Düzce University
Journal of Science & Technology

Research Article

Special Pell And Pell Lucas Matrices Of Size 3×3

Fikri KÖKEN

Eregli Kemal Akman Vocational School, Necmettin Erbakan University, Konya, TURKEY
Corresponding author's e-mail address: kokenfikri@gmail.com
DOI: 10.29130/dubited.582108

ABSTRACT

In this study, we study on special Pell and Pell Lucas matrices of size 3×3, entries of whose $n\,th$ powers are related specific Pell and Pell Lucas numbers with indices certain positive integer according to powers these matrices.

Keywords: *Pell numbers, Pell Lucas numbers, Pell matrices*

3x3 Boyutlu Özel Pell Ve Pell Lucas Matrisleri

ÖZET

Bu çalışmada, *3x3* boyutlu matrisler için, matrisin *n.* kuvvetinin elemanları bu matrislerin kuvvetlerine göre belirli pozitif tamsayı indisli Pell ve Pell Lucas sayıları ile ilişkilendirilerek, özel Pell ve Pell Lucas matrisleri üzerinde çalışıldı.

Anahtar Kelimeler: *Pell sayılar, Pell Lucas sayılar, Pell matrisler*

Received: 25/06/2019, Revised: 18/10/2019, Accepted: 22/10/2019

I. INTRODUCTION

Pell numbers P_n and Pell Lucas numbers Q_n satisfy same recurrence relation $X_n = 2X_{n-1} + X_{n-2}$, $n \geq 2$ with different initial conditions: $P_0 = 0$, $P_1 = 1$, $Q_0 = 2$, $Q_1 = 2$. Also, by solving characteristic equation of this recurrence, $x^2 - 2x - 1 = 0$, two distinct characteristic roots can be obtained: $\varphi = 1 + \sqrt{2}$ and $\phi = 1 - \sqrt{2}$, and then, let us recall that the $n\,th$ Pell and Pell Lucas numbers can be defined explicitly by Binet's formulas:

$$P_n = \frac{\varphi^n - \phi^n}{\varphi - \phi},\; Q_n = \varphi^n + \phi^n,\; n \in Z\,. \tag{1}$$

Herein, it is seen that Pell and Pell Lucas numbers with negative subscripts can be derived from positive subscripts in the way that $P_{-n} = (-1)^{n+1} P_n$ and $Q_{-n} = (-1)^n Q_n$, $n > 0$. And also, the numbers $\varphi = 1 + \sqrt{2}$ and $\phi = 1 - \sqrt{2}$ are related to the P_n and P_{n-1} numbers, the powers of φ and ϕ reveal as

$$\varphi^n = P_n\varphi + P_{n-1},\; \phi^n = P_n\phi + P_{n-1}\,. \tag{2}$$

In addition, by some observations from the Pell Lucas recurrence and initial conditions of the Pell Lucas number, it is seen that all of them are even, $Q_n = 2q_n$, and q_n satisfies the same recurrence. That is, q_n is defined by $q_n = 2q_{n-1} + q_{n-2}$ with initial conditions $q_0 = 1$, $q_1 = 1$. All of these numbers are related to each other with the following identities [2-4], [6-8]:

$$P_n + P_{n-1} = q_n,\quad P_n + q_n = P_{n+1},\quad 3P_n - P_{n+1} = q_{n-1}, \tag{3}$$

$$P_{n+1} + P_{n-1} = Q_n,\quad q_{n+1} + q_{n-1} = 4P_n,\quad 3P_n + P_{n-1} = q_{n+1},\quad 3q_n + 4P_n = q_{n+2}\,. \tag{4}$$

The P_n and Q_n numbers are also connected with a integer $n\,th$ powers of certain matrices of size 2×2, 3×3, …, $n\times n$. In [2], the author develops a matrix method for generating the Pell sequence by using the first three elements of the Pell sequence as elements of a symmetric matrix;

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}, \tag{5}$$

and also, in [6], the authors give a Pell Lucas matrix M_Q of size 2×2, entries of which are the first three elements of the Pell Lucas sequence, it holds

$$\begin{pmatrix} Q_{n+1} \\ Q_n \end{pmatrix} = M_Q \begin{pmatrix} P_n \\ P_{n-1} \end{pmatrix} \text{ and } \begin{pmatrix} 8P_{n+1} \\ 8P_n \end{pmatrix} = M_Q \begin{pmatrix} Q_n \\ Q_{n-1} \end{pmatrix}, \tag{6}$$

it is seen that

$$M_Q = 2\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix},\; M_Q^{2n} = 2^{3n}\begin{pmatrix} P_{2n+1} & P_{2n} \\ P_{2n} & P_{2n-1} \end{pmatrix} \text{ and } M_Q^{2n+1} = 2^{3n}\begin{pmatrix} Q_{2n+2} & Q_{2n+1} \\ Q_{2n+1} & Q_{2n} \end{pmatrix}. \tag{7}$$

