tensor. From (2.6), it easily follows that

$$R(W,\xi)X_1 = \alpha^2[g(W, X_1)\xi - \eta(X_1)W], \tag{2.9}$$

$$R(W,\xi)\xi = \alpha^2[\eta(W)\xi - W], \tag{2.10}$$

for any vector fields W, X_1 where R denotes the Riemannian curvature tensor of M. An α-cosymplectic manifold is said to be an η-Einstein manifold if the Ricci tensor S satisfies condition

$$S(W, X_1) = \lambda_1 g(W, X_1) + \lambda_2\eta(W)\eta(X_1), \tag{2.11}$$

where λ_1, λ_2 are certain scalars.

3. Generalized weakly symmetric α-cosymplectic manifold

In this section, let us consider a generalized weakly symmetric α-cosymplectic manifold (M^{2n+1}, g) $(n \geq 1)$. Now, contracting X_1 over X_4 in both sides of (1.2), we get

$$\begin{aligned}(\nabla_W S)(X_2, X_3) = {} & \mathcal{A}_1(W)S(X_2, X_3) + \mathcal{B}_1(R(W, X_2)X_3) + \mathcal{B}_1(X_2)S(W, X_3)\\ & +\mathcal{D}_1(X_3)S(X_2, W) + \mathcal{D}_1(R(W, X_3)X_2) + 2n\mathcal{A}_2(W)g(X_2, X_3)\\ & +2n\mathcal{B}_2(X_2)g(W, X_3) + 2n\mathcal{D}_2(X_3)g(W, X_2)\\ & +\mathcal{B}_2(W)g(X_2, X_3) - \mathcal{B}_2(X_2)g(W, X_3) + \mathcal{D}_2(W)g(X_2, X_3)\\ & -\mathcal{D}_2(X_3)g(X_2, W).\end{aligned} \tag{3.1}$$

Putting $X_3 = \xi$ in (3.1), we obtain

$$\begin{aligned}(\nabla_W S)(X_2, \xi) = {} & \mathcal{A}_1(W)S(X_2, \xi) + \mathcal{B}_1(R(W, X_2)\xi) + \mathcal{B}_1(X_2)S(W, \xi)\\ & +\mathcal{D}_1(\xi)S(X_2, W) + \mathcal{D}_1(R(W, \xi)X_2) + 2n\mathcal{A}_2(W)\eta(X_2)\\ & +2n\mathcal{B}_2(X_2)\eta(W) + 2n\mathcal{D}_2(\xi)g(X_2, W) + \mathcal{B}_2(W)\eta(X_2)\\ & -\mathcal{B}_2(X_2)\eta(W) + \mathcal{D}_2(W)\eta(X_2) - \mathcal{D}_2(\xi)g(X_2, W).\end{aligned} \tag{3.2}$$

Using (2.6), (2.7) and (2.9) in (3.2), we get

$$\begin{aligned}(\nabla_W S)(X_2, \xi) = {} & -2n\alpha^2\mathcal{A}_1(W)\eta(X_2) + \alpha^2\mathcal{B}_1(X_2)\eta(W) - \alpha^2\mathcal{B}_1(W)\eta(X_2)\\ & -2n\alpha^2\mathcal{B}_1(X_2)\eta(W) + \mathcal{D}_1(\xi)S(X_2, W) + \alpha^2\mathcal{D}_1(\xi)g(X_2, W)\\ & -\alpha^2\mathcal{D}_1(W)\eta(X_2) + 2n\mathcal{A}_2(W)\eta(X_2) + 2n\mathcal{B}_2(X_2)\eta(W)\\ & +2n\mathcal{D}_2(\xi)g(X_2, W) + \mathcal{B}_2(W)\eta(X_2) - \mathcal{B}_2(X_2)\eta(W)\\ & +\mathcal{D}_2(W)\eta(X_2) - \mathcal{D}_2(\xi)g(X_2, W).\end{aligned} \tag{3.3}$$

We know that

$$(\nabla_W S)(X_2, X_3) = \nabla_W S(X_2, X_3) - S(\nabla_W X_2, X_3) - S(X_2, \nabla_W X_3). \tag{3.4}$$

