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A Study on Approximate Analytic Solutions of the Combined KdV-mKdV Equation

Birleştirilmiş KdV-mKdV Denkleminin Yaklaşık Analitik Çözümleri Üzerine Bir Çalışma

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Abstract
In this paper, the investigation focuses on solitary wave solutions of the combined KdV-mKdV equation by using reduced differential transform method (RDTM). To prove validity of the proposed method, the approximate analytic solutions and exact solutions of the equation are compared via absolute errors. The obtained results are represented by graphics. The effects of the time and dispersion parameter (μ) on analytic solutions are investigated. As a result, it can be said that the applied method is quite precise and successful for similar type equations.

Keywords: Combined KdV-mKdV Equation, Gardner Equation, RDTM, Reduced Differential Transform Method, Solitary Wave

Öz
Bu çalışmada, araştırma indirgenmiş diferansiyel dönüşüm metodunu (İDDM) kullanarak birleştirilmiş KdV-mKdV denkleminin solitary dalga çözümleri üzerine odaklanmaktadır. Önerilen metodun geçerliliğini kanıtlamak için, mutlak hata vasıtasıyla yaklaşık analitik çözümler ve tam çözümler karşılaştırılmıştır. Elde edilen sonuçlar grafiklerle temsil edilmiştir. Zaman ve dispersiyon parametresinin (μ) analitik çözümler üzerindeki etkileri araştırılmıştır. Sonuç olarak, uygulanan yöntemin benzer tipteki denklemler için oldukça hassas ve başarılı olduğu söylenebilir.

***Anahtar kelimeler**: Birleştirilmiş KdV-mKdV Denklemi, Gardner Denklemi, İDDM, İndirgenmiş Diferansiyel Dönüşüm Metodu, Solitary Dalga*

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1. Introduction

In science, many significant events in varied fields are generally modelled by the nonlinear partial differential equations. One of these equations is the combined KdV-mKdV equation (Hirota, 1980; Hoover and Grant, 1983; Mohamad, 1992; Lou and Chen, 1994; Zhang et al., 2000). The equation defines various interesting physical phenomena, such as the weakly nonlinear long waves in various physical applications (Nakoulima et al., 2004). Many analytic and numerical methods have been proposed and implemented for solution of the equation (Zhang, 1998; Kaya and Inan, 2005; Wazwaz, 2007; Biswas and Zerrad, 2008; Lu and Shi, 2010; Triki et al., 2010; Ak et al., 2018; Ak, 2019).

On the other part, the reduced differential transform method for approximate analytic solution of partial differential equations was proposed in 2009 (Keskin and Oturanc, 2009). After then, RLW equation (Keskin, 2010), the generalized Korteweg-de Vries equation (Keskin and Oturanc, 2010a) and KdV equation (Keskin and Oturanc, 2010b) were solved by using the method. To solve Fornberg-Whitham type equations, the RDTM was applied by Hesam et al. (2012). Abazari and Soltanalizadeh implemented the method to Kawahara and modified Kawahara equations (Abazari and Soltanalizadeh, 2013). Burgers equation, Burgers-Huxley equation, Huxley equation and Burgers-Fisher equation are solved via reduced differential transform method (Abazari and Abazari, 2013). Saravanan and Magesh solved the Newell-Whitehead-Segel equation by the method (Saravanan and Magesh, 2013). Al-Amr obtained solution of the generalized Drinfeld-Sokolov equation and the Kaup-Kupershmidt equation by using the method (Al-Amr, 2014).

In this paper, we aimed to define a new approximate method to the equation. This paper is organized as follows: In Section 2, firstly we reminded reduced differential transform method. In Section 3, the proposed method is implemented to the equation and efficiency of the method is investigated. Finally, we closed this work with Section 4 including conclusion remarks.

