Journal of Universal Mathematics
Vol.4 No.1 pp.51-61 (2021)
ISSN-2618-5660
DOI: 10.33773/jum.679843

NEW APPROACHES ON SYMMETRIC GENERALIZED INTUITIONISTIC FUZZY METRIC SPACES

M. JEYARAMAN AND M. SUGANTHI

Abstract. In this paper, we prove the existence and uniqueness of a common fixed point in symmetric generalized intuitionistic fuzzy metric spaces using property (E.A.) or CLRg property. We introduce the new notion for a pair of mappings (f, g) on a generalized intuitionistic fuzzy metric space called weakly commuting of type (J_f) and R-weakly commuting of type (J_f).

1. Introduction

In 2006, Mustafa and Sims [9] presented a definition of G-metric space. After that, several fixed point results were proved in G-metric spaces. On the other hand, Atanassov [3] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Park [12] has introduced and studied the notion of intuitionistic fuzzy metric spaces. In 2010, Sun and Yang [14] introduced the concept of generalized fuzzy metric space using the concept of continuous t-norm. In 2017, Muthuraj, Jeyaraman et al. [11], proved two unique common coupled fixed point theorems for Junck type and for three mappings in symmetric generalized intuitionistic fuzzy metric spaces. We prove the existence and uniqueness of a common fixed point in symmetric generalized intuitionistic fuzzy metric spaces using property (E.A.) or CLRg property. We introduce the new notion for a pair of mappings (f, g) on a generalized intuitionistic fuzzy metric space called weakly commuting of type (J_f) and R-weakly commuting of type (J_f).

2. Preliminaries

Definition 2.1. A 5-tuple $(X, G, H, *, \diamond)$ is said to be a generalized intuitionistic fuzzy metric space, if X is an arbitrary nonempty set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, G and H are fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions:
For every $x, y, z, a \in X$ and $t, s > 0$.
(i) $G(x, y, z, t) + H(x, y, z, t) \leq 1$,
(ii) $G(x, x, y, t) > 0 \quad \text{for} \quad x \neq y$,

Date: January, 2020.
2000 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. Weakly commuting of type (J_f), R-weakly commuting of type (J_f), Generalized metric space, Intuitionistic fuzzy metric space.

(iii) $G(x,x,y,t) \geq G(x,y,z,t)$ for $y \neq z$,
(iv) $G(x,y,z,t) = 1$ if and only if $x = y = z$,
(v) $G(x,y,z,t) = G(p(x,y,z),t)$, where p is a permutation function,
(vi) $G(x,a,a,t) * G(a,y,z,s) \leq G(x,y,z,t+s)$,
(vii) $G(x,y,z,.) : (0,\infty) \to [0,1]$ is continuous,
(viii) G is a non-decreasing function on R^+,

$$\lim_{t\to\infty} G(x,y,z,t) = 1 \quad \text{and} \quad \lim_{t\to 0} H(x,y,z,t) = 0$$

(ix) $H(x,x,y,t) < 1$ for $x \neq y$,
(x) $H(x,x,y,t) \leq H(x,y,z,t)$ for $y \neq z$,
(xi) $H(x,y,z,t) = 0$ if and only if $x = y = z$,
(xii) $H(x,y,z,t) = H(p(x,y,z),t)$, where p is a permutation function,
(xiii) $H(x,a,a,t) \diamond H(a,y,z,s) \geq H(x,y,z,t+s)$,
(xiv) $H(x,y,z,.) : (0,\infty) \to [0,1]$ is continuous,
(xv) H is a non-increasing function on R^+,

$$\lim_{t\to\infty} G(x,y,z,t) = 0 \quad \text{and} \quad \lim_{n\to\infty} H(x,y,z,t) = 1$$

In this case, the pair (G,H) is called a generalized intuitionistic fuzzy metric on X.

Example 2.2. Let X be a nonempty set and G and H be metrics on X^3 where t-norm is defined by $a * b = \min\{a,b\}$ and t-conorm is defined by $a \diamond b = \max\{a,b\}$. For all $x,y,z \in X$ and $t > 0$,

$$G(x,y,z,t) = \frac{t}{t + G(x,y,z)} \quad \text{and} \quad H(x,y,z,t) = \frac{G(x,y,z)}{t + G(x,y,z)}.$$

Then $(X,G,H,*,\diamond)$ is a generalized intuitionistic fuzzy metric space.