In addition, in [3], R.S. Melham study the 3×3 matrix $R(p,q)$ for the sequence $\{U_n(p,q)\}_{n=0}^{\infty}$, it is seen the Pell matrix $R_p = R(2,-1)$ in the Pell sequence $\{P_n\}_{n=0}^{\infty} := \{U_n(2,-1)\}_{n=0}^{\infty}$, which contains the first three rows of the Pascal's like triangle on its secondary diagonal and below as follows;

$$R_p^n = \begin{pmatrix} 0\,0\,1 \\ 0\,1\,4 \\ 1\,2\,4 \end{pmatrix}^n = \begin{pmatrix} P_{n-1}^2 & P_{n-1}P_n & P_n^2 \\ 2P_{n-1}P_n & P_n^2 + P_{n-1}P_{n+1} & 2P_nP_{n+1} \\ P_n^2 & P_nP_{n+1} & P_{n+!}^2 \end{pmatrix}. \tag{8}$$

In [7], F. Koken define two matrices P_q and $(P_q - 8I)$, and give representations of the Pell and Pell Lucas matrices; such that P_q^n matrix contain their square powers and multiplications of the Pell and Pell Lucas numbers with indices certain integer; rows or columns of $(P_q - 8I)^n$ matrix involve consecutive elements in the Pell Lucas sequence. And also, for n th powers of these matrices, the Pell and Pell Lucas polynomial expressions with matrix coefficients are given by

$$P_q^n = \begin{cases} 2^{3n-2}\left[\frac{1}{32}P_n q_{n-1}P_q^2 - P_nP_{n-2}P_q + 2P_{n-2}q_{n-1}I\right], & if\ n\ is\ even \\ 2^{3n-2}\left[\frac{1}{32}P_{n-1}q_nP_q^2 - \frac{1}{2}q_nq_{n-2}P_q + 2P_{n-1}q_{n-2}I\right], & otherwise \end{cases}, \tag{9}$$

and

$$\begin{aligned}(P_q - 8I)^n &= 2^{4(n-2)}(P_q - 8I)\left[P_{n-1}(P_q - 8I) + 2^4 P_{n-2}I\right] \\ &= 2^{4(n-2)}\left[P_{n-1}P_q^2 - 2^4 q_{n-2}P_q + 2^6 P_{n-3}I\right].\end{aligned} \tag{10}$$

Our aims are to develop some new ideas for special Pell and Pell Lucas matrices, by taking advantage of the properties of the matrices $(P_q - 8I)$ and P_q, a positive integer $n\,th$ powers of special Pell and Pell Lucas matrices are established by using methods in the matrix theory.

II. SPECIAL PELL AND PELL LUCAS MATRICES

Let us suppose that $A = [a_{ij}]_{3\times 3}$ is any 3×3 matrix such that all eigenvalues of the matrix A are $\lambda_1 = 2^4\varphi$, $\lambda_2 = 0$ and $\lambda_3 = 2^4\phi$, where $\varphi = 1+\sqrt{2}$ and $\phi = 1-\sqrt{2}$, and all entries of the A are integers. And also, let us represent eigenvectors t_i corresponding the eigenvalues λ_i, $i = 1,2,3$ of the matrix A as

$$t_1 = \begin{pmatrix} t_{11} \\ t_{21} \\ t_{31} \end{pmatrix}, t_2 = \begin{pmatrix} t_{12} \\ t_{22} \\ t_{32} \end{pmatrix}, t_3 = \begin{pmatrix} t_{13} \\ t_{23} \\ t_{33} \end{pmatrix}. \tag{11}$$