2. Mathematical Modelling

If the Korteweg-de Vries (KdV) equation

$$u_t + auu_x + u_{xxx} = 0, \tag{1}$$

and the modified KdV equation (mKdV)

$$u_t + bu^2u_x + u_{xxx} = 0, \tag{2}$$

are combined, the resulting equation

$$u_t + auu_x + bu^2u_x + \mu u_{xxx} = 0, \tag{3}$$

is called the combined KdV-mKdV equation. Also, it is known as the Gardner equation. Here, a, b and μ are nonzero, arbitrary and real parameters. $u(x,t)$ represents solitary wave profile, while x and t are independent variables. In addition, u_{xxx} is dispersion term, uu_x and u^2u_x are nonlinear terms.

3. Analysis of the Method

In this section, we introduce basic knowledges related to reduced differential transformation method.

3.1. Reduced differential transform method

Let us consider $w(x,t)$ function which can be written as $w(x,t) = f(x)g(t)$. From the one dimensional differential transform method (Keskin and Oturanc, 2009), $w(x,t)$ can be expressed as

$$\begin{aligned} w(x,t) &= \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} W(i,j)x^i t^j \end{aligned} \tag{4}$$

where $W(i,j) = F(i)G(j)$ is called the spectrum of $w(x,t)$.

If $w(x,t)$ is an analytic function, then the spectrum function is reduced to

$$W_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k} w(x,t)]_{t=t_0}. \quad (5)$$

Besides, the inverse differential transform of $W_k(x)$ is described as

$$(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^k. \quad (6)$$

From (5) and (6), we get

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}[\frac{\partial^k}{\partial t^k} w(x,t)]_{t=t_0}(t-t_0)^k. \quad (7)$$

Table 1. Basic operations in RDTM (Keskin and Oturanc, 2009).

Original Function	Transformed Function
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{d}{dx} U_k(x)$
$w(x,t) = \frac{\partial}{\partial t} u(x,t)$	$W_k(x) = (k+1)U_{k+1}(x)$
$w(x,t) = \frac{\partial^{r+s}}{\partial x^r \partial t^s} u(x,t)$	$W_k(x) = \frac{(k+s)!}{k!}\frac{d^r}{dx^r} U_{k+s}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = u(x,t)v(x,t)z(x,t)$	$W_k(x) = \sum_{r=0}^{k} \sum_{s=0}^{k-r} U_r(x)V_s(x)Z_{k-r-s}(x)$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n) = \begin{cases} x^m & k = n \\ 0 & otherwise \end{cases}$

4. Numerical Applications

In this section, we consider approximate solitary wave solution of the combined KdV-mKdV equation in order to show performance of the method (Keskin and Oturanc, 2009). The accuracy of the method is measured by using the absolute error norms.

The solitary wave solution of (3) is given by Hamdi et al. (2011)

$$u(x,t) = A\, sech\,[B(x-ct)] \quad (8)$$

where A is amplitude (Hamdi et al., 2011), B is inverse width (Hamdi et al., 2011), c is velocity (Hamdi et al., 2011) and

$$A = \frac{6c}{a\left(1+\sqrt{1+\frac{6bc}{a^2}}\right)} \quad , \quad B = \sqrt{\frac{c}{\mu}}. \quad (9)$$

We consider (3) subject to initial condition

$$u(x,0) = A\, sech\,(Bx). \quad (10)$$

If the reduced differential transform method is applied on the combined KdV-mKdV equation, the recursive equation is obtained as

$$\begin{aligned}(k+1)U_{k+1}(x) &+ a\sum_{r=0}^{k} U_r(x)\frac{d}{dx}U_{k-r}(x)\\ &+b\sum_{r=0}^{k}\sum_{s=0}^{k-r} U_r(x)U_s(x)\frac{d}{dx}U_{k-r-s}(x)\\ &+\mu\frac{d^2}{dx^3}U_k(x)=0 \quad , \quad k=0,1,2\ldots \end{aligned} \tag{11}$$

