Generalization of $z$-ideals in right duo rings

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Abstract

The aim of this paper is to generalize the notion of $z$-ideals to arbitrary noncommutative rings. A left (right) ideal $I$ of a ring $R$ is called a left (right) $z$-ideal if $M_a \subseteq I$, for each $a \in I$, where $M_a$ is the intersection of all maximal ideals containing $a$. For every two left ideals $I$ and $J$ of a ring $R$, we call $I$ a left $z_J$-ideal if $M_a \cap J \subseteq I$, for every $a \in I$, whenever $J \nsubseteq I$ and $I$ is a $z_J$-ideal, we say that $I$ is a left relative $z$-ideal. We characterize the structure of them in right duo rings. It is proved that a duo ring $R$ is von Neumann regular ring if and only if every ideal of $R$ is a $z$-ideal. Also, every one sided ideal of a semisimple right duo ring is a $z$-ideal. We have shown that if $I$ is a left $z_J$-ideal of a $p$-right duo ring, then every minimal prime ideal of $I$ is a left $z_J$-ideal. Moreover, if every proper ideal of a $p$-right duo ring $R$ is a left relative $z$-ideal, then every ideal of $R$ is a $z$-ideal.

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1. Introduction

Throughout this article all rings are associative with identity. The notion of $z$-ideals which are both algebraic and topological objects was first introduced in [6] by Kohls. These ideals play a fundamental role in studying the ideal structure of $C(X)$, the ring of real-valued continuous functions on a completely regular Hausdorff space $X$, see [6]. Although in [6], he defined these ideals topologically, in terms of zero-sets, he showed that they can be characterized algebraically. Gillman and Jerison in [4], have proved it to be a powerful tool in the study of both algebraic properties of function rings and topological properties of Tychonoff spaces.

It was Mason [11], who initiated the study of $z$-ideals in arbitrary commutative rings with identity. An ideal $I$ of a commutative ring $R$ is called a $z$-ideal ($z^\circ$-ideal) if for each $a \in I$, the intersection of all maximal ideals (minimal prime ideals) containing $a$
is contained in $I$. A. Rezaei Aliabad and R. Mohamadian in [12], characterized the $z$-ideals and $z^0$-ideals of formal power series ring on a commutative ring. They showed that if $R$ is a commutative ring, then an ideal $I$ of formal power series ring $R[[x]]$ is a $z$-ideal if and only if $I = (J, x)$, where $J$ is a $z$-ideal of $R$. Also, they characterized a relation between the set of $z^0$-ideals of $R[[x]]$ and the set of $z^0$-ideals of $R$.

Let $I$ and $J$ be two ideals of a commutative ring $R$. $I$ is said to be a $z_J$-$ideal$ if $M_a \cap J \subseteq I$, for every $a \in I$, where $M_a$ is the intersection of all maximal ideals containing $a$. Whenever $J \not\subseteq I$ and $I$ is a $z_J$-$ideal$, we say that $I$ is a $relative \ z$-$ideal$. This special kind of $z$-ideals introduced and investigated by F. Azarpanah and A. Taherifar in [2]. They have shown that for any ideal $J$ in $C(X)$, the sum of every two $z_J$-ideals is a $z_J$-$ideal$ if and only if $X$ is an $F$-space, where the $F$-space is a space for which every finitely generated ideal of $C(X)$ is principal. A space $X$ is called $P$-$space$ if every prime ideal in $C(X)$ is a $z$-$ideal$. It is in [2] shown that every principal ideal in $C(X)$ is a relative $z$-$ideal$ if and only if $X$ is a $P$-space. Also, they characterized the space $X$ for which the sum of every two relative $z$-ideals of $C(X)$ is a relative $z$-$ideal$. If $I$ is an ideal of a semisimple ring and $Ann(I) \neq 0$, A. R. Aliabad and F. Azarpanah and A. Taherifar in [1], have shown that $I$ is a relative $z$-$ideal$ and the converse is also true for each finitely generated ideal in $C(X)$.

