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Ideals and Filters in Neutrosophic Lattices

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Abstract: In this paper, we give ideal and filter definitions for neutrosophic lattices for the first time. We extend neutrosophic ideals and filters of neutrosophic lattices to pure neutrosophic lattices to obtain more flexible definitions.

Key words: Neutrosophic sets, Neutrosophic lattice, Neutrosophic ideal, Neutrosophic filter.

1. Introduction

In classical set theory [1], set concepts with strict boundaries are used. In this theory, there are two possibilities, whether or not an object is a member of a set. In 1965, fuzzy sets were formed by Zadeh [2] in order to examine the accuracy of expressions in more detail. The truth values of expressions are interpreted as 'true' or 'false' in propositional logic. On the other hand, Zadeh expresses the truth value of an expression in the range [0,1] and places the accuracy value or in other words the truth level on a much more sensitive ground. With fuzzy set interpretation, the degree of belonging of an element to a set is extended to the range [0,1] and more sensitive measurements are emphasized. Later, Atanassov [3] introduced the more accurate belonging measurements and systems by adding the degree of belonging and the degree of non-belonging in his intuitionistic fuzzy sets. Smarandache [4] introduced neutrosophic clusters that further expanded the systems of Zadeh and Atanassov. This system is based on measuring and modeling an element's belonging, not belonging, and indeterminacy. Lattice theory is one of the important cornerstones of abstract algebra in mathematics. It is a structure used to put forward the general theories in the order of various sets and objects belonging to these sets. The lattices frequently encountered in the field of logic and algebra were put forward by Birkhoff and turned into theory [5]. The lattice structures of neutrosophic sets [6] and neutrosophic semi-lattices are introduced [7]. It accepts true, false

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and indeterminacy expressions of these lattices as elements. Since the working date of this type of lattices is very new, it does not have a wide literature network. On the other hand, since there are only two studies on the properties of these lattices, the topic has many features that can be improved. The purpose of this paper is to examine the lattices of neutrosophic clusters in detail. In particular, the properties of filters and their ideals will be examined.

2. Fundamental Definitions

In this section, basic definitions will be given. All definitions in this section are taken from lattice theory [5].

De inition 2.1 *Let P be a set. If the following properties are provided for each x, y, z element in P, the binary "$\leq$" relation defined on the set P is called partial ordering:*

(i) $x \leq x$ *(Reflexive),*

(ii) if $x \leq y$ *and* $y \leq x$ *then* $y = x$ *(Antisymmetric),*

(iii) if $x \leq y$ *and* $y \leq z$ *then* $x \leq z$ *(Transitive).*

The set P is also called a partially ordered set (poset) and is represented by $(P, \leq)$.

Definition 2.2 *Let P be a poset and $S \subseteq P$. An element x in P is said to be an upper bound for a subset S of X if for every u in S, we have $u \leq x$. Similarly, a lower bound for a subset S is an element l such that for every s in S, $l \leq s$. If there is an upper bound and a lower bound for P, then the poset is said to be bounded. The set of upper bounds of S denoted by S^u and the set of lower bounds of S denoted by S^l:*

$$S^u = \{x \in P;\ \forall s \in S, s \leq x\}$$

$$S^l = \{x \in P;\ \forall s \in S, \leq s\}$$

Definition 2.3 *Let P be a poset and $S \subseteq P$. If S^u has a smallest element, then this element is called the smallest upper bound (or supremum) of S and is denoted by $supS = \vee S$. Equivalently,*

(i) x *is an upper bound of* S*,*

(ii) for every upper bound y of S, if $x \leq y$, then x is the smallest upper bound of S.

If $supS$ exists, then it is unique. Particularly, if $S = \{x, y\}$, then $sup\{x, y\}$ is denoted by $sup\{x, y\} = x \vee y$.

Definition 2.4 *Let P be a poset and $S \subseteq P$. If S^l has a greatest element, then this element is called the greatest lower bound (or infimum) of S and is denoted by $infS = \wedge S$. Equivalently,* *(i)* x *is an lower bound of* S,

(ii) *for every lower bound* y *of* S, *if* $y \leq x$, *then* x *is the greatest lower bound of* S.

If $infS$ *exists, then it is unique. Particularly, if* $S = \{x, y\}$, *then* $inf\{x, y\}$ *is denoted by* $inf\{x, y\} = x \wedge y$.

Definition 2.5 *Let* $(L, \leq)$ *be a poset. If for all* $x, y \in L$ $sup\{x, y\}$ *and* $inf\{x, y\}$ *exist, then* (L, sup, inf) *is called a lattice.*

Definition 2.6 *Let* A *be a lattice and* $B \subseteq A$. *If* B *is closed under the operation defined over* A, *then* B *is called a sublattice.*

Definition 2.7 *Let* L *be a lattice. A nonempty subset* K *of* L *is called an ideal if*

(i) K *is a sublattice of* L,

(ii) $a \in K, b \in L, a \vee b \in K$ *imply* $b \in K$.