For $k=0,1$, the first two terms of $U_{k+1}(x)$ are calculated by utilizing the initial condition in recursive equation (11) as follows

$$U_0(x)=Asech(Bx), \tag{12}$$

$$U_1(x)=\frac{1}{2}AB(2A^2b-11B^2\mu+2aA\cosh[Bx]+B^2\mu\, cosh\,[2Bx])\, sech\,^3[Bx]\, tanh\,[Bx], \tag{13}$$

$$\begin{aligned}U_2(x)=\frac{1}{2}AB^2\, sech[Bx]\,(B^4\mu^2 &+ sech[Bx]\,(10aAB^2\mu\\ &+sech[Bx]\,(3a^2A^2+2B^2\mu(15A^2b-91B^2\mu)+sech[Bx]\,(aA(8A^2b-87B^2\mu)\\ &+sech[Bx](-4a^2A^2+5(A^2b-28B^2\mu)(A^2b-6B^2\mu)\\ &-10aA(A^2b-9B^2\mu)\, sech[Bx]-6(A^2b-20B^2\mu)(A^2b-6B^2\mu)\, sech^2[Bx]))))).\end{aligned} \tag{14}$$

Substituting (12), (13) and (14) into inverse differential transform, the approximate analytic solution of the combined KdV-mKdV equation in the Poisson series form are:

$$U_2(x,t)=U_0(x)+U_1(x)t+U_2(x)t^2, \tag{15}$$

$$\begin{aligned}U_2(x,t)=\frac{1}{2}A\, sech[Bx](\,2 &+ B^2t^2(B^4\mu^2\, sech[Bx](10aAB^2\,\mu\\ &+sech[Bx](\,3a^2A^2+2B^2\mu(15A^2b-91B^2\mu)+sech[Bx](\,aA(8A^2b-87B^2\mu)\\ &+sech[Bx]\,(-4a^2A^2+5(A^2b-28B^2\mu)(A^2b-6B^2\mu)\\ &-10aA(A^2b-9B^2\mu)\, sech[Bx]-6(A^2b-20B^2\mu)(A^2b-6B^2\mu)sech^2[Bx])))))\\ &+Bt(2A^2b-11B^2\mu+2aA\, cosh[Bx]+B^2\,\mu\, cosh[2Bx])sech^2[Bx]\, tanh[Bx])\end{aligned} \tag{16}$$

which is the first three terms of the Poisson series of the exact solution (8).

In this section, we have done some computations to examine the accuracy and reliability of the method for the combined KdV-mKdV equation. We set $a=b=1$ and $c=0.1$ and make computation for $\mu=5$ and $\mu=1$, respectively. After $2-$approximate solutions obtained from RDTM, exact solutions and absolute errors are given with respect to different values of t at $x=0$ in Table 2. It is observed from the table that the absolute errors increase as time increases. Also, it can be seen from the table that the absolute errors for $\mu=5$ slightly smaller than for $\mu=1$. Table 3 shows exact solutions, $2-$ approximate solutions and absolute errors obtained by RDTM at $t=3$ in some points of the intervals $-40\le x\le 40$. It can be noticed from Table 3 that the absolute errors decrease while away from $x=0$ both $\mu=5$ and $\mu=1$. Besides, approximate solutions are better for $\mu=1$.

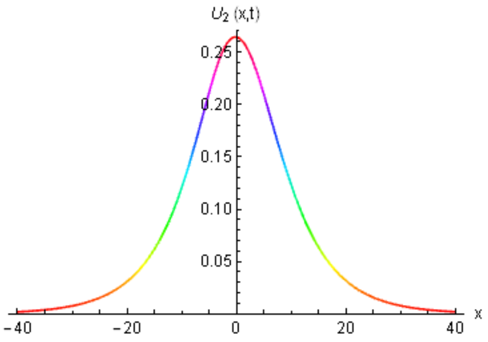
Table 2. Solutions for different values of μ at $x = 0$.