Definition 2.3. Let $(X,G,H,*,\diamond)$ be a generalized intuitionistic fuzzy metric space.
(i) A sequence $\{x_n\}$ in X is said to be convergent to x if

$$\lim_{n\to\infty} G(x_n,x_n,x,t) = 1 \quad \text{and} \quad \lim_{n\to\infty} H(x_n,x_n,x,t) = 0$$

(ii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if

$$\lim_{n,m\to\infty} G(x_n,x_n,x_m,t) = 1 \quad \text{and} \quad \lim_{n,m\to\infty} H(x_n,x_n,x_m,t) = 0,$$

that is, for any $\epsilon > 0$ and for each $t > 0$, there exists $n_0 \in N$ such that

$$G(x_n,x_n,x_m,t) > 1 - \epsilon \quad \text{and} \quad H(x_n,x_n,x_m,t) < \epsilon \text{ for } n,m \geq n_0.$$

(iii) A generalized intuitionistic fuzzy metric space $(X,G,H,*,\diamond)$ is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.4. Let $(X,G,H,*,\diamond)$ be a generalized intuitionistic fuzzy metric space. If the conditions

$$\lim_{n\to\infty} G(x_n,y_n,z_n,t_n) = G(x,y,z,t) \quad \text{and} \quad \lim_{n\to\infty} H(x_n,y_n,z_n,t_n) = H(x,y,z,t)$$

are satisfied whenever $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$ and

$$\lim_{n\to\infty} G(x,y,z,t_n) = G(x,y,z,t) \quad \text{and} \quad \lim_{n\to\infty} H(x,y,z,t_n) = H(x,y,z,t),$$

then G an d H are called convergent functions on $X^3 \times (0, \infty)$.

Lemma 2.5. *Let $(X, G, H, *, \diamond)$ be a generalized intuitionistic fuzzy metric space. Then G and H are continuous function on $X^3 \times (0, \infty)$.*

Proof.

$$\text{Since} \lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$$

$$\text{and} \lim_{n\to\infty} G(x, y, z, t_n) = G(x, y, z, t), \lim_{n\to\infty} H(x, y, z, t_n) = H(x, y, z, t),$$

there is $n_0 \in N$ such that $|t - t_n| < \epsilon$ and $|t - t_n| > \delta$ for $n \geq n_0$ and $\epsilon < \frac{t}{2}$ and $\delta > \frac{t}{2}$.
We know that $G(x, y, z, t)$ is non-decreasing and $H(x, y, z, t)$ is non-increasing with respect to t. Hence we have

$$\begin{aligned}
G(x_n, y_n, z_n, t) &\geq G(x_n, y_n, z_n, t - \epsilon) \\
&\geq G(x_n, x, x, \frac{\epsilon}{3}) * G(x, y_n, z_n, t - \frac{4\epsilon}{3}) \\
&\geq G(x_n, x, x, \frac{\epsilon}{3}) * G(y_n, y, y, \frac{\epsilon}{3}) * G(y, x, z_n, t - \frac{5\epsilon}{3}) \\
&\geq G(x_n, x, x, \frac{\epsilon}{3}) * G(y_n, y, y, \frac{\epsilon}{3}) * G(z_n, z, z, \frac{\epsilon}{3}) * G(z, y, z, t - 2\epsilon) \\
&\text{and} \\
H(x_n, y_n, z_n, t) &\leq H(x_n, y_n, z_n, t - \delta) \\
&\leq H(x_n, x, x, \frac{\delta}{3}) \diamond H(x, y_n, z_n, t - \frac{4\delta}{3}) \\
&\leq H(x_n, x, x, \frac{\delta}{3}) \diamond H(y_n, y, y, \frac{\delta}{3}) \diamond H(y, x, z_n, t - \frac{5\delta}{3}) \\
&\leq H(x_n, x, x, \frac{\delta}{3}) \diamond H(y_n, y, y, \frac{\delta}{3}) \diamond H(z_n, z, z, \frac{\delta}{3}) \diamond H(z, y, z, t - 2\delta). \\
G(x, y, z, t + 2\epsilon) &\geq G(x, y, z, t_n + \epsilon) \\
&\geq G(x, x_n, x_n, \frac{\epsilon}{3}) * G(x_n, y, z, t_n + \frac{2\epsilon}{3}) \\
&\geq G(x, x_n, x_n, \frac{\epsilon}{3}) * G(y, y_n, y_n, \frac{\epsilon}{3}) * G(y_n, x_n, z, t_n + \frac{\epsilon}{3}) \\
&\geq G(x, x_n, x_n, \frac{\epsilon}{3}) * G(y, y_n, y_n, \frac{\epsilon}{3}) * G(z, z_n, z_n, \frac{\epsilon}{3}) * G(z, y, x, t_n) \\
&\text{and} \\
H(x, y, z, t + 2\delta) &\leq H(x, y, z, t_n + \delta) \\
&\leq H(x, x_n, x_n, \frac{\delta}{3}) \diamond H(x_n, y, z, t_n + \frac{2\delta}{3}) \\
&\leq H(x, x_n, x_n, \frac{\delta}{3}) \diamond H(y, y_n, y_n, \frac{\delta}{3}) \diamond H(y_n, x_n, z, t_n + \frac{\delta}{3}) \\
&\leq H(x, x_n, x_n, \frac{\delta}{3}) \diamond H(y, y_n, y_n, \frac{\delta}{3}) \diamond H(z, z_n, z_n, \frac{\delta}{3}) \diamond H(z, y, x, t_n)
\end{aligned}$$

