Öneri.C.6.S.23.
Ocak 2005.ss.113-117.

MORE ON THE PROBABILITY-GENERATING FUNCTIONS

Özcan BAYTEKİN

Marmara University, Faculty of Economics and Administrative Science, Yardımcı Doçent Dr.

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***Absract:** In this paper, we shall introduce an important mathematical concept which has many applications to the probabilistics models we are considering. In order to present a rigorous development of this topic, mathematics of a considerably higher level than we are assuming here would be required. However if are willing to avoid certain mathematical difficulties that arise and if we are willing to accept that certain operations are valid, then we can obtain a sufficient understanding of the main ideas involved in order to use them intelligently.*

In this paper, we made a brief summary of moment generating functions, and also we considered an application of this theory to a probability problem whose solution is impossible by hand calculations. We made also a comparison of binomial distribution and Poisson distribution with the aid of moment generating functions.

Keywords: Moment Generating Funtions, Factorial Generating Functions, Probability, Random Variable, Poisson Distribution, Binomial Distribution

OLASILIK MOMENT OLUŞUM FONKSİYONLARININ İLERİ BİR AŞAMASI VE UYGULAMA ÖRNEĞİ

***Özet:** Bu araştırmada, göz önüne alınarak incelenen olasalık modellerinde, bir çok tatbikat alanı bulunan, çok önemli bir matematiksel kavramı inceleyeceğiz. Diğer bir ifade ile , bazı olasılık problemlerinin çözümünde, eğer moment gelişim fonksiyonları kullanılmaz ise, bu durumda çok ileri bir matematik bilgisi gerekecektir. Bu makalede dikkate alınan nokta, bazı olasılık problemlerinin, eğer moment gelişim fonksiyonları kullanılmamışsa, çözümün hemen hemen el ile gerçekleştirilmesini imkansız oluşunun anlaşılmasıdır. Yukarda bahsedilen özellikleri açıklayabilmek için, makalede önce moment gelişim fonksiyonlarının tanıtımı ve matematiksel özelliklerin açıklaması yapılmış daha sonrada ilgi çekici bir olasılık problemi uygulaması moment gelişim fonksiyonlarının özellikleri kullanılarak kolayca çözülmüştür. Moment gelişim fonksiyonlarının işleyişindeki temel mantığı açıklayabilmek için, logoritmik hesapların işleyiş mantığı açıklanmıştır.*

Anahtar Kelimeler: Moment Oluşum Fonksiyonu, Faktöryel Oluşum Fonksiyonu, Olasılık, Tesadüfi Değişken, Poisson Dağılımı, Binom Dağılımı

I. INTRODUCTION

The factorial moment generating function of a discrete integer-valued random variable x is defined to be

$$n(t)=E\left(t^{x}\right)=\sum_{x=0}^{\infty}t^{x}\quad \Pr(X=x)$$

where t is a constant [1].

This function is useful in that it produces the factorial moments of X.

$$\frac{d^{n}(n(t))}{dt^{n}}=\frac{d^{n}}{dt^{n}}\left[\sum_{x=0}^{\infty}t^{n}\quad \Pr(X=x)\right]$$

$$=\sum_{x=0}^{\infty}\Pr(X=x)\frac{d^{n}t^{x}}{dt^{n}}$$

$$=\sum_{x=0}^{\infty}x(x-1).....(x-n+1)\quad \Pr(X=x).t^{x-n}$$

Thus;

$$\frac{d^{n}(n(1))}{dt^{n}}=\sum_{x=0}^{\infty}x(x-1).....(x-n+1)\quad \Pr(X=x).(1)^{x-n}$$

$$=\sum_{x=0}^{\infty}x(x-1).....(x-n+1)\quad \Pr(X=x)$$

$$=E(X(X-1)....(X-n+1))$$

$$=E\left(\frac{X!}{(X-n)!}\right)$$

The factorial moment, because

X(X-1)....(X-n+1)(X-n) equals X(X-1)...(X-n+1).

Another useful observation is that the coefficients of $E(t^x)$ provide the probabilities of X. Thus given the factorial generating function one can find the Pr(X=x). Pr(X=k) is the coefficient of the term t^k.