Next we take $X_3 = \xi$ in (3.4) and then using (2.2), (2.4) and (2.7), we obtain

$$(\nabla_W S)(X_2, \xi) = -2n\alpha^3 g(X_2, W) - \alpha S(X_2, W). \tag{3.5}$$

Now, using (3.5) in (3.3), we have

$$\begin{aligned}&-2n\alpha^3 g(X_2,W)-\alpha S(X_2,W)\\&=-2n\alpha^2\mathcal{A}_1(W)\eta(X_2)-(2n-1)\alpha^2\mathcal{B}_1(X_2)\eta(W)+\mathcal{D}_1(\xi)S(X_2,W)\\&-\alpha^2\mathcal{B}_1(W)\eta(X_2)-\alpha^2\mathcal{D}_1(W)\eta(X_2)+\alpha^2\mathcal{D}_1(\xi)g(X_2,W)\\&+2n[\mathcal{A}_2(W)\eta(X_2)+\mathcal{B}_2(X_2)\eta(W)+\mathcal{D}_2(\xi)g(X_2,W)]\\&+\mathcal{B}_2(W)\eta(X_2)-\mathcal{B}_2(X_2)\eta(W)+\mathcal{D}_2(W)\eta(X_2)-\mathcal{D}_2(\xi)g(X_2,W),\end{aligned}\tag{3.6}$$

which results in

$$\alpha^2[\mathcal{A}_1(\xi)+\mathcal{B}_1(\xi)+\mathcal{D}_1(\xi)]=[\mathcal{A}_2(\xi)+\mathcal{B}_2(\xi)+\mathcal{D}_2(\xi)]\tag{3.7}$$

for $W=X_2=\xi$. In particular, if $\mathcal{A}_2(\xi)=\mathcal{B}_2(\xi)=\mathcal{D}_2(\xi)=0$, equation (3.7) turns into

$$\alpha^2[\mathcal{A}_1(\xi)+\mathcal{B}_1(\xi)+\mathcal{D}_1(\xi)]=0.\tag{3.8}$$

Theorem 3.1. *In a generalized weakly symmetric α-cosymplectic manifold (M^{2n+1},g), the relation (3.7) is hold.*

Putting $X_2=\xi$ in (3.1) and using (3.4), we obtain

$$\begin{aligned}&-2n\alpha^3 g(W,X_3)-\alpha S(W,X_3)\\&=-2n\alpha^2\mathcal{A}_1(W)\eta(X_3)+\alpha^2 g(W,X_3)\mathcal{B}_1(\xi)-\alpha^2\mathcal{B}_1(W)\eta(X_3)\\&+\mathcal{B}_1(\xi)S(W,X_3)-(2n-1)\alpha^2\mathcal{D}_1(X_3)\eta(W)-\alpha^2\mathcal{D}_1(W)\eta(X_3)\\&+2n[\mathcal{A}_2(W)\eta(X_3)+\mathcal{B}_2(\xi)g(W,X_3)+\mathcal{D}_2(X_3)\eta(W)]\\&+\mathcal{B}_2(W)\eta(X_3)-\mathcal{B}_2(\xi)g(W,X_3)+\mathcal{D}_2(W)\eta(X_3)-\mathcal{D}_2(X_3)\eta(W).\end{aligned}\tag{3.9}$$

Taking $X_3=\xi$ in (3.9) and also using (2.2) and (2.7), we get

$$\begin{aligned}&\alpha^2[2n\mathcal{A}_1(W)+\mathcal{B}_1(W)+\mathcal{D}_1(W)+(2n-1)\eta(W)(\mathcal{B}_1(\xi)+\mathcal{D}_1(\xi))]\\&=2n\mathcal{A}_2(W)+(2n-1)\eta(W)[\mathcal{B}_2(\xi)+\mathcal{D}_2(\xi)]+\mathcal{B}_2(W)+\mathcal{D}_2(W).\end{aligned}\tag{3.10}$$

Now putting $W=\xi$ in (3.9) and using (2.1), (2.2) and (2.7), we obtain

$$\begin{aligned}&\alpha^2[2n(\mathcal{A}_1(\xi)+\mathcal{B}_1(\xi))\eta(X_3)+(2n-1)\mathcal{D}_1(X_3)+\mathcal{D}_1(\xi)\eta(X_3)]\\&=2n[(\mathcal{A}_2(\xi)+\mathcal{B}_2(\xi))\eta(X_3)+\mathcal{D}_2(X_3)]+\mathcal{D}_2(\xi)\eta(X_3)-\mathcal{D}_2(X_3).\end{aligned}\tag{3.11}$$

Replacing X_3 by W in (3.11) and using (3.7), we have

$$\alpha^2\mathcal{D}_1(W)-\alpha^2\mathcal{D}_1(\xi)\eta(W)=\mathcal{D}_2(W)-\mathcal{D}_2(\xi)\eta(W).\tag{3.12}$$

