DOI:10.25092/baunfbed.680597 **J. BAUN Inst. Sci. Technol., 22(1), 193-203, (2020)**

A bridge construction from Sheffer stroke basic algebras to MTL-algebras

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Geliş Tarihi (Received Date): 22.04.2019
Kabul Tarihi (Accepted Date): 25.10.2019

Abstract

In this study, we bridge over from Sheffer stroke basic algebras to MTL-algebras by means of defining all operations only via Sheffer stroke operation. We also give some equalities and inequalities which are used in this construction. Furthermore, we deal with construction relations between other algebraic structures as BL-algebras, MV-algebras and Gödel algebras and Sheffer stroke basic algebras.

Keywords: *Sheffer stroke basic algebras, MTL-algebras, MV-algebras, BL-algebras, Gödel algebras.*

Sheffer stroke temel cebirlerinden MTL-cebirlerine bir köprü inşası

Öz

Bu çalışmada, MTL-cebirlerini oluşturan tüm operatörleri sadece Sheffer stroke operatörü yardımıyla tanımlayarak, Sheffer stroke temel cebirlerden MTL-cebirlerine bir köprü inşası oluşturduk. Ayrıca, bu inşa süresince kullanılacak bazı eşitlikler ve eşitsizlikleri verdik. Bunun yanı sıra, BL-cebirleri, MV-cebirleri ve Gödel cebirleri gibi cebirsel yapılar ile Sheffer stroke temel cebirleri arasındaki yapı ilişkilerini de ele aldık.

Anahtar kelimeler: *Sheffer stroke temel cebirleri, MTL-cebirleri, BL-cebirleri, MV-cebirleri, Gödel cebirleri.*

1. Introduction

When we construct a structure as a mathematical model, the first thing to do is always to get rid of unnecessary expressions. For that reason, we attempt to express equivalent

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statements as possible as with the least number of axioms or the least number of operations and so on. In the first instance, Tarski dealt with a problem which was about the least number of axioms for Abelian groups. He shown that Abelian groups can be characterized from the point of divisor operator via a single axiom [1].

By the time we analyze this reduction, we think about Sheffer stroke operation for algebraic structures. Oner and Senturk introduced a reduction of basic algebras by means of only Sheffer stroke operation which is called Sheffer stroke basic algebra [2]. Sheffer stroke basic algebras play an important role in great numbers of logics as many-valued Łukasiewicz logics, non-classical logics, fuzzy logics and etc.

Esteva and Godo introduced MTL-algebras in [3]. In recent times, the theory of MTL-algebras has been strengthened by the help of some theorems [4, 5]. Almost all these works are an important influence on algebraic structures. For instance, Vetterlein showed that most of MTL-algebras is embeddable into the positive cone of a partially ordered group [4]. He verified that an MTL-algebra is an integral, commutative, bounded, prelinear residuated lattice [5]. Furthermore, MTL-algebras are the most basis residuated structures having all algebras induced by their residua and continuous t-norms. So, MTL-algebras have an important position in working fuzzy logics and their related structures [6].

In this paper we would like to characterize MTL-algebras by means of Sheffer stroke basic algebras. In addition to this, we investigate construction relations between other algebraic structures as BL-algebras, MV-algebras and Gödel algebras and Sheffer stroke basic algebras.

2. Preliminaries

Throughout this section, we will give some definitions and lemmas about Sheffer stroke basic algebras and MTL-algebras, which are used in the rest of this paper.

2.1. Definition [7] Let L be a nonempty set. If the binary operations $\wedge$ and $\vee$ verify the following identities on L:

(L_1) $u \vee v = v \vee u$ and $u \wedge v = v \wedge u$,
(L_2) $u \vee (v \vee z) = (u \vee v) \vee z$ and $u \wedge (v \wedge z) = (u \wedge v) \wedge z$,
(L_3) $u \vee u = u$ and $u \wedge u = u$,
(L_4) $u \vee (u \wedge v) = u$ and $u \wedge (u \vee v) = u$,

then the algebraic structure $\mathcal{L} = (L; \vee, \wedge)$ is called a lattice.

