Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 478 – 493
DOI : 10.15672/hujms.475318

Research Article

Robust confidence intervals for the difference of two independent population variances

Hayriye Esra Akyüz*[1], Hamza Gamgam[2]

[1] *Bitlis Eren University, Faculty of Science and Arts, Department of Statistics, Bitlis, Turkey*
[2] *Gazi University, Faculty of Science, Department of Statistics, Ankara, Turkey*

Abstract

In this study, we propose confidence intervals and their bootstrap versions for the difference of variances of two independent population using some robust variance estimators. The proposed confidence intervals are compared with Herbert confidence interval in terms of coverage probability and average width. A simulation study is conducted to evaluate performances of the proposed confidence intervals under different scenarios. The simulation results indicate that the coverage probabilities for the proposed confidence intervals are very close to nominal confidence levels when the difference of population variances is zero. Confidence interval based on binary distance produces the narrowest average widths. Herbert confidence interval have not perform well for skewed distribution populations. Confidence interval based on comedian is generally recommended when the difference of population variances for skewed distributions is not zero. Average widths of bootstrap percentile confidence intervals are smaller, and decreases as sample size and nominal size increases, as expected. Consequently, we recommend bootstrap percentile confidence interval based on binary distances for skewed distributions.

Mathematics Subject Classification (2010). 62F25, 62F35, 62F40

Keywords. average width, bootstrap, confidence interval, coverage probability, independent populations, robust estimator

1. Introduction

Confidence intervals for the ratio of population variances are well known for the case where the populations are normally distributed. Although interpretation of variance differences in randomized and clinical trials is critical, few statistical methods consider the variance differences. It is very important to examine variance differences between experimental and control groups in randomized experiments. Information about the variance of effects of intervention may be of substantive interest. For example, the effect of change in clinical trials allows the study of different factors in an investigation. Variability in intervention effects in health care can be explained by interpretation of variance differences. Interpreting of this difference with interval estimation provides a more realistic view of population parameters than point estimates [13], since it provides the desired information

*Corresponding Author.
Email addresses: heakyuz@beu.edu.tr (H.E. Akyüz), gamgam@gazi.edu.tr (H. Gamgam)
Received: 26.10.2018; Accepted: 09.01.2020

within a range, and each statement is based on an assumption that is unquestionably true. This forms the basis for robust statistics, allows to be checked assumptions and provides a bridge between predicted models and real-life data.

The sample variance S^2 is the maximum likelihood estimation for the population variance, and it is widely used to estimation normal population variances. The variance distribution converges to the asymptotically normal distribution [20]. The Chi-square distribution is a special form of Gamma distribution with parameters $\alpha = (n-1)/2$ and $\beta = 2$. When the sample sizes and variances of two normal populations are equal, the difference of two random Gamma variables with parameters (α, β) have McKay Type II distribution with parameters $a = (\alpha - 0.5)$, $b = \beta^2$, and $c = 0$, which is close to normal even when the sample size is too small [20].

Nevertheless, S^2 is not robust for estimation of non-normal population variance and coverage probabilities of confidence interval for this estimators are much smaller than nominal confidence levels [13,26]. In such cases, it is essential to use robust scale estimators to estimate population variance.

Barham and Jeyeratnam [7] proposed robust confidence intervals based on the L-estimators. Cojbasica and Tomovica [14] obtained nonparametric confidence intervals based on the bootstrap method for variance differences of exponential distribution. Cojbasica and Loncar [15] determined one-sided confidence intervals for population variance of skewed distributions. Abu-Shawiesh et al. [1] conducted a simulation study for confidence interval of population standard deviation. Niwitpong [21] studied the generalized and closed form confidence intervals for normally distributed population. Niwitpong [22] proposed an analytical definition of coverage probabilities (CP) and average widths (AW) of closed form confidence intervals and compared with confidence intervals suggested by Niwitpong [21]. Burch [9] proposed a nonparametric bootstrap confidence interval for variance components. Suwan and Niwitpong [27] studied interval estimation methods for a linear function of variances of non-normal populations using the kurtosis coefficient. Burch [10] considered asymptotic variance of the natural logarithm of sample variance to construct approximate confidence intervals for population variance. Akyüz et al. [4] focused on interval estimation with sample variance estimators based on winsorized and trimmed mean for the differences of non-normal population variances. Burch [11] obtained confidence intervals based on a power transformation of the sample variance. Thangjai and Niwitpong [28] proposed the simultaneous confidence intervals, based on generalized confidence interval approach and simulation-based approach, for all differences of variances of log-normal distributions. Herbert et al. [20] obtained confidence intervals based on the sample variance difference estimator for variance differences of normal populations. Skewed distributions are very common in applications. Therefore;we consider the squares of some scale estimators to estimate population variance.

The focus of this article is on calculating confidence intervals, because methods for interval estimation of the variances difference based on robust estimators have not previously been described. Robust estimators were used to estimate the difference of population variances based on binary distance (Q_n^2), median binary distance (S_n^2), and comedian (COM) which is equal to the square of median absolute deviation from the median (MAD). We proposed confidence intervals for the difference of variances of two independent populations based on three robust estimators, and compared these and their bootstrap percentile versions with Herbert confidence intervals in terms of the CP and AW. The AWs were obtained by dividing the total differences of the lower limit and upper limits found for each replication to the number of replications. Also, the CPs were determined as the proportion of cases where the variance difference was between the lower and upper interval limit [3].

The rest of this article is organized as follows. In Section 2, the Herbert confidence interval is presented for normal population variance differences. The existing robust variance estimators are presented in Section 3. Section 4 describes the proposed and bootstrap percentile confidence intervals, and Section 5 details a Monte Carlo simulation study. The study findings are summarized in Section 6. Section 7 presents real-life data set examples. Finally, Section 8 summarizes and concludes the paper.

2. Herbert confidence interval

Herbert et al. [20] estimated $\delta = \sigma_A^2 - \sigma_B^2$, where σ^2 is population variance and subscripts A and B denote independent populations. The natural unbiased estimator of δ is $D = S_A^2 - S_B^2$, where S_A^2 and S_B^2 are the sample variances. The sampling distribution variance of S^2 can be expressed as:

$$\mathrm{Var}(S^2) = \frac{1}{n}\left\{\eta - \left(\frac{n-3}{n-1}\right)\sigma^4\right\} \tag{2.1}$$

where η is the fourth central moment and n is sample size [23]. Thus, the sampling variance of the difference between independent population A and B variances can be expressed as follows:

$$\begin{aligned}\mathrm{Var}(D) &= \mathrm{Var}(S_A^2 - S_B^2) = \mathrm{Var}(S_A^2) + \mathrm{Var}(S_B^2)\\ &= \frac{1}{n_A}\left\{\eta_A - \left(\frac{n_A-3}{n_A-1}\right)\sigma_A^4\right\} + \frac{1}{n_B}\left\{\eta_B - \left(\frac{n_B-3}{n_B-1}\right)\sigma_B^4\right\}\end{aligned} \tag{2.2}$$

which is more conveniently expressed in terms of the standardized fourth central moment. For $\gamma = \frac{\eta}{\sigma^4}$, it can be obtained as:

$$\begin{aligned}\mathrm{Var}(D) &= \frac{1}{n_A}\left\{\gamma_A\sigma_A^4 - \left(\frac{n_A-3}{n_A-1}\right)\sigma_A^4\right\} + \frac{1}{n_B}\left\{\gamma_B\sigma_B^4 - \left(\frac{n_B-3}{n_B-1}\right)\sigma_B^4\right\}\\ &= \sigma_A^4\left\{\frac{\gamma_A}{n_A} - \left[\frac{n_A-3}{n_A(n_A-1)}\right]\right\} + \sigma_B^4\left\{\frac{\gamma_B}{n_B} - \left[\frac{n_B-3}{n_B(n_B-1)}\right]\right\},\end{aligned} \tag{2.3}$$

where n_A and n_B defines the sample sizes of independent populations A and B, respectively. This expression does not depend on any distributional assumptions, so the values of parameters are unknown and must be estimated from sample data. Herbert et al. [20] used the estimator G proposed by Bonett [8] to estimate population kurtosis coefficient. It is defined as:

$$G = (n_A + n_B)\frac{\left\{\sum\left(y_{i(B)} - m'_{(B)}\right)^4 + \sum\left(y_{i(A)} - m'_{(A)}\right)^4\right\}}{\left\{\sum\left(y_{i(B)} - m_{(B)}\right)^2 + \sum\left(y_{i(A)} - m_{(A)}\right)^2\right\}^2} \tag{2.4}$$

where m is the sample mean, and m' is the trimmed mean with trimmed proportion $0.5/\sqrt{(n-4)}$. $y_i(A)$ and $y_i(B)$ is the observed values of independent populations A and B with sample sizes n_A and n_B , respectively. When the estimator G is used to estimate the kurtosis coefficient, the sampling variance of variance differences for independent populations is obtained as follows:

$$\widehat{\mathrm{Var}}(D) = S_A^4\left\{\frac{G}{n_A} - \frac{n_A-3}{n_A(n_A-1)}\right\} + S_B^4\left\{\frac{G}{n_B} - \frac{n_B-3}{n_B(n_B-1)}\right\} \tag{2.5}$$