Putting $W=\xi$ in (3.6), we get

$$\begin{aligned}&-2n\alpha^3\eta(X_2)+2n\alpha^3\eta(X_2)\\&=-2n\alpha^2\mathcal{A}_1(\xi)\eta(X_2)-(2n-1)\alpha^2\mathcal{B}_1(X_2)-2n\alpha^2\mathcal{D}_1(\xi)\eta(X_2)\\&-\alpha^2\mathcal{B}_1(\xi)\eta(X_2)-\alpha^2\mathcal{D}_1(\xi)\eta(X_2)+\alpha^2\mathcal{D}_1(\xi)\eta(X_2)\\&+2n[\mathcal{A}_2(\xi)\eta(X_2)+\mathcal{B}_2(X_2)+\mathcal{D}_2(\xi)\eta(X_2)]\\&+\mathcal{B}_2(\xi)\eta(X_2)-\mathcal{B}_2(X_2)+\mathcal{D}_2(\xi)\eta(X_2)-\mathcal{D}_2(\xi)\eta(X_2).\end{aligned}\tag{3.13}$$

Replacing X_2 by W in (3.13) and then using (3.7), (3.12), we obtain

$$\alpha^2\mathcal{B}_1(W)-\alpha^2\mathcal{B}_1(\xi)\eta(W)=\mathcal{B}_2(W)-\mathcal{B}_2(\xi)\eta(W).\tag{3.14}$$

Using (3.10), (3.12) and (3.14), we get

$$\alpha^2[\mathcal{A}_1(W)+(\mathcal{B}_1(\xi)+\mathcal{D}_1(\xi))\eta(W)]=\mathcal{A}_2(W)+(\mathcal{B}_2(\xi)+\mathcal{D}_2(\xi))\eta(W).\tag{3.15}$$

In view of (3.12), (3.14) and (3.15), we obtain

$$\alpha^2[\mathcal{A}_1(W)+\mathcal{B}_1(W)+\mathcal{D}_1(W)]=[\mathcal{A}_2(W)+\mathcal{B}_2(W)+\mathcal{D}_2(W)].\tag{3.16}$$

Next, for the choice of $\mathcal{A}_2=\mathcal{B}_2=\mathcal{D}_2=0$, the relation in equation (3.16) yields the following:

$$\alpha^2[\mathcal{A}_1(W)+\mathcal{B}_1(W)+\mathcal{D}_1(W)]=0.\tag{3.17}$$

This motivates us to state the following theorems.

Theorem 3.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a $(GWS)_{2n+1}$ α-cosymplectic manifold, the sum of the associated is given by (3.16).*

Theorem 3.3. *There does not exist an α-cosymplectic manifold which is*

(i) *recurrent,*

(ii) *generalized recurrent provided the 1-forms associated to the vector fields are colliner,*

(iii) *pseudo symmetric,*

(iv) *generalized semi-pseudo symmetric provided the 1-forms associated to the vector fields are collinear.*

Again from (3.6), putting $W = \xi$, we have

$$\begin{aligned} &2n[\alpha^2\mathcal{A}_1(\xi) - \mathcal{A}_2(\xi) + \alpha^2\mathcal{D}_1(\xi) - \mathcal{D}_2(\xi)]\eta(X_2) \\ &= [-\alpha^2\mathcal{B}_1(\xi) + \mathcal{B}_2(\xi)]\eta(X_2) + (2n-1)[-\alpha^2\mathcal{B}_1(X_2) + \mathcal{B}_2(X_2)]. \end{aligned} \tag{3.18}$$

Using (3.7), the above equation becomes

$$[\alpha^2\mathcal{B}_1(\xi) - \mathcal{B}_2(\xi)]\eta(X_2) = \alpha^2\mathcal{B}_1(X_2) - \mathcal{B}_2(X_2). \tag{3.19}$$

Taking $X_2 = \xi$ in (3.6) and using (3.7), we obtain

$$\begin{aligned} &2n[\alpha^2\mathcal{A}_1(W) - \mathcal{A}_2(W)] + [\alpha^2\mathcal{B}_1(W) - \mathcal{B}_2(W)] + [\alpha^2\mathcal{D}_1(W) - \mathcal{D}_2(W)] \\ &= (2n-1)[\alpha^2\mathcal{A}_1(\xi) - \mathcal{A}_2(\xi)]\eta(W). \end{aligned} \tag{3.20}$$

Putting (3.16) in (3.20), we have

$$[\alpha^2\mathcal{A}_1(\xi) - \mathcal{A}_2(\xi)]\eta(W) = \alpha^2\mathcal{A}_1(W) - \mathcal{A}_2(W). \tag{3.21}$$

