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On a fourth-order elliptic Kirchhoff type problem with critical Sobolev exponent

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Abstract

This work is concerned with a class of fourth-order elliptic Kirchhoff type problems involving the critical term. By means of the truncation and the concentration compact argument, for each positive integer k, the existence of k pairs nontrivial solutions is established.

Keywords: Elliptic equations Kirchhoff type problem variational methods.
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1. Introduction and main result

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, with $N \geq 5$. Consider the problem

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = \lambda f(x,u) + |u|^{2^{\star\star}-2}u & \text{in} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$, $2^{\star\star} := \frac{2N}{N-4}$ and M, f are continuous functions satisfying some hypothesis which will be given later.

The presence of the nonlocal term $M\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)$ in (1) causes some mathematical difficulties and so the study of such a class of problems is of much interest. This type of problems are closely related to the following hyperbolic equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

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which was proposed by Kirchhoff [9] as a model to describe the transversal vibrations of a stretched string by considering the subsequent change in string length during the vibrations. Recently, the Kirchhoff type problems with or without critical growth have been investigated by many researchers, we cite here [1, 2, 5, 6, 7, 8, 10, 13, 14, 16]. In [17], Wand and An have used the mountain pass theorem to prove the existence of solutions for the problem

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) & \text{in} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

A more general problem

$$\begin{cases} \Delta(|\Delta|^{p-2}\Delta u) - M\left(\int_\Omega |\nabla u|^p dx\right)\Delta_p u = \lambda f(x,u) & \text{in} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

was considered in [4, 12]. Particularly, the critical case is studied in our previous paper [8]. By the concentration compactness principle of Lions [11] and the ideas of Brezis and Nirenberg [3], sufficient conditions were obtained to the existence of a least one nontrivial solution of the perturbed problem (1) for λ large enough. To our knowledge, the existence of multiple solutions for problem (1) has not studied until now. Motivated by the above results, in this note we are interested in finding multiple solutions by using the variational method, the truncation technique and the concentration compact argument.

Throughout the paper, we assume the following conditions on the Kirchhoff function and the nonlinearity:

(m_1) $M : [0, +\infty) \to (0, +\infty)$ is continuous and increasing;

(f_1) $f(x,t) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is odd in t;

(f_2) $f(x,t) = o(|t|)$ as $t \to 0$, uniformly in Ω;

(f_3) There exists $q \in (2, 2^{\star\star})$ such that $\displaystyle\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{q-2}t} = 0$, uniformly in Ω;

(f_4) There exists $\theta \in (2, 2^{\star\star})$ such that

$$0 < \theta F(x,t) := \theta \int_0^t f(x,s)ds \leq tf(x,t) \text{ for all } x \in \Omega \text{ and } t \in \mathbb{R}\backslash\{0\}.$$

The main result is the following theorem.

Theorem 1.1. *Suppose that (m_1) and $(f_1) - (f_4)$ hold. Then for each positive integer k, there exists $\lambda_k^\star > 0$ such that problem (1) admits a least k pairs nontrivial solutions provided that $\lambda \geq \lambda_k^\star$.*

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2. Auxiliary Results

We look for solutions in the Hilbert space $H := H_0^1(\Omega) \cap H^2(\Omega)$ with the inner product

$$\langle u, v\rangle_H = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v)dx$$

and the norm $||u||_H^2 = \int_\Omega (|\Delta u|^2 + |\nabla u|^2)dx$. Denote $|u|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$ for $u \in L^p(\Omega)$.

Let $a \in \left(M(0), \frac{\theta}{2}M(0)\right)$. Then by (m_1), there is $t_0 > 0$ such that $M(t_0) = a$, so let us define

$$M_a(t) = \begin{cases} M(t) & \text{if} \quad 0 \leq t \leq t_0 \\ a & \text{if} \quad t \geq t_0. \end{cases}$$

Replacing M with M_a, problem (1) turns into

$$\begin{cases} \Delta^2 u - M_a\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) + |u|^{2^{\star\star}-2}u & \text{in} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3}$$