Let $n \to \infty$. By continuity of the functions G and H, with respect to t, we can get

$$G(x, y, z, t + 2\epsilon) \geq G(z, y, x, t) \geq G(z, y, x, t - 2\epsilon) \text{ and}$$

$$H(x,y,z,t+2\delta) \leq H(z,y,x,t) \leq H(z,y,x,t-2\delta).$$

Therefore G and H are continuous functions on $X^3 \times (0,\infty)$. □

3. Weakly Commuting of Type (J_f)

Definition 3.1. A pair of self mapping (f,g) of generalized intuitionistic fuzzy metric space $(X,G,H,*,\diamond)$ is said to be weakly commuting of type (J_f) if

$$G(fgx,gfx,ffx,t) \geq G(fx,gx,fx,t)$$

and

$$H(fgx,gfx,ffx,t) \leq H(fx,gx,fx,t)$$

for all $x \in X$ and $t > 0$.

Definition 3.2. A pair of self mapping (f,g) of generalized intuitionistic fuzzy metric space $(X,G,H,*,\diamond)$ is said to be R-weakly commuting of type (J_f) if there exists some positive real number R such that

$$G(fgx,gfx,ffx,t) \geq G(fx,gx,fx,t/R)$$

and

$$H(fgx,gfx,ffx,t) \leq H(fx,gx,fx,t/R)$$

for all $x \in X$ and $t > 0$.

Remark 3.3. If we interchange f and g in above definitions, the pair of self mapping (g,f), of generalized intuitionistic fuzzy metric space $(X,G,H,*,\diamond)$, is said to be weekly commuting of type (J_g) and R-weakly commuting of type (J_g), respectively.

Example 3.4. Let $X = [0,1]$ be endowed with a standard generalized intuitionistic fuzzy metric defined by

$$G(x,y,z,t) = \frac{t}{t+|x-y|+|y-z|+|z-x|}$$

and

$$H(x,y,z,t) = \frac{|x-y|+|y-z|+|z-x|}{t+|x-y|+|y-z|+|z-x|}$$

for all $x,y,z \in X$ and $t > 0$.
Define $f,g : X \to X$ by $fx = \frac{x^2}{4}$, $gx = x^2$, for all $x \in X$. Clearly, $x = 0$ is the only coincidence point of f and g.
So f and g are weakly compatible. It should be noted that

$$G(fgx,gfx,ffx,t) \geq G(fx,gx,fx,t), H(fgx,gfx,ffx,t) \leq H(fx,gx,fx,t)$$

for all $x \in X$ and $t > 0$.
Then the pair (f,g) is weakly commuting of type (J_f) but not weakly commuting of type (J_g).