Now, let a matrix T be transformation matrix of the A, $T = [t_1, t_2, t_3]$, with each column is an eigenvector given in (11) of the A associated with an eigenvalue of the A, and $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$ is the diagonal matrix located in the (i,i) position of eigenvalues of the matrix A. Thus, $A=T\Lambda T^{-1}$ can be written by using $T^{-1}=1/\det(T)adj(T)$, due to three distinct eigenvectors of the matrix A are chosen. That is, the matrix A is diagonalizable $A=T\Lambda T^{-1}$ for an invertible matrix T with having its columns as eigenvectors of the matrix A. Therefore, for all integers $n\geq 1$, we get $A^n=T\Lambda^n T^{-1}$;

$$A^n=T\begin{pmatrix} 2^{4n}\varphi^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^{4n}\phi^n \end{pmatrix}T^{-1}, \text{ where } T^{-1}=\frac{1}{\det(A)}\begin{pmatrix} t_{22}t_{33}-t_{23}t_{32} & t_{13}t_{32}-t_{12}t_{33} & t_{12}t_{23}-t_{13}t_{22} \\ t_{31}t_{23}-t_{21}t_{33} & t_{11}t_{33}-t_{13}t_{31} & t_{21}t_{13}-t_{11}t_{23} \\ t_{21}t_{32}-t_{22}t_{31} & t_{12}t_{31}-t_{11}t_{32} & t_{11}t_{22}-t_{12}t_{21} \end{pmatrix} \tag{12}$$

By using the equations $\varphi^n=P_n\varphi+P_{n-1}$ and $\phi^n=P_n\phi+P_{n-1}$ given in (2), we have

$$\begin{aligned} A^n &= 2^{4n}T\begin{pmatrix} P_n\varphi+P_{n-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_n\phi+P_{n-1} \end{pmatrix}T^{-1} \\ &= 2^{4n-4}\left(P_n\left(T\Lambda T^{-1}\right)+2^4P_{n-1}I-2^4P_{n-1}\begin{pmatrix} 0 & t_{12} & 0 \\ 0 & t_{22} & 0 \\ 0 & t_{32} & 0 \end{pmatrix}T^{-1} \right) \end{aligned} \tag{13}$$

Thus, the matrix A^n is associated with the Pell numbers in the final equation as

$$A^n=2^{4(n-1)}P_nA+2^{4n}P_{n-1}\left(I-F\right), \tag{14}$$

where

$$F=\begin{pmatrix} 0 & t_{12} & 0 \\ 0 & t_{22} & 0 \\ 0 & t_{32} & 0 \end{pmatrix}T^{-1}. \tag{15}$$

Now, we derive some special matrices $A=\left[a_{ij}\right]_{3\times 3}$, which occur with solve of linear equation system $A\,t_i=\lambda_i\,t_i$ for the special cases of the eigenvectors t_1, t_2 and t_3 given in (11).

Firstly, if two eigenvectors t_1 and t_3 given in (11) are chosen such that

$$t_1=\begin{pmatrix} \phi^2 \\ -2\phi \\ 1 \end{pmatrix}, t_3=\begin{pmatrix} \varphi^2 \\ -2\varphi \\ 1 \end{pmatrix}, \tag{16}$$

then, it is necessary to hold the system $At_i=\lambda_i t_i$, $i=1,2,3$. For $i=1,3$, by using $\varphi+\phi=2$, $\varphi-\phi=2\sqrt{2}$ and $\varphi^2+\phi^2=6$, the following equations are achieved as

$$\begin{cases} a_{11}\varphi^2-2a_{12}\varphi+a_{13}=-16\varphi \\ a_{11}\phi^2-2a_{12}\phi+a_{13}=-16\phi \end{cases} \Rightarrow \begin{cases} 3a_{11}-2a_{12}+a_{13}=-16 \\ a_{11}-a_{12}=-8 \end{cases}, \tag{17}$$

$$\begin{cases} a_{21}\varphi^2 - 2a_{22}\varphi + a_{23} = 32 \\ a_{21}\phi^2 - 2a_{22}\phi + a_{23} = 32 \end{cases} \Rightarrow \begin{cases} 3a_{21} - 2a_{22} + a_{23} = 32 \\ a_{21} - a_{22} = 0 \end{cases}, \tag{18}$$

$$\begin{cases} a_{31}\varphi^2 - 2a_{32}\varphi + a_{33} = 16\phi \\ a_{31}\phi^2 - 2a_{32}\phi + a_{33} = 16\varphi \end{cases} \Rightarrow \begin{cases} 3a_{31} - 2a_{32} + a_{33} = 16 \\ a_{31} - a_{32} = -8 \end{cases}. \tag{19}$$

