

Alternative characterizations of some linear combinations of an idempotent matrix and a tripotent matrix that commute

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Abstract

In this work, first, Theorem 2 in [1] [Yao, H., Sun, Y., Xu, C., and Bu, C., A note on linear combinations of an idempotent matrix and a tripotent matrix, *J. Appl. Math. Informatics*, 27 (5-6), 1493-1499, 2009] and Theorem 2.2 in [2][Özdemir H., Sarduvan M., Özban A.Y., Güler N., On idempotency and tripotency of linear combinations of two commuting tripotent matrices, *Appl. Math. Comput.*, 207 (1), 197-201, 2009] are reconsidered in different ways under the condition that the matrices involved in the linear combination are commutative. Thus, it is seen that there are some missing results in Theorem 2 in [1]. Then, by considering the obtained results and doing some detailed investigations, it is given a new characterization, without any restriction on the involved matrices except for commutativity, of a linear combination of an idempotent and a tripotent matrix that commute.

Keywords: Idempotent matrix, tripotent matrix, linear combination, commutativity.

Değişmeli bir idempotent ve bir tripotent matrisin bazı lineer kombinasyonlarının alternatif karakterizasyonları

Öz

Bu çalışmada ilk olarak [1][Yao, H., Sun, Y., Xu, C., and Bu, C., A note on linear combinations of an idempotent matrix and a tripotent matrix, *J. Appl. Math. Informatics*, 27 (5-6), 1493-1499, 2009]'deki Teorem 2 ve [2][Özdemir H., Sarduvan M., Özban A.Y., Güler N., On idempotency and tripotency of linear combinations of two

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commuting tripotent matrices, Appl. Math. Comput., 207 (1), 197-201, 2009]'deki Teorem 2.2, lineer kombinasyonda içerilen matrislerin değişmeli olması koşulu altında farklı tarzlarda yeniden ele alınmaktadır. Böylece, [1]'deki Teorem 2'de bazı eksik sonuçların mevcut olduğu görülmüştür. Daha sonra elde edilen sonuçları göz önüne alarak ve bazı detaylı incelemeler yaparak, değişmeli bir idempotent ve bir tripotent matrisin bir lineer kombinasyonunun, içerilen matrisler üzerinde değişmelilik dışında herhangi bir kısıtlama olmaksızın, yeni bir karakterizasyonu verilmektedir.

Anahtar kelimeler: İdempotent matris, tripotent matris, lineer kombinasyon, değişmelilik.

1. Introduction

Let \mathbb{C} be the field of complex numbers and \mathbb{C}^* be the set of nonzero complex numbers. For a positive integer n , let \mathbb{C}_n be the set of all $n \times n$ complex matrices over \mathbb{C} . The symbols I and $\mathbf{0}$ stand for the identity and zero matrices of appropriate sizes, respectively. Also, the similarity and the direct sum of two matrices A and B are denoted by $A \sim B$ and $A \oplus B$, respectively.

Let us recall some definitions and concepts from the matrix algebra. A matrix $A \in \mathbb{C}_n$ is called an idempotent matrix if $A^2 = A$. Also, if $A^2 = -A$, then the matrix A is defined as a skew-idempotent matrix. Note that A is idempotent if and only if $-A$ is a skew-idempotent. In addition, a matrix $A \in \mathbb{C}_n$ is called a tripotent matrix if $A^3 = A$. And also, if $A^3 = A$ and $A^2 \neq \pm A$, then the matrix $A \in \mathbb{C}_n$ is called an essentially tripotent matrix. So, an essentially tripotent matrix is a tripotent matrix which is not idempotent or skew-idempotent.

It has been extensively studied the problem of characterization of a linear combination of special types of matrices since 2000 [3]. Baksalary et al. have established all situations for idempotency of linear combinations of an idempotent matrix and an essentially tripotent matrix in 2002 [4]. In that work, it has been used the fact that for an essentially tripotent matrix A , there exist two disjoint idempotent matrices B_1 and B_2 such that $A = B_1 - B_2$. So, in that study, the results have been stated in terms of the idempotents. Then Yao et al. have considered the same problem in a different way in 2009 [1]. They have stated the results in terms of the matrices involved in the linear combination. Özdemir et al. in 2009, have obtained the sets of necessary and sufficient conditions for the idempotency or tripotency of a linear combination of two commuting tripotent matrices [2]. Actually, the first two studies above consist of different characterizations of the same problem. In the third study, unlike previous ones, both of the matrices in the linear combination belong to the set of tripotent matrices while in the preceding two works, one of the matrices is idempotent and the other is tripotent. There are a lot of studies related to these subjects in the literature. For details, it can be seen the references [1-14].