When the observations are normally distributed, variance differences are also nearly normal [2,20]. Also, it may be reasonable to generate variance difference confidence intervals by assuming an approximate normal distribution. Thus, the confidence interval for the difference of variances of two independent populations can be expressed as:

$$P\left(D - z_{1-\alpha/2}\sqrt{S_A^4\left\{\frac{G}{n_A} - \frac{n_A-3}{n_A(n_A-1)}\right\} + S_B^4\left\{\frac{G}{n_B} - \frac{n_B-3}{n_B(n_B-1)}\right\}} \le \sigma_A^2 - \sigma_B^2 \le D \right.$$
$$\left. + z_{1-\alpha/2}\sqrt{S_A^4\left\{\frac{G}{n_A} - \frac{n_A-3}{n_A(n_A-1)}\right\} + S_B^4\left\{\frac{G}{n_B} - \frac{n_B-3}{n_B(n_B-1)}\right\}}\right) = 1-\alpha \tag{2.6}$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$th quantile of the standard normal distribution [10].

3. The robust estimators of the variance

In this section, we consider variance estimators based on binary distance, median binary distance and MAD. Also, it is included simulation results for the distributions of these estimators.

The estimator Q_n^2 for a random sample $Y_1, Y_2, ..., Y_n$ with model distribution F can be expressed as [1]:

$$Q_n^2 = (d_n)^2 \times \left(2.2219(|y_i - y_j|)_{(g)}\right)^2, \tag{3.1}$$

where $i<j;\ \ i = 1,2,...,n;\ \ j = 1,2,...,n; g = \binom{h}{2}$ and $h = [\frac{n}{2}] + 1$. Also, $[.]$ expression indicates the largest integer function. Thus, Q_n^2 is the square of the g-th order statistic of the $\binom{n}{2}$ interpoint distance [1].

The estimator S_n^2, which is an appropriate estimator for asymmetric distributions, can be expressed as:

$$S_n^2 = (c_n)^2 \times (1.1926\mathrm{med_i}\,(\mathrm{med_j}\,|y_i - y_j|))^2, i = 1,2,...,n;\ \ j = 1,2,...,n \tag{3.2}$$

where med is the sample median [25]. Comedian is a robust estimator for population variance [19].

$$\mathrm{COM}\,(Y,Y) = \mathrm{MAD}^2(Y). \tag{3.3}$$

It can be defined as $\mathrm{MAD}^2(Y) = \mathrm{COM} = (b_n)^2 \times (1.4826\mathrm{med_i}\,(|y_i - \mathrm{med_j} y_j|))^2$. It is known that the comedian is used as a robust estimator for the covariance. When X=Y for the independent random samples X_i and Y_i, i=1,2,...,n , it is a robust estimator for variance.

The values for constants d_n, c_n and b_n are given by Croux ve Rousseeuw [16]. The constants d_n, c_n and b_n are unbiasing factor so that they become an unbiased estimator of standard deviation for the Normal distribution.

A simulation study was conducted in MATLAB R-2018b to determine of distributions of estimators Q_n^2, S_n^2, and COM under 10,000 replications, $\alpha = 0.05$, and sample sizes $n = 10, 20, 50, 100$. Random samples were generated from a normal distribution. Shapiro-Wilk goodness of fit test was used to determine whether estimators complied with Gamma or Weibull distributions. stats::swGOFT function was used for this purpose. Also, it was obtained the average p-values for each sample. According to simulation results; it is seen that histograms of distribution of estimators resemble Gamma and Weibull distributions in various situations. However, it is essential to use the results of goodness of fit test to identify the distribution. These results are as in Table 1.

Table 1. Average p-values for the Shapiro-Wilk goodness of fit test for robust estimators

		p-values	
Estimator	Sample size(n)	Gamma	Weibull
Q_n^2	10	0.3055	0.1726
	20	0.3137	0.1710
	50	0.3110	0.1701
	100	0.3110	0.1700
S_n^2	10	0.4452	0.0107
	20	0.4047	0.0100
	50	0.4062	0.0101
	100	0.4060	0.0101
COM	10	0.5299	0.0487
	20	0.5045	0.0423
	50	0.5000	0.0429
	100	0.5000	0.0430

Table 1 shows the average p-values for 10,000 replications. When Table 1 is examined, it is seen that all estimators conform to the Gamma distribution for $\alpha = 0.05(p-\text{value} > \alpha)$. S_n^2 and COM estimators do not conform to the Weibull distribution ($p-$value $< \alpha$). Also, the average p-values based on Gamma distribution for the estimator Q_n^2 were higher than those of the Weibull distribution. Therefore, it was obtained that the estimators conform to the Gamma distribution.

4. Confidence intervals for the difference of variances of two independent populations

The proposed confidence intervals and their bootstrap versions for the difference of variances of two independent populations are defined in the following sections.

4.1. Proposed confidence intervals

While the sample standard deviation had the highest efficiency in the normal distribution, the efficiency in the skewed distribution was very poor. In such cases, it is known that the variance, the square of the standard deviation, is not a robust estimator. Since population variance was studied in this study, the squares of the scale estimators were calculated. Because, the scale estimators are robust estimators for population standard deviation with certain unbiased coefficients. It is known in the literature that they are efficiency estimators, although these estimators are biased estimators for population variance [1]. We propose robust confidence intervals based on estimators $(Q_{n_1}^2 - Q_{n_2}^2)$, $(S_{n_1}^2 - S_{n_2}^2)$ and $(COM_1 - COM_2)$ for the variance difference of independent populations.

Section 3 showed that sampling distributions of estimators Q_n^2, S_n^2, and COM conform to the Gamma distribution. Thus, the estimators are distributed as the difference between two independent Gamma random variables with parameters (α, β) which have McKay Type II distributions with parameters $a = \alpha - 0.5$, $b = \beta^2$, and $c = 0$. Consider a random sample $Y_1, Y_2, ..., Y_n$ of size n, then the McKay Type II probability density function is defined as [20]:

$$f_Y(y) = \frac{(1-c^2)^{a+0.5}|y|^a}{\sqrt{\pi}2^a b^{a+1}\Gamma(a+0.5)} e^{-yc/b} K_\alpha\left(\frac{|y|}{b}\right)\Bigg| y \neq 0, \tag{4.1}$$

$$f_Y(y) = \frac{(1-c^2)^{a+0.5}\Gamma(a)}{\sqrt{\pi}2b\Gamma(a+0.5)}\Bigg| y = 0. \tag{4.2}$$

where K_α is the modified Bessel function of the second kind of order α [20]. The McKay Type II distribution is very nearly normal, even when sample size is very small.