μ	t	$Exact\ Sol.$	$Approx. Sol.$	$Absolute\ Error$
5	1.0	0.2648846	0.2642074	6.771563×10^{-4}
	1.5	0.2648515	0.2633279	1.523608×10^{-3}
	2.0	0.2648051	0.2620965	2.708652×10^{-3}
	2.5	0.2647456	0.2605133	4.232299×10^{-3}
	3.0	0.2646728	0.2585783	6.094565×10^{-3}
	3.5	0.2645869	0.2562914	8.295468×10^{-3}
	4.0	0.2644878	0.2536527	1.083503×10^{-2}
	4.5	0.2643755	0.2506622	1.371327×10^{-2}
	5.0	0.2642502	0.2473199	1.693023×10^{-2}
1	1.0	0.2647787	0.2613928	3.385825×10^{-3}
	1.5	0.2646133	0.2569951	7.618262×10^{-3}
	2.0	0.2643821	0.2508382	1.354396×10^{-2}
	2.5	0.2640854	0.2429222	2.116321×10^{-2}
	3.0	0.2637234	0.2332470	3.047639×10^{-2}
	3.5	0.2632967	0.2218128	4.148393×10^{-2}
	4.0	0.2628058	0.2086195	5.418636×10^{-2}
	4.5	0.2622513	0.1936670	6.858429×10^{-2}
	5.0	0.2616338	0.1769554	8.467840×10^{-2}

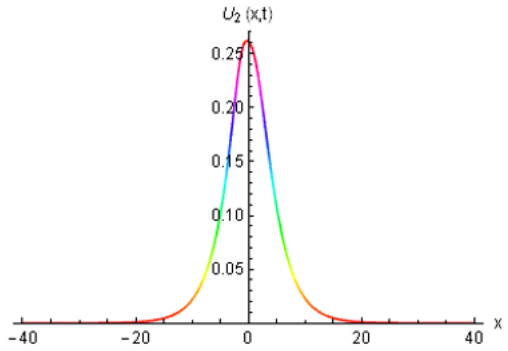
Table 3. Solutions for different values of μ at $t = 3$.

μ	x	$Exact\ Sol.$	$Approx. Sol.$	$Absolute\ Error$
5	-40	0.0017740	0.0017729	1.112823×10^{-6}
	-30	0.0072957	0.0072770	1.866102×10^{-5}
	-20	0.0299187	0.0296333	2.854101×10^{-4}
	-10	0.1170998	0.1157518	1.348054×10^{-3}
	0	0.2646728	0.2585783	6.094565×10^{-3}
	10	0.1262671	0.1258319	4.352047×10^{-4}
	20	0.0325490	0.0328900	3.411201×10^{-4}
	30	0.0079415	0.0079690	2.746466×10^{-5}
	40	0.0019311	0.0019328	1.706311×10^{-6}
1	-40	0.0000015	0.0000015	2.349729×10^{-10}
	-30	0.0000365	0.0000366	4.779814×10^{-9}
	-20	0.0008634	0.0008631	3.189492×10^{-7}
	-10	0.0203670	0.0200931	2.738487×10^{-4}
	0	0.2637234	0.2332470	3.047639×10^{-2}
	10	0.0246055	0.0251129	5.074350×10^{-4}
	20	0.0010438	0.0010449	1.108403×10^{-6}
	30	0.0000442	0.0000442	3.599020×10^{-9}
	40	0.0000019	0.0000019	2.438539×10^{-10}

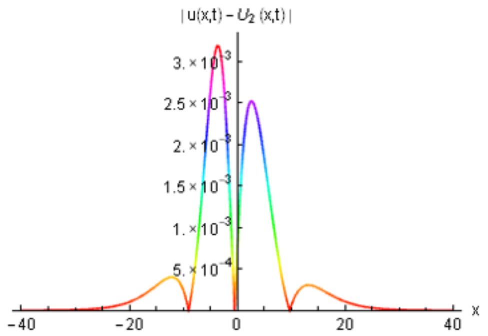
The details of approximate solutions and absolute errors are shown in Figure 1 for $\mu = 5$ and $\mu = 1$ at time $t = 3$, respectively. The wave takes shape of a bell when μ increases as observed from Figure 1a and Figure 1b. Besides, it can be seen from Figure 1c and Figure 1d that absolute errors decline when μ is increased. In addition, their three dimensional states are indicated by Figure 2. $2-$approximate solutions of the problem are depicted for $\mu = 5$ and $\mu = 1$ at different values of t in Figure 3. It is observed from Figure 3 that shape of the wave have deformation as time increases. Also, the wave takes shape of a bell when μ increases as in Figure 1.