In this work, first, Theorem 2 in [1] is given by restricting to the case $A_1A_2 = A_2A_1$ in a different way. And then, implicit results in this theorem are stated in explicit forms. To do so, block forms of matrices are used, and then all items of the theorem are partitioned to the subcases. The purpose of doing so is to do the characterization more distinctive. Next, Theorem 2.2 of [2] is stated by restricting to the case $A_2^2 = A_2$. This restriction is

done by keeping in mind the different hypotheses of the theorems. And then, Theorem 1 (a) of [4] is considered. This theorem is stated in terms of an idempotent and a tripotent while Baksalary et al. have stated in terms of three idempotents. To do so, it is used the fact that a tripotent matrix A can be written as $A = \frac{1}{2}(A^2 + A) - \frac{1}{2}(A^2 - A)$. It is seen that the results emerging after these discussions are compatible with each other. Finally, it is established a result related to the idempotency of a linear combination of an idempotent and a tripotent matrix without any conditions other than commutativity. And the items of the theorem is partitioned to the subcases. These subcases divide the results according to the form of tripotent matrix involved in the linear combination. Thanks to this, the results about the characterization can be examined in detail.

2. Results

Yao et al. have established in [1] all cases for the characterization of the idempotency of linear combinations of an idempotent and a tripotent matrix as follows:

Theorem 2.1. ([1], Page 1496) Let A_1 and A_2 be a nonzero complex tripotent matrix and an idempotent matrix, respectively. Let A be a linear combination of the form $a_1A_1 + a_2A_2$ with $a_1, a_2 \in \mathbb{C}^*$. Then the following list comprises characterization of all cases in which A is an idempotent matrix:

- (i) $A_1A_2 + A_2A_1 + a_1A_1^2 - A_1 = \mathbf{0}$ holds along with $a_1 \in \mathbb{C}^*$, $a_2 = 1$, and when $a_1 \neq \pm 1, p = q$ where $p + q = rank A_1$;
- (ii) $A_1A_2 + A_2A_1 = A_2 + \frac{1}{2}(A_1^2 + A_1)$ holds along with $a_1 = -1, a_2 = 2$;
- (iii) $A_1A_2 + A_2A_1 = A_2 + 2A_1 - A_1^2$ holds along with $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}$;
- (iv) $A_1A_2 + A_2A_1 = -A_2 + \frac{1}{2}(A_1 - A_1^2)$ holds along with $a_1 = 1, a_2 = 2$;
- (v) $A_1A_2 + A_2A_1 = A_1^2 + 2A_1 - A_2$ holds along with $a_1 = -\frac{1}{2}, a_2 = \frac{1}{2}$.

If $A_1A_2 = A_2A_1$, then the preceding theorem turns into the following.

Theorem 2.2. Let A_1 and A_2 be a nonzero complex tripotent matrix and an idempotent matrix, respectively. Let A be a linear combination of the form $a_1A_1 + a_2A_2$ with $a_1, a_2 \in \mathbb{C}^*$. If $A_1A_2 = A_2A_1$, then the following list comprises characterization of all cases in which A is an idempotent matrix:

- (i) $2A_1A_2 + a_1A_1^2 - A_1 = \mathbf{0}$ holds along with $a_1 \in \{-1, 1\}, a_2 = 1$;
- (ii) $2A_1A_2 = A_2 + \frac{1}{2}(A_1^2 + A_1)$ holds along with $a_1 = -1, a_2 = 2$;
- (iii) $2A_1A_2 = A_2 + 2A_1 - A_1^2$ holds along with $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}$;
- (iv) $2A_1A_2 = -A_2 + \frac{1}{2}(A_1 - A_1^2)$ holds along with $a_1 = 1, a_2 = 2$;
- (v) $2A_1A_2 = A_1^2 + 2A_1 - A_2$ holds along with $a_1 = -\frac{1}{2}, a_2 = \frac{1}{2}$.

Notice that $a_1 = -1$ or $a_1 = 1$ in Theorem 2.2 (i), while $a_1 \in \mathbb{C}^*$ in Theorem 2.1 (i). Now, we shall prove this case.