These ideals are also studied further by others in commutative rings. In the following, we present a generalization of $z$-ideals to noncommutative rings and investigate the structure of them in right duo rings, which are rings in which every right ideal is a two-sided ideal. In fact, we generalize the results in [1] to right duo rings. This paper is organized as follows:

In the second section, we study some properties of ideals in right duo rings. In the third section, we shall generalize the concept of $z$-ideal to noncommutative rings and we study their structure in right duo rings. We show that every $z$-ideal of a right duo ring is semiprime. Mason in [10], showed that if $I$ is a $z$-ideal of a semisimple commutative ring, then every minimal prime ideal of $I$ is also a $z$-ideal. In a right duo ring, we consider sufficient conditions that every minimal prime ideal of a $z$-ideal is also a $z$-ideal. We will show that every ideal of von Neumann right duo rings is a $z$-ideal. Also, if every left ideal of a right duo ring $R$ is a $z$-ideal, then $R$ is a von Neumann ring. Furthermore, every left ideal of a semisimple right duo ring is a $z$-ideal.

In the fourth section, we generalize left relative $z$-ideals to noncommutative rings. We define the concept of $p$-right duo rings to obtain equivalent condition to minimal prime ideals of an ideal, and then study left relative $z$-ideals of their rings. We will present sufficient conditions in order that if every proper ideal of a ring $R$ is a left relative $z$-ideal, then every ideal of $R$ is a $z$-ideal.

Let us close this section by mentioning some symbols. Let $R$ be a ring and $I$ an ideal of $R$. The set of all prime ideals of $R$ is denoted by $\text{Spec}(R)$. Also, $\text{Min}(I)$ is the set of all minimal prime ideals containing $I$, for each ideal $I$ of $R$, and the Jacobson radical of $R$ is denoted by $\text{rad}(R)$.

### 2. Some properties of structure of right duo rings

Recall that a ring $R$ is called a right duo ring if each right ideal of $R$ is a two sided ideal. We can similarly define the notion of a left duo ring. A ring $R$ is said to be a duo ring if $R$ is a right and left duo ring. Commutative rings and division rings are clearly duo ring. Furthermore, any valuation ring arising from a Krull valuation of a division ring is always duo ring, see [8, Exercise 19.9]. It is easily seen that any finite direct product of a right duo ring is a right duo ring. Proposition 1.1 of [3] says that any homomorphic image of a right duo ring is a right duo ring, and so is any factor ring of it. Gerg Marks in Proposition 5 of [9] shows that any power series ring of a right self injective von Neumann right duo ring is a right duo ring. In particular, the power series ring of a division ring is a right
Let $2.1$ be a right duo ring and $x \in R$. Then

1. $RxR = xR$.
2. $Rx \subseteq xR$.

We know that a ring $R$ is called a Dedekind-finite ring if whenever $x, y \in R$ and $xy = 1$, then $yx = 1$. Now, we assume that $R$ is a right duo ring and $ab = 1$, for some $a, b \in R$. Then there exists an element $r \in R$ such that $ab = br$, by Lemma 2.1. Hence, we have $a = a.1 = a(ab) = a(br) = (ab)r = 1.r = r$, and so $1 = ba$. Therefore, every right duo ring is Dedekind-finite, see [8, Theorem 3.2].

It is well known that if $P$ is a prime ideal of a right duo ring and $xy \in P$, then $x \in P$ or $y \in P$, because $xy \in P$ implies that $xRy \subseteq xyR \subseteq P$, by Lemma 2.1. Since $P$ is a prime ideal, we have $x \in P$ or $y \in P$. Therefore, we have the following Lemma:

**Lemma 2.2.** Let $R$ be a right duo ring and $P$ be a proper ideal of $R$. Then the following statements are equivalent:

1. $P$ is a prime ideal.
2. For every $x, y \in R$, if $xy \in P$ then $x \in P$ or $y \in P$.

Therefore, if $P$ is a prime ideal of $R$ and $x^n \in P$, for some $x \in R$ and $n \in \mathbb{N}$, then $x \in P$.

Let $R$ be a ring and $I$ be an ideal of $R$. We denote by $\sqrt{I}$ the subset

$$\{r \in R \mid \exists n \in \mathbb{N}, r^n \in I\}$$

of $R$. It is easily seen from Lemma 2.1 that if $P$ is a prime ideal of a right duo ring $R$, then $\sqrt{P} = P$.