Thus, we propose the confidence interval based on estimator $(Q^2_{n_1} - Q^2_{n_2})$ with constant d_n as follows:

$$P\Big((d_n)^2 (Q^2_{n_1} - Q^2_{n_2}) - z_{\alpha/2}(d_n)^2\sqrt{\mathrm{Var}\,(Q^2_{n_1} - Q^2_{n_2})} \le \sigma_1^2 - \sigma_2^2 \le (d_n)^2 (Q^2_{n_1} - Q^2_{n_2}) + z_{\alpha/2}(d_n)^2\sqrt{\mathrm{Var}\,(Q^2_{n_1} - Q^2_{n_2})}\Big) = 1-\alpha \tag{4.3}$$

where

$$\left(Q^2_{n_1} - Q^2_{n_2}\right) = \left\{2.2219^2\left(\left(|y_{1i}-y_{1j}|_{(g)}\right)^2 - \left(|y_{2i}-y_{2j}|_{(g)}\right)^2\right)\right\}, \tag{4.4}$$

and $z_{\alpha/2}$ is the $(1-\alpha/2)$th quantile of the standard normal distribution. d_n is unbiasing factor.

Similarly, confidence intervals for the estimators $(S^2_{n_1} - S^2_{n_2})$ and $(COM_1 - COM_2)$ with constants c_n and b_n are as:

$$P\Big((c_n)^2 (S^2_{n_1} - S^2_{n_2}) - z_{\alpha/2}(c_n)^2\sqrt{\mathrm{Var}\,(S^2_{n_1} - S^2_{n_2})} \le \sigma_1^2 - \sigma_2^2 \le (c_n)^2 (S^2_{n_1} - S^2_{n_2}) + z_{\alpha/2}(c_n)^2\sqrt{\mathrm{Var}\,(S^2_{n_1} - S^2_{n_2})}\Big) = 1-\alpha, \tag{4.5}$$

and

$$P\Big((b_n)^2 (COM_1 - COM_2) - z_{\alpha/2}(b_n)^2\sqrt{\mathrm{Var}\,(COM_1 - COM_2)} \le \sigma_1^2 - \sigma_2^2 \le (b_n)^2 (COM_1 - COM_2) + z_{\alpha/2}(b_n)^2\sqrt{\mathrm{Var}\,(COM_1 - COM_2)}\Big) = 1-\alpha, \tag{4.6}$$

where

$$\left(S^2_{n_1} - S^2_{n_2}\right) = \left\{1,1926^2\left(\left(\mathrm{med_{1i}}(\mathrm{med_{1j}}\,|y_{1i}-y_{1j}|)^2\right) - \left(\mathrm{med_{2i}}(\mathrm{med_{2j}}\,|y_{2i}-y_{2j}|)^2\right)\right)\right\} \tag{4.7}$$

and

$$(COM_1 - COM_2) = \left\{1,4826^2\left((\mathrm{med_{1i}}\,(|y_{1i} - \mathrm{med_{1j}}y_{1j}|))^2 - (\mathrm{med_{2i}}\,(|y_{2i} - \mathrm{med_{2j}}y_{2j}|))^2\right)\right\}. \tag{4.8}$$

Analytical expressions for $\widehat{\mathrm{Var}}(Q^2_{n_1} - Q^2_{n_2})$, $\widehat{\mathrm{Var}}(S^2_{n_1} - S^2_{n_2})$ and $\widehat{\mathrm{Var}}(COM_1 - COM_2)$ are not available in the literature, and are difficult to calculate. Thus, the estimated variances were obtained using Monte Carlo simulation or bootstrap methods [4, 12].

If the difference estimator that is obtained in the i^{th} replication of the T repeated simulation with sample data of size n is $D_i = (Q^2_{n_1} - Q^2_{n_2})$, $i = 1, 2, ..., T$, the estimated variance with Monte Carlo simulation for $Var(Q^2_{n_1} - Q^2_{n_2})$ can be expressed as:

$$\widehat{\mathrm{Var}}\left(Q^2_{n_1} - Q^2_{n_2}\right) = \frac{\sum_{i=1}^{T}\left(D_i - \bar{D}\right)^2}{T-1} \tag{4.9}$$

where the $\bar{D}$ is the mean of the differences $D_i = (Q^2_{n_1} - Q^2_{n_2})$, $i = 1, 2, ..., T$.

Bootstrapping has been applied widely in statistics when analytical derivations of the distribution of an estimator are intractable, and it can be found to generate distributions close to the underlying true distributions. If analytical expressions for some parameter estimators or statistical properties of these estimators are not available, estimated values of these estimators can be obtained by bootstrapping [6,29]. Variance may also be estimated with the bootstrap method [17].

Bootstrap samples of size n are generated by simple random sampling with replacement, and obtained the estimated variance for $(Q^2_{n_1} - Q^2_{n_2})$. Thus, the bootstrap estimator for variance of $(Q^2_{n_1} - Q^2_{n_2})$ is as:

$$\widehat{\mathrm{Var}}\left(Q^2_{n_1} - Q^2_{n_2}\right) = \frac{\sum_{i=1}^{B}\left(D_i - \bar{D}\right)^2}{B-1} \tag{4.10}$$

where B and D_i, i=1,2,..., B are the bootstrap replication and the estimated values $(Q^2_{n_1} - Q^2_{n_2})$, respectively.

Similarly, estimated variances for $\widehat{\mathrm{Var}}(S^2_{n_1} - S^2_{n_2})$ and $\widehat{\mathrm{Var}}(COM_1 - COM_2)$ are obtained by Monte Carlo and bootstrap methods.

Table 2. Variance estimation values for the robust estimators

		Simulation method	
Variance estimation	Sample size (n)	Monte Carlo	Bootstrap
$\widehat{\mathrm{Var}}(Q^2_{n_1} - Q^2_{n_2})$	10	0.7066	0.7195
	20	0.1081	0.0998
	50	0.0465	0.0460
	100	0.0320	0.0300
$\widehat{\mathrm{Var}}(S^2_{n_1} - S^2_{n_2})$	10	2.4812	2.4569
	20	2.0625	2.0045
	50	1.2695	1.2598
	100	0.8991	0.8001
$\widehat{\mathrm{Var}}(COM_1 - COM_2)$	10	1.4974	1.4550
	20	0.7570	0.7571
	50	0.5391	0.5299
	100	0.2658	0.2514

Table 2 shows estimated variances for $\widehat{\mathrm{Var}}(Q^2_{n_1} - Q^2_{n_2})$, $\widehat{\mathrm{Var}}(S^2_{n_1} - S^2_{n_2})$ and $\widehat{\mathrm{Var}}(COM_1 - COM_2)$ where the populations were assumed to be normal distributed, based on 10,000 replications by Monte Carlo simulation and bootstrap. The two methods yield similar results for all sample sizes.

4.2. Bootstrap percentile confidence interval

The bootstrap percentile confidence interval is based on the percentile of the distribution of bootstrap replications, which includes the percentile and adjusted percentile intervals. Here, we only consider percentile type confidence intervals because they are compatible with the transformations [6]. Theoretically, this method has better coverage probability than the bootstrap-t method [18]. The lower and upper limits of the confidence interval are obtained as follows:

1. Calculate estimators $(Q^2_{n_1} - Q^2_{n_2})$, $(S^2_{n_1} - S^2_{n_2})$, and $(COM_1 - COM_2)$ based on random samples $\{x_1, x_2, ..., x_{n1}\}$ and $\{y_1, y_2, ..., y_{n2}\}$ for sizes n_1 and n_2 .

2. Obtain bootstrap samples $x^{*b} = \left\{x_1^{*b}, x_2^{*b}, ..., x_{n1}^{*b}\right\}$ and $y^{*b} = \left\{y_1^{*b}, y_2^{*b}, ..., y_{n2}^{*b}\right\}$ of sample sizes n_1 and n_2 with simple random sampling under replacement.

3. Calculate $\left(Q_{n_1}^{2^{*b}} - Q_{n_2}^{2^{*b}}\right)$, $\left(S_{n_1}^{2^{*b}} - S_{n_2}^{2^{*b}}\right)$, and $\left(COM_1^{*b} - COM_2^{*b}\right)$ based on bootstrap samples.

4. Repeat steps (2) to (3) 1000 times (B).

5. Sort the estimators $\left(Q_{n_1}^{2^{*b}} - Q_{n_2}^{2^{*b}}\right)$, $\left(S_{n_1}^{2^{*b}} - S_{n_2}^{2^{*b}}\right)$, and $\left(COM_1^{*b} - COM_2^{*b}\right)$ in ascending order.