2.2. Definition [7] An algebraic structure $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ is called bounded lattice if it satisfies the following properties for each $u \in L$:

(L_{B1}) $u \vee 1 = 1$ and $u \wedge 1 = u$,
(L_{B2}) $u \vee 0 = u$ and $u \wedge 0 = 0$,

where 1 and 0 are said to be the greatest and the least elements of the structure, respectively.

2.3. Definition [8] Let $\mathcal{B}$= (B; |) be a groupoid. If the operation | satisfies the following conditions

(S_1) $u \mid v = v \mid u$,
(S_2) $(u \mid u) \mid (u \mid v) = u$,
(S_3) $u \mid ((v \mid z) \mid (v \mid z)) = ((u \mid v) \mid (u \mid v)) \mid z$,
(S_4) $(u \mid ((u \mid u) \mid (v \mid v))) \mid (u \mid ((u \mid u) \mid (v \mid v))) = u$,

then it is called Sheffer stroke operation. Furthermore, if it verifies the following identity

(S_5) $v \mid (u \mid (u \mid u)) = v \mid v$

then it is called ortho-Sheffer stroke operation.

2.4. Lemma [9] Let $\mathcal{B}$= (B; |) be a groupoid with Sheffer stroke operation. Then the following identities are satisfied for each $u, v, z \in B$:

(i) $(u \mid v) \mid (u \mid (v \mid z)) = u$,
(ii) $(u \mid u) \mid v = v \mid (u \mid v)$,
(iii) $u \mid ((v \mid v) \mid u) = v \mid u$.

2.5. Definition [8] Let $\mathcal{B}$= (B; |) be a groupoid with Sheffer stroke operation. The relation $\leq$ is defined on B as

$u \leq v$ if and only if $u \mid v = v \mid v$

then, it is an order on B.

2.6. Lemma [8] Let | be a Sheffer stroke operation on B and $\leq$ the order of $\mathcal{B}$= (B; |). The following properties are satisfied:

(i) $u \leq v$ if and only if $v \mid v \leq u \mid u$,
(ii) $u \mid (v \mid (u \mid u)) = u \mid u$ is the identity of $\mathcal{B}$,
(iii) $u \leq v$ implies $v \mid z \leq u \mid z$ for all $z \in B$,
(iv) $m \leq u$ and $m \leq v$ imply $u \mid v \leq m \mid m$.

2.7. Definition [2] If the algebra $\mathcal{B}$= (B; |) satisfies the following identities

(SH_1) $(u \mid (u \mid u)) \mid (u \mid u) = u$,
(SH_2) $(u \mid (v \mid v)) \mid (v \mid v) = (v \mid (u \mid u)) \mid (u \mid u)$,
(SH_3) $\big((u \mid (v \mid v)) \mid (v \mid v)\big) \mid (z \mid z)) \mid ((u \mid (z \mid z)) \mid (u \mid (z \mid z))) = u \mid (u \mid u)$

for each $u, v, z \in B$, then it is called Sheffer stroke basic algebra.

We will give some fundamental properties of Sheffer stroke basic algebras.

2.8. Lemma [2] Let $\mathcal{B}$= (B; |) be a Sheffer stroke basic algebra. The elements $1 \in B$ is an algebraic constant of $\mathcal{B}$ and this structure verifies the following identities:

(i) $u \mid (u \mid u) = 1$,

(ii) $u \mid (1 \mid 1) = 1$,
(iii) $1 \mid (u \mid u) = u$,
(iv) $\big((u \mid (v \mid v)) \mid (v \mid v)\big) \mid (v \mid v) = u \mid (v \mid v)$,
(v) $(v \mid (u \mid (v \mid v))) \mid (u \mid (v \mid v)) = 1$.

2.9. Lemma [2] Let $\mathcal{B} = (B; \mid)$ be a Sheffer stroke basic algebra. A binary relation $\leq$ is defined on B as follows

$u \leq v$ if and only if $u \mid (v \mid v) = 1$.

Then it is a partial order on B such that $u \leq 1$ for each $u \in B$.