Example 3.5. Let $X = [0,2]$ be endowed with a standard generalized intuitionistic fuzzy metric. Define $fx = 2-x$, $gx = x$, then by an easy calculation, one can show that the pair (f,g) is weakly commuting of type (J_f) and R-weakly commuting of type (J_f).

Lemma 3.6. *If f and g are weakly commuting of type (J_f) or R-weakly commuting of type (J_f), then f and g are weakly compatible.*

Proof. Let x be a coincidence point of f and g, i.e., $fx = gx$. If the pair (f, g) of generalized intuitionistic fuzzy metric space $(X, G, H, *, \diamond)$ is weakly commuting of type (J_f), we have

$$G(fgx, gfx, fgx, t) = G(fgx, gfx, ffx, t) \geq G(fx, gx, gx, t) \geq 1,$$

and

$$H(fgx, gfx, fgx, t) = H(fgx, gfx, ffx, t) \leq H(fx, gx, gx, t) \leq 0.$$

It follows that $fgx = gfx$, and then they commute at their coincidence point. Similarly, if the pair (f, g) of generalized intuitionistic fuzzy metric space $(X, G, H, *, \diamond)$ is R-weakly commuting of type (J_f), we have, for all $x \in X$,

$$G(fgx, gfx, fgx, t) \geq G(fgx, gfx, ffx, t) \geq G(fx, gx, fx, t/R) = 1,$$

$$H(fgx, gfx, fgx, t) \leq H(fgx, gfx, ffx, t) \leq H(fx, gx, fx, t/R) = 0.$$

Thus $fgx = gfx$, which implies that f and g are weakly compatible. □

Definition 3.7. [1] A pair of self mapping (f, g) on X is said to satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \text{ for all } z \in X$$

.

Definition 3.8. A pair of self mapping (f, g) on X is said to satisfy CLRg property if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gz \text{ for all } z \in X.$$

For proving our main results, we use the following relation:
Define $\Phi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+\}$ where $\mathbb{R}^+ = [0, \infty)$ and each $\phi \in \Phi$ satisfies the following conditions:
$(\Phi - 1)$: ϕ is strictly increasing.
$(\Phi - 2)$: ϕ is upper semi-continuous from the right.
$(\Phi - 3)$: $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.

Lemma 3.9. *Let $(X, G, H, *, \diamond)$ be a generalized intuitionistic fuzzy metric space. If there exists $\phi \in \Phi$ such that*

$$G(x, y, z, \phi(t)) \geq G(x, y, z, t) \text{ and } H(x, y, z, \phi(t)) \leq H(x, y, z, t) \text{ for all } t > 0,$$

then $x = y = z$.

Lemma 3.10. *Let $(X, G, H, *, \diamond)$ be a generalized intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in X. If there exists $\phi \in \Phi$ such that*

$$G(y_n, y_n, y_{n+1}, \phi(t)) \geq G(y_{n-1}, y_{n-1}, y_n, t) * G(y_n, y_n, y_{n+1}, t)$$

and

$$H(y_n, y_n, y_{n+1}, \phi(t)) \leq H(y_{n-1}, y_{n-1}, y_n, t) \diamond H(y_n, y_n, y_{n+1}, t)$$

for all $t > 0$ and $n = 1, 2, \ldots,$
then $\{y_n\}$ is a Cauchy sequence in X.

4. Main Results

Theorem 4.1. *Let $(X, G, H, *, \diamond)$ be a symmetric generalized intuitionistic fuzzy metric space and the mappings $f, g : X \to X$ satisfy the following conditions:*
(4.1.1) f and g are weakly commuting of type (J_f),
(4.1.2) $f(X) \subseteq g(X)$,
(4.1.3) $g(X)$ is a complete subspace of X,
(4.1.4) there exists a $\phi \in \Phi$ such that for all $x, y, z \in X$ and $t > 0$,

$$G(fx, fy, fz, \phi(t)) \geq G(gx, gx, fx, t) * G(gy, gy, fy, t) * G(gz, gz, fz, t),$$

$$H(fx, fy, fz, \phi(t)) \leq H(gx, gx, fx, t) \diamond H(gy, gy, fy, t) \diamond H(gz, gz, fz, t).$$

Then f and g have common fixed point.