6. The lower and upper values of the bootstrap confidence interval based on $\left(Q_{n_1}^{2^{*b}} - Q_{n_2}^{2^{*b}}\right)$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of the estimator, $\left(Q_{n_1}^{2^{*b}} - Q_{n_2}^{2^{*b}}\right)_{[\alpha/2\,]}$ and $\left(Q_{n_1}^{2^{*b}} - Q_{n_2}^{2^{*b}}\right)_{[1-\alpha/2\,]}$, respectively.

7. The lower and upper values of the bootstrap confidence interval based on $\left(S_{n_1}^{2^{*b}} - S_{n_2}^{2^{*b}}\right)$ are $\left(S_{n_1}^{2^{*b}} - S_{n_2}^{2^{*b}}\right)_{[\alpha/2\,]}$ and $\left(S_{n_1}^{2^{*b}} - S_{n_2}^{2^{*b}}\right)_{[1-\alpha/2\,]}$.

8. Similarly, the lower and upper limits of the bootstrap confidence interval based on $\left(COM_1^{*b} - COM_2^{*b}\right)$ are $\left(COM_1^{*b} - COM_2^{*b}\right)_{[\alpha/2\,]}$ and $\left(COM_1^{*b} - COM_2^{*b}\right)_{[1-\alpha/2\,]}$, respectively.

5. Simulation study

Performances of the proposed confidence intervals, bootstrap percentile confidence intervals, and Herbert confidence interval for variance difference of independent populations were compared with a simulation study in MATLAB R-2018b, since theoretical comparison was not possible. The most common 95% and 90% confidence levels ($\alpha = 0.05, 0.10$) was used, and confidence intervals were compared in terms of CP and AW.

If the data are from a symmetric distribution, CP will be exactly same or close to $(1-\alpha) = 0.95$. Therefore, CP is a useful criterion to evaluate the confidence interval. On the other hand, smaller AW implies a better confidence interval. In particular, smaller AW indicates the more appropriate method when CP is the same.

We used Monte Carlo simulation to obtain $\widehat{\mathrm{Var}}(Q_{n_1}^2 - Q_{n_2}^2)$, $\widehat{\mathrm{Var}}(S_{n_1}^2 - S_{n_2}^2)$, and $\widehat{\mathrm{Var}}(COM_1 - COM_2)$. Tables 3-6 summarize CP and AW results for variance differences of two independent populations.

The simulation process was as follows: - We used sample sizes n = 10, 20, 50, and 100, - Type I errors $\alpha = 0.05$ and $\alpha = 0.10$, - Simulation study is 10,000 replications, - Random samples were generated from symmetric and skewed distributions: N(0,1); N(10,1); N(10,3); Gamma (1,1); Gamma (3,1); Weibull (1, 0.5); Weibull (1,2); Weibull (1,3); Beta (1,1); Beta (2,2); Beta (3,1); Beta (5,1); Chi-square (1); Chi-square (5); Rayleigh (3); Rayleigh (5); Student-t (3), Student-t (10); Uniform (0,1); Uniform (2,4).

6. Results

Figure 1 shows the distributions of robust estimators for the variance difference of independent populations. A simulation study with 10,000 replications was performed to determine the distributions of estimators. We used the Shapiro-Wilk goodness of fit test to determine whether the estimator distributions were normal under $\alpha = 0.05$. The p-values of the estimators $(Q_{n_1}^2 - Q_{n_2}^2)$, $(S_{n_1}^2 - S_{n_2}^2)$, and $(COM_1 - COM_2)$ were as 0.0945, 0.0857, and 0.0647 respectively. Thus, the distributions of the robust estimators were confirmed to normal ($p - \text{values} > \alpha$).

Table 3 compares the values of CP and AW of the all confidence intervals for normally distributed samples. The values of CPs of confidence interval are close to nominal confidence levels even for small sample sizes. Thus, the performance of proposed confidence intervals are comparable to Herbert confidence interval. The AWs of confidence intervals reduced as sample size increased for the both type I error levels. The estimator

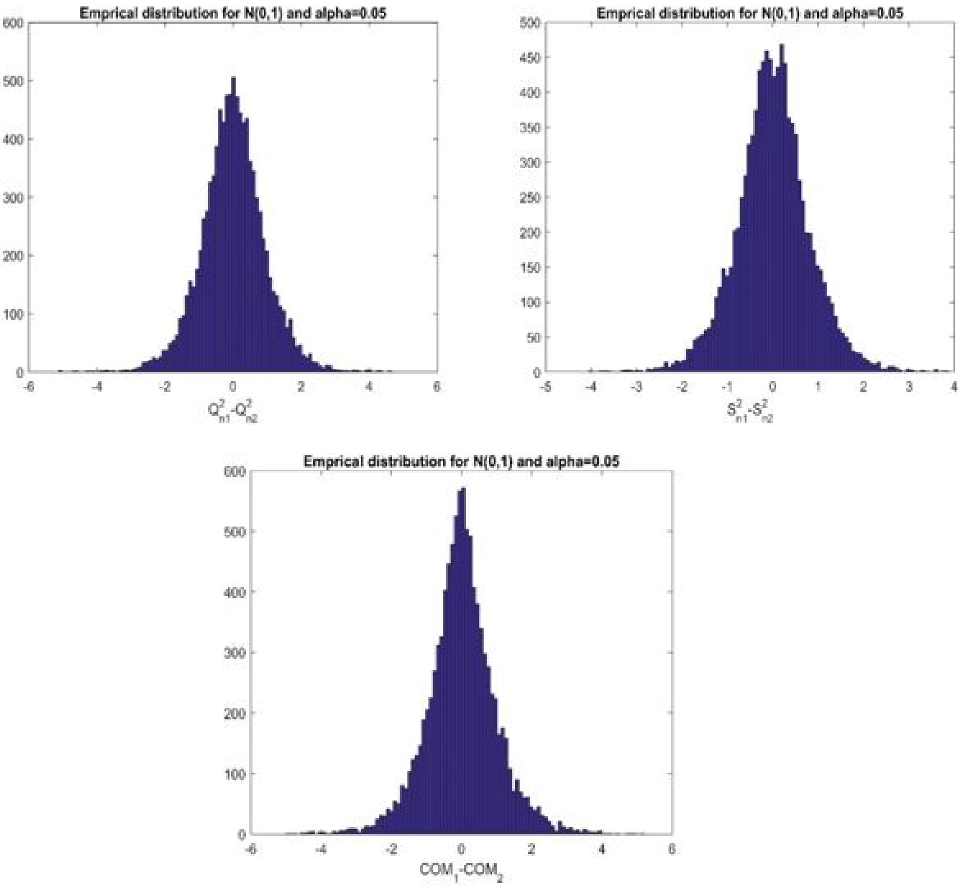

Figure 1. Empirical distributions of robust estimators for variance difference between independent populations

$(Q^2_{n_1} - Q^2_{n_2})$ has the narrowest widths for similar CPs. Thus, the confidence interval based on $(Q^2_{n_1} - Q^2_{n_2})$ produced better results for normal populations (Table 3).

Table 4 shows the values of CP and AW for the bootstrap percentile and Herbert confidence intervals under normal distribution when the difference of population variance equals to zero.

The CPs of bootstrap percentile confidence interval are closer to nominal confidence levels than those of the Herbert confidence interval. Similar to the previous case, the AWs of confidence intervals reduced as sample size increased. AWs based on bootstrap percentiles were much lower than those of the Herbert confidence intervals when the difference of population variance was zero (Table 4).

Table 5 shows the values of CP and AW under Gamma, Weibull, Beta, Chi-squared, Rayleigh, Student-t, Uniform distributions, and $n = 10, 20, 50, 100$ when the population variance difference is zero. The CPs of confidence intervals for all robust estimators are very close to nominal confidence levels, but not for the Herbert confidence interval, for all cases considered. The AWs of confidence interval based on the estimator $(Q^2_{n_1} - Q^2_{n_2})$ are narrower than those of the other confidence intervals. Thus, bootstrap percentile confidence interval based on the estimator $(Q^2_{n_1} - Q^2_{n_2})$ has the best results for skewed populations.