2.10. Definition [6] Let A be a nonempty set, the operations $\vee$, $\wedge$, $\circledast$ and $\rightarrow$ be binary operations on A and the elements 0 and 1 be algebraic constant of B. If an algebraic structure $\mathcal{B} = (B; \vee, \wedge, \circledast, \rightarrow, 0, 1)$ is satisfied the follows

(MTL_1) $(A; \vee, \wedge, 0, 1)$ is a bounded lattice,
(MTL_2) $(A; \circledast, 1)$ is a commutative monoid,
(MTL_3) $u \circledast v \leq z$ if and only if $u \leq v \rightarrow z$,
(MTL_4) $(u \rightarrow v) \vee (v \rightarrow u) = 1$

for any $u, v, z \in B$, then it is called an MTL-algebra.

2.11. Definition [6] Let $\mathcal{B} = (B; \vee, \wedge, \circledast, \rightarrow, 0, 1)$ be an MTL-algebra. Then $\mathcal{B}$ is called

(i) a BL-algebra if $u \wedge v = u \circledast (u \rightarrow v)$ for each $u, v \in B$.
(ii) an MV-algebra if $(u \rightarrow v) \rightarrow v = (v \rightarrow u) \rightarrow u$ for each $u, v \in B$.
(iii) a Gödel algebra if $u \circledast u = u$ for each $u \in B$.

3. A construction of MTL-algebras by means of Sheffer stroke basic algebras

In this part of the paper, we give a construction of MTL-algebras by the help of Sheffer stroke basic algebras. For this aim, we introduce the binary operations $\vee$, $\wedge$, $\rightarrow$, and $\circledast$ only via Sheffer stroke operation. On the other hand, we deal some other algebraic structures construction as BL-algebras, MV-algebras and Gödel.

3.1. Lemma Let $\mathcal{B} = (B; \mid)$ be a Sheffer stroke basic algebra. The inequality

$(v \mid z) \mid (v \mid z) \leq z$

is verified for every $v, z \in B$.

Proof We know that $v \leq 1$ is satisfied for each $v \in B$. Therefore,

$v \leq 1 \Rightarrow 1 \mid z \leq v \mid z$ (by using Lemma 2.6 (iii))
$\Rightarrow z \mid z \leq v \mid z$ (by using Lemma 2.8 (iii))
$\Rightarrow (v \mid z) \mid (v \mid z) \leq (z \mid z) \mid (z \mid z)$ (by using Lemma 2.6 (i))
$\Rightarrow (v \mid z) \mid (v \mid z) \leq z$ (by using Definition 2.3 (S_2)). ■

3.2. Lemma Let $\mathcal{B}=(B;|)$ be a Sheffer stroke basic algebra. The inequality

$u \leq v \mid (u \mid v)$

is satisfied for each $u, v \in B$.

Proof From the definition of Sheffer stroke basic algebra, we have $v \leq 1$ for each $v \in B$. Then

$v \leq 1 \Rightarrow 1 \mid (u \mid u) \leq (u \mid u) \mid v$ (by using Lemma 2.6 (iii))
$\Rightarrow u \leq (u \mid u) \mid v$ (by using Lemma 2.8 (iii))
$\Rightarrow u \leq v \mid (u \mid v)$ (by using Lemma 2.4(ii)). ■

3.3. Lemma Let $\mathcal{B}=(B;|)$ be a Sheffer stroke basic algebra. The biconditional statement

$(u \mid v) \mid (u \mid v) \leq z$ if and only if $u \leq v \mid (z \mid z)$

holds for each $u, v, z \in B$.

Proof ($\Rightarrow$:) Assume that $(u \mid v) \mid (u \mid v) \leq z$. Applying Lemma 2.6 (i), we have $z \mid z \leq ((u \mid v) \mid (u \mid v)) \mid ((u \mid v) \mid (u \mid v))$. By using Definition 2.3 (S_2), we get $z \mid z \leq u \mid v$ and then $v \mid (u \mid v) \leq v \mid (z \mid z)$ from the Lemma 2.6 (iii). So, by the Lemma 3.2, the inequalities $u \leq v \mid (u \mid v) \leq v \mid (z \mid z)$ are obtained. From the transitivity of $\leq$, we obtain $u \leq v \mid (z \mid z)$ for each $u, v, z \in B$.