Since A_2 is a nonzero complex idempotent matrix, there exists a nonsingular matrix S such that

$$A_2 = S \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} S^{-1}. \quad (1)$$

Let us write the matrix A_1 as follows:

$$A_1 = S \begin{pmatrix} K & L \\ M & N \end{pmatrix} S^{-1}, \quad (2)$$

where the sizes of the blocks of A_1 are suitable with the sizes of the blocks of A_2 . Considering the condition $A_1A_2 = A_2A_1$ together with (1) and (2) leads to $L = \mathbf{0}$ and $M = \mathbf{0}$. So, we get

$$A_1 = S \begin{pmatrix} K & \mathbf{0} \\ \mathbf{0} & N \end{pmatrix} S^{-1} \quad (3)$$

if $A_1A_2 = A_2A_1$. Notice that $K^3 = K$ and $N^3 = N$ since $A_1^3 = A_1$. On the other hand, from (i), we know that $2A_1A_2 + a_1A_1^2 - A_1 = \mathbf{0}$. If (1) and (3) are substituted into the last equality, then it is obtained

$$K + a_1K^2 = \mathbf{0} \text{ and } a_1N^2 - N = \mathbf{0}. \quad (4)$$

If we postmultiply the first equality of (4) by a_1K and use the equality $K^3 = K$, then we get

$$a_1K^2 + a_1^2K = \mathbf{0}. \quad (5)$$

Since $a_1K^2 = -K$ by (4), it is obvious from (5) that $-K + a_1^2K = \mathbf{0}$, that is $K = a_1^2K$. On the other hand, postmultiplying the second equality of (4) by a_1N in view of $N^3 = N$ leads to

$$a_1^2N - a_1N^2 = \mathbf{0}. \quad (6)$$

From the second equality of (4) and (6), we get $a_1N^2 = N$. So, from the equalities of (4), we arrive at

$$K = a_1^2K \text{ and } N = a_1^2N. \quad (7)$$

Then, we have $K \neq \mathbf{0}$ or $N \neq \mathbf{0}$ since $A_1 \neq \mathbf{0}$. If $K \neq \mathbf{0}$, then $a_1^2 = 1$, that is $a_1 = \pm 1$ from the first equality of (7). Similarly, if $N \neq \mathbf{0}$, then $a_1^2 = 1$, therefore $a_1 = \pm 1$ from the second equality of (7). So, in both situation, we get $a_1 = \pm 1$.

Now, we shall examine, in great detail, the matrix equalities in Theorem 2.2.

In case $a_1 = 1$, Theorem 2.2 (i) turns into the equality

$$2A_1A_2 + A_1^2 - A_1 = \mathbf{0}. \quad (8)$$

Since $a_1 = 1$, it is obvious that

$$K^2 = -K \text{ and } N^2 = N \tag{9}$$

from the equalities of (4). The equalities of (9) state that K is a skew-idempotent matrix and N is an idempotent matrix. It is seen that studying with diagonal forms of these matrices is enough since the matrices K and I (of A_2) are commutative, and similarly, the matrices N and $\mathbf{0}$ (of A_2) are commutative. For that reason, we shall proceed by taking the diagonal forms of K and N .

The diagonal form of K can be $\mathbf{0}$, $-I$, or $-I \oplus \mathbf{0}$. Similarly, the diagonal form of N can be $\mathbf{0}$, I , or $I \oplus \mathbf{0}$. On the other hand, we know that at least one of the matrices K and N must be nonzero. So, the possible cases for the pair (K, N) are

$$\begin{aligned} (K, N) &\sim (\mathbf{0}, I), \\ (K, N) &\sim (\mathbf{0}, I \oplus \mathbf{0}), \\ (K, N) &\sim (-I, \mathbf{0}), \\ (K, N) &\sim (-I, I), \\ (K, N) &\sim (-I, I \oplus \mathbf{0}), \\ (K, N) &\sim (-I \oplus \mathbf{0}, \mathbf{0}), \\ (K, N) &\sim (-I \oplus \mathbf{0}, I), \\ (K, N) &\sim (-I \oplus \mathbf{0}, I \oplus \mathbf{0}). \end{aligned}$$

If each of these cases together with (1), (3), and (8) are considered, then the following matrix equalities, respectively, are obtained:

$$\begin{aligned} A_1 + A_2 &= I, A_1^2 = A_1, \\ A_1^2 &= A_1, A_1 A_2 = \mathbf{0}, \\ A_1^2 &= -A_1 = A_2, \\ A_1^2 &= I, A_2 = \frac{1}{2} (I - A_1), \\ A_2 &= \frac{1}{2} (A_1^2 - A_1), \\ A_1^2 &= -A_1 = -A_1 A_2, \\ -A_1 A_2 &= \frac{1}{2} (A_1^2 - A_1), \\ -A_1 A_2 &= \frac{1}{2} (A_1^2 - A_1). \end{aligned} \tag{10}$$

Thus, in Theorem 2.2 (i), in case $(a_1, a_2) = (1, 1)$, we get the matrix equalities given above.