The energy functional associated to (3) is given by

$$I_{\lambda,a}(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \frac{1}{2}\widehat{M}_a\left(\int_\Omega |\nabla u|^2 dx\right) - \lambda\int_\Omega F(x,u)dx - \frac{1}{2^{\star\star}}\int_\Omega |u|^{2^{\star\star}} dx,$$

where $\widehat{M}_a(t) = \int_0^t M_a(s)ds$. By the above assumptions, $I_{\lambda,a} \in C^1(H)$ and for all $u, v \in H$

$$\begin{aligned} \langle I'_{\lambda,a}(u), v\rangle &= \int_\Omega \Delta u \Delta v dx + M_a\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla v dx \\ &\quad -\lambda\int_\Omega f(x,u)v dx - \int_\Omega |u|^{2^{\star\star}-2}uv dx. \end{aligned}$$

Lemma 2.1. *Suppose that (m_1), (f_1) and $(f_3)-(f_4)$) hold. Then $I_{\lambda,a}$ satisfies the $(PS)_c$ condition at every level $c < c^\star$, where*

$$c^\star = \min\left\{\left(\frac{1}{\theta} - \frac{1}{2^{\star\star}}\right)S_\star^{\frac{N}{4}}, \left(\frac{M(0)}{2} - \frac{a}{\theta}\right)t_0\right\},$$

where $S_\star := \inf_{u\in H\setminus\{0\}} \frac{||u||_H^2}{|u|_{2^{\star\star}}^2}$.

Proof. Let $\{u_n\} \subset H$ be a sequence such that $I_{\lambda,a}(u_n) \to c < c^\star$ and $I'_{\lambda,a}(u_n) \to 0$. By definition of M_a, we have $M_a(t) \le a$ and $M(0)t \le \widehat{M}_a(t)$ for all $t \ge 0$. Therefore by (f_4)

$$\begin{aligned} c + o_n(1) + o_n(1)||u_n||_H &= I_{\lambda,a}(u_n) - \frac{1}{\theta}\langle I'_{\lambda,a}(u_n), u_n\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_\Omega |\Delta u_n|^2 dx + \left(\frac{M(0)}{2} - \frac{a}{\theta}\right)\int_\Omega |\nabla u_n|^2 dx \\ &\ge \min\left\{\left(\frac{1}{2} - \frac{1}{\theta}\right), \left(\frac{M(0)}{2} - \frac{a}{\theta}\right)\right\}||u_n||_H^2. \end{aligned}$$

This shows that $\{u_n\}$ is bounded in H. Then up to subsequence, for some $u \in H$,

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H, \\ &u_n \to u \text{ a.e. in } \Omega, \\ &u_n \to u \text{ in } L^r(\Omega) \text{ for all } r \in [1, 2^{\star\star}), \\ &|\Delta u_n|^2 \rightharpoonup \mu \text{ weakly in the sense of measures}, \\ &|u_n|^{2^{\star\star}} \rightharpoonup \nu \text{ weakly in the sense of measures}, \end{aligned} \tag{4}$$

where μ and ν are nonnegative bounded measures on $\overline{\Omega}$. Applying concentration compact result due to Lions [11], we can find at most countable index set J and elements $\{x_j\}_{j\in J}$ of $\overline{\Omega}$ such that

$$\begin{cases} \nu = |u|^{2^{\star\star}} + \sum_{j\in J}\nu_j\delta_{x_j}, & \nu_j > 0, \\ \mu \ge |\Delta u|^2 + \sum_{j\in J}\mu_j\delta_{x_j}, & \mu_j > 0 \\ S_\star \nu_j^{2/2^{\star\star}} \le \mu_j \text{ for all } j \in J. \end{cases} \tag{5}$$

We claim that $\nu_j \geq S_\star^{\frac{N}{4}}$ for all $j \in J$. Let $j \in J$ be fixed and for an arbitrary $\varepsilon > 0$, choose ϕ_ε of $C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \phi_\varepsilon \leq 1$,

$$\phi_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in B(x_j, \varepsilon) \\ 0 & \text{if} \quad x \notin B(x_j, 2\varepsilon) \end{cases}$$

and $|\nabla \phi_\varepsilon|_\infty \leq \frac{2}{\varepsilon}$ and $|\Delta \phi_\varepsilon|_\infty \leq \frac{2}{\varepsilon^2}$. Clearly $\langle I'_{\lambda,a}(u_n), \phi_\varepsilon u_n \rangle = o_n(1)$, that is

$$\begin{aligned} &\int_\Omega |\Delta u_n|^2 \phi_\varepsilon dx + \int_\Omega \Delta u_n \left(2\nabla u_n \nabla \phi_\varepsilon + u_n \Delta \phi_\varepsilon\right) dx \\ &+ M_a \left(\int_\Omega |\nabla u_n|^2 dx\right) \left(\int_\Omega u_n \nabla u_n \nabla \phi_\varepsilon dx + \int_\Omega \phi_\varepsilon |\nabla u_n|^2 dx\right) \\ &= \lambda \int_\Omega f(x, u_n) u_n \phi_\varepsilon dx + \int_\Omega |u_n|^{2^{\star\star}} \phi_\varepsilon dx + o_n(1). \end{aligned} \tag{6}$$