($\Leftarrow$:) Let $u \leq v \mid (z \mid z)$ for each $u, v, z \in B$. Then we have $v \mid (v \mid (z \mid z) \leq u \mid v$ from the Lemma 2.6 (iii). By Lemma 2.6 (i), the inequality $(u \mid v) \mid (u \mid v) \leq (v \mid (v \mid (z \mid z))) \mid (v \mid (v \mid (z \mid z)))$ is obtained. Using Lemma 2.4 (iii), we get $(u \mid v) \mid (u \mid v) \leq (v \mid z) \mid (v \mid z)$. From Lemma 3.1, we conclude that $(u \mid v) \mid (u \mid v) \leq (v \mid z) \mid (v \mid z) \leq z$. Using transitivity of $\leq$, we reach $(u \mid v) \mid (u \mid v) \leq z$ for each $u, v, z \in B$. ■

3.4. Lemma Let $\mathcal{B}=(B;|)$ be a Sheffer stroke basic algebra. The following identity

$(u \mid (v \mid v)) \mid ((v \mid (u \mid u) \mid (v \mid (u \mid u)) = v \mid (u \mid u)$

is satisfied for each $u, v \in B$.

Proof In Definition 2.3 (S_3), we assign $[u \coloneqq u \mid (v \mid v)]$, $[v \coloneqq v]$ and $[z \coloneqq u \mid u]$ simultaneously. Then we get

$(u \mid (v \mid v)) \mid ((v \mid (u \mid u)) \mid (v \mid (u \mid u))) = (((u \mid (v \mid v)) \mid v) \mid ((u \mid (v \mid v)) \mid v)) \mid (u \mid u).$

By the commutativity of Sheffer stroke operator

$(u \mid (v \mid v)) \mid ((v \mid (u \mid u)) \mid (v \mid (u \mid u))) = ((v \mid (u \mid (v \mid v))) \mid (v \mid (u \mid (v \mid v)))) \mid (u \mid u).$

From Lemma 2.6 (ii), we obtain

$$(u \mid (v \mid v)) \mid ((v \mid (u \mid u)) \mid (v \mid (u \mid u))) = ((v \mid v) \mid (v \mid v)) \mid (u \mid u).$$

By using Definition 2.3 (S_2), we conclude that

$(u \mid (v \mid v)) \mid ((v \mid (u \mid u)) \mid (v \mid (u \mid u))) = v \mid (u \mid u).$ ■

3.5. Theorem Let $\mathcal{B}$= (B; |) be a Sheffer stroke basic algebra. The identity

$((v \mid v) \mid u) \mid u = ((u \mid u) \mid v) \mid v$

is verified for each $u, v \in B$.

Proof Using the Definition 2.3 (S_2) and the Definition 2.7 (SH_2), we get

$$\begin{aligned} ((v \mid v) \mid u) \mid u &= ((v \mid v) \mid ((u \mid u) \mid (u \mid u))) \mid ((u \mid u) \mid (u \mid u)) \\ &= ((u \mid u) \mid ((v \mid v) \mid (v \mid v))) \mid ((v \mid v) \mid (v \mid v)) \\ &= ((u \mid u) \mid v) \mid v. \end{aligned}$$ ■

3.6. Theorem Let $\mathcal{B}$= (B; |) be a Sheffer stroke basic algebra. If the operations are defined as

$u \vee v := (u \mid (v \mid v)) \mid (v \mid v)$
$u \wedge v := (((v \mid v) \mid u) \mid u) \mid (((v \mid v) \mid u) \mid u)$
$u \to v := u \mid (v \mid v)$
$u \circledast v := (u \mid v) \mid (u \mid v)$

then $\mathcal{B}$= (B; $\vee, \wedge, \circledast, \to, 0, 1$) is an MTL-algebra.

Proof
($\boldsymbol{MTL_1}$) We demonstrate that $\mathcal{B}$= (B; $\vee, \wedge, 0, 1$) is a bounded lattice. First of all, we verify the lattice structure conditions.