Again from (3.6), we get

$$\begin{aligned} S(X_2, W) = \\ &- \frac{[-2n\alpha^2\mathcal{A}_1(W) - \alpha^2\mathcal{B}_1(W) - \alpha^2\mathcal{D}_1(W) + 2n\mathcal{A}_2(W) + \mathcal{B}_2(W) + \mathcal{D}_2(W)]}{\alpha + \mathcal{D}_1(\xi)}\eta(X_2) \\ &- \frac{[2n\alpha^3 + \alpha^2\mathcal{D}_1(\xi) + 2n\mathcal{D}_2(\xi) - \mathcal{D}_2(\xi)]}{\alpha + \mathcal{D}_1(\xi)}g(X_2, W) \\ &+ \frac{[(2n-1)\alpha^2\mathcal{B}_1(X_2) - 2n\mathcal{B}_2(X_2) + \mathcal{B}_2(X_2)]}{\alpha + \mathcal{D}_1(\xi)}\eta(W). \end{aligned} \tag{3.22}$$

In view of (3.16), (3.19) and (3.21), we obtain

$$\begin{aligned} S(X_2, W) = &- \frac{[2n(\alpha^3 + \mathcal{D}_2(\xi)) + \alpha^2\mathcal{D}_1(\xi) - \mathcal{D}_2(\xi)]}{\alpha + \mathcal{D}_1(\xi)}g(X_2, W) \\ &+ \frac{(2n-1)[\alpha^2\mathcal{A}_1(\xi) - \mathcal{A}_2(\xi) + \alpha^2\mathcal{B}_1(\xi) - \mathcal{B}_2(\xi)]}{\alpha + \mathcal{D}_1(\xi)}\eta(X_2)\eta(W). \end{aligned} \tag{3.23}$$

Theorem 3.4. *A generalized weakly symmetric α-cosymplectic manifold is an η-Einstein space provided $\mathcal{D}_1(\xi) \neq -\alpha$.*

Theorem 3.5. *In an α-cosymplectic manifold the following table is hold.*

Type of curvature restriction	*Nature of the space corresponding to curvature restriction*
locally symmetric space	*Einstein space*
locally recurrent space	*η- Einstein space*
generalized recurrent space	*η- Einstein space*
pseudo symmetric space	*η- Einstein space*
generalized pseudo symmetric space	*η- Einstein space*
semi-pseudo symmetric space	*η- Einstein space*
generalized semi-pseudo symmetric space	*η- Einstein space*
almost pseudo symmetric space	*η- Einstein space*
almost generalized pseudo symmetric space	*η- Einstein space*
weakly symmetric space	*η- Einstein space*

4. Existence of Generalized weakly symmetric α-cosymplectic manifold

Let $M^3 = \{(x, y, z) \in R^3\}$ be a 3-dimensional manifold, where (x, y, z) are the standard coordinates in R^3. The vector fields are

$$e_1 = e^{-2z}\frac{\partial}{\partial x},$$
$$e_2 = e^{-2z}\frac{\partial}{\partial y},$$
$$e_3 = \frac{\partial}{\partial z}.$$

It is obvious that $\{e_1, e_2, e_3\}$ are linearly independent at each point of M^3. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0,$$

and given by the tensor product

$$g = \frac{1}{e^{-4z}}(dx \otimes dx + dy \otimes dy) + dz \otimes dz).$$

Let η be the 1-form defined by $\eta(W) = g(W, e_3)$ for any vector field W on M^3 and $\varphi(e_1) = e_2$, $\varphi(e_2) = -e_1$, $\varphi(e_3) = 0$. Then using the linearity of g and φ, we have

$$\varphi^2 W = -W + \eta(W)e_3, \quad \eta(e_3) = 1, \quad g(\varphi W, \varphi X_1) = g(W, X_1) - \eta(W)\eta(X_1),$$

for any vector fields on M^3. It remains to prove that $d\Phi = 2\alpha\eta \wedge \Phi$ and the Nijenhuis torsion tensor of φ is zero. It follows that $\Phi(e_1, e_2) = -1$ and otherwise $\Phi(e_i, e_j) = 0$ for $i \le j$. Therefore, the essential non-zero component of Φ is as follows:

$$\Phi(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = -\frac{1}{e^{-4z}}$$

and hence

$$\Phi = -\frac{1}{e^{-4z}} dx \wedge dy. \tag{4.1}$$

Consequently, the exterior derivative $d\Phi$ is given by

$$d\Phi = -\frac{4}{e^{-4z}} dx \wedge dy \wedge dz. \tag{4.2}$$

Since $\eta = dz$, by (4.1) and (4.2), we find

$$d\Phi = 4\eta \wedge \Phi.$$

Then,

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 2e_1, \quad [e_2, e_3] = 2e_2.$$