One property of moment generatin function is that $Y=X_1 + X_2 + ... + X_n$ where Xi, i = 1, ... , n are independent random variables has moment generatig function:

$$M_Y(t) = M_{x1}(t).M_{x2}(t).M_{x3}(t)...M_{xn}(t)$$

This type of behaviour will be extended to the factorial generating function.

To see that this is true for moment generating functions.

$$M_Y(t) = E(e^{tY})$$

$$= E(e^{t(x1 + x2 + ... + xn)})$$

$$= E(e^{tx1 + tx2 + ... + txn})$$

$$= E(e^{tx1}.e^{tx2}....e^{txn})$$

and by the independence of x_i,

$$M_Y(t) = E(e^{tx1}).E(e^{tx2})...E(e^{txn})$$

$$= M_{x1}(t).M_{x2}(t)...M_{xn}(t)$$

If $X_1, X_2, ... ,X_n$ have identical distributions, they will also have identical moment generating functions, say $M_x(t)$. Then $M_Y(t) = [M_x(t)]^n$

These results are also true for the factorial moment generating function. Namely if $Y = X_1 + ... + X_n$ and $X_1, X_2, ... , X_n$ are independent random variables then

$$m_Y(t) = E(t^Y) = E(t^{x_1+....+x_n})$$

$$= E(t^{x_1})E(t^{x_2})....E(t^{x_n})$$

$$= m_{x_1}(t).m_{x_2}(t)....m_{x_n}(t)$$

II. ILLUSTRATIVE EXAMPLE

We illustrate some of these concepts with one problem.

Let X denote the number of spots on a fair die.[2] First, we will determine the factorial generating function of X. If four dices are tossed four times, with results X_1, X_2, X_3 and X_4 and if the X_i are random variables, find the factor geneartying function for Y, Y represents the total number of spots observed in four dices what is the Pr(Y = 7).

If X represents the number of spots on a fair die, the X has a probability distribution;

$$\Pr(X = x) = \begin{cases} 1/6 & for\ x = 1,2,3...6 \\ 0 & otherwise \end{cases}$$

Thus the factorial moment generating for X is

$$m(t) = \sum_{x=0}^{\infty} t^x \Pr(X = x) = E(t^x)$$

$$= \sum_{x=0}^{6} t^x.\frac{1}{6} = \frac{(t+t^2+t^3+t^4+t^5+t^6)}{6}$$

$$= \frac{t(1+t+t^2+t^3+t^4+t^5)}{6}$$

We have here a geometric progression

Let $S_5 = 1 + t + t^2 + t^3 + t^4 + t^5$

$$S_5 - tS_5 = 1 - t^6$$

$$S_5 = \frac{1-t^6}{1-t}$$

Thus

$$m(t) = \frac{t}{6}\left(\frac{1-t^6}{1-t}\right)$$

Let Y be the number of spots observed in four independently tossed dices. Let X_1, X_2, X_3, X_4 be random variables indicating the number of spots observed in the first, second, third and fourth die respectively [3]. Thus, $Y = X_1 + X_2 + X_3 + X_4$ and each of the X_i are independent and have moment generating function,

$$m_x(t) = m_{x_1}(t) = m_{x_2}(t) = m_{x_3}(t) = m_{x_4}(t)$$

$$= \frac{t}{6}\left(\frac{1-t^6}{1-t}\right)$$

Thus,

$$m_Y(t) = [m_x(t)]^4 = \frac{t^4}{6^4}\left(\frac{1-t^6}{1-t}\right)^4$$

To find the Pr(Y = 7) we manipulate this expression in order to read off the coefficient on the term t^7. Thus,

$$m_y(t) = \frac{t^4}{6^4}\left(1-t^6\right)^4\left(1-t\right)^{-4} \quad \text{but}$$

$$(1-t^6)^4 = \sum_{k=0}^{4}\binom{4}{k}.(-1)^k.t^{6k}$$

and for |t| < 1

$$(1-t)^{-4} = \sum_{j=0}^{\infty}\binom{-4}{j}.(-t)^j$$

By a generalization of the binomial theorem

Here

$$\binom{-4}{j} = \frac{(-4).(-4-1)...(-4-j+1)}{(j)!}$$

Thus;