(L_1) By the Definition 2.7 (SH_2), we get

$u \vee v = (u \mid (v \mid v)) \mid (v \mid v) = (v \mid (u \mid u)) \mid (u \mid u) = v \vee u.$

From the Theorem 3.5, we obtain

$u \wedge v = (((v \mid v) \mid u) \mid u) \mid (((v \mid v) \mid u) \mid u) = (((u \mid u) \mid v) \mid v) \mid (((u \mid u) \mid v) \mid v) = v \wedge u.$

(L_2) From the definition of $\vee$-operation, we have

$(u \vee v) \vee z = (((u \mid (v \mid v)) \mid (v \mid v)) \mid (z \mid z)) \mid (z \mid z).$

Using Lemma 2.4 (ii), we obtain

$(((u \mid (v \mid v)) \mid (v \mid v)) \mid (z \mid z)) \mid (z \mid z) = (((u \mid u) \mid (v \mid v)) \mid (z \mid z)) \mid (z \mid z).$

One more time applying Lemma 2.4 (ii),

$(((u \mid (v \mid v)) \mid (v \mid v)) \mid (z \mid z)) \mid (z \mid z) = (z \mid z) \mid (((u \mid u) \mid (v \mid v)) \mid ((u \mid u) \mid (v \mid v))).$

From the Definition 2.3 (S_3), it is obtained

$(z \mid z) \mid (((u \mid u) \mid (v \mid v)) \mid ((u \mid u) \mid (v \mid v))) = (u \mid u) \mid (((z \mid z) \mid (v \mid v)) \mid ((z \mid z) \mid (v \mid v)))$.

Again we carry out Lemma 2.4 (ii) twice in succession,

$(u \mid u) \mid (((z \mid z) \mid (v \mid v)) \mid ((z \mid z) \mid (v \mid v))) = (((z \mid (v \mid v)) \mid (v \mid v)) \mid (u \mid u)) \mid (u \mid u)$.

By the definition of $\vee$-operation and using (L_1), we conclude that

$(((z \mid (v \mid v)) \mid (v \mid v)) \mid (u \mid u)) \mid (u \mid u) = (z \vee v) \vee u = u \vee (v \vee z)$.

Therefore, the equality $(u \vee v) \vee z = u \vee (v \vee z)$ is obtained for each $u, v, z \in B$.

From the definition of $\wedge$-operation and by Lemma 2.8 (i), we obtain $((u \mid u) \mid u) = 1$, then by substuting $u \mid u := u$ in Lemma 2.8 (iii), we get $(1 \mid u) = (u \mid u)$. By the help of Definition 2.3 (S_2), we have $(u \mid u) \mid (u \mid u) = u$. So, we attain the following equality

$(u \wedge v) \wedge z = (((((v \mid v) \mid u) \mid u) \mid z) \mid z) \mid (((((v \mid v) \mid u) \mid u) \mid z) \mid z)$.

By using Theorem 3.5 and the same method which is applied for $\vee$-operation in above, it is obtained

$(((((v \mid v) \mid u) \mid u) \mid z) \mid z) \mid (((((v \mid v) \mid u) \mid u) \mid z) \mid z)$
$= (((((u \mid u) \mid v) \mid v) \mid z) \mid z) \mid (((((u \mid u) \mid v) \mid v) \mid z) \mid z)$
$= (((((z \mid z) \mid v) \mid v) \mid u) \mid u) \mid (((((z \mid z) \mid v) \mid v) \mid u) \mid u)$.

By (L_1) and the $\wedge$-operation definition,

$(((((z \mid z) \mid v) \mid v) \mid u) \mid u) \mid (((((z \mid z) \mid v) \mid v) \mid u) \mid u) = (v \wedge z) \wedge u = u \wedge (v \wedge z)$.

Hence, the equality $(u \wedge v) \wedge z = u \wedge (v \wedge z)$ is satisfied for each $u, v, z \in B$.

(L_3) The following identities are obtained from the properties of Sheffer stroke basic algebra:

$u \vee u = (u \mid (u \mid u)) \mid (u \mid u) = 1 \mid (u \mid u) = u$.

$u \wedge u = (((u \mid u) \mid u) \mid u) \mid (((u \mid u) \mid u) \mid u) = (1 \mid u) \mid (1 \mid u) = (u \mid u) \mid (u \mid u) = u$.