Now, we shall consider the case $a_1 = -1$. In this case, we have

$$K^2 = K \text{ and } N^2 = -N \tag{11}$$

from the equalities of (4). The last equalities states that K is an idempotent matrix and N is a skew-idempotent matrix. So, if we consider the fact that at least one of the matrices K and N is nonzero, then all possible diagonal forms of K and N are as in the following:

$$K \sim \mathbf{0} \quad ; N \sim -I,$$

$$\begin{aligned}
 K \sim \mathbf{0} & \quad ; N \sim (-I \oplus \mathbf{0}), \\
 K \sim I & \quad ; N \sim \mathbf{0}, \\
 K \sim I & \quad ; N \sim -I, \\
 K \sim I & \quad ; N \sim (-I \oplus \mathbf{0}), \\
 K \sim (I \oplus \mathbf{0}) & ; N \sim \mathbf{0}, \\
 K \sim (I \oplus \mathbf{0}) & ; N \sim -I, \\
 K \sim (I \oplus \mathbf{0}) & ; N \sim (-I \oplus \mathbf{0}).
 \end{aligned} \tag{12}$$

Considering these cases together with (1), (3), and (8) yield the following matrix equalities, respectively.

$$\begin{aligned}
 -A_1 + A_2 &= I, \quad A_1^2 = -A_1, \\
 A_1^2 &= -A_1, \quad A_1 A_2 = \mathbf{0}, \\
 A_1^2 &= A_1 = A_2, \\
 A_1^2 &= I, \quad A_2 = \frac{1}{2} (I + A_1), \\
 A_2 &= \frac{1}{2} (A_1^2 + A_1), \\
 A_1^2 &= A_1 = A_1 A_2, \\
 A_1 A_2 &= \frac{1}{2} (A_1^2 + A_1), \\
 A_1 A_2 &= \frac{1}{2} (A_1^2 + A_1).
 \end{aligned} \tag{13}$$

Thus, if $(a_1, a_2) = (-1, 1)$ in Theorem 2.2 (i), then we get the matrix equalities given above. Note that the last two equalities of (10) are the same. And, similarly, the last two equalities of (13) are the same.

Now, we shall consider Theorem 2.2 (ii). If the equality

$$2A_1 A_2 = A_2 + \frac{1}{2}(A_1^2 + A_1) \tag{14}$$

is premultiplied by $2A_1$ leads to

$$4A_1^2 A_2 = 2A_1 A_2 + A_1 + A_1^2. \tag{15}$$

If we postmultiply the equality (15) by A_2 , then we arrive at $4A_1^2 A_2 = 2A_1 A_2 + A_1 A_2 + A_1^2 A_2$, that is, $A_1^2 A_2 = A_1 A_2$. From (1) and (3), we obtain $K^2 = K$ since $A_1^2 A_2 = A_1 A_2$. On the other hand, from (1), (3), and (14), it is seen that $K = I$ (because $K^2 = K$) and $N^2 = -N$. Thus, we get

$$A_1 = S \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & N \end{pmatrix} S^{-1}, \quad N^2 = -N. \tag{16}$$

Since the matrix N is a skew-idempotent matrix, all the possible diagonal forms of N are

$$\mathbf{0}, \quad -I, \quad -I \oplus \mathbf{0}. \tag{17}$$

Considering (14) together with (1), (16), and (17) yields the following matrix equalities, respectively.

$$A_1^2 = A_1 = A_2, \quad A_1^2 = I, A_2 = \frac{1}{2} (I + A_1), \quad A_2 = \frac{1}{2} (A_1^2 + A_1). \quad (18)$$

Thus, the item (ii) of Theorem 2.2 is partitioned into the subequalities of (18).

Now, we deal with Theorem 2.2 (iii). If (1) and (3) are substituted into the equality

$$2A_1A_2 = A_2 + 2A_1 - A_1^2, \quad (19)$$

then we get $K^2 = I$ and $N^2 = 2N$. On the other hand, since the matrix N is tripotent, the equalities $N^3 = N$ and $N^2 = 2N$ lead to $N = \mathbf{0}$. Hence, we can write

$$A_1 = S \begin{pmatrix} K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} S^{-1}, \quad K^2 = I. \quad (20)$$

Because of the involutiveness of the matrix K , all the possible diagonal forms of N are

$$I, \quad -I, \quad I \oplus -I. \quad (21)$$

If we consider (19) together with (1), (20), and (21), then we arrive at the following matrix equalities, respectively.

$$A_1^2 = A_1 = A_2, \quad A_1^2 = -A_1 = A_2, \quad A_1^2 = A_2. \quad (22)$$

Notice that the equalities of (22) are the partitioned subversions of the item (iii) of Theorem 2.2.