Let us consider different choices of eigenvector t_2 for t_1 and t_3 given in (16), if it is chosen as $t_2 = (-k, -2k, k)^T$, or $t_2 = (k, 2k, -k)^T$ where $k \in \mathrm{Z} \setminus \{0\}$ is arbitrary, then, for $k = 1$, the matrices T_1, T_1^{-1} and F_1 with are obtained from equations (12), (15) and (16), by using $\varphi\phi = -1$, $\varphi - \phi = 2\sqrt{2}$ and $\varphi^2 - \phi^2 = 4\sqrt{2}$ as

$$T_1 = \begin{pmatrix} \phi^2 & -1 & \varphi^2 \\ -2\phi & -2 & -2\varphi \\ 1 & 1 & 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix} T_1^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & 1 \end{pmatrix}, \tag{20}$$

and also, $At_2 = 0$ turns to the system

$$\begin{pmatrix} a_{13} - 2a_{12} - a_{11} \\ a_{23} - 2a_{22} - a_{21} \\ a_{33} - 2a_{32} - a_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{21}$$

From the solutions of the equation systems given in (17), (18), (19) and (21), we get

$$A_1 = \begin{pmatrix} -4 & 4 & 4 \\ 8 & 8 & 24 \\ 4 & 12 & 28 \end{pmatrix}. \tag{22}$$

If we use the equality (14) for the matrix A_1 in (22), then we obtain

$$A_1^n = 2^{4n-2}\begin{pmatrix} -P_n + 3P_{n-1} & P_n - P_{n-1} & P_n + P_{n-1} \\ 2(P_n - P_{n-1}) & 2(P_n + P_{n-1}) & 2(3P_n + P_{n-1}) \\ P_n + P_{n-1} & 3P_n + P_{n-1} & 7P_n + 3P_{n-1} \end{pmatrix}, \tag{23}$$

$$= 2^{4n-2}\begin{pmatrix} q_{n-2} & q_{n-1} & q_n \\ 2q_{n-1} & 2q_n & 2q_{n+1} \\ q_n & q_{n+1} & q_{n+2} \end{pmatrix}, \tag{24}$$

then, we rewrite as

$$A_1^n = 2^{2(2n-1)}\left[\binom{2}{i-1} q_{n+i+j-4}\right]_{3\times 3}. \tag{25}$$

It is seen that matrix A_1^n given in (25) equal the matrix $(P_q - 8I)^n$ given in [7].

Now, let us suppose that $t_2=(-k,2k,-k)^T$ or $t_2=(k,-2k,k)^T$, $k\in Z\setminus\{0\}$, then for $k=1$, matrices T_2, T_2^{-1} and F_2 are obtained from equations (12), (15) and (16) in the way that

$$T_2=\begin{pmatrix}\phi^2 & -1 & \varphi^2\\ -2\phi & 2 & -2\varphi\\ 1 & -1 & 1\end{pmatrix},\quad F_2=\frac{1}{2}\begin{pmatrix}-1 & -1 & 1\\ 2 & 2 & -2\\ -1 & -1 & 1\end{pmatrix} \tag{26}$$

and also, $At_2=0$ turns to the system

$$\begin{pmatrix}2a_{12}-a_{11}-a_{13}\\ 2a_{22}-a_{21}-a_{23}\\ 2a_{32}-a_{31}-a_{33}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}. \tag{27}$$

From the solutions of the equations in (17), (18), (19) and (27), we get

$$A_2=\begin{pmatrix}-8 & 0 & 8\\ 16 & 16 & 16\\ 8 & 16 & 24\end{pmatrix}. \tag{28}$$

Theorem 1. Let A_2 be a matrix given in (28), for all integers $n\geq 1$, then

$$A_2^n=2^{4n-2}\begin{pmatrix}q_{n-2} & P_{n-1} & q_{n-1}\\ 2q_{n-1} & 2P_n & 2q_n\\ q_n & P_{n+1} & q_{n+1}\end{pmatrix}. \tag{29}$$

Proof By using the equality (14) for the matrix A_2 in (28), then we have

$$A_2^n=2^{4n-1}\begin{pmatrix}3P_{n-1}-P_n & P_{n-1} & P_n-P_{n-1}\\ 2(P_n-P_{n-1}) & 2P_n & 2(P_n+P_{n-1})\\ P_n+P_{n-1} & 2P_n+P_{n-1} & 3P_n+P_{n-1}\end{pmatrix}, \tag{30}$$

by using the identities given in (3) and (4) such as, $P_n+P_{n-1}=q_n$, $P_n-P_{n-1}=q_{n-1}$, $3P_{n-1}-P_n=q_{n-2}$ and $3P_n+P_{n-1}=q_{n+1}$, desired result is obtained.