In conclusion, it can be noted that Nijenhuis torsion of φ is zero. Thus, the manifold is a 2-cosymplectic manifold. Using Koszul's formula, we can get the ∇ operator as follows:

$$\begin{aligned} &\nabla_{e_1}e_3 = 2e_1, && \nabla_{e_1}e_2 = 0, && \nabla_{e_1}e_1 = -2e_3, \\ &\nabla_{e_2}e_3 = 2e_2, && \nabla_{e_2}e_2 = -2e_3, && \nabla_{e_2}e_1 = 0, \\ &\nabla_{e_3}e_3 = 0, && \nabla_{e_3}e_2 = 0, && \nabla_{e_3}e_1 = 0. \end{aligned} \tag{4.3}$$

Then using equation (4.3), the non-vanishing components of $\tilde{R}$ (skew-symmetry and up to symmetry) can clearly be seen:

$$\tilde{R}(e_1, e_3, e_1, e_3) = \tilde{R}(e_2, e_3, e_2, e_3) = 4 = \tilde{R}(e_1, e_2, e_1, e_2).$$

Since $\{e_1, e_2, e_3\}$ forms a basis, any vector field $W, X_1, X_2, X_3 \in \chi(M)$ can be written as

$$W = \sum_{i=1}^{3} a_i e_i, \quad X_1 = \sum_{i=1}^{3} b_i e_i, \quad X_2 = \sum_{i=1}^{3} c_i e_i, \quad X_3 = \sum_{i=1}^{3} d_i e_i$$

and the components can be obtained from the following relations by the symmetry properties,

$$\begin{aligned}
&\tilde{R}(W, X_1, X_2, X_3) = T_1 = \frac{1}{4}[(a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)] \\
&+ (a_1b_3 - a_3b_1)(c_1d_3 - c_3d_1) + (a_2b_3 - a_3b_2)(c_2d_3 - c_3d_2) \\
&\tilde{R}(e_1, X_1, X_2, X_3) = \lambda_1 = \frac{1}{4}[b_3(c_1d_3 - c_3d_1) + b_2(c_1d_2 - c_2d_1)] \\
&\tilde{R}(e_2, X_1, X_2, X_3) = \lambda_2 = \frac{1}{4}[b_3(c_2d_3 - c_3d_2) - b_1(c_1d_2 - c_2d_1)] \\
&\tilde{R}(e_3, X_1, X_2, X_3) = \lambda_3 = \frac{1}{4}[b_1(c_3d_1 - c_1d_3) + b_2(c_3d_2 - c_2d_3)] \\
&\tilde{R}(W, e_1, X_2, X_3) = \lambda_4 = \frac{1}{4}[a_3(c_1d_3 - c_3d_1) + a_2(c_1d_2 - c_2d_1)] \\
&\tilde{R}(W, e_2, X_2, X_3) = \lambda_5 = \frac{1}{4}[a_3(c_2d_3 - c_3d_2) + a_1(c_2d_1 - c_1d_2)] \\
&\tilde{R}(W, e_3, X_2, X_3) = \lambda_6 = \frac{1}{4}[a_1(c_3d_1 - c_1d_3) + a_2(c_3d_2 - c_2d_3)] \\
&\tilde{R}(W, X_1, e_1, X_3) = \lambda_7 = \frac{1}{4}[d_3(a_1b_3 - a_3b_1) + d_2(a_1b_2 - a_2b_1)] \\
&\tilde{R}(W, X_1, e_2, X_3) = \lambda_8 = \frac{1}{4}[d_3(a_2b_3 - a_3b_2) + d_1(a_2b_1 - a_1b_2)] \\
&\tilde{R}(W, X_1, e_3, X_3) = \lambda_9 = \frac{1}{4}[d_1(a_3b_1 - a_1b_3) + d_2(a_3b_2 - a_2b_3)] \\
&\tilde{R}(W, X_1, X_2, e_1) = \lambda_{10} = \frac{1}{4}[c_3(a_1b_3 - a_3b_1) + c_2(a_1b_2 - a_2b_1)] \\
&\tilde{R}(W, X_1, X_2, e_2) = \lambda_{11} = \frac{1}{4}[c_3(a_2b_3 - a_3b_2) + c_1(a_2b_1 - a_1b_2)] \\
&\tilde{R}(W, X_1, X_2, e_3) = \lambda_{12} = \frac{1}{4}[c_1(a_3b_1 - a_1b_3) + c_2(a_3b_2 - a_2b_3)] \\
&\tilde{G}(W, X_1, X_2, X_3) = T_2 = (b_1c_1 + b_2c_2 - b_3c_3)(a_1d_1 + a_2d_2 - a_3d_3) \\
&- (a_1c_1 + a_2c_2 - a_3c_3)(b_1d_1 + b_2d_2 - b_3d_3).
\end{aligned}$$