**Lemma 2.3.** Let $R$ be a right duo ring and $I$ and $J$ be ideals of $R$. Then

$$\sqrt{I} + \sqrt{J} \subseteq \sqrt{I + J}.$$

**Proof.** Let $a \in \sqrt{I}$ and $b \in \sqrt{J}$. Then there exist $n, m \in \mathbb{N}$ such that $a^m \in I$ and $b^n \in J$. Now, we claim that $(a + b)^{m+n} \in I + J$. In fact, $(a + b)^{m+n}$ is the sum of $2^{m+n}$ elements of the form $f = c_1c_2 \cdots c_{m+n}$ where each $c_i = a$ or $b$. If at least $m$ of these $c_i$s are $a$, then there exists $a' \in R$ such that $f = a^ma'$, by Lemma 2.1, and so $f \in I$, because $a^m \in I$. If the number of the $c_i = a$ is smaller than $m$, then at least $n$ of them are $b$, and hence there exists $b' \in R$ such that $f = b^nb'$, by Lemma 2.1. Thus $f \in J$, because $b^n \in J$. Therefore $(a + b)^{m+n} \in I + J$.

**Proposition 2.4.** Let $R$ be a right duo ring and $I$ be an ideal of $R$. Then $\sqrt{I}$ is an ideal of $R$.

**Proof.** Clearly, $0 \in \sqrt{I}$. If $a, b \in \sqrt{I}$, then $a + b \in \sqrt{I}$, by Lemma 2.3. Now, assume that $a \in \sqrt{I}$ and $r \in R$. Hence there exists $n \in \mathbb{N}$ such that $a^n \in I$, and so there exists an element $r' \in R$ such that $(ra)^n = a^n r' \in I$, by Lemma 2.1. Therefore $ra \in \sqrt{I}$ and similarly we show that $ar \in \sqrt{I}$.

Let $R$ be a ring and $a \in R$. The intersection of all maximal ideals of $R$ containing $a$ will be denoted by $M_a$. We set $M_a = R$ when $a$ is a unit.

**Lemma 2.5.** Let $R$ be a right duo ring and $a, b \in R$. Then $M_{ab} = M_a \cap M_b$. In particular, if $a \in R$ we conclude that $M_{an} = M_a$, for every $n \in \mathbb{N}$. 

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Proof. Clearly, for every \( x \in M_a \) we have \( M_x \subseteq M_a \). Thus \( M_{ab} \subseteq M_a \) and \( M_{ab} \subseteq M_b \), and so \( M_{ab} \subseteq M_a \cap M_b \), for each \( a, b \in R \). Conversely, let \( x \in M_a \cap M_b \). We show that every maximal ideal containing \( ab \) is also containing \( x \). Assume that \( N \) is a maximal ideal of \( R \) such that \( ab \in N \). Then \( a \in N \) or \( b \in N \), by Lemma 2.2. If \( a \in N \), then \( x \in M_a \cap M_b \subseteq M_a \subseteq N \). If \( b \in N \), then \( x \in M_b \subseteq N \). Therefore \( x \in N \). Thus \( M_a \cap M_b \subseteq M_{ab} \), and consequently \( M_{ab} = M_a \cap M_b \). \( \square \)

3. Generalization of \( z \)-ideals in a right duo ring

The \( z \)-ideals are studied further in commutative rings. These ideals are useful concept in studying the ideal structure of the ring \( C(X) \) of continuous real-valued functions on a topological space \( X \). In the following, we shall present a generalization of \( z \)-ideals to noncommutative rings.

Definition 3.1. A left (right) ideal \( I \) of a ring \( R \) is called a left (right) \( z \)-ideal if \( M_a \subseteq I \), for all \( a \in I \).

In the following we show that every one sided \( z \)-ideal is an ideal.

Proposition 3.2. Let \( R \) be a ring and \( I \) be a left (right) \( z \)-ideal of \( R \). Then \( I \) is an ideal of \( R \).

Proof. Let \( I \) be a left \( z \)-ideal of \( R \) and \( a \in I \). If \( N \) is a maximal ideal of \( R \) containing \( a \), then \( ar \in N \) for every \( r \in R \). Thus \( M_{ar} \subseteq M_a \). On the other hand, since \( I \) is a left \( z \)-ideal, we have \( M_a \subseteq I \). Therefore, \( ar \in M_{ar} \subseteq M_a \subseteq I \), and so \( ar \in I \). Hence \( I \) is an ideal. \( \square \)

Here in after a left (right) \( z \)-ideal of a ring is called a \( z \)-ideal, by Proposition 3.2.

Example 3.3. Every intersection of maximal ideals of a ring \( R \) is a \( z \)-ideal. In fact, every intersection of \( z \)-ideals is a \( z \)-ideal.

Lemma 3.4. Let \( R \) be a ring and \( I \) be a left (right) ideal of \( R \). Then the