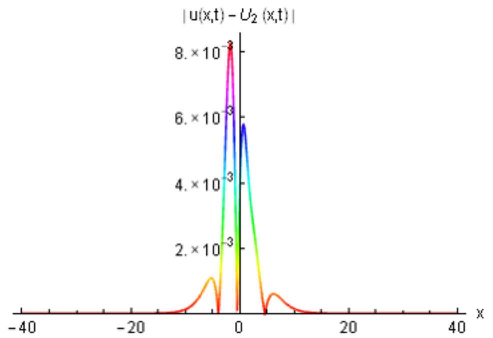
(a) $\boldsymbol{\mu} = \mathbf{5}$



(b) $\boldsymbol{\mu} = \mathbf{1}$

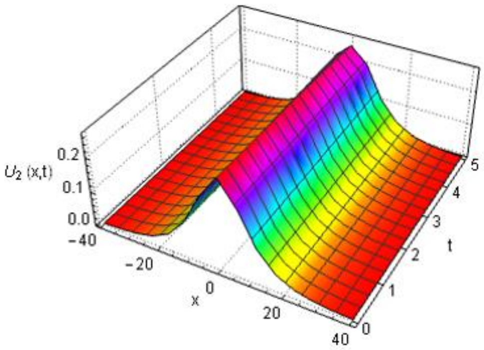


(c) $\boldsymbol{\mu} = \mathbf{5}$

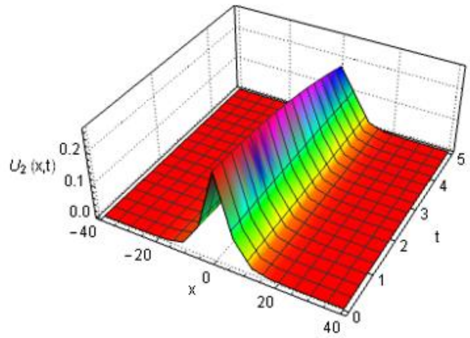


(d) $\boldsymbol{\mu} = \mathbf{1}$

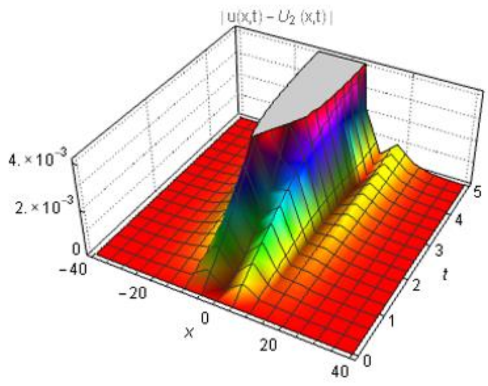
Figure 1. 2 −approximate solutions and absolute errors for different values of μ at $t = 3$



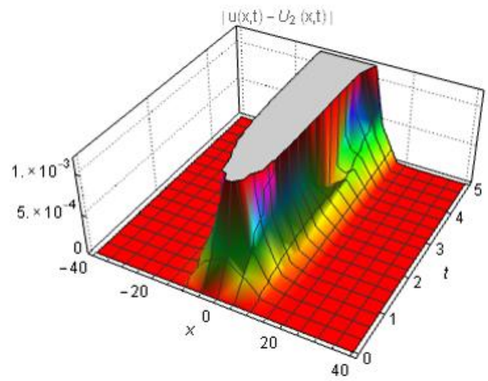
(a) $\boldsymbol{\mu} = \mathbf{5}$



(b) $\boldsymbol{\mu} = \mathbf{1}$



(c) $\boldsymbol{\mu} = \mathbf{5}$



(d) $\boldsymbol{\mu} = \mathbf{1}$

Figure 2. 2 −approximate solutions and absolute errors for different values of μ at $0 \leq t \leq 5$