Observe that

$$\int_\Omega |\nabla(u_n - u)|^2 dx = -\int_\Omega (u_n - u)\Delta(u_n - u) dx \leq |u_n - u|_2 ||u_n - u||_H,$$

thus (4) yields

$$\nabla u_n \to \nabla u \text{ in } L^2(\Omega). \tag{7}$$

Set

$$A^1_{n,\varepsilon} := \int_\Omega u_n \nabla u_n \nabla \phi_\varepsilon dx, \; A^2_{n,\varepsilon} := \int_\Omega \Delta u_n \nabla u_n \nabla \phi_\varepsilon dx, \; A^3_{n,\varepsilon} := \int_\Omega u_n \Delta u_n \Delta \phi_\varepsilon dx.$$

By the Hölder inequality, we have

$$\begin{aligned} \limsup_{n\to\infty} \left|A^1_{n,\varepsilon}\right| &\leq \limsup_{n\to\infty} \left(\int_\Omega |\nabla u_n|^2\right)^{\frac{1}{2}} \left(\int_\Omega |u_n|^2 |\nabla \phi_\varepsilon|^2 dx\right)^{\frac{1}{2}} \\ &\leq C \left(\int_\Omega |u|^2 |\nabla \phi_\varepsilon|^2 dx\right)^{\frac{1}{2}} \\ &\leq C \left(\int_{B(x_j,\varepsilon)} |\nabla \phi_\varepsilon|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \left(\int_{B(x_j,\varepsilon)} |u|^{2^{\star\star}} dx\right)^{\frac{1}{2^{\star\star}}} \\ &\leq C |\nabla \phi_\varepsilon|_\infty w_N^{\frac{2}{N}} \varepsilon^2 \left(\int_{B(x_j,\varepsilon)} |u|^{2^{\star\star}} dx\right)^{\frac{1}{2^{\star\star}}} \to 0 \text{ as } \varepsilon \to 0 \end{aligned} \tag{8}$$

where w_N is the volume of $B(0,1)$. In the same way

$$\begin{aligned} \limsup_{n\to\infty} \left|A^2_{n,\varepsilon}\right| &\leq \limsup_{n\to\infty} \left(\int_\Omega |\Delta u_n|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_n|^2 |\nabla \phi_\varepsilon|^2 dx\right)^{\frac{1}{2}} \\ &\leq C \left(\int_\Omega |\nabla u|^2 |\nabla \phi_\varepsilon|^2 dx\right)^{\frac{1}{2}} \\ &\leq C \left(\int_{B(x_j,\varepsilon)} |\nabla \phi_\varepsilon|^N dx\right)^{\frac{1}{N}} \left(\int_{B(x_j,\varepsilon)} |\nabla u|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}} \\ &\leq C |\nabla \phi_\varepsilon|_\infty w_N^{\frac{1}{N}} \varepsilon \left(\int_{B(x_j,\varepsilon)} |\nabla u|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}} \to 0 \text{ as } \varepsilon \to 0 \end{aligned} \tag{9}$$

and

$$\begin{aligned}
\limsup_{n\to\infty}\left|A^3_{n,\varepsilon}\right| &\le \limsup_{n\to\infty}\left(\int_\Omega|\Delta u_n|^2\right)^{\frac{1}{2}}\left(\int_\Omega|u_n|^2|\Delta\phi_\varepsilon|^2dx\right)^{\frac{1}{2}}\\
&\le C\left(\int_\Omega|u|^2|\Delta\phi_\varepsilon|^2dx\right)^{\frac{1}{2}}\\
&\le C\left(\int_{B(x_j,\varepsilon)}|\Delta\phi_\varepsilon|^{\frac{N}{2}}dx\right)^{\frac{2}{N}}\left(\int_{B(x_j,\varepsilon)}|u|^{2^{\star\star}}dx\right)^{\frac{1}{2^{\star\star}}}\\
&\le C|\Delta\phi_\varepsilon|_\infty w_N^{\frac{2}{N}}\varepsilon^2\left(\int_{B(x_j,\varepsilon)}|u|^{2^{\star\star}}dx\right)^{\frac{1}{2^{\star\star}}}\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}\tag{10}$$