If the eigenvalue $t_2=(k,2k,k)^T$ or $t_2=(-k,-2k,-k)^T$, $k\in Z\setminus\{0\}$ is chosen, then for $k=1$ the matrices T_3 and F_3 are

$$T_3=\begin{pmatrix}\phi^2 & 1 & \varphi^2\\ -2\phi & 2 & -2\varphi\\ 1 & 1 & 1\end{pmatrix},\quad F_3=\frac{1}{2}\begin{pmatrix}1 & 1 & -1\\ 2 & 2 & -2\\ 1 & 1 & -1\end{pmatrix}, \tag{31}$$

and also, $At_2=0$ turns to the system

$$a_{11}+2a_{12}+a_{13}=0,\ a_{21}+2a_{22}+a_{23}=0,\ a_{31}+2a_{32}+a_{33}=0\,. \tag{32}$$

From the solutions of the equations in (17), (18), (19) and (32), we get

$$A_3=\begin{pmatrix}-8 & 0 & 8\\ -16 & -16 & 48\\ -24 & -16 & 56\end{pmatrix}. \tag{33}$$

Theorem 2. Let A_3 be a matrix given in (33), for all integers $n\geq 1$, then

$$A_3^n=2^{4n-1}\begin{pmatrix}-q_{n-1} & -P_{n-1} & q_n\\ -2q_n & -2P_n & 2q_{n+1}\\ -q_{n+1} & -P_{n+1} & q_{n+2}\end{pmatrix}. \tag{34}$$

Proof From the equality (14) for the matrix A_3 in (33), we have

$$A_3^n=2^{4n-1}\begin{pmatrix}P_{n-1}-P_n & -P_{n-1} & P_n+P_{n-1}\\ -2\left(P_n+P_{n-1}\right) & -2P_n & 2\left(3P_n+P_{n-1}\right)\\ -\left(3P_n+P_{n-1}\right) & -2P_n-P_{n-1} & 7P_n+3P_{n-1}\end{pmatrix}. \tag{35}$$

From $P_n+P_{n-1}=q_n$, $P_n-P_{n-1}=q_{n-1}$, $3P_{n-1}-P_n=q_{n-2}$ and $3P_n+P_{n-1}=q_{n+1}$, we obtain desired result.

Secondly, if two eigenvectors t_1 and t_3 are chosen such that

$$t_1=\begin{pmatrix}\phi^2\\ 2\phi\\ -1\end{pmatrix},\ t_3=\begin{pmatrix}\varphi^2\\ 2\varphi\\ -1\end{pmatrix}, \tag{36}$$

then, it is necessary to hold the system $At_i=\lambda_i t_i$, $i=1,2,3$. By using $\varphi+\phi=2$, $\varphi-\phi=2\sqrt{2}$ and $\varphi^2+\phi^2=6$, the following equations are achieved as

$$\begin{cases}a_{11}\varphi^2+2a_{12}\varphi-a_{13}=-16\varphi\\ a_{11}\phi^2+2a_{12}\phi-a_{13}=-16\phi\end{cases}\Rightarrow\begin{cases}3a_{11}+2a_{12}-a_{13}=-16\\ a_{11}+a_{12}=-8\end{cases}, \tag{37}$$

$$\begin{cases}a_{21}\phi^2+2a_{22}\phi-a_{23}=-32\\ a_{21}\varphi^2+2a_{22}\varphi-a_{23}=-32\end{cases}\Rightarrow\begin{cases}3a_{21}+2a_{22}-a_{23}=-32\\ a_{21}+a_{22}=0\end{cases}, \tag{38}$$

$$\begin{cases}g\phi^2+2h\phi-i=-16\varphi\\ g\varphi^2+2h\varphi-i=-16\phi\end{cases}\Rightarrow\begin{cases}3a_{31}+2a_{32}-a_{33}=-16\\ a_{21}+a_{22}=8\end{cases}. \tag{39}$$

Now, let us consider the above similar choices of vector t_2 according to the eigenvectors t_1 and t_3 given in (36) For example, if it is chosen as $t_2=\left(-k,-2k,k\right)^T$ or $t_2=\left(k,2k,-k\right)^T$ where $k\in \mathrm{Z}\setminus\{0\}$ is arbitrary, then for $k=1$, matrices T_4, T_4^{-1} and F_4 are obtained from equations (12), (15) and (36) as

$$T_4 = \begin{pmatrix} \phi^2 & -1 & \varphi^2 \\ 2\phi & -2 & 2\varphi \\ -1 & 1 & -1 \end{pmatrix}, \quad F_4 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ -2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}, \tag{40}$$

and also, $At_2 = 0$ turns to the system

$$a_{13} - 2_{12} - a_{11} = 0,\ a_{23} - 2a_{22} - a_{21} = 0,\ a_{33} - 2a_{32} - a_{31} = 0\ . \tag{41}$$