Table 6 shows how increased difference between population variances affected CP. As expected, CPs were lower when compared to the case where the difference of variances was zero. Herbert confidence interval does not perform well for skewed distributions. It is seen

Table 3. Coverage probability (CP) and average width (AW) of confidence intervals for normal distributions

			CP (AW)			
Distribution	Type I error	n	$(Q^2_{n_1} - Q^2_{n_2})$	$(S^2_{n_1} - S^2_{n_2})$	$(COM_1 - COM_2)$	Herbert CI
N(0,1)	$\alpha = 0.05$	10	0.9444 (1.9285)	0.9422 (3.2890)	0.9401 (4.5082)	0.9510 (2.5677)
		20	0.9479 (1.6134)	0.9447 (2.5619)	0.9425 (3.0446)	0.9510 (1.7712)
		50	0.9483 (1.1518)	0.9480 (1.6513)	0.9445 (1.8833)	0.9515 (1.1107)
		100	0.9489 (0.8247)	0.9493 (1.0443)	0.9489 (1.2946)	0.9515 (0.8847)
	$\alpha = 0.10$	10	0.9054 (1.6011)	0.8995 (2.8531)	0.9000 (3.7944)	0.9099 (2.1521)
		20	0.9061 (1.3392)	0.9004 (2.1506)	0.8984 (2.5753)	0.9098 (1.4899)
		50	0.9069 (0.9071)	0.8997 (1.3982)	0.9010 (1.5816)	0.9099 (0.9296)
		100	0.9069 (0.6278)	0.9016 (0.9594)	0.9016 (1.0898)	0.9096 (0.6581)
N(10,1)	$\alpha = 0.05$	10	0.9446 (1.9129)	0.9424 (3.1544)	0.9418 (4.9157)	0.9506 (2.5593)
		20	0.9465 (1.6064)	0.9464 (2.5768)	0.9453 (3.5741)	0.9506 (1.7726)
		50	0.9495 (1.1027)	0.9490 (1.4853)	0.9450 (1.9914)	0.9506 (1.1100)
		100	0.9505 (0.7381)	0.9501 (1.0665)	0.9489 (1.3967)	0.9510 (0.7830)
	$\alpha = 0.10$	10	0.9018 (1.8132)	0.9020 (2.8854)	0.9035 (4.8606)	0.9097 (2.1563)
		20	0.9025 (1.3808)	0.9037 (2.2170)	0.9057 (3.2108)	0.9097 (1.4863)
		50	0.8994 (0.9782)	0.9026 (1.4041)	0.9027 (1.7716)	0.9095 (0.9322)
		100	0.9011 (0.6965)	0.9015 (0.8991)	0.9094 (1.1979)	0.9098 (0.6571)
N(10,3)	$\alpha = 0.05$	10	0.9441 (17.3853)	0.9414 (28.6852)	0.9399 (41.4839)	0.9515 (23.2479)
		20	0.9479 (14.3232)	0.9454 (20.3212)	0.9441 (27.3287)	0.9520 (15.9532)
		50	0.9482 (9.2258)	0.9470 (12.9875)	0.9470 (17.0630)	0.9520 (9.9790)
		100	0.9497 (6.4420)	0.9486 (9.1380)	0.9492 (11.6996)	0.9520 (7.0582)
	$\alpha = 0.10$	10	0.9058 (14.4646)	0.9008 (23.9410)	0.9007 (34.7127)	0.9089 (19.4067)
		20	0.9042 (11.9871)	0.9026 (16.9650)	0.9009 (22.9343)	0.9088 (13.3763)
		50	0.8987 (8.0601)	0.9020 (10.9457)	0.9001 (14.3242)	0.9090 (8.3885)
		100	0.9065 (5.2713)	0.9020 (7.6458)	0.9007 (9.8672)	0.9090 (5.9318)

Table 4. Coverage probability (CP) and average width (AW) of Bootstrap percentile confidence intervals for normal distributions

			CP (AW)			
Distribution	Type I error	n	$(Q^2_{n_1} - Q^2_{n_2})$	$(S^2_{n_1} - S^2_{n_2})$	$(COM_1 - COM_2)$	Herbert CI
N(0,1)	$\alpha = 0.05$	10	0.9433 (1.8788)	0.9377 (3.2449)	0.9451 (5.2786)	0.8930 (2.0381)
		20	0.9573 (1.4048)	0.9385 (2.5289)	0.9442 (3.6056)	0.8640 (1.4688)
		50	0.9379 (1.0037)	0.9485 (1.5611)	0.9436 (2.1159)	0.8951 (1.0361)
		100	0.9572 (0.7012)	0.9443 (1.0277)	0.9456 (1.3558)	0.8968 (0.7434)
	$\alpha = 0.10$	10	0.8937 (1.5952)	0.8908 (2.7532)	0.8636 (4.3471)	0.8421 (1.7958)
		20	0.9092 (1.1363)	0.8942 (2.0997)	0.8688 (2.9584)	0.8157 (1.2911)
		50	0.8891 (0.8750)	0.9036 (1.3264)	0.9008 (1.7859)	0.8535 (0.9062)
		100	0.8969 (0.6043)	0.9039 (0.9039)	0.9030 (1.1956)	0.8548 (0.6606)
N(10,1)	$\alpha = 0.05$	10	0.9481 (1.7429)	0.9453 (3.0383)	0.9440 (4.8908)	0.8819 (2.0112)
		20	0.9516 (1.3971)	0.9441 (2.4865)	0.9432 (3.5446)	0.8571 (1.4410)
		50	0.9534 (1.0003)	0.9480 (1.4669)	0.9530 (1.9843)	0.8877 (1.0008)
		100	0.9559 (0.6118)	0.9505 (1.0455)	0.9529 (1.3761)	0.9079 (0.7532)
	$\alpha = 0.10$	10	0.8971 (1.7271)	0.8928 (2.8567)	0.8928 (4.4839)	0.8146 (1.7557)
		20	0.8982 (1.3712)	0.9039 (2.2072)	0.8962 (3.1277)	0.8375 (1.3769)
		50	0.8952 (0.8609)	0.9057 (1.3006)	0.9022 (1.7502)	0.8397 (0.8908)
		100	0.9017 (0.5804)	0.9053 (0.8737)	0.9023 (1.1505)	0.8538 (0.6520)
N(10,3)	$\alpha = 0.05$	10	0.9466 (17.3101)	0.9470 (28.5836)	0.9432 (40.3177)	0.8559 (17.3453)
		20	0.9458 (13.0787)	0.9415 (21.2026)	0.9426 (27.3048)	0.8613 (13.2294)
		50	0.9455 (7.1509)	0.9488 (12.0339)	0.9532 (16.9924)	0.8893 (8.9892)
		100	0.9524 (5.3821)	0.9491 (9.0995)	0.9543 (11.5801)	0.8987 (6.7049)
	$\alpha = 0.10$	10	0.8940 (13.8750)	0.8908 (22.5881)	0.8900 (33.3747)	0.7725 (14.4333)
		20	0.8976 (11.1245)	0.8981 (16.8917)	0.8985 (22.1774)	0.8020 (11.3278)
		50	0.8980 (8.1678)	0.9064 (10.6366)	0.9057 (14.2444)	0.8492 (8.2544)
		100	0.8975 (5.1445)	0.9059 (7.6031)	0.9057 (9.7335)	0.8557 (5.9128)

that the AWs of confidence intervals based on estimator $(Q^2_{n_1} - Q^2_{n_2})$ are the narrowest, but the CPs of confidence intervals based on estimator $(COM_1 - COM_2)$ are more closer

Table 5. Coverage probability (CP) and average width (AW) of bootstrap percentile confidence intervals for some non-normal distributions and $\alpha = 0.05$