By (4), (6) -(7), continuity of f and $(f_2)-(f_3)$, we have

$$\begin{aligned}
&\int_\Omega\phi_\varepsilon d\mu+\limsup_{n\to\infty}\left(2A^2_{n,\varepsilon}+A^3_{n,\varepsilon}+M_a\left(\int_\Omega|\nabla u_n|^2dx\right)A^1_{n,\varepsilon}\right)+M(0)\int_\Omega\phi_\varepsilon|\nabla u|^2dx\\
&\le\lambda\int_\Omega f(x,u)u\phi_\varepsilon+\int_\Omega\phi_\varepsilon d\nu.
\end{aligned}\tag{11}$$

From (8)-(10), we see that

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\left(2A^2_{n,\varepsilon}+A^3_{n,\varepsilon}+M_a\left(\int_\Omega|\nabla u_n|^2dx\right)A^1_{n,\varepsilon}\right)=0.$$

Letting $\varepsilon\to0$ in (11), we obtain $\mu_j\le\nu_j$. Therefore (5) implies $S_\star^{\frac{N}{4}}\le\nu_j$.

Now we prove that J is empty. Assume by contradiction that there is some $j\in J$. Then

$$\begin{aligned}
c+o_n(1) &= I_{\lambda,a}(u_n)-\frac{1}{\theta}\langle I'_{\lambda,a}(u_n),u_n\rangle\ge\left(\frac{1}{\theta}-\frac{1}{2^{\star\star}}\right)\int_\Omega|u_n|^{2^{\star\star}}dx\\
&\ge \left(\frac{1}{\theta}-\frac{1}{2^{\star\star}}\right)\int_\Omega\phi_\varepsilon|u_n|^{2^{\star\star}}dx,
\end{aligned}$$

therefore let $n\to+\infty$

$$c\ge\left(\frac{1}{\theta}-\frac{1}{2^{\star\star}}\right)\nu_j\ge\left(\frac{1}{\theta}-\frac{1}{2^{\star\star}}\right)S_\star^{\frac{N}{4}}\ge c^\star,$$

which is impossible and hence $J=\emptyset$. It follows that $u_n\to u$ in $L^{2^{\star\star}}(\Omega)$, thus

$$\lim_{n\to\infty}\int_\Omega|u_n|^{2^{\star\star}-2}u_n(u_n-u)dx=0.\tag{12}$$

On the other hand, it not difficult to see that

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)dx=0.\tag{13}$$

Since $\langle I'(u_n),u_n-u\rangle=o_n(1)$, by continuity of M, (7) and (12)-(13) we deduce that

$$\lim_{n\to\infty}\int_\Omega\Delta u_n\Delta(u_n-u)dx=0.$$

Similarly, we also obtain

$$\lim_{n\to\infty}\int_\Omega\Delta u\Delta(u_n-u)dx=0.$$

So that $\Delta u_n\to\Delta u$ in $L^2(\Omega)$. From this and (7) we conclude that $||u_n||_H\to||u||_H$. Finally $u_n\to u$ in H. $\square$

3. Proof of Theorem 1.1

To this end, we need to ensure that $I_{\lambda,a}$ satisfies the conditions of the following version of Symmetric Mountain Pass theorem [15].

Theorem 3.1. *Let $H = V \oplus W$ be a real Banach space with $dimV < \infty$. Assume that $I \in C^1(H, \mathbb{R})$ is an even functional verifying $I(0) = 0$ and*

(i) there exist $\alpha, \rho > 0$ such that

$$\inf_{u \in \partial B_\rho(0) \cap W} I(u) \geq \alpha;$$

(ii) there exists a subspace $E \subset H$ such that $dimV < dimE$ and

$$\max_{u \in E} I(u) \leq \beta \text{ for some } \beta > 0;$$

(iii) the functional I satisfies $(PS)_c$ for every $c \in (0, \beta)$.

Then I admits at least $dimE - dimV$ pairs nontrivial critical points.