Now, we consider item (iv) of Theorem 2.2. From the equality

$$2A_1A_2 = -A_2 + \frac{1}{2}(A_1 - A_1^2), \quad (23)$$

we get $4A_1A_2 = -2A_2 + A_1 - A_1^2$. Premultiplying the last equality by A_1 and considering $A_1^3 = A_1$; and similarly, postmultiplying by A_2 on account of $A_2^2 = A_2$ lead to $4A_1^2A_2 = -2A_1A_2 + A_1^2A_2 - A_1A_2$, that is $A_1^2A_2 = -A_1A_2$. Substituting the equalities (1) and (3) into the equality $A_1^2A_2 = -A_1A_2$ yields $K^2 = -K$. On the other hand, by taking into account (23) together with (1), (3), and $K^2 = -K$, we arrive at $K = -I$ and $N^2 = N$. So, we get

$$A_1 = S \begin{pmatrix} -I & \mathbf{0} \\ \mathbf{0} & N \end{pmatrix} S^{-1}, \quad N^2 = N. \quad (24)$$

Since N is an idempotent, all the possible cases are

$$N \sim \mathbf{0}, \quad N \sim I, \quad N \sim (I \oplus \mathbf{0}). \quad (25)$$

If we first put the forms of (25) into (24), and then, consider (1) together with (23), then we get the following matrix equalities, respectively.

$$A_1^2 = -A_1 = A_2, \quad A_1^2 = I, A_2 = \frac{1}{2}(I - A_1), \quad A_2 = \frac{1}{2}(A_1^2 - A_1). \quad (26)$$

The matrix equalities (26) are the partitioned statements of (23).

Finally, we shall consider the item (v) of Theorem 2.2. If the statements of A_1 and A_2 in (3) and (1), respectively, are used in the equality

$$2A_1A_2 = A_1^2 + 2A_1 - A_2, \tag{27}$$

then it is obtained $K^2 = I$ and $N^2 = -2N$. Hence, we get $N = \mathbf{0}$ because of $N^3 = N$. Since K is an involutive matrix, all the possible cases for K are

$$K \sim I, \quad K \sim -I, \quad K \sim (I \oplus -I). \tag{28}$$

Since $N = \mathbf{0}$, from (3), we have

$$A_1 = S \begin{pmatrix} K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} S^{-1}, \quad K^2 = I. \tag{29}$$

If, we first consider statements (28) and (29) together and then use expressions (1) and (27), then we get the following matrix equalities, respectively.

$$A_1^2 = A_1 = A_2, \quad A_1^2 = -A_1 = A_2, \quad A_1^2 = A_2. \tag{30}$$

Note that the equalities of (30) are the partitioned version of the item (v) of Theorem 2.2.

The detailed examination of the results of Theorem 2 in [1] under the condition $A_1A_2 = A_2A_1$ are listed in Table 1.

We notice that there are some missing results in Theorem 2 in [1], and therefore in Table 1. To see one of these missing results, we consider Theorem 2.2 of [2].

Özdemir et al. have established the necessary and sufficient conditions for the idempotency of two nonzero commuting tripotent matrices A_1 and A_2 with $A_1 \neq \pm A_2$ in Theorem 2.2 of [2]. On the other hand, we know that every idempotent matrix is a tripotent matrix. So, if we take $A_2^2 = A_2$ in Theorem 2.2 of [2], then we get the following theorem:

Theorem 2.3. Let $A_1, A_2 \in \mathbb{C}_n$ be a tripotent and an idempotent matrix, respectively, such that $A_1A_2 = A_2A_1$, $A_1 \neq \pm A_2$, and let $A = a_1A_1 + a_2A_2$ where $a_1, a_2 \in \mathbb{C}^*$. Then A is an idempotent matrix if and only if

- (a) $(a_1, a_2) \in \left\{ \left(\frac{1}{2}, \frac{1}{2} \right), \left(-\frac{1}{2}, \frac{1}{2} \right) \right\}$ and $A_1^2 = A_2$,
- (b) $(a_1, a_2) = (-1, -1)$ and $\frac{1}{2}(A_1^2 + A_1) + A_2 + A_1A_2 = \mathbf{0}$,
- (c) $(a_1, a_2) = (1, 1)$ and $\frac{1}{2}(A_1 - A_1^2) = A_1A_2$,
- (d) $(a_1, a_2) = (-1, 1)$ and $\frac{1}{2}(A_1^2 + A_1) = A_1A_2$,
- (e) $(a_1, a_2) = (1, -1)$ and $\frac{1}{2}(A_1^2 - A_1) + A_2 - A_1A_2 = \mathbf{0}$,
- (f) $(a_1, a_2) = (1, 2)$ and $A_2 = \frac{1}{2}(A_1^2 - A_1)$,

(g) $(a_1, a_2) = (-1, 2)$ and $A_2 = \frac{1}{2}(A_1^2 + A_1)$.