Proof. Let $x_0, x_1, x_2 \in X$ be such that $fx_0 = gx_1$ and $fx_1 = gx_2$. Then by induction, we can define a sequence $\{y_n\} \in X$ as follows:

$$y_n = fx_n = gx_{n+1},\ n \in N.$$

We will prove $\{y_n\}$ is a Cauchy sequence in X.

$$\begin{aligned} &G(y_n, y_n, y_{n+1}, \phi(t)) = G(fx_n, fx_n, fx_{n+1}, \phi(t)) \\ &\geq G(gx_n, gx_n, fx_n, t) * G(gx_n, gx_n, fx_n, t) * G(gx_{n+1}, gx_{n+1}, fx_{n+1}, t) \\ &\geq G(gx_n, gx_n, gx_{n+1}, t) * G(gx_n, gx_n, gx_{n+1}, t) * G(gx_{n+1}, gx_{n+1}, gx_{n+2}, t) \\ &= G(gx_n, gx_n, gx_{n+1}, t) * G(gx_{n+1}, gx_{n+1}, gx_{n+2}, t) \quad \text{and} \\ &H(y_n, y_n, y_{n+1}, \phi(t)) = H(fx_n, fx_n, fx_{n+1}, \phi(t)) \\ &\leq H(gx_n, gx_n, fx_n, t) \diamond H(gx_n, gx_n, fx_n, t) \diamond H(gx_{n+1}, gx_{n+1}, fx_{n+1}, t) \\ &\leq H(gx_n, gx_n, gx_{n+1}, t) \diamond H(gx_n, gx_n, gx_{n+1}, t) \diamond H(gx_{n+1}, gx_{n+1}, gx_{n+2}, t) \\ &= H(gx_n, gx_n, gx_{n+1}, t) \diamond H(gx_{n+1}, gx_{n+1}, gx_{n+2}, t). \end{aligned}$$

This gives

$$G(y_n, y_n, y_{n+1}, \phi(t)) \geq G(y_{n-1}, y_{n-1}, y_n, t) * G(y_n, y_n, y_{n+1}, t)$$

and

$$H(y_n, y_n, y_{n+1}, \phi(t)) \leq H(y_{n-1}, y_{n-1}, y_n, t) \diamond H(y_n, y_n, y_{n+1}, t).$$

By the lemma (3.10), the sequence $\{y_n\}$ is a Cauchy sequence.
Since $y_n = gx_{n+1}$, $\{gx_{n+1}\}$ is a Cauchy sequence in $g(X)$.
By (4.1.3) hypotheses, we know that
$g(X)$ is complete and then there exists $u \in g(X)$ such that

$$\lim_{n\to\infty} gx_n = u = \lim_{n\to\infty} fx_n.$$

Now $u \in g(X)$, so there exists $p \in X$ such that $u = gp$.
Therefore

$$\lim_{n\to\infty} gx_n = gp = \lim_{n\to\infty} fx_n.$$

We will prove that $fp = gp$.

$$G(fp, fp, fx_n, \phi(t)) \geq G(gp, gp, fp, t) * G(gp, gp, fp, t) * G(gx_n, gx_n, fx_n, t) \text{ and}$$

$$H(fp, fp, fx_n, \phi(t)) \leq H(gp, gp, fp, t) \diamond H(gp, gp, fp, t) \diamond H(gx_n, gx_n, fx_n, t).$$

Taking limit as $n \to \infty$, we have

$$G(fp, fp, gp, \phi(t)) \geq G(gp, gp, fp, t) * G(gp, gp, fp, t) * G(gp, gp, fp, t)$$

or

$$G(fp, fp, gp, \phi(t)) \geq G(gp, gp, fp, t) \text{ and}$$

$$H(fp, fp, gp, \phi(t)) \leq H(gp, gp, fp, t) \diamond H(gp, gp, fp, t) \diamond H(gp, gp, gp, t)$$

or

$$H(fp, fp, gp, \phi(t)) \leq H(gp, gp, fp, t).$$

Since generalized intuitionistic fuzzy metric space is symmetric, we have

$$G(fp, fp, gp, \phi(t)) \geq G(gp, gp, fp, t) = G(fp, fp, gp, t),$$

$$H(fp, fp, gp, \phi(t)) \leq H(gp, gp, fp, t) = H(fp, fp, gp, t)$$

which implies $fp = gp$ (by Lemma (3.9)).
Since the pair (f, g) is weakly commuting of type (J_f), then