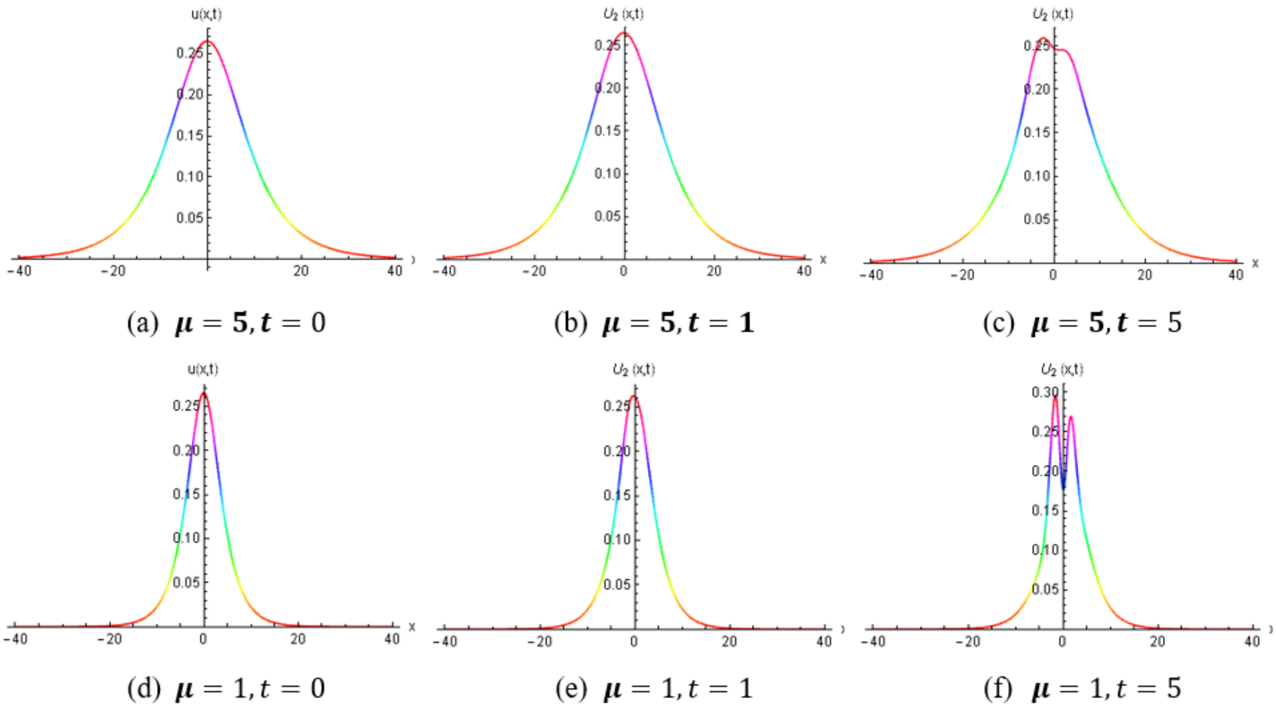


Figure 3. Exact and 2 −approximate solutions for different values of μ and t

5. Conclusion

In this paper, we investigate reduced differential transform method for the solitary wave solution of the combined KdV-mKdV equation. The effects of the time and dispersion parameter (μ) on behaviors of solitary wave solutions are discussed. From the obtained results which are showed in the tables conclude that the absolute errors are satisfactorily small. So, the proposed method has reliability and offer high accuracy for the approximate analytic solution of the equation. Also, the method have some advantages such as fast and computational cost is quite small. Consequently, the method can be recommended as an alternative method for numerical solutions of these type non-linear equations.

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