Note that in case $A_2^2 = A_2$, the remaining cases of Theorem 2.2 in [2] lead to some contradictions such as $A_1 = \pm A_2$, $A_1 = \mathbf{0}$, or $A_2 = \mathbf{0}$. So, we eliminate these cases in the theorem above. Observe that the item (a) of Theorem 2.3 is an implicit form of the matrix equalities, except for $A_1 = \pm A_2$, in the 5th cell in Table 1. Similarly, the items (c) and (d) of Theorem 2.3 correspond to the matrix equalities, except for $A_1 = \pm A_2$, in the cells 1 and 2, respectively, in Table 1. And finally, the items (f) and (g) of Theorem 2.3 are implicit forms of the matrix equalities, except for $A_1 = \pm A_2$, in the cells 4 and 3, respectively, in Table 1. Notice that the items (b) and (e) in Theorem 2.3 are not seen in Table 1. So, it is seen that these results have been overlooked in Theorem 2 of [1].

Now, let us consider the item (b) of Theorem 2.3. We know that the matrices A_1 and A_2 can be written as in (3) and (1), respectively.

On the other hand, since

$$\frac{1}{2}(A_1^2 + A_1) + A_2 + A_1A_2 = \mathbf{0}, \tag{31}$$

we obtain $A_2 = A_1^2A_2$. If (1) and (3) are substituted into the equality $A_2 = A_1^2A_2$, then we get $K = I$. Also, using the equality (31) together with $K^2 = I$, (1), and (3) yields $K = -I$ and $N^2 = -N$. So, it can be written

$$A_1 = S \begin{pmatrix} -I & \mathbf{0} \\ \mathbf{0} & N \end{pmatrix} S^{-1}, N^2 = -N. \tag{32}$$

Since N is an involutive matrix and $A_1 \neq -A_2$, all the possible cases for N are

$$N \sim -I, \quad N \sim (-I \oplus \mathbf{0}). \tag{33}$$

At first, if the statements of (33) are considered together with (32), and then (1) and (31) are used, then we get the following matrix equalities, respectively.

$$A_1 = -I, \quad A_1A_2 = -A_2, A_1^2 = -A_1. \tag{34}$$

Similarly, let us handle item (e) of Theorem 2.3. Since

$$\frac{1}{2}(A_1^2 - A_1) + A_2 - A_1A_2 = \mathbf{0}, \tag{35}$$

we get $A_1^2A_2 = A_2$. Considering (1), (3), and the equality $A_1^2A_2 = A_2$ leads to $K^2 = I$. And then, if it is used the equality (35) together with $K^2 = I$, (1) and (3), then it is obtained $K = I$ and $N^2 = N$. So, we can write

$$A_1 = S \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & N \end{pmatrix} S^{-1}, N^2 = N. \tag{36}$$

From the idempotency of N and $A_1 \neq A_2$, we have

$$N \sim I, \quad N \sim (I \oplus \mathbf{0}). \quad (37)$$

Considering the statements (37) in view of (36), and then using (1) and (35) yield the following matrix equalities, respectively.

$$A_1 = I, \quad A_1 A_2 = A_2, \quad A_1^2 = A_1 \quad (38)$$

Consequently, (34) and (38) are the partitioned versions of the matrix equalities in the items (b) and (e), respectively, of Theorem 2.3. Notice that while the condition $A_1 \neq \pm A_2$ must hold in Theorem 2.2 of [2], it is not necessary to be $A_1 \neq \pm A_2$ in Theorem 2 of [1].

So, in Theorem 2.2, which is derived from Theorem 2 of [1] in case $A_1 A_2 = A_2 A_1$, it is not considered the condition $A_1 \neq \pm A_2$.

Now, what would happen in cases $A_1 = A_2$ or $A_1 = -A_2$?

Let us find $a_1, a_2 \in \mathbb{C}^*$ such that

$$(a_1 A_1 + a_2 A_2)^2 = a_1 A_1 + a_2 A_2 \quad (39)$$

in case $A_1 = A_2$. Since $A_1 = A_2$, from (39), we get

$$((a_1 + a_2)A_1)^2 = (a_1 + a_2)A_1. \quad (40)$$

Idempotency of A_1 and (40) lead to $(a_1 + a_2)^2 A_1 = (a_1 + a_2)A_1$. From this, we obtain $(a_1 + a_2)(a_1 + a_2 - 1)A_1 = \mathbf{0}$. In view of $A_1 \neq \mathbf{0}$, the last equality yields