Lemma 3.1. *Assume that $(f_1) - (f_3)$ hold. Then for each $\lambda > 0$, there exist $\alpha_\lambda, \rho_\lambda > 0$ such that*

$$\inf_{u \in \partial B_{\rho_\lambda}(0)} I(u) \geq \alpha_\lambda.$$

Proof. By $(f_2) - (f_3)$ and the continuity of f, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon |t|^2 + C_\varepsilon |t|^q \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

It follows from Sobolev's embeddings that

$$\begin{aligned} I_{\lambda,a}(u) &\geq \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \frac{M(0)}{2}\int_\Omega |\nabla u|^2 dx - \lambda\varepsilon |u|_2^2 - \lambda C_\varepsilon |u|_q^q - \frac{1}{2^{\star\star}}|u|_{2^{\star\star}}^{2^{\star\star}} \\ &\geq \left(\frac{\min(1, M(0))}{2} - \lambda d_1 \varepsilon\right) ||u||_H^2 - \lambda d_2 C_\varepsilon ||u||_H^q - d_3 ||u||_H^{2^{\star\star}}. \end{aligned}$$

Since $2 < q < 2^{\star\star}$, the desired result follows by choosing ε small enough. □

Lemma 3.2. *Assume that $(f_1) - (f_4)$ hold. Then, for each positive integer k and $\beta > 0$, there exists $\lambda_k > 0$ such that for any $\lambda \geq \lambda_k$, there is a k-dimensional subspace $E_{k,\lambda} \subset H$ satisfying*

$$\max_{u \in E_{k,\lambda}} I(u) < \beta.$$

Proof. Let $x_i \in \Omega,\ i = 1, ..., k$ and $0 < \varepsilon < 1$ such that $B(x_i, \varepsilon) \subset \Omega$ for all $i = 1, ..., k$ and $B(x_i, \varepsilon) \cap B(x_j, \varepsilon) = \emptyset$ for all $i \neq j$. Let $\phi \in C_0^\infty(B(0,1))$ and put $\phi_\varepsilon^i(x) = \phi\left(\frac{x - x_i}{\varepsilon}\right)$. Then

$$\frac{||\phi_\varepsilon^i||_H^2}{|\phi_\varepsilon^i|_\theta^2} = \frac{\varepsilon^{N-4}|\Delta\phi|_2^2 + \varepsilon^{N-2}|\nabla\phi|_2^2}{\varepsilon^{\frac{2N}{\theta}}|\phi|_\theta^2} \leq \varepsilon^{N-4-\frac{2N}{\theta}} \frac{||\phi||_H^2}{|\phi|_\theta^2} =: \zeta_\varepsilon. \tag{14}$$

Denote $V_{k,\varepsilon} = \text{span}\{\phi_\varepsilon^1, ..., \phi_\varepsilon^k\}$. Since all norms in $\mathbb{R}^k$ are equivalent, there is $C_e > 0$ such that for any $u = \sum_{i=1}^k \gamma_i \phi_\varepsilon^i \in V_{k,\varepsilon}$,

$$|u|_\theta^\theta = \int_{\bigcup_{i=1}^k B(x_i,\varepsilon)} \left|\sum_{i=1}^k \gamma_i \phi_\varepsilon^i\right|^\theta dx = \sum_{i=1}^k |\gamma_i \phi_\varepsilon^i|_\theta^\theta \geq C_e \left(\sum_{i=1}^k |\gamma_i \phi_\varepsilon^i|_\theta^2\right)^{\frac{\theta}{2}}.$$

Combining this last inequality with (14), we get

$$|u|_\theta^\theta \geq C_e \left(\frac{1}{\zeta_\varepsilon} \sum_{i=1}^{k} ||\gamma_i \phi_\varepsilon^i||_H^2 \right)^{\frac{\theta}{2}} = \frac{C_e}{\zeta_\varepsilon^{\frac{\theta}{2}}} ||u||_H^\theta = C_e \varepsilon^{N+2\theta-\frac{N\theta}{2}} \frac{|\phi|_\theta^\theta}{||\phi||_H^\theta} ||u||_H^\theta$$
$$=: \sigma \varepsilon^{N+2\theta-\frac{N\theta}{2}} ||u||_H^\theta \tag{15}$$