$$G(fgp, gfp, ffp, \phi(t)) \geq G(fp, gp, fp, t) = 1$$

and

$$H(fgp, gfp, ffp, \phi(t)) \leq H(fp, gp, fp, t) = 0,$$

implies that $ffp = fgp = gfp = ggp$. Hence $fu = fgp = gfp = gu$.
Finally, we show that $u = gp$ is common fixed point of f and g.
Suppose $fu \neq u$, then

$$\begin{aligned}
G(fu, fp, fp, \phi(t)) &\geq G(gu, gu, fu, t) * G(gp, gp, fp, t) * G(gp, gp, fp, t), \\
G(fu, fp, fp, \phi(t)) &\geq G(fu, fu, fu, t) * G(fp, fp, fp, t) * G(fp, fp, fp, t), \\
G(fu, u, u, \phi(t)) &\geq 1 * 1 * 1 = 1, \\
H(fu, fp, fp, \phi(t)) &\leq H(gu, gu, fu, t) \diamond H(gp, gp, fp, t) \diamond H(gp, gp, fp, t), \\
H(fu, fp, fp, \phi(t)) &\leq H(fu, fu, fu, t) \diamond H(fp, fp, fp, t) \diamond H(fp, fp, fp, t), \\
H(fu, u, u, \phi(t)) &\leq 0 \diamond 0 \diamond 0 = 0, \text{ a contradiction}
\end{aligned}$$

Hence $fu = gu = u$.

To prove the uniqueness, suppose we have u and v such that u $\neq v$ and

$$\text{fu} = \text{gu} = \text{u} \quad \text{and} \quad \text{fv} = \text{gv} = \text{v}$$

Using condition (4.1.4), we have

$$\begin{aligned}
G(u, v, v, \phi(t)) &= G(fu, fv, fv, \phi(t)) \\
&\geq G(gu, gu, fu, t) * G(gv, gv, fv, t) * G(gv, gv, fv, t) \\
&= 1 * 1 * 1 = 1, \\
H(u, v, v, \phi(t)) &= H(fu, fv, fv, \phi(t)) \\
&\leq H(gu, gu, fu, t) \diamond H(gv, gv, fv, t) \diamond H(gv, gv, fv, t) \\
&= 0 \diamond 0 \diamond 0 = 0.
\end{aligned}$$

$$\text{Hence } G(u, v, v, \phi(t)) \geq 1 \quad \text{and} \quad H(u, v, v, \phi(t)) \leq 0,$$

which gives contradiction. Hence $u = v$. Therefore u is a unique common fixed point. □

Example 4.2. Let $X = [0,1]$ be endowed with standard generalized intuitionistic fuzzy metric. Define $f, g : X \to X$ by $f(x) = \frac{x^2}{4}$, $g(x) = x^2$, $x \in X$. We see that $x = 0$ is the only coincidence point.
So f and g are weakly compatible.

$$\text{Also } G(fgx, gfx, ffx, t) \geq G(fx, gx, fx, t)$$

$$\text{and } H(fgx, gfx, ffx, t) \leq H(fx, gx, fx, t).$$

Then the pair (f, g) is weakly commuting of type (J_f) but not weakly commuting of type (J_g).

Example 4.3. Let $X = [-1,1]$ be endowed with standard generalized intuitionistic fuzzy metric. Let $\phi(t) = \frac{t}{2}$ and define $f, g : X \to X$ by $f(x) = \frac{x}{6}$, $g(x) = \frac{x}{2}(x+1)$, $x \in X$. We see that $x = 0$ is the only coincidence point and f and g are weakly compatible.
Let $\{x_n = \frac{1}{n}\}$ be a sequence such that

$$G(fp, fp, fx_n, \phi(t)) \geq G(fp, fp, gp, t)$$

and

$$H(fp, fp, fx_n, \phi(t)) \leq H(fp, fp, gp, t)$$

where p is a coincidence point.
Then the pair (f, g) is weakly commuting of type (J_f) and f and g have a unique common fixed point.

Corollary 4.4. Theorem (4.1) remains true if we replace weakly commuting of type (J_f) by weakly commuting of type (J_g) or replace R-weakly commuting of type (J_g).