		CP (AW)			
Distribution of both population	n	$(Q^2_{n_1} - Q^2_{n_2})$	$(S^2_{n_1} - S^2_{n_2})$	$(COM_1 - COM_2)$	Herbert CI
Gamma (3,1)	10	0.9407 (5.0482)	0.9414 (8.6089)	0.9408 (12.0548)	0.9164 (10.4039)
	20	0.9466 (4.0243)	0.9442 (6.2027)	0.9436 (8.1122)	0.9172 (7.2370)
	50	0.9482 (2.7860)	0.9449 (4.0077)	0.9460 (4.8912)	0.9159 (4.6116)
	100	0.9493 (2.0532)	0.9453 (2.8416)	0.9471 (3.4143)	0.9164 (3.2816)
Weibull (1,2)	10	0.9441 (0.3866)	0.9418 (0.6674)	0.9418 (0.9760)	0.9154 (0.6108)
	20	0.9451 (0.3099)	0.9491 (0.4820)	0.9428 (0.6689)	0.9126 (0.4142)
	50	0.9467 (0.2208)	0.9472 (0.3086)	0.9467 (0.4121)	0.9164 (0.2566)
	100	0.9486 (0.1595)	0.9501 (0.2170)	0.9527 (0.2868)	0.9182 (0.1802)
Beta (3,1)	10	0.9416 (0.0636)	0.9382 (0.1166)	0.9392 (0.1717)	0.9140 (0.0876)
	20	0.9443 (0.0513)	0.9470 (0.0885)	0.9434 (0.1210)	0.9187 (0.0639)
	50	0.9494 (0.0358)	0.9476 (0.0600)	0.9485 (0.0751)	0.9183 (0.0416)
	100	0.9505 (0.0269)	0.9490 (0.0443)	0.9484 (0.0542)	0.9157 (0.0298)
Beta (2, 2)	10	0.9476 (0.0833)	0.9451 (0.1537)	0.9415 (0.2471)	0.9106 (0.1103)
	20	0.9514 (0.0674)	0.9516 (0.1103)	0.9431 (0.1766)	0.9125 (0.0717)
	50	0.9495 (0.0465)	0.9468 (0.0682)	0.9473 (0.1123)	0.9151 (0.0431)
	100	0.9494 (0.0338)	0.9502 (0.0471)	0.9483 (0.0816)	0.9150 (0.0300)
Chi-squared (1)	10	0.9437 (2.5056)	0.9427 (3.7187)	0.9415 (5.0694)	0.9173 (10.6621)
	20	0.9408 (1.55549	0.9417 (2.2545)	0.9409 (2.9274)	0.9142 (8.1081)
	50	0.9413 (0.9182)	0.9399 (1.2183)	0.9393 (1.5298)	0.9194 (5.3431)
	100	0.9476 (0.6272)	0.9453 (0.8056)	0.9443 (0.9975)	0.9190 (3.9023)
Rayleigh (3)	10	0.9453 (6.9394)	0.9427 (12.1430)	0.9403 (17.9107)	0.9150 (11.0672)
	20	0.9442 (5.5309)	0.9430 (8.5767)	0.9442 (11.8982)	0.9199 (7.4779)
	50	0.9518 (3.9274)	0.9496 (5.5097)	0.9491 (7.3994)	0.9161 (4.6091)
	100	0.9506 (2.9465)	0.9490 (3.9649)	0.9462 (5.2175)	0.9184 (3.2358)
Student-t (3)	10	0.9419 (4.3296)	0.9414 (6.2949)	0.9454 (7.8396)	0.9181 (13.6772)
	20	0.9437 (3.2928)	0.9428 (4.1713)	0.9427 (4.7374)	0.9135 (13.0315)
	50	0.9458 (2.3343)	0.9462 (2.6055)	0.9450 (2.7566)	0.9135 (10.3312)
	100	0.9484 (1.6538)	0.9482 (1.7981)	0.9462 (1.8615)	0.9194 (7.3724)
Uniform (0,1)	10	0.9496 (0.1211)	0.9499 (0.2432)	0.9446 (0.4151)	0.9168 (0.1676)
	20	0.9525 (0.0896)	0.9493 (0.1797)	0.9481 (0.3171)	0.9116 (0.1052)
	50	0.9501 (0.0544)	0.9482 (0.1085)	0.9479 (0.2052)	0.9140 (0.0616)
	100	0.9518 (0.0356)	0.9509 (0.0745)	0.9500 (0.1493)	0.9141 (0.0424)

to nominal confidence level compared to that of others when the difference between the variances is not zero.

7. Real Data Examples

7.1. Example I

Hemoglobin data were obtained from 170 persons, including 85 healthy and 85 coronary artery patients, from the Cardiology Department of Bitlis State Hospital, as shown in Figure 2 [5].

The Kolmogorov-Smirnov (KS) goodness of fit test for normality showed that the data has a normal distribution (disease group: KS test statistic = 0.1450, p-value = 0.0505; control group: KS test statistic = 0.0878, p-value = 0.5013).

Table 7 shows some descriptive statistics, 95% confidence intervals and their's AWs for Example I. The width of confidence interval based on estimator $\left(Q^2_{n_1} - Q^2_{n_2}\right)$ is narrower than that of the other intervals, which is consistent with the simulation outcomes.

7.2. Example II

We considered the study of honey as a cough remedy for children who were ill with an upper respiratory tract infection [24]. Parents were instructed to give their child liquid medicine prior to bedtime. Unknown to the parents, some were provided with dextromethorphan (DM), an over the counter cough medicine, while others were provided a

Table 6. Coverage probability (CP) and average width (AW) of bootstrap percentile confidence intervals for distributions with unequal variances and $\alpha = 0.05$

		CP (AW)			
Distribution of both population	n	$(Q^2_{n_1} - Q^2_{n_2})$	$(S^2_{n_1} - S^2_{n_2})$	$(COM_1 - COM_2)$	Herbert CI
Gamma (3,1) and Gamma (1,1)	10	0.5959 (3.4504)	0.8451 (6.1793)	0.9263 (9.6799)	0.7313 (4.9087)
	20	0.6966 (2.8426)	0.9115 (4.8809)	0.9509 (6.6891)	0.7937 (4.0660)
	50	0.7790 (1.9914)	0.9181 (3.0238)	0.9390 (3.8004)	0.8569 (3.0646)
	100	0.7959 (1.3902)	0.9043 (2.0221)	0.9231 (2.4614)	0.8733 (2.3472)
Weibull (1,3) and Weibull (1,0.5)	10	0.3954 (8.8729)	0.4515 (12.5464)	0.4884 (17.8421)	0.3507 (41.8777)
	20	0.4108 (2.9456)	0.4662 (4.8458)	0.4619 (6.5059)	0.4544 (44.8977)
	50	0.4100 (1.3670)	0.4503 (1.6654)	0.5008 (2.2583)	0.3942 (41.0439)
	100	0.4100 (0.8680)	0.4710 (0.9243)	0.5000 (1.2743)	0.4611 (36.1003)
Beta (5,1) and Beta (1,1)	10	0.6595 (0.0890)	0.8668 (0.1644)	0.8783 (0.2716)	0.8657 (0.1031)
	20	0.7932 (0.0749)	0.8957 (0.1427)	0.8906 (0.2293)	0.8900 (0.0684)
	50	0.7381 (0.0494)	0.8820 (0.0914)	0.8900 (0.1580)	0.8816 (0.0437)
	100	0.7545 (0.0319)	0.8962 (0.0597)	0.8942 (0.1124)	0.8969 (0.0315)
Chi-squared (1) and Chi-squared (5)	10	0.5363 (11.2544)	0.8209 (19.8830)	0.9136 (30.9610)	0.6936 (15.5072)
	20	0.5635 (8.3347)	0.8590 (14.2904)	0.9225 (19.4034)	0.7662 (13.5317)
	50	0.6255 (6.2044)	0.8931 (9.4598)	0.9276 (11.8353)	0.8264 (9.8399)
	100	0.6207 (4.4948)	0.8829 (6.6234)	0.9211 (7.9865)	0.8627 (7.7075)
Rayleigh (3) and Rayleigh (5)	10	0.6725 (13.4485)	0.8968 (23.6476)	0.9014 (37.3312)	0.7693 (15.0666)
	20	0.7488 (11.0432)	0.9325 (18.5358)	0.9087 (26.5315)	0.8255 (11.9156)
	50	0.8407 (8.4380)	0.9328 (12.4924)	0.9487 (17.0241)	0.8722 (8.4660)
	100	0.8433 (5.8667)	0.9358 (8.2907)	0.9435 (11.1386)	0.8809 (6.1577)
Student-t (3) and Student-t (10)	10	0.4571 (3.5489)	0.6809 (5.8039)	0.8027 (8.6065)	0.5632 (5.6865)
	20	0.4314 (2.7049)	0.6394 (4.0373)	0.8183 (5.1798)	0.6247 (5.5038)
	50	0.4592 (1.8747)	0.6802 (2.3601)	0.8286 (2.7592)	0.6656 (4.4714)
	100	0.4551 (1.2905)	0.7842 (1.5077)	0.8424 (1.7133)	0.7128 (3.6506)
Uniform (0,1) and Uniform (2,4)	10	0.6853 (0.3668)	0.9292 (0.6535)	0.9470 (1.0977)	0.8382 (0.3525)
	20	0.7590 (0.2954)	0.9303 (0.5667)	0.9402 (0.9278)	0.8700 (0.2465)
	50	0.7601 (0.1941)	0.9419 (0.3599)	0.9419 (0.6367)	0.8931 (0.1634)
	100	0.7678 (0.1233)	0.9482 (0.2314)	0.9462 (0.4473)	0.9100 (0.1181)