(L_4) We will show that this system is satisfied absorption laws. By the definition of $\wedge$ and $\vee$ operations, we have

$u \wedge (u \vee v) = u \wedge (u \mid (v \mid v)) \mid (v \mid v)$.

Substituting $[k := (u \mid (v \mid v)) \mid (v \mid v)]$ in the above equation, it is obtained

$u \wedge k = (((k \mid k) \mid u) \mid u) \mid (((k \mid k) \mid u) \mid u)$.

By using Lemma 2.4 (ii) and Definition 2.3 (S_2), we obtain

$(((k \mid k) \mid (k \mid k)) \mid u) \mid (((k \mid k) \mid (k \mid k)) \mid u) = (k \mid u) \mid (k \mid u)$.

We write $[k := (u \mid (v \mid v)) \mid (v \mid v)]$ in the last equation, then applying Definition 2.7 (SH_2)

$$\begin{aligned}(k \mid u) \mid (k \mid u) &= (((u \mid (v \mid v)) \mid (v \mid v)) \mid u) \mid (((u \mid (v \mid v)) \mid (v \mid v)) \mid u) \\ &= (((v \mid (u \mid u)) \mid (u \mid u)) \mid u) \mid (((v \mid (u \mid u)) \mid (u \mid u)) \mid u).\end{aligned}$$

By the Definition 2.3 (S_3) and using (S_2) twice, we obtain

$$\begin{aligned}&(((v \mid (u \mid u)) \mid (u \mid u)) \mid ((u \mid u) \mid (u \mid u))) \mid (((v \mid (u \mid u)) \mid (u \mid u)) \mid ((u \mid u) \mid (u \mid u))) \\ &= (u \mid u) \mid (u \mid u) \\ &= u.\end{aligned}$$

Hence, we get $u \wedge (u \vee v) = u$. Similarly, we obtain $u \vee (u \wedge v) = u$. Therefore, $\mathcal{B} = (B; \vee, \wedge)$ is a lattice. Furthermore, we have the following identities:

$$\begin{aligned}u \vee 1 &= (u \mid (1 \mid 1) \mid (1 \mid 1) = 1, \\ u \vee 0 &= (u \mid (0 \mid 0) \mid (0 \mid 0) = u, \\ u \wedge 1 &= (((1 \mid 1) \mid u) \mid u) \mid (((1 \mid 1) \mid u) \mid u) = u, \\ u \wedge 0 &= (((0 \mid 0) \mid u) \mid u) \mid (((0 \mid 0) \mid u) \mid u) = 0.\end{aligned}$$

Consequently, we verify that $\mathcal{B} = (B; \vee, \wedge, 0, 1)$ is a bounded lattice.

$(\boldsymbol{MTL_2})$ We show that $\mathcal{B} = (B; \circledast, 1)$ is a commutative monoid.

Let a, b and c be elements of B. Then,

$$\begin{aligned}(u \circledast v) \circledast z &= (((u \mid v) \mid (u \mid v)) \mid z) \mid (((u \mid v) \mid (u \mid v)) \mid z) \\ &= (u \mid ((v \mid z) \mid (v \mid z))) \mid (u \mid ((v \mid z) \mid (v \mid z))) \\ &= u \circledast (v \circledast z).\end{aligned}$$

The commutativity is clear from the definition $\circledast$ operation. As a result, $\mathcal{B} = (B; \circledast, 1)$ is a commutative monoid.

$(\boldsymbol{MTL_3})$ The biconditional statement is obtained from Lemma 3.3.

$(\boldsymbol{MTL_4})$

$$\begin{aligned}(u \to v) \vee (v \to u) &= (u|(v|v)) \vee (v|(u|u)) \\ &= ((u \mid (v \mid v)) \mid ((v \mid (u \mid u)) \mid (v \mid (u \mid u)))) \mid ((v \mid (u \mid u)) \mid (v \mid (u \mid u))) \\ &= (v \mid (u \mid u))) \mid ((v \mid (u \mid u)) \mid (v \mid (u \mid u))) \\ &= 1.\end{aligned}$$

As a consequence, $\mathcal{B} = (B; \vee, \wedge, \circledast, \to, 0, 1)$ is an MTL-algebra. ■

3.7. Theorem Let $\mathcal{B} = (B; \vee, \wedge, \circledast, \to, 0, 1)$ be an MTL-algebra where the operations are defined as Theorem 3.6. Then

(i) $\mathcal{B}$ is also a BL-algebra,

(ii) $\mathcal{B}$ is also an MV-algebra,
(iii) $\mathcal{B}$ is also a Gödel algebra.