$$m_4(t) = E(t^Y) = \frac{t^4}{6^4}\left[\sum_{k=0}^{4}\binom{4}{k}.(-1)^k(t)^{6k}\right]\left[\sum_{j=0}^{\infty}\binom{-3}{j}.(-t)^j\right]$$

or writing this as a double sum

$$E(t^Y) = \frac{1}{1296}\sum_{k=0}^{4}\sum_{j=0}^{\infty}\binom{4}{k}\binom{-4}{j}.(-1)^{k+j}(t)^{6k+j+4}$$

to find the Pr(Y = 7), we first determine the value of k and j so that 6k + j + 4 = 7, k = 0,1,2,3 and j = 0,1,2,3,...

If k is any number but 0, 6k + j + 3 will be greater than 7. Thud k = 0 and 6.0 + j + 4 = 7 or j = 3.

Thus Pr(Y = 7) is

$$= \frac{1}{6^4}\binom{4}{0}\binom{-4}{3}.(-1)^{0+3}$$

$$= \frac{1}{6^4}\;\frac{4!}{0!4!}\cdot\frac{(-4)(-5)(-6)(-1)}{3!} = \frac{5}{324}$$

III. MEAN VALUE OF THE RANDOM VARIABLE Y

The factorial generating function of the random variable Y was found to be

$$m_y(t) = \frac{t^4}{6^4}\left(\frac{1-t^6}{1-t}\right)^4 = \frac{t^4}{1296}\left(t^5+t^4+t^3+t^2+t+1\right)^4$$

and $$m_y'(t) = \frac{4(t^6+t^5+t^4+t^3+t^2+t)^4}{1296}$$

$$m_y'(t) = \frac{4(t^6+t^5+t^4+t^3+t^2+t)^3(6t^5+5t^4+4t^3+3t^2+2t+1)}{1296}$$

$$m_y'(1) = \frac{4(6)^3 21}{6^4} = 14$$

IV. APPLICATION OF MOMENT GENERATING FUNCTIONS TO BINOMIAL DISTRIBUTION

Let Y_1 be a binomial random variable with n_1 trials and probability of success given by p. Let Y_2 be another binomial random variable with n_2 trials and probability of success given by p. If Y_1 and Y_2 are independent, find the probability function of $Y_1 + Y_2$.

Using the moment generating function approach :

$$m_{Y_1}(t) = (p+qe^t)^{n_1}$$

$$m_{Y_2}(t) = (p+qe^t)^{n_2}$$

and, since Y_1 and Y_2 are independent

$$m_{Y_1+Y_2}(t) = m_{Y_1}(t).m_{Y_2}(t) = (p+qe^t)^{n_1+n_2}$$

which is the moment generating function of the binomial random variable with parameters n_1+n_2 and p. Using the following theorem:

Theorem I: Suppose that each of the two random variables X and Y, moment generating functions exists and are given by $m_x(t)$ and $m_y(t)$ respectively [4]. If

$m_x(t)=m_y(t)$ and for all values of t, then X and Y have the same probability distribution, finally we can have

$$P[Y_1+Y_2+k]=\binom{n_1+n_2}{k}p^k q^{n_1+n_2-k}$$

k= 0,1,2,...,n_1+n_2

V. APPLICATION OF MOMENT GENERATING FUNCTIONS TO POISSON DISTRIBUTION

Let Y_1 and Y_2 be independent Poisson random variables with mean λ_1 and λ_2 respectively. Our aim is to find the probability function of Y_1+Y_2:

Similar to section IV

$$m_{Y_1}(t)=e^{-\lambda_1(1-e^t)}$$

$$m_{Y_2}(t)=e^{-\lambda_2(1-e^t)}$$

so that.

$$m_{Y_1+Y_2}(t)=e^{-(\lambda_1+\lambda_2)(1-e^t)}$$

Which is the moment generating function of a Poisson random variable with mean $\lambda_1 + \lambda_2$. By Theorem I then

$$P[Y_1+Y_2=k]=\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^k}{k!} \qquad k=0,1,\ldots$$