Table 7. Descriptive statistics and confidence intervals for hemoglobin

Descriptive statistics	Group I	Group II
Sample size (n)	85	85
Mean	13.55	13.62
Variance	3.87	2.31
Median	14	13.5
Kurtosis	2.48	2.43
Skewness	0.52	0.06
Q^2_n	2.13	1.71
S^2_n	2.14	1.66
COM	1.63	1.93
Method	**95% confidence interval**	**Width**
$(Q^2_{n_1} - Q^2_{n_2})$	[0, 2.10]	2.10
$(S^2_{n_1} - S^2_{n_2})$	[-1.15, 3.71]	4.86
$(COM_1 - COM_2)$	[-2.80, 4.23]	7.03
Herbert confidence interval	[0.44, 2.60]	2.16

similar dose of honey. Parents then rated their child's cough symptoms and an improvement in total cough symptoms score was determined for each child.

The KS test confirmed the Figure 3 that these data were not normal distributed (honey: KS test statistic = 0.1316, p-value=0.0061; DM: KS test statistic = 0.0977; p-value=0.0086).

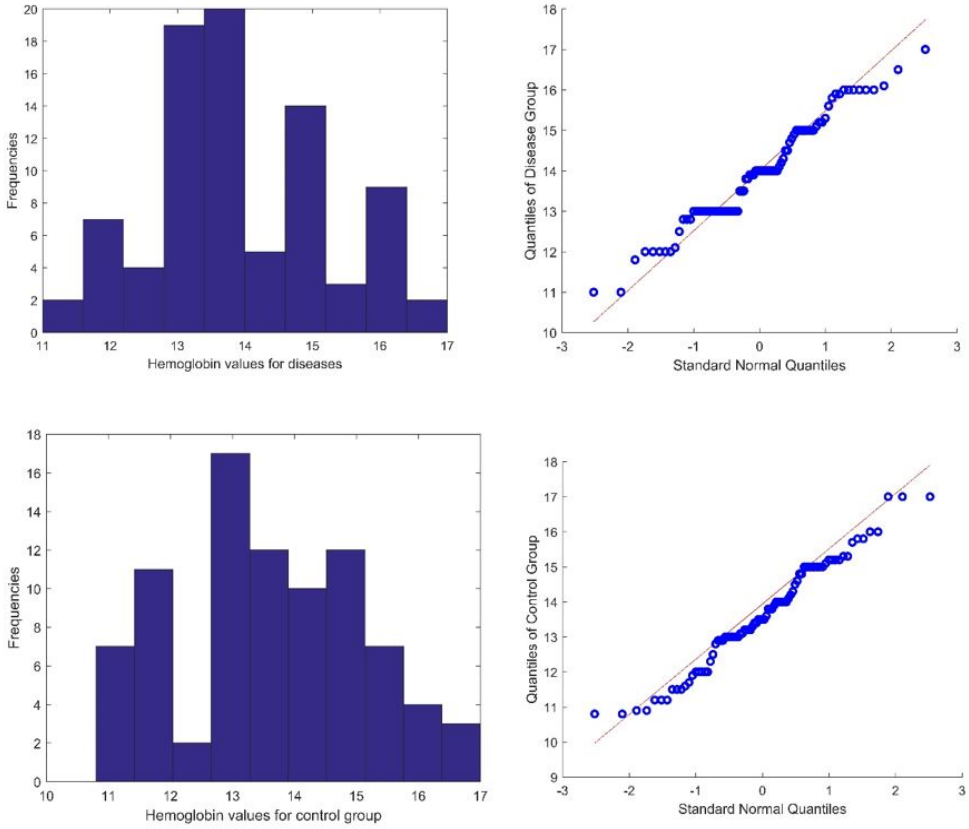

Figure 2. Histograms and Q-Q plots of coronary artery disease

Table 8. Descriptive statistics and confidence intervals for dosage

Descriptive statistics	Group I	Group II
Sample size (n)	35	35
Mean	10.71	8.33
Variance	8.15	10.60
Median	11	9
Kurtosis	2.80	1.98
Skewness	-0.20	0.08
Q_n^2	2.02	4.05
S_n^2	2.38	3.57
COM	2.99	2.99
Method	**95% confidence interval**	**Width**
$(Q_{n_1}^2 - Q_{n_2}^2)$	[0, 10.14]	10.14
$(S_{n_1}^2 - S_{n_2}^2)$	[-11.37, 7.11]	18.48
$(COM_1 - COM_2)$	[-18.41, 5.90]	24.31
Herbert confidence interval	[-7.63, 17.23]	24.86

Table 7 shows that width of confidence interval based on estimator $(Q_{n_1}^2 - Q_{n_2}^2)$ is smaller than that of the others, and the Herbert confidence interval has the largest width for skewed population, which is consistent with the simulation results.

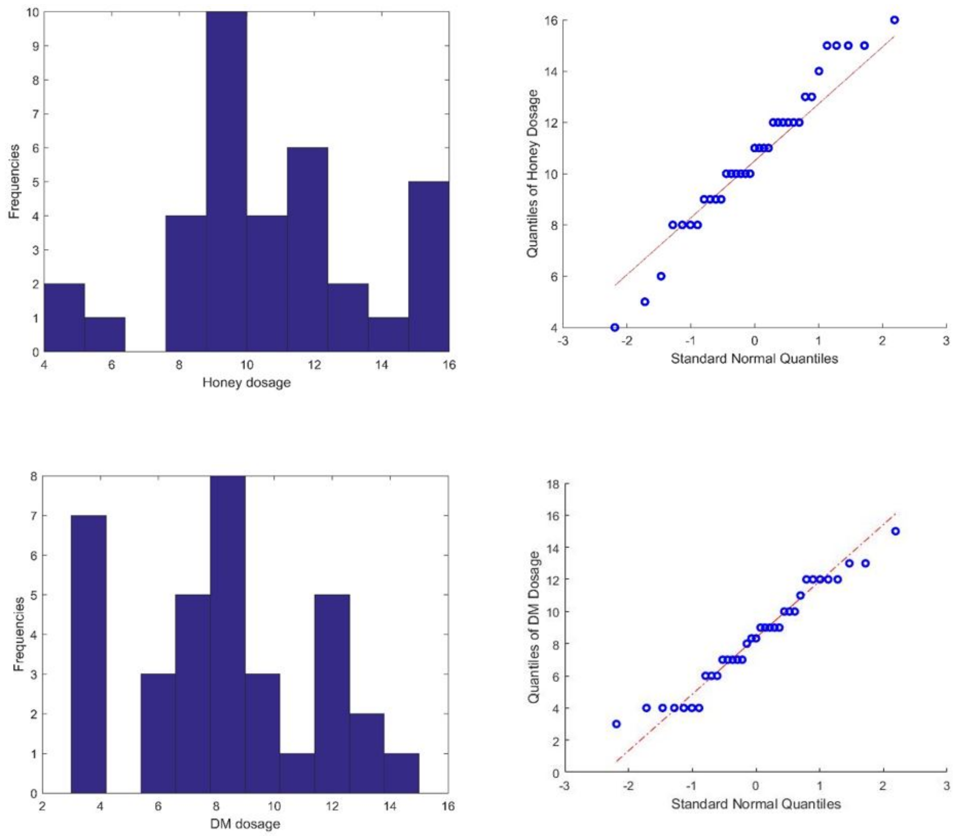

Figure 3. Histograms and Q-Q plots of Honey and DM dosages

8. Conclusion

Although making interpretation of variance differences in randomized and clinical trials is an important issue, just a few statistical methods deal with the variance differences. We proposed confidence intervals for the difference of variances of two independent populations based on three robust estimators. These intervals and their bootstrap percentile versions were compared with Herbert confidence intervals in terms of CP and AW.