From the solutions of the equations (37), (38), (39) and (41), we get

$$A_4 = \begin{pmatrix} -8 & 0 & -8 \\ -16 & 16 & 16 \\ -8 & 16 & 24 \end{pmatrix}. \tag{42}$$

Theorem 3. Let A_4 be a matrix given in (42), for all integers $n \geq 1$, then

$$A_4^n = 2^{4n-1}\begin{pmatrix} q_{n-2} & -P_{n-1} & -q_n \\ -2q_{n-1} & P_{n+1} & 2q_n \\ -q_n & P_{n+1} & q_{n+1} \end{pmatrix}. \tag{43}$$

Proof By using the equality (14) for the matrix A_4 in (42), we achieve

$$A_4^n = 2^{4n-1}\begin{pmatrix} 3P_{n-1} - P_n & -P_{n-1} & -P_n - P_{n-1} \\ 2\left(P_{n-1} - P_n\right) & 2P_n + P_{n-1} & 2\left(P_n + P_{n-1}\right) \\ -\left(P_n + P_{n-1}\right) & 2P_n + P_{n-1} & 3P_n + P_{n-1} \end{pmatrix}. \tag{44}$$

The matrix A_4^n given in (43) can be obtained by using the well-known identities, $P_n + P_{n-1} = q_n$, $P_n - P_{n-1} = q_{n-1}$, $3P_{n-1} - P_n = q_{n-2}$ and $3P_n + P_{n-1} = q_{n+1}$ given in (3) and (4).

The eigenvector $t_2 = \left(-k, 2k, -k\right)^T$ or $t_2 = \left(k, -2k, k\right)^T$, $k \in \mathrm{Z} \setminus \{0\}$ is chosen, and matrices T_5 and F_5 are obtained as

$$T_5 = \begin{pmatrix} \phi^2 & -1 & \varphi^2 \\ 2\phi & 2 & 2\varphi \\ -1 & -1 & -1 \end{pmatrix}, \quad F_5 = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & -2 \\ 1 & -1 & 1 \end{pmatrix}, \tag{45}$$

and also, $At_2 = 0$ turns to the system

$$2a_{12} - a_{11} - a_{13} = 0,\ 2a_{22} - a_{21} - a_{23} = 0,\ 2a_{32} - a_{31} - a_{33} = 0\ . \tag{46}$$

From the solutions of the equations (37), (38), (39) and (46), we get

$$A_5 = \begin{pmatrix} -4 & -4 & -4 \\ -8 & 8 & 24 \\ -4 & 12 & 28 \end{pmatrix}. \tag{47}$$

Theorem 4. Let A_5 be a matrix given in (47), for all integers $n \geq 1$, then

$$A_5^n = 2^{4n-2} \begin{pmatrix} q_{n-2} & -q_{n-1} & -q_{n-1} \\ -2q_{n-1} & 2q_n & 2q_{n+1} \\ -q_n & q_{n+1} & q_{n+2} \end{pmatrix}. \tag{48}$$

Proof From the equality (14) for the matrix A in (47), we have

$$A_5^n = 2^{4n-2} \begin{pmatrix} 3P_{n-1} - P_n & P_{n-1} - P_n & P_{n-1} - P_n \\ 2\left(P_{n-1} - P_n\right) & 2\left(P_n + P_{n-1}\right) & 2\left(3P_n + P_{n-1}\right) \\ -\left(P_n + P_{n-1}\right) & 3P_n + P_{n-1} & 7P_n + 3P_{n-1} \end{pmatrix}. \tag{49}$$

The matrix in (48) is derived by $P_n + P_{n-1} = q_n$, $P_n - P_{n-1} = q_{n-1}$, $3P_{n-1} - P_n = q_{n-2}$ and $3P_n + P_{n-1} = q_{n+1}$.