VI. RELATIONS BETWEEN POISSON PROBABILITY DISTRIBUTION AND BINOMIAL PROBABILITY DISTRIBUTION

For this purpose, let us begin by finding the conditional probability of Y_1 of the previous section, given $Y_1+Y_2=$ n. By definition:

$$P[Y_1=k \mid Y_1+Y_2=m]=\frac{P[Y_1=k \mid Y_1+Y_2=m]}{P[Y_1+Y_2=m]}$$

$$=\frac{P[Y_1=k, Y_2=m-k]}{P[Y_1+Y_2=m]}$$

$$=\frac{\dfrac{e^{-\lambda_1}\lambda_1^{\,k}}{k!}\dfrac{e^{-\lambda_2}\lambda_2^{\,m-k}}{(m-k)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^m}{m!}}$$

$$=\binom{m}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{m-k} \qquad k=0,1,\ldots m$$

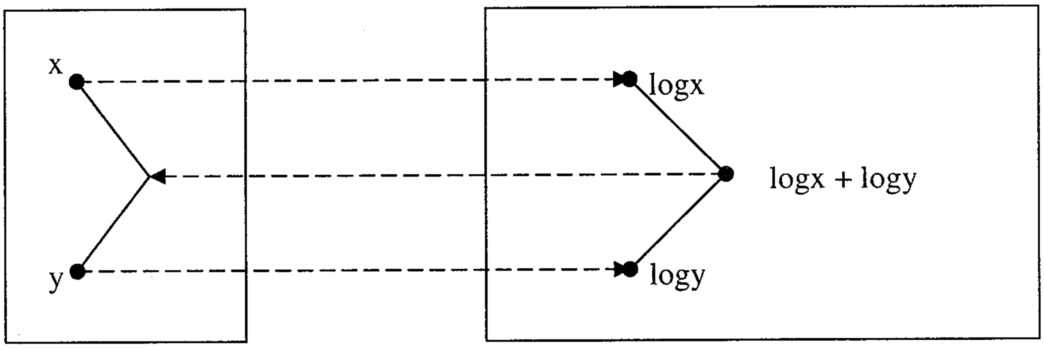
Which is the probability distribution function for a binomial random variable with parameters m and $\dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

VII. MORE INTERPRETATION ON FACTORIAL GENERATING FUNCTION

In order to motivate what explained in the previous sections, let us recall our earliest encounter with the logarithm. It was introduced purely as a computational aid. With each positive number x, we associated another number, denoted by logx. In order to compute xy, for example, we obtain the values of logx and of logy and then evaluate logx + logy, which represents logxy [5]. From, the knowledge of logxy, we were then able to obtain the value of xy(with the aid of tables). In a similar way we may simplify the computation of other arithmetic calculations with the aid of the logarithm. The above approach is useful for the following reasons.

- To each positive number x there corresponds exactly one number, logx, and this number is easily obtained from tables.
- To each value of logx there corresponds exactly one value of x, and this value is again available from tables (That is the relationship between x and logx is one to one).
- Certain arithmetic operations involving x and y such as multiplication and division, may be replaced by simpler operations, such as addition and subtraction, by means of the "transformed" numbers logx and logy.

Instead of performing the arithmetic directly with the numbers x and y we first obtain the numbers logx and logy, do our arithmetic with these numbers and then transform back.



VIII. CONCLUSIONS

In this section III and IV we calculated the probability and mean value of a given problem by the aid of factorial moment generating function or in some books [5] it is also called the probability generating function P(t). Which has the advantages explained in section IV.

It should be noted that mean computation of the example considered is almost impossible by hand calculations if the probability generating function is not used. If the number of dices increases, the difficulty mentioned above can be realized more easily which is the philosophy which lies behind the idea.

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Özcan BAYTEKİN (ozcanbaytekin@hotmail.com) has his B.S degree from Bosphorus University from mathematics department and his Masters degree from Middle East Technical University from Civil Engineering Department. His Ph.D degree is obtained from University of İstanbul from Business Department. His research areas are mathematics, statistics and operations research.