$$a_1 + a_2 = 0 \quad \text{or} \quad a_1 + a_2 = 1. \quad (41)$$

In case $A_1 = -A_2$, from (39), we get $((a_2 - a_1)A_2)^2 = (a_2 - a_1)A_2$. From this, considering that $A_2^2 = A_2$, it is obtained $(a_2 - a_1)^2 A_2 = (a_2 - a_1)A_2$. The last equality leads to $(a_2 - a_1)(a_2 - a_1 - 1)A_2 = \mathbf{0}$. Hence, we get

$$a_2 - a_1 = 0 \quad \text{or} \quad a_2 - a_1 = 1 \quad (42)$$

since $A_2 \neq \mathbf{0}$. However, in Theorem 2.2, which is obtained from Theorem 2 of [1], it is not seen the results (41) and (42), while there is no the condition $A_1 \neq \pm A_2$. If it is considered Table 1 together with (34), (38), (41), and (42), then it is arrived at the following result.

Theorem 2.4. Let $A_1, A_2 \in \mathbb{C}_n$ be a tripotent and an idempotent matrix, respectively, such that $A_1 A_2 = A_2 A_1$, and let $A = a_1 A_1 + a_2 A_2$, where $a_1, a_2 \in \mathbb{C}^*$. Then A is an idempotent matrix if and only if any of the following sets of conditions holds:

(a) $(a_1, a_2) = (1, 1)$ and one of the following matrix equalities:

(a1) $A_1 + A_2 = I, \quad A_1^2 = A_1$

(a2) $A_1^2 = A_1, \quad A_1 A_2 = \mathbf{0}$

(a3) $A_1^2 = I, A_2 = \frac{1}{2}(I - A_1)$

(a4) $A_2 = \frac{1}{2}(A_1^2 - A_1)$

(a5) $A_1^2 = -A_1 = -A_1A_2$

(a6) $-A_1A_2 = \frac{1}{2}(A_1^2 - A_1)$

(b) $(a_1, a_2) = (-1, 1)$ and one of the following matrix equalities:

(b1) $-A_1 + A_2 = I, A_1^2 = -A_1$

(b2) $A_1^2 = -A_1, A_1A_2 = \mathbf{0}$

(b3) $A_1^2 = I, A_2 = \frac{1}{2}(I + A_1)$

(b4) $A_2 = \frac{1}{2}(A_1^2 + A_1)$

(b5) $A_1^2 = A_1 = A_1A_2$

(b6) $A_1A_2 = \frac{1}{2}(A_1^2 + A_1)$

(c) $(a_1, a_2) = (-1, -1)$ and one of the following matrix equalities:

(c1) $A_1 = -I$

(c2) $A_1A_2 = -A_2, A_1^2 = -A_1$

(d) $(a_1, a_2) = (1, -1)$ and one of the following matrix equalities:

(d1) $A_1 = I$

(d2) $A_1A_2 = A_2, A_1^2 = A_1$

(e) $(a_1, a_2) = (-1, 2)$ and one of the following matrix equalities:

(e1) $A_1^2 = I, A_2 = \frac{1}{2}(I + A_1)$

(e2) $A_2 = \frac{1}{2}(A_1^2 + A_1)$

(f) $(a_1, a_2) = (1, 2)$ and one of the following matrix equalities:

(f1) $A_1^2 = I, A_2 = \frac{1}{2}(I - A_1)$

(f2) $A_2 = \frac{1}{2}(A_1^2 - A_1)$

(g) $(a_1, a_2) \in \left\{ \left(\frac{1}{2}, \frac{1}{2} \right), \left(-\frac{1}{2}, \frac{1}{2} \right) \right\}$ and $A_1^2 = A_2$

(h) $a_1, a_2 \in \mathbb{C}^*$ with $a_1 + a_2 = 0$ or $a_1 + a_2 = 1$; $A_1 = A_2$

(i) $a_1, a_2 \in \mathbb{C}^*$ with $a_1 - a_2 = 0$ or $a_1 - a_2 = -1$; $A_1 = -A_2$.

Note that there are matrix equalities such as $A_1 = A_2$ or $A_1 = -A_2$ in the each cells of Table 1. However, these matrix equalities are special cases of Theorem 2.4 (h) and (i). So, it is not taken into consideration these type of matrix equalities when it is moved the results in Table 1 to Theorem 2.4.

Baksalary et al. have established all situations for the idempotency of a linear combination of as essentially tripotent matrix and an idempotent matrix in [4]. Now, we shall state that theorem in case the matrices involved in the linear combination are commutative.