By (f_4), for some $C_1, C_2 > 0$ we have

$$F(x,t) \geq C_1 |t|^\theta - C_2 \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

Then from (15) we entail

$$\begin{aligned} I_{\lambda,a}(u) &\leq \frac{\max(1,a)}{2} ||u||_H^2 - \lambda \int_\Omega F(x,u)dx = \frac{\max(1,a)}{2} ||u||_H^2 - \lambda \sum_{i=1}^{k} \int_{B(x_i,\varepsilon)} F(x,u)dx \\ &\leq \frac{\max(1,a)}{2} ||u||_H^2 - \lambda C_1 \sigma \varepsilon^{N+2\theta-\frac{N\theta}{2}} ||u||_H^\theta + \lambda C_2 k w_N \varepsilon^N. \end{aligned} \tag{16}$$

Let $\eta \in \left(N + 2\theta - \frac{N\theta}{2}, N\right)$ and set

$$g_{k,\varepsilon}(t) = \frac{\max(1,a)}{2} t^2 - C_1 \sigma \varepsilon^{\left(N+2\theta-\frac{N\theta}{2}\right)-\eta} t^\theta + C_2 k w_N \varepsilon^{N-\eta}.$$

The function $g_{k,\varepsilon}$ attains the maximal value at $t_\varepsilon := \left(\frac{\max(1,a)}{\theta C_1 \sigma} \varepsilon^{\eta-\left(N+2\theta-\frac{N\theta}{2}\right)} \right)^{\frac{1}{\theta-2}}$. Therefore for all $t \geq 0$,

$$g_{k,\varepsilon}(t) \leq g_{k,\varepsilon}(t_\varepsilon) \to 0 \text{ as } \varepsilon \to 0^+,$$

thus for given $\beta > 0$, we can find $\varepsilon_k \in (0,1)$ such that

$$g_{k,\varepsilon}(t) < \beta \text{ for all } t \geq 0 \text{ and } \varepsilon \in (0, \varepsilon_k]. \tag{17}$$

Let now $\lambda \geq \lambda_k := \varepsilon_k^{-\eta}$ and consider the k-subspace $E_{k,\lambda} := V_{k,\varepsilon}$ with $\varepsilon = \lambda^{-\frac{1}{\eta}}$. Since $\varepsilon \leq \varepsilon_k$, by (16)-(17), for all $u \in E_{k,\lambda}$ we have

$$I_{\lambda,a}(u) \leq \frac{\max(1,a)}{2} ||u||_H^2 - \varepsilon^{-\eta} C_1 \sigma \varepsilon^{N+2\theta-\frac{N\theta}{2}} ||u||_H^\theta + \varepsilon^{-\eta} C_2 k w_N \varepsilon^N = g_{k,\varepsilon}(||u||_H) < \beta.$$

□

Proof of Theorem 1.1. Let $V = \{0\}$ and $W = H$. Obviously $I_{\lambda,a}$ is an even functional and $I_{\lambda,a}(0) = 0$. In view of Lemma 3.1, $I_{\lambda,a}$ satisfies condition Theorem 3.1 (i). Let $k \in \mathbb{N}^*$ and $0 < \beta < c^\star$ with $c^\star$ is given in Lemma 2.1. According to Lemma 2.1, for any $\lambda \geq \lambda_k$, $I_{\lambda,a}$ verifies the $(PS)_c$ for all $c \in (0, \beta)$, so condition Theorem 3.1 (iii) follows. Moreover, the condition Theorem 3.1 (ii) holds true for the k-subspace $E_{k,\lambda}$. Applying Theorem 3.1, $I_{\lambda,a}$ has k pairs nontrivial critical points. Let u a critical point of $I_{\lambda,a}$. Then

$$\left(\frac{M(0)}{2} - \frac{a}{\theta} \right) t_0 \geq c^\star > \beta \geq I_{\lambda,a}(u) - \frac{1}{\theta} \langle I'_{\lambda,a}(u), u \rangle \geq \left(\frac{M(0)}{2} - \frac{a}{\theta} \right) \int_\Omega |\nabla u|^2 dx.$$

Therefore $\int_\Omega |\nabla u|^2 dx < t_0$ and hence $M_a \left(\int_\Omega |\nabla u|^2 dx \right) = M \left(\int_\Omega |\nabla u|^2 dx \right)$. It follows that u is a solution of (1). The proof of Theorem 1.1 is complete.

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