Definition 2.8 *Let* L *be a lattice. If* $x \wedge a \in F$ *for all* $x \in F$ *and for all* $a \in L$, *then* F *sublattice is a filter of* L.

Definition 2.9 *Let* L *be a lattice. If* $x \vee a \in I$ *for all* $x \in I$ *and for all* $a \in L$, *then* I *sublattice is an ideal of* L.

3. Neutrosophic Definitions

In this section, basic definitions of neutrosophic lattices and some explanatory examples will be given. Definitions are underlined from the paper neutrosophic lattices [6].

Definition 3.1 *Let* $N(P)$ *be a poset and* $0, 1, I, 1+I \in N(P)$. $max\{x, y\}, min\{x, y\} \in N(P)$ *are defined max and min elements on* $N(P)$, *respectively.* 0 *is the smallest element and* $I \cup 1 = 1 + I$ *is the largest element.* $(N(P), min, max)$ *is called a neutrosophic lattice.*

Example 3.2 *Let* $N(P) = \{0, 1, I, 1+I, a, aI\}$ *be a poset and* $N(P)$ *be a neutrosophic lattice.* $N(P)$ *is represented by a Hasse neutrosophic diagram as in Figure 1.*

Example 3.3 *Figure 2 shows* $N(P) = \{0, 1, I, 1+I, a_1, a_2, a_1I, a_2I\}$ *is a neutrosophic lattice.*

Example 3.4 $N(P) = \{0, 1, I, I \cup 1\}$ *neutrosophic lattice is the smallest neutrosophic lattice.*

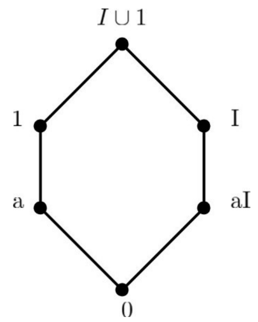

Figure 1: A Hasse neutrosophic diagram of Example 3.2

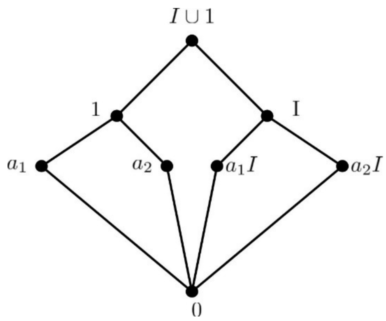

Figure 2: A Hasse neutrosophic diagram of Example 3.3

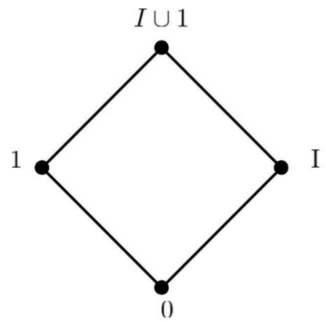

Figure 3: A Hasse neutrosophic diagram of Example 3.4

Example 3.5 *Figure 4 shows neutrosophic lattice* $N(P) = \{0, 1, a_1, a_2, a_1I, a_2I, I, 1 \cup I\}$ *with* $1 > a_1 > a_2 > 0$.

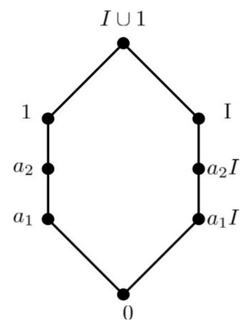

Figure 4: A Hasse neutrosophic diagram of Example 3.5

Example 3.6 *Let* $N(P) = \{0, 1, I, 1+I, a_1, a_2, a_3, a_4, a_1I, a_2I, a_3I, a_4I\}$ *be a neutrosophic lattice. If* $i \neq j, 1 \leq i, j \leq 4$ *then* a_i *and* a_j *are incompatible.*

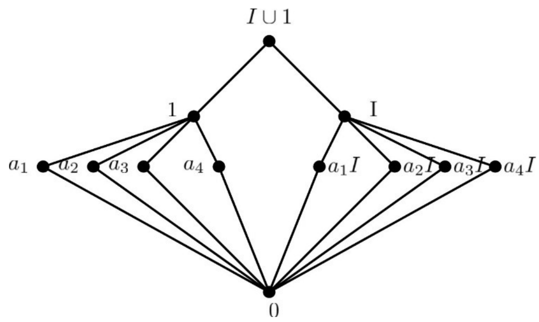

Figure 5: A Hasse neutrosophic diagram of Example 3.6

4. Neutrosophic Ideals and Filters

In this section, we give new definitions for neutrosophic lattices and prove some fundamental theorems for the definitions.

Definition 4.1 *Let* N *be a neutrosophic lattice and* F *be a neutrosophic sublattice. If* $x \wedge y \in F$ *for all* $x \in N$ *and for all* $y \in F$*, then* F *is called a neutrosophic filter.*

Theorem 4.2 *Every neutrosophic lattice is the neutrosophic filter itself.*

Proof Suppose N is a neutrosophic lattice and F be a neutrosophic filter. From the definition, it is clear that $F \subseteq N$. Now we show that $N \subseteq F$.