Now, we calculate the components of $\tilde{R}$ which are the non-vanishing covariant derivatives:

$$\begin{aligned}\nabla_{e_1}\tilde{R}(W,X_1,X_2,X_3) &= +2a_1\lambda_3 - 2a_3\lambda_1 + 2b_1\lambda_6 - 2b_3\lambda_4 \\ &\quad + 2c_1\lambda_9 - 2c_3\lambda_7 + 2d_1\lambda_{12} - 2d_1\lambda_{10} \\ \nabla_{e_2}\tilde{R}(W,X_1,X_2,X_3) &= +2a_2\lambda_3 - 2a_3\lambda_2 + 2b_2\lambda_6 - 2b_3\lambda_5 \\ &\quad + 2c_2\lambda_9 - 2c_3\lambda_8 + 2d_2\lambda_{12} - 2d_3\lambda_{11} \\ \nabla_{e_3}\tilde{R}(W,X_1,X_2,X_3) &= 0.\end{aligned}$$

Depending on the following choice of the 1-forms

$$\begin{aligned}\mathcal{A}_1(e_1) &= \frac{2a_1\lambda_3 - 2a_3\lambda_1 + 2b_1\lambda_6 - 2b_3\lambda_4}{T_1} \\ \mathcal{A}_2(e_1) &= \frac{2c_1\lambda_9 - 2c_3\lambda_7 + 2d_1\lambda_{12} - 2d_1\lambda_{10}}{T_2} \\ \mathcal{A}_1(e_2) &= \frac{2a_2\lambda_3 - 2a_3\lambda_2 + 2b_2\lambda_6 - 2b_3\lambda_5}{T_1} \\ \mathcal{A}_2(e_2) &= \frac{2c_2\lambda_9 - 2c_3\lambda_8 + 2d_2\lambda_{12} - 2d_3\lambda_{11}}{T_2}\end{aligned}$$

one can easily verify the following relations that follow

$$\begin{aligned}\nabla_{e_i}\tilde{R}(W,X_1,X_2,X_3) =& \mathcal{A}_1(e_i)\tilde{R}(W,X_1,X_2,X_3) + \mathcal{B}_1(W)\tilde{R}(e_i,X_1,X_2,X_3) \\ &+ \mathcal{B}_1(X_1)\tilde{R}(W,e_i,X_2,X_3) + \mathcal{D}_1(X_2)\tilde{R}(W,X_1,e_i,X_3) \\ &+ \mathcal{D}_1(X_3)\tilde{R}(W,X_1,X_2,e_i) + \mathcal{A}_2(e_i)\tilde{G}(W,X_1,X_2,X_3) \\ &+ \mathcal{B}_2(W)\tilde{G}(e_i,X_1,X_2,X_3) + \mathcal{B}_2(X_1)\tilde{G}(W,e_i,X_2,X_3) \\ &+ \mathcal{D}_2(X_2)\tilde{G}(W,X_1,e_i,X_3) + \mathcal{D}_2(X_3)\tilde{G}(W,X_1,X_2,e_i)\end{aligned}$$

for $i = 1,2,3$. From the above, we can state the following the theorem.

Theorem 4.1. *There exists an α-cosymplectic manifold (M^3, g) which is a generalized weakly symmetric α-cosymplectic manifold.*

5. Generalized weakly Ricci-symmetric α-cosymplectic manifold

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an α-cosymplectic manifold. If the manifold is generalized weakly Ricci symmetric manifold then there exists 1-forms $\tilde{\mathcal{A}}_i$, $\tilde{\mathcal{B}}_i$ and $\tilde{\mathcal{D}}_i$ that satisfy the condition

$$\begin{aligned}(\nabla_W S)(X_2,X_3) = \tilde{\mathcal{A}}_1(W)S(X_2,X_3) + \tilde{\mathcal{B}}_1(X_2)S(W,X_3) + \tilde{\mathcal{D}}_1(X_3)S(X_2,W) \\ + \tilde{\mathcal{A}}_2(W)g(X_2,X_3) + \tilde{\mathcal{B}}_2(X_2)g(W,X_3) + \tilde{\mathcal{D}}_2(X_3)g(X_2,W).\end{aligned} \tag{5.1}$$

Putting $X_3 = \xi$ in (5.1), we obtain

$$\begin{aligned}(\nabla_W S)(X_2,\xi) = -2n\alpha^2[\tilde{\mathcal{A}}_1(W)\eta(X_2) + \tilde{\mathcal{B}}_1(X_2)\eta(W)] + \tilde{\mathcal{D}}_1(\xi)S(X_2,W) \\ + \tilde{\mathcal{A}}_2(W)\eta(X_2) + \tilde{\mathcal{B}}_2(X_2)\eta(W) + \tilde{\mathcal{D}}_2(\xi)g(X_2,W).\end{aligned} \tag{5.2}$$