The simulation results showed that all proposed confidence intervals had higher CPs than the Herbert confidence interval under normal distribution. Also, confidence interval based on estimator $\left(Q^2_{n_1} - Q^2_{n_2}\right)$ had the smaller AWs than the other confidence intervals. On the other hand, AWs of confidence intervals based on estimators $\left(S^2_{n_1} - S^2_{n_2}\right)$ and $(COM_1 - COM_2)$ are as narrow as those of the Herbert confidence interval.

When the populations were normal distributed, bootstrap percentile confidence intervals had narrower AWs than classical confidence intervals. Higher CPs were obtained wtih variance difference which is zero. CPs decreased with the increase in variance differences as in Herbert confidence interval.

The CPs of proposed confidence intervals were higher than those of the Herbert confidence interval when the populations were skewed distributed. Herbert's bootstrap percentile confidence interval had less than nominal level of coverage. The AWs of confidence interval based on estimator $(COM_1 - COM_2)$ were similar to AW of the Herbert confidence interval, but AWs for estimator $(Q^2_{n_1} - Q^2_{n_2})$ were narrower than those of the other intervals.

CPs were lower in the case of the variance which is not zero. In this case, CPs of confidence interval based on estimator $(COM_1 - COM_2)$ were closer to the nominal level than those of the others. However, the estimator $(Q^2_{n_1} - Q^2_{n_2})$ performed better in terms of AWs. Consequently, we recommend bootstrap percentile confidence interval based on estimator $(Q^2_{n_1} - Q^2_{n_2})$ for the difference of variances of two independent populations.

Two real data sets were analysed to illustrate the findings of the study and the simulation results were verified.

Acknowledgment. The authors are grateful to the reviewers for their suggestions and valuable comments on the paper.

References

[1] M.O.A. Abu-Shawiesh, S. Banik and , B.M.G. Kibria, *A simulation study on some confidence intervals for the population standard deviation*, SORT-Stat. Oper. Res. Trans. **35** (2), 83-102, 2011.

[2] H.E. Akyüz, *Interval estimation based on robust estimators for the difference of two independent population variances*, Gazi University, Graduate School of Natural and Applied Sciences, PhD Thesis, Ankara, Turkey, 2017.

[3] H.E. Akyüz, H. Gamgam and A. Yalçınkaya, *Interval estimation for the difference of two independent nonnormal population variances*, Gazi University Journal of Science **30** (3),117-129, 2017.

[4] H.E. Akyüz and H. Gamgam, *Interval estimation for nonnormal population variance with kurtosis coefficient based on trimmed mean*, Turkiye Klinikleri Journal of Biostatistics **9** (3), 213-221, 2017.

[5] H.E. Akyüz and H. Gamgam, *Comparison of binary logistic regression models based on bootstrap method: an application on coronary artery disease data*, Gazi University Journal of Science **32** (1), 318-331, 2019.

[6] C. Atakan, *Bootstrap percentile confidence intervals for actual error rate in linear discriminant analysis*, Hacet. J. Math. Stat. **38** (3), 357-372, 2009.

[7] A.M. Barham and S. Jeyaratnam *Robust confidence interval for the variance*, J. Stat. Comput. Simul. **62** (3), 189-205, 1999.

[8] D.G. Bonett, *Approximate confidence interval for standard deviation of nonnormal distributions*, Comput. Statist. Data Anal. **50** (3), 775-782, 2006.

[9] B.D. Burch, *Nonparametric bootstrap confidence intervals for variance components applied to interlaboratory comparisons*, J. Agric. Biol. Environ. Stat. **17** (2), 228-245, 2012.

[10] B.D. Burch, *Estimating kurtosis and confidence ntervals for the variance under nonnormality*, J. Stat. Comput. Simul. **84** (12), 2710-2720, 2014.

[11] B.D. Burch, *Distribution-dependent and distribution-free confidence intervals for the variance*, Stat. Methods Appl. **26** (4), 629-648, 2017.

[12] J. Carpenter and J. Bithell, *Bootstrap confidence intervals: when, which, what? A practical guide for medical statisticians*, Stat. Med. **19** (9), 1141-1164, 2000.

[13] G. Casella and R.L. Berger *Statistical Inference*, Duxbury Thomson Learning, USA, 2002.

[14] V. Cojbasica and A. Tomovica, *Nonparametric confidence intervals for population variance of one sample and the difference of variances of two samples*, Comput. Statist. Data Anal. **51** (12), 5562-5578, 2007.

[15] V. Cojbasica and V. Loncar, *One-sided confidence intervals for population variances of skewed distributions*, J. Statist. Plann. Inference **141** (5), 1667-1672, 2011.

[16] C. Croux and P.J. Rousseeuw, *A class of high-breakdown scale estimators based on subranges*, Comm. Statist. Theory Methods **21** (7), 1935-1951, 1992.

[17] B. Efron, *Bootstrap methods: Another look at the jackknife*, Ann. Statist. **7** (1), 1-26, 1979.

[18] B. Efron and R.J. Tibshirani, *An Introduction to the Bootstrap*, Chapman & Hall/CRC, USA, 1993.

[19] M. Falk, *Asymptotic independence of median and MAD*, Statist. Probab. Lett. **34** (4), 341-345, 1997.

[20] R.D. Herbert, A. Hayen, P. Macaskill and S.D. Walter, *Interval estimation for the difference of two independent variances*, Comm. Statist. Simulation Comput. **40** (5), 744-758, 2011.

[21] S. Niwitpong, *Confidence intervals for the difference of two normal population variances*, World Academy of Science, Engineering and Technology **5** (8), 602-605, 2011.

[22] S. Niwitpong, *A note on coverage probability of confidence interval for the difference between two normal variances*, Appl. Math. Sci. **6** (67), 3313-3320, 2012.

[23] M.J. Panik, *Advanced Statistics from an Elementary Point of View*, 1st edition, USA: Academic Press, Elsevier, 2005.

[24] I.M. Paul, J. Beiler, A. McMonagle, M.L. Shaffer, L. Duda and C.M. Berlin, *Effect of honey, dextromethorphan, and no treatment on nocturnal cough and sleep quality for coughing children and their parents*, Archives of Pediatrics and Adolescent Medicine **161** (12), 1140-1146, 2007.

[25] P.J. Rousseeuw and C. Croux, *Alternatives to the median absolute deviation*, J. Amer. Statist. Assoc. **88** (424), 1273-1283, 1993.

[26] H. Scheffe, *The Analysis of Variance*, Wiley, New York, 1959.

[27] S. Suwan and S. Niwitpong, *Interval estimation for a linear function of variances of nonnormal distributions that utilize the kurtosis*, Appl. Math. Sci. **7** (99), 4909-4918, 2013.

[28] W. Thangjai and S. Niwitpong, *Simultaneous Confidence Intervals for All Differences of Variances of Log-Normal Distributions*, In International Conference of the Thailand Econometrics Society (pp. 235-244). Springer, Cham.,2019.

[29] V. Zardasht, *A bootstrap test for symmetry based on quantiles*, Hacet. J. Math. Stat., **47** (4), 1061-1069, 2018.