If it is chosen as $t_2 = \left(k, -2k, -k\right)^T$ or $t_2 = \left(-k, 2k, k\right)^T$, then matrices T_6 and F_6 are

$$T_6 = \begin{pmatrix} \phi^2 & 1 & \varphi^2 \\ 2\phi & -2 & 2\varphi \\ -1 & -1 & -1 \end{pmatrix}, \quad F_6 = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ -2 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}, \tag{50}$$

and also, $At_2 = 0$ turns to the system

$$a_{11} - 2a_{12} - a_{13} = 0, \quad a_{21} - 2a_{22} - a_{23} = 0, \quad a_{31} - 2a_{32} - a_{33} = 0. \tag{51}$$

From the solutions of the equations (37), (38), (39) and (51), we get

$$A_6 = \begin{pmatrix} -8 & 0 & -8 \\ 16 & -16 & 48 \\ 24 & -16 & 56 \end{pmatrix}. \tag{52}$$

Theorem 5. Let A_6 be a matrix given in (52), for all integers $n \geq 1$, then

$$A_6^n = 2^{4n-1} \begin{pmatrix} -q_{n-1} & P_{n-1} & -q_n \\ 2q_n & -2P_n & 2q_{n+1} \\ q_{n+1} & -P_{n+1} & q_{n+2} \end{pmatrix}. \tag{53}$$

Proof By using the equality (14) for the matrix A_6 in (52), we obtain

$$A_6^n = 2^{4n-1} \begin{pmatrix} P_{n-1} - P_n & P_{n-1} & -P_n - P_{n-1} \\ 2\left(P_n + P_{n-1}\right) & -2P_n & 2\left(3P_n + P_{n-1}\right) \\ 3P_n + P_{n-1} & -2P_n - P_{n-1} & 7P_n + 3P_{n-1} \end{pmatrix}. \tag{54}$$

The matrix given in (53) is obtained by using $P_n+P_{n-1}=q_n$, $P_n-P_{n-1}=q_{n-1}$ and $3P_n+P_{n-1}=q_{n+1}$.

Thirdly, if two eigenvectors t_1 and t_3 are chosen such that

$$t_1=\begin{pmatrix}-\phi^2\\ 2\phi\\ 1\end{pmatrix},\ t_3=\begin{pmatrix}-\varphi^2\\ 2\varphi\\ 1\end{pmatrix}, \tag{55}$$

then, it is necessary to hold the system $At_i=\lambda_i t_i$. By using $\varphi+\phi=2$, $\varphi-\phi=2\sqrt{2}$ and $\varphi^2+\phi^2=6$, the following equations are established as

$$\begin{cases}-a_{11}\varphi^2+2a_{12}\varphi+a_{13}=16\varphi\\ -a_{11}\phi^2+2a_{12}\phi+a_{13}=16\phi\end{cases}\Rightarrow\begin{cases}-3a_{11}+2a_{12}+a_{13}=16\\ -a_{11}+a_{12}=8\end{cases}, \tag{56}$$

$$\begin{cases}-a_{21}\varphi^2+2a_{22}\varphi+a_{23}=-32\\ -a_{21}\phi^2+2a_{22}\phi+a_{23}=-32\end{cases}\Rightarrow\begin{cases}-3a_{21}+2a_{22}+a_{23}=-32\\ -a_{21}+a_{22}=0\end{cases}, \tag{57}$$

$$\begin{cases}-a_{31}\varphi^2+2a_{32}\varphi+a_{33}=16\phi\\ -a_{31}\phi^2+2a_{32}\phi+a_{33}=16\varphi\end{cases}\Rightarrow\begin{cases}-3a_{31}+2a_{32}+a_{33}=16\\ -a_{31}+a_{32}=-8\end{cases}. \tag{58}$$

The cases are omitted for sake of brevity, but they can be carried out by means of analogous arguments such that if $t_2=(-k,-2k,k)^T$ or $t_2=(k,2k,-k)^T$ is chosen, where $k\in \mathrm{Z}\setminus\{0\}$ is arbitrary, then T_7 and F_7 matrices are obtained as

$$T_7=\begin{pmatrix}-\phi^2 & -1 & -\varphi^2\\ 2\phi & -2 & 2\varphi\\ 1 & 1 & 1\end{pmatrix},\quad F_7=\frac{1}{2}\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ -1 & -1 & -1\end{pmatrix}. \tag{59}$$

From the solutions of the equations (56), (57), (58) and system $At_2=0$, we get

$$A_7=\begin{pmatrix}-8 & 0 & -8\\ -16 & -16 & -48\\ 24 & 16 & 56\end{pmatrix},\quad A_7^n=2^{4n-1}\begin{pmatrix}-q_{n-1} & -P_{n-1} & -q_n\\ -2q_n & -2P_n & -2q_{n+1}\\ q_{n+1} & P_{n+1} & q_{n+2}\end{pmatrix}. \tag{60}$$