In view of (3.5) the relation (5.2) becomes

$$\begin{aligned}-2n\alpha^3 g(X_2,W) - \alpha S(X_2,W) \;=\; & -2n\alpha^2[\tilde{\mathcal{A}}_1(W)\eta(X_2) + \tilde{\mathcal{B}}_1(X_2)\eta(W)] \\ & + \tilde{\mathcal{D}}_1(\xi)S(X_2,W) + \tilde{\mathcal{A}}_2(W)\eta(X_2) \\ & + \tilde{\mathcal{B}}_2(X_2)\eta(W) + \tilde{\mathcal{D}}_2(\xi)g(X_2,W).\end{aligned} \tag{5.3}$$

Setting $W = X_2 = \xi$ in (5.3) and using (2.1), (2.2) and (2.7), we get

$$2n\alpha^2[\tilde{\mathcal{A}}_1(\xi) + \tilde{\mathcal{B}}_1(\xi) + \tilde{\mathcal{D}}_1(\xi)] = \tilde{\mathcal{A}}_2(\xi) + \tilde{\mathcal{B}}_2(\xi) + \tilde{\mathcal{D}}_2(\xi). \tag{5.4}$$

Again, putting $W = \xi$ in (5.3), we get

$$2n\alpha^2[\tilde{\mathcal{A}}_1(\xi)\eta(X_2) + \tilde{\mathcal{B}}_1(X_2) + \tilde{\mathcal{D}}_1(\xi)\eta(X_2)] = \tilde{\mathcal{A}}_2(\xi)\eta(X_2) + \tilde{\mathcal{B}}_2(X_2) + \tilde{\mathcal{D}}_2(\xi)\eta(X_2). \quad (5.5)$$

Setting $X_2 = \xi$ in (5.3) and then using (2.1), (2.2) and (2.7), we obtain

$$2n\alpha^2[\tilde{\mathcal{A}}_1(W) + \tilde{\mathcal{B}}_1(\xi)\eta(W) + \tilde{\mathcal{D}}_1(\xi)\eta(W)] = \tilde{\mathcal{A}}_2(W) + \tilde{\mathcal{B}}_2(\xi)\eta(W) + \tilde{\mathcal{D}}_2(\xi)\eta(W). \quad (5.6)$$

Replacing X_2 by W in (5.5) and then adding the resultant with (5.6), we obtain

$$\begin{aligned} &2n[\alpha^2\tilde{\mathcal{A}}_1(W) + \alpha^2\tilde{\mathcal{B}}_1(W)] - [\tilde{\mathcal{A}}_2(W) + \tilde{\mathcal{B}}_2(W)] \\ &= -2n[\alpha^2\tilde{\mathcal{A}}_1(\xi) + \alpha^2\tilde{\mathcal{B}}_1(\xi) + \alpha^2\tilde{\mathcal{D}}_1(\xi)]\eta(W) \\ &+[\tilde{\mathcal{A}}_2(\xi) + \tilde{\mathcal{B}}_2(\xi) + \tilde{\mathcal{D}}_2(\xi)]\eta(W) - 2n\alpha^2\tilde{\mathcal{D}}_1(\xi)\eta(W) + \tilde{\mathcal{D}}_2(\xi)\eta(W). \end{aligned} \quad (5.7)$$

Due to (5.4), equation (5.7) turns into

$$\begin{aligned} &2n\alpha^2[\tilde{\mathcal{A}}_1(W) + \tilde{\mathcal{B}}_1(W)] + 2n\alpha^2\tilde{\mathcal{D}}_1(\xi)\eta(W) \\ &\quad = [\tilde{\mathcal{A}}_2(W) + \tilde{\mathcal{B}}_2(W)] + \tilde{\mathcal{D}}_2(\xi)\eta(W). \end{aligned} \quad (5.8)$$

Then taking, $X_2 = W = \xi$ in (5.1), we obtain

$$\begin{aligned} &2n\alpha^2[\tilde{\mathcal{A}}_1(\xi) + \tilde{\mathcal{B}}_1(\xi)]\eta(X_3) + 2n\alpha^2\tilde{\mathcal{D}}_1(X_3) \\ &\quad = [\tilde{\mathcal{A}}_2(\xi) + \tilde{\mathcal{B}}_2(\xi)]\eta(X_3) + \tilde{\mathcal{D}}_2(X_3). \end{aligned} \quad (5.9)$$