Proof By using the Definition 2.11, we have the following conclusions.

(i) For each $a, b \in B$, we get the following equality:

$$\begin{aligned} u \wedge v &= (((v \mid v) \mid u) \mid u) \mid (((v \mid v) \mid u) \mid u) \\ &= (((u \mid u) \mid v) \mid v) \mid (((u \mid u) \mid v) \mid v) \\ &= u \circledast (u \rightarrow v). \end{aligned}$$

Hence, we conclude that $\mathcal{B}$ is also a BL-algebra.

(ii) By using Definition 2.7 (SH_2), it is obtained

$$(u \rightarrow v) \rightarrow v = (u \mid (v \mid v)) \mid (v \mid v) = (v \mid (u \mid u)) \mid (u \mid u) = (v \rightarrow u) \rightarrow u$$

for each $u, v \in B$. Thus, $\mathcal{B}$ is also an MV-algebra.

(iii) We have $u \circledast u = (u \mid u) \mid (u \mid u) = u$ for each $u \in B$. As a result, $\mathcal{B}$ is also a Gödel algebra. ■

3.8. Example Let $\mathcal{B} = (\{p, q, r, s\}; |, 0, 1)$ be a Sheffer stroke basic algebra having the following Hasse diagram:

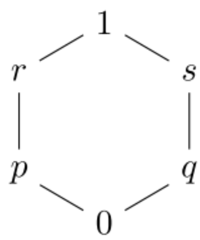

Figure 1. Hasse diagram of $\mathcal{B}$.

The operation on this structure is defined as the below Cayley table:

Table 1. Sheffer stroke operation on the structure $\mathcal{B}$.

$\mid$	0	p	q	r	s	1
0	1	1	1	1	1	1
p	1	s	1	s	1	s
q	1	1	r	1	r	r
r	1	s	1	q	1	q
s	1	1	r	1	P	p
1	1	s	r	q	P	0

If the binary operations $\vee$, $\wedge$, $\circledast$ and $\rightarrow$ are given as Theorem 3.6, the operations on $\mathcal{B}$ have the following Cayley tables:

Table 2. Cayley tables of the binary operations $\vee$, $\wedge$, $\circledast$ and $\rightarrow$ on the structure $\mathcal{B}$.

$\vee$	0	p	q	r	s	1
0	1	p	q	r	s	1
p	p	p	1	r	1	1
q	q	1	q	1	s	1
r	r	r	1	r	1	1
s	s	1	s	1	s	1
1	1	1	1	1	1	1

$\wedge$	0	p	q	r	s	1
0	0	0	0	0	0	0
p	0	p	0	p	0	p
q	0	0	q	0	q	q
r	0	p	0	r	0	r
s	0	0	q	0	s	s
1	0	p	q	r	s	1

$\circledast$	0	p	q	r	s	1
0	0	0	0	0	0	0
p	0	p	0	p	0	p
q	0	0	q	0	q	q
r	0	s	0	r	0	r
s	0	0	q	0	s	s
1	0	p	q	r	s	1

$\rightarrow$	0	p	q	r	s	1
0	1	1	1	1	1	1
p	s	1	s	1	s	1
q	r	r	1	r	1	1
r	q	1	q	1	s	1
s	p	p	1	r	1	1
1	0	p	q	r	s	1

Hence, we obtain that the algebraic structure $\mathcal{B} = (\{p, q, r, s\}; \vee, \wedge, \circledast, \rightarrow, 0, 1)$ is an MTL-algebra. Furthermore, $\mathcal{B}$ is also a BL-algebra, an MV-algebra and also a Gödel algebra by the Theorem 3.7.

Acknowledgments

The author thanks the academic editor for their valuable comments and suggestions and the anonymous referees for his/her remarks which helped them to improve the presentation of the paper.

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