Theorem 4.5. *Let $(X, G, H, *, \diamond)$ be a symmetric generalized intuitionistic fuzzy metric space and suppose mappings $f, g : X \to X$ are weakly commuting of type (J_f) satisfying the following conditions:*
(4.5.1) f and g satisfy the property (E.A.),
(4.5.2) $g(X)$ is a closed subspace of X,
(4.5.3) there exists a $\phi \in \Phi$ such that for all $x, y, z \in X$ and $t > 0$,

$$G(fx, fy, fz, \phi(t)) \geq G(gx, gx, fx, t) * G(gy, gy, fy, t) * G(gz, gz, fz, t)$$

and

$$H(fx, fy, fz, \phi(t)) \leq H(gx, gx, fx, t) \diamond H(gy, gy, fy, t) \diamond H(gz, gz, fz, t).$$

Then f and g have a unique common fixed point.

Proof. The mappings f and g satisfy the property (E.A.), then there exists a sequence $\{x_n\} \in X$ satisfying

$$\lim_{n\to\infty} fx_n = u = \lim_{n\to\infty} gx_n$$

for some $u \in X$.
Since $g(X)$ is a closed subspace of X and

$$\lim_{n\to\infty} gx_n = u,$$

then there exists $p \in X$ such that $gp = u$.

$$\text{Also } \lim_{n\to\infty} gx_n = gp = \lim_{n\to\infty} fx_n$$

.
We will prove, $fp = gp$.

$$G(fp, fp, fx_n, \phi(t)) \geq G(gp, gp, fp, t) * G(gp, gp, fp, t) * G(gx_n, gx_n, fx_n, t)$$

and

$$H(fp, fp, fx_n, \phi(t)) \leq H(gp, gp, fp, t) \diamond H(gp, gp, fp, t) \diamond H(gx_n, gx_n, fx_n, t),$$

taking limit as $n \to \infty$, we have

$$\begin{aligned}
&G(fp, fp, gp, \phi(t)) \geq G(gp, gp, fp, t) * G(gp, gp, fp, t) * G(gp, gp, fp, t)\\
&G(fp, fp, gp, \phi(t)) \geq G(gp, gp, fp, t), \text{and}\\
&H(fp, fp, gp, \phi(t)) \leq H(gp, gp, fp, t) \diamond H(gp, gp, fp, t) \diamond H(gp, gp, fp, t)\\
&H(fp, fp, gp, \phi(t)) \leq H(gp, gp, fp, t).
\end{aligned}$$

Since generalized intuitionistic fuzzy metric space is symmetric, we have

$$G(fp, fp, gp, \phi(t)) \geq G(fp, fp, gp, t)$$

and

$$H(fp, fp, gp, \phi(t)) \leq H(fp, fp, gp, t),$$

which implies $fp = gp = u$.
Since the pair (f, g) is weakly commuting of type (J_f), then

$$G(fgp, gfp, ffp, \phi(t)) \geq G(fp, gp, fp, t) = 1$$

and

$$H(fgp, gfp, ffp, \phi(t)) \leq H(fp, gp, fp, t) = 0,$$

which implies $ffp = fgp = gfp = ggp$. Hence $fu = fgp = gfp = gu$.
Now, we will show that $fp = u$ is a common fixed point of f and g.
Suppose $fu \neq u$, then

$$\begin{aligned}
G(fu, u, u, \phi(t)) &= G(fu, fp, fp, \phi(t))\\
&\geq G(gu, gu, fu, t) * G(gp, gp, fp, t) * G(gp, gp, fp, t)\\
&= G(fu, fu, fu, t) * 1 * 1\\
&= 1 * 1 * 1 = 1,\\
H(fu, u, u, \phi(t)) &= H(fu, fp, fp, \phi(t))\\
&\leq H(gu, gu, fu, t) \diamond H(gp, gp, fp, t) \diamond H(gp, gp, fp, t)\\
&= H(fu, fu, fu, t) \diamond 0 \diamond 0\\
&= 0 \diamond 0 \diamond 0 = 0,
\end{aligned}$$

a contradiction. Hence $fu = u = gu$.
To prove the uniqueness, suppose we have u and v such that $u \neq v, fu = gu = u$ and $fv = gv = v$, then again using condition (4.5.3), we have,