If A_1 is an essentially tripotent matrix, there exist two nonzero disjoint idempotent matrices B_1 and B_2 such that $A_1 = B_1 - B_2$. Let A_2 be an idempotent matrix and suppose that $A_1A_2 = A_2A_1$. We know that B_1 and B_2 can be written as

$$B_1 = \frac{1}{2}(A_1^2 + A_1) \text{ and } B_2 = \frac{1}{2}(A_1^2 - A_1). \quad (43)$$

So, the condition $A_1A_2 = A_2A_1$ is equivalent to the equalities $A_2B_1 = B_1A_2$ and $A_2B_2 = B_2A_2$. So, in case $A_1A_2 = A_2A_1$, we shall only considered Theorem 1 (a) in [4]. If the six items of Theorem 1(a) are stated in terms of the matrices A_1 and A_2 in view of (43), then the following result is obtained:

Corollary 2.5. Let A_1 be an essentially tripotent matrix and let A_2 be an idempotent matrix such that $A_1A_2 = A_2A_1$. The linear combination of the form $a_1A_1 + a_2A_2$ is an idempotent matrix if and only if any of the following sets of conditions holds:

- (i) $(a_1, a_2) = (1,1)$ and $A_1A_2 = \frac{1}{2}(A_1 - A_1^2)$,
- (ii) $(a_1, a_2) = (1,2)$ and $A_2 = \frac{1}{2}(A_1^2 - A_1)$,
- (iii) $(a_1, a_2) = (-1,1)$ and $A_1A_2 = \frac{1}{2}(A_1 + A_1^2)$,
- (iv) $(a_1, a_2) = (-1,2)$ and $A_2 = \frac{1}{2}(A_1^2 + A_1)$,
- (v) $(a_1, a_2) = \left\{ \left(\frac{1}{2}, \frac{1}{2} \right), \left(-\frac{1}{2}, \frac{1}{2} \right) \right\}$ and $A_2 = A_1^2$.

Observe that the items (i) and (iii) of Corollary 2.5 correspond to the items (a6) and (b6) in Theorem 2.4, respectively. (Notice that the items (a3) and (a4) are special cases of the item (a6). And also, in the remaining cases of (a), the matrix A_1 is an idempotent or a skew-idempotent matrix, that is, A_1 is not an essentially tripotent matrix. Similarly, (b3) and (b4) in Theorem 2.4 are special cases of (b6). And also, in the remaining cases of (b), the matrix A_1 is an idempotent or a skew-idempotent, that is, A_1 is not an essentially tripotent matrix.)

On the other hand, items (ii) and (iv) of Corollary 2.5 correspond to items (f2) and (e2), respectively, in Theorem 2.4. (Notice that (f1) and (e1) are special cases of (f2) and (e2), respectively). The item (v) of Corollary 2.5 corresponds to the item (g) of Theorem 2.4. Note that since $A_1 = \pm A_2$, that is the tripotent matrix A_1 is particularly an idempotent or a skew-idempotent matrix, in the items (h) and (i), there is no items in Corollary 2.5 corresponding to these items.

Table 1. The summary of the results (10), (13), (18), (22), (26), and (30).

(a_1, a_2)	Matrix Equalities
(1,1)	$A_1 + A_2 = I, A_1^2 = A_1$ $A_1^2 = A_1, A_1 A_2 = \mathbf{0}$ $A_1^2 = -A_1 = A_2$ $A_1^2 = I, A_2 = \frac{1}{2} (I - A_1)$ $A_2 = \frac{1}{2} (A_1^2 - A_1)$ $A_1^2 = -A_1 = -A_1 A_2$ $-A_1 A_2 = \frac{1}{2} (A_1^2 - A_1)$
(-1,1)	$-A_1 + A_2 = I, A_1^2 = -A_1$ $A_1^2 = -A_1, A_1 A_2 = \mathbf{0}$ $A_1^2 = A_1 = A_2$ $A_1^2 = I, A_2 = \frac{1}{2} (I + A_1)$ $A_2 = \frac{1}{2} (A_1^2 + A_1)$ $A_1^2 = A_1 = A_1 A_2$ $A_1 A_2 = \frac{1}{2} (A_1^2 + A_1)$
(-1,2)	$A_1^2 = A_1 = A_2$ $A_1^2 = I, A_2 = \frac{1}{2} (I + A_1)$ $A_2 = \frac{1}{2} (A_1^2 + A_1)$
(1,2)	$A_1^2 = -A_1 = A_2$ $A_1^2 = I, A_2 = \frac{1}{2} (I - A_1)$ $A_2 = \frac{1}{2} (A_1^2 - A_1)$
$\left(\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$	$A_1^2 = A_1 = A_2$ $A_1^2 = -A_1 = A_2$ $A_1^2 = A_2$

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