If $t_2=(k,-2k,-k)^T$ or $t_2=(-k,2k,k)^T$ is chosen, then T_8 and F_8 matrices are

$$T_8=\begin{pmatrix}-\phi^2 & 1 & -\varphi^2\\ 2\phi & -2 & 2\varphi\\ 1 & -1 & 1\end{pmatrix},\quad F_8=\frac{1}{2}\begin{pmatrix}-1 & -1 & -1\\ 2 & 2 & 2\\ 1 & 1 & 1\end{pmatrix}. \tag{61}$$

From the solutions of the equations (56), (57), (58) and system $At_2=0$, we get

$$A_8 = \begin{pmatrix} -8 & 0 & -8 \\ 16 & 16 & -16 \\ -8 & -16 & 24 \end{pmatrix}, \quad A_8^n = 2^{4n-1} \begin{pmatrix} q_{n-2} & P_{n-1} & -q_{n-1} \\ 2q_{n-1} & 2P_n & -2q_n \\ -q_n & -P_{n+1} & q_{n+1} \end{pmatrix}. \tag{62}$$

If $t_2 = (k, 2k, k)^T$ or $t_2 = (k, 2k, k)^T$ is chosen, then T_9 and F_9 matrices are

$$T_9 = \begin{pmatrix} -\phi^2 & 1 & -\varphi^2 \\ 2\phi & 2 & 2\varphi \\ 1 & 1 & 1 \end{pmatrix}, \quad F_9 = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}. \tag{63}$$

From the solutions of the equations (56), (57), (58) and system $At_2 = 0$, we get

$$A_9 = \begin{pmatrix} -4 & 4 & -4 \\ 8 & 8 & -24 \\ -4 & -12 & 28 \end{pmatrix}, \quad A_9^n = 2^{4n-2} \begin{pmatrix} q_{n-2} & q_{n-1} & -q_n \\ 2q_{n-1} & 2q_n & -2q_{n+1} \\ -q_n & -q_{n+1} & q_{n+2} \end{pmatrix}. \tag{64}$$

Also, based on the other values given for the eigenvectors t_2 except those used, similar and related results can be written by using this way. The other cases, except $k = 1$ used, are omitted for sake of brevity, but they can be carried out by means of analogous arguments.

III. CONCLUSION

Special matrices of 3×3 dimensions, whose $n\,th$ powers are related to the $n\,th$ Pell and Pell Lucas numbers, are derived by using methods in the matrix theory, according to the properties of the matrices $(P_q - 8I)$ and P_q given in [7]. The matrix A_i, $i = 1 - 9$, are Pell and Pell Lucas matrices, have been found via the equation

$$A_i^n = 2^{4(n-1)} P_n A_i + 2^{4n} P_{n-1} (I - F_i). \tag{65}$$

These matrices are important in terms of their role in the study related to the Pell and Pell Lucas numbers, since these subjects for the Fibonacci and Lucas numbers have been studied from polynomial sequences to quaternions in the literature [1,5].

IV. REFERENCES

[1] B. D. Bitim, "Some Identities of Fibonacci and Lucas Quaternions by Quaternion Matrices", *Düzce University Journal of Science & Technology*, vol.7, no.1, pp.606–615, 2019.

[2] J. Ercolano, "Matrix generator of Pell sequence", *The Fibonacci Quarterly*, vol.17, no.1, pp. 71–77, 1979.

[3] R.S. Melham, "*Lucas Sequences and Functions of a 3-by-3 Matrix*", The Fibonacci Quarterly, 37.2, 111–116, 1999.

[4] A. F. Horadam, “Pell identities”, *The Fibonacci Quarterly*, vol.9, no.3, pp. 245–252, 263, 1971.

[5] S. Karakaya, H. Özdemir, T. Petik, “3×3 Dimensional Special Matrices Associated with Fibonacci and Lucas Numbers”, *Sakarya University Journal of Science*, vol.22, no.6, 1917–1922, 2018.

[6] F. Koken ve S. Arslan, “Pell ve Pell Lucas sayılarının Özellikleri”, XXIII. Ulusal Matematik Sempozyumu, Kayseri,Türkiye, ss:85, 2010.

[7] F. Koken, “The Representations of the Pell and Pell Lucas Matrices”, International Marmara Science And Social Sciences Congress, Kocaeli, Türkiye, Proceedings Book I (Science), ss: 466–473, 2019.

[8] T. Koshy, *Pell and Pell-Lucas numbers with application*, Springer New York Heidelberg Dordrecht London, pp. 115–172, 2014.