$$\begin{aligned}
G(u, v, v, \phi(t)) &= G(fu, fv, fv, \phi(t))\\
&\geq G(gu, gu, fu, t) * G(gv, gv, fv, t) * G(gv, gv, fv, t)\\
&= 1 * 1 * 1 = 1,\\
H(u, v, v, \phi(t)) &= H(fu, fv, fv, \phi(t))\\
&\leq H(gu, gu, fu, t) \diamond H(gv, gv, fv, t) \diamond H(gv, gv, fv, t)\\
&= 0 \diamond 0 \diamond 0 = 0,
\end{aligned}$$

which is a contradiction. Hence $u = v$. Therefore u is a unique common fixed point of f and g. □

Theorem 4.6. *Let $(X, G, H, *, \diamond)$ be a symmetric generalized intuitionistic fuzzy metric space and suppose mappings $f, g : X \to X$ are weakly commuting of type (J_f) satisfying the following conditions:*
(4.6.1) f and g satisfy CLRg property,
(4.6.2) there exists a $\phi \in \Phi$ such that for all $x, y, z \in X$ and $t > 0$,

$$G(fx, fy, fz, \phi(t)) \geq G(gx, gx, fx, t) * G(gy, gy, fy, t) * G(gz, gz, fz, t)$$
$$H(fx, fy, fz, \phi(t)) \leq H(gx, gx, fx, t) \diamond H(gy, gy, fy, t) \diamond H(gz, gz, fz, t).$$

Then f and g have a unique common fixed point.

Proof. The proof follows on the same lines of Theorem(4.5) and by definition of CLRg property. □

5. CONCLUSION

This paper introduced new kinds of pair of self mappings viz. weakly commuting of type (J_f) and R-weakly commuting of type (J_f) on generalized intuitionistic fuzzy metric spaces. Common fixed point theorems are proved based on these newly introduced pair of self mappings. This work provides a new approach to analyze intuitionistic fuzzy metric spaces and paves a path that leads to extensions of such conceptual work.

References

[1] M. Aamri and D. El. Moutawakil, Some new common fixed point theorems under strict contractive conditions", Math. Anal. Appl. 270 (2002), 181 - 188.

[2] C. Alaca, D. Turkoglu and C. Yildiz, Fixed points in intuitionistic fuzzy metric space, Chaos, Soliton and Fractals, 29 (5) (2006), 1073 - 1078.

[3] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and System, 20 (1986), 87 - 96.

[4] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994), 395 - 399.

[5] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets Syst. 90 (1997), 365 - 368.

[6] G. Jungck, Commuting mappings and fixed points, Am. Math. Mon. 83 (1976),261 - 263.

[7] O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 326 - 334.

[8] S. Manro, S. S. Bhatia and S. Kumar, Common fixed point theorems in fuzzy metric spaces, Annals of Fuzzy Mathematics and Informatics, 3(1) (2012), 151 - 158.

[9] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289 - 297.

[10] Z. Mustafa, H. Aydi and E. Karapinar, On common fixed points G - metric spaces using (E.A.) property, Computer and mathematics with applications 64 (2012), 1944 - 1956.

[11] R. Muthuraj, M. Sornavalli and M. Jeyaraman, Common coupled fixed point theorems in generalized intuitionistic fuzzy metric spaces, Notes On Intuitionistic Fuzzy sets, 23(1) (2017), 57 - 69

[12] J. H. Park, Intuitionistic fuzzy metric space, Chaos, Solitions & Fractals, 22 (2004), 1039 - 1046.

[13] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. 32 (46) (1982), 149 - 153.

[14] G. Sun and K. Yang, Generalized fuzzy metric spaces with properties, Res. J. Appl. Sci. 2 (2010), 673 - 678.

[15] L. A. Zadeh, Fuzzy sets, Inf. Control 8 (1965), 338 - 353.

Department of Mathematics, Raja Doraisingam Government Arts College, Sivagangai - 630561, Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India
Email address: jeya.math@gmail.com

Research Scholar, Department of Mathematics, Raja Doraisingam Government Arts College, Sivagangai - 630561, Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India.
Email address: vimalsugan@gmail.com