(i) Suppose $N = \{1+I, 1, I, 0\}$. F must be $\{1+I, 1, I, 0\}$ since a neutrosophic filter F is also a neutrosophic lattice. Therefore, $N = F$. (ii) Suppose $N = \{1+I, 1, I, 0, a_i, a_iI\}(1 \leq i \leq n)$ and $a_iI \in F$ but $a_i \notin F$. a_i must be in F since $a_iI \in F$ and F is a neutrosophic lattice. On the other hand, a_iI must be in F since $a_i \in F$ and F is a neutrosophic lattice. Therefore, $N = F$. □

Definition 4.3 *Let* N *be a neutrosophic lattice and* L *be a neutrosophic sublattice. If* $x \vee y \in L$ *for all* $x \in N$ *and for all* $y \in L$*, then* L *is called a neutrosophic ideal.*

Theorem 4.4 *The neutrosophic ideal of every neutrosophic lattice is itself.*

Proof Suppose N be a neutrosophic lattice and L be a neutrosophic ideal of N.

(i) Suppose N is $\{1+I, 1, I, 0\}$. L must be $\{1+I, 1, I, 0\}$ since neutrosophic ideal N is also a neutrosophic lattice. Therefore, $N = L$.

(ii) Suppose $N = \{1+I, 1, I, 0, a_i, a_iI\}, (1 \leq i \leq n)$ and $a_iI \in L$ but $a_i \notin L$. Since $a_iI \in L$ and L is a neutrosophic lattice, $a_i \in L$ must hold. On the other hand, $a_iI \in L$ must hold since $a_i \in L$ and L is a neutrosophic lattice. Therefore, $N = L$.

□

Remark 4.5 *As can be seen the above, the definitions of neutrosophic ideal and filter are not flexible definitions to improve new properties and structures. For this reason, we will use the definition of pure neutrosophic lattice to give more flexible neutrosophic filter and ideal definitions.*

Definition 4.6 *A neutrosophic lattice only has neutrosophic coordinates, or if its coordinates other than* 0 *are neutrosophic, it is a pure neutrosophic lattice [6].*

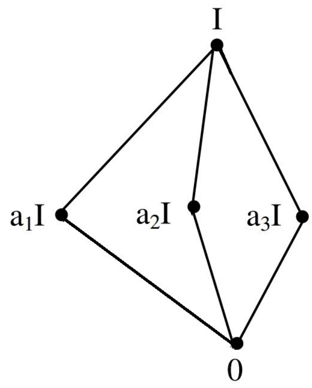

Figure 6: A Hasse neutrosophic diagram of Definition 4.6

The lattice shown in Figure 6 is a pure neutrosophic lattice but it is not a neutrosophic lattice. On the other hand, Figure 6 is a pure neutrosophic lattice of the neutrosophic lattice in Figure 5.

Definition 4.7 *Let* N *be a neutrosophic lattice and* F *be a pure neutrosophic sublattice. If* $x \wedge y \in F$ *for all* $x \in N$ *and for all* $y \in F$*, then* F *is called a neutrosophic pure filter of* N*.*

Example 4.8 *Let* $N(P) = \{0, 1, I \cup 1, a_1, a_2, a_3, a_4, a_1I, a_2I, a_3I, a_4I, I\}$ *be a neutrosophic lattice.* $F = \{0, a_2I, a_3I, a_4I, I\}$ *is a pure neutrosophic sublattice. On the one hand,* F *is a filter and so is a pure neutrosophic filter.*

Definition 4.9 *Let* N *be a neutrosophic lattice and* L *be a pure sublattice. If* $x \vee y \in L$ *for all* $x \in N$ *and for all* $y \in L$*, then* L *is called a neutrosophic pure ideal of* N*.*

In Definition 4.9, the only neutrosophic subideal containing element 0 must also be the lattice itself.

Corollary 4.10 *Every neutrosophic filter is a pure neutrosophic filter, but not vice versa.*

Corollary 4.11 *Every neutrosophic ideal is a pure neutrosophic ideal, but not vice versa.*

5. Conclusion and Future Studies

In this paper, ideal and filter concepts for neutrosophic lattices are introduced. The defined neutrosophic ideal and filter is provided only for the neutrosophic lattice itself. In order to bring more flexible definitions, neutrosophic pure filter and ideal concepts on pure neutrosophic lattices are introduced. Pure neutrosophic lattices provide a rich working area for pure neutrosophic filters because they have to include the 0 element, but they are still rigid for ideals.

In the future, we hope to develop new definitions and gain new fields of study by adhering to the neutrosophic structure for neutrosophic pure ideals.

6. Declaration

This paper is extracted from master thesis entitled "On Lattices of Neutrosophic Sets" in Bitlis Eren University, Türkiye, 2017.

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