Replacing X_3 by W in (5.9) and adding with (5.8), we find out

$$\begin{aligned} &2n\alpha^2[\tilde{\mathcal{A}}_1(W) + \tilde{\mathcal{B}}_1(W) + \tilde{\mathcal{D}}_1(W)] + 2n\alpha^2[\tilde{\mathcal{A}}_1(\xi) + \tilde{\mathcal{B}}_1(\xi) + \tilde{\mathcal{D}}_1(\xi)]\eta(W) \\ &= [\tilde{\mathcal{A}}_2(W) + \tilde{\mathcal{B}}_2(W) + \tilde{\mathcal{D}}_2(W)] + [\tilde{\mathcal{A}}_2(\xi) + \tilde{\mathcal{B}}_2(\xi) + \tilde{\mathcal{D}}_2(\xi)]\eta(W). \end{aligned} \quad (5.10)$$

Using equation (5.4), we get from the (5.10) equation

$$2n\alpha^2[\tilde{\mathcal{A}}_1(W) + \tilde{\mathcal{B}}_1(W) + \tilde{\mathcal{D}}_1(W)] = [\tilde{\mathcal{A}}_2(W) + \tilde{\mathcal{B}}_2(W) + \tilde{\mathcal{D}}_2(W)]. \quad (5.11)$$

This leads to the following theorem.

Theorem 5.1. *In a generalized weakly Ricci symmetric α-cosymplectic manifold* (M^{2n+1}, g) $(n \geq 1)$*, the sum of the associated* 1*-forms is related by (5.11).*

Again from (5.3), we have

$$\begin{aligned} S(X_2, W) = &-\frac{[2n\alpha^3 + \tilde{\mathcal{D}}_2(\xi)]}{\alpha + \tilde{\mathcal{D}}_1(\xi)}g(X_2, W) + \frac{[2n\alpha^2\tilde{\mathcal{A}}_1(W) - \tilde{\mathcal{A}}_2(W)]}{\alpha + \tilde{\mathcal{D}}_1(\xi)}\eta(X_2) \\ &+ \frac{[2n\alpha^2\tilde{\mathcal{B}}_1(X_2) - \tilde{\mathcal{B}}_2(X_2)]}{\alpha + \tilde{\mathcal{D}}_1(\xi)}\eta(W). \end{aligned} \quad (5.12)$$

From (5.6), we get

$$2n\alpha^2\tilde{\mathcal{A}}_1(W) - \tilde{\mathcal{A}}_2(W) = [-2n\alpha^2(\tilde{\mathcal{B}}_1(\xi) + \tilde{\mathcal{D}}_1(\xi)) + (\tilde{\mathcal{B}}_2(\xi) + \tilde{\mathcal{D}}_2(\xi))]\eta(W). \quad (5.13)$$

Using (5.4) in (5.5), we obtain

$$[2n\alpha^2\tilde{\mathcal{B}}_1(\xi) - \tilde{\mathcal{B}}_2(\xi)]\eta(X_2) = 2n\alpha^2\tilde{\mathcal{B}}_1(X_2) - \tilde{\mathcal{B}}_2(X_2). \quad (5.14)$$

In view of (5.12), (5.13) and (5.14), we have

$$S(X_2, W) = -\frac{[2n\alpha^3 + \tilde{\mathcal{D}}_2(\xi)]}{\alpha + \tilde{\mathcal{D}}_1(\xi)}g(X_2, W) + \frac{[-2n\alpha^2\tilde{\mathcal{D}}_1(\xi) + \tilde{\mathcal{D}}_2(\xi)]}{\alpha + \tilde{\mathcal{D}}_1(\xi)}\eta(X_2)\eta(W).$$

This leads to the following theorems.

Theorem 5.2. *A generalized weakly Ricci symmetric α-cosymplectic manifold is an η-Einstein space provided* $\tilde{\mathcal{D}}_1(\xi) \neq -\alpha$

Theorem 5.3. *In an α-cosymplectic manifold the following table is hold.*

Type of curvature restriction	*Nature of the space corresponding to curvature restriction*
locally symmetric space	*Einstein space*
locally recurrent space	η*- Einstein space*
generalized recurrent space	η*- Einstein space*
pseudo symmetric space	η*- Einstein space*
generalized pseudo symmetric space	η*- Einstein space*
semi-pseudo symmetric space	η*- Einstein space*
generalized semi-pseudo symmetric space	η*- Einstein space*
almost pseudo symmetric space	η*- Einstein space*
almost generalized pseudo symmetric space	η*- Einstein space*
weakly symmetric space	η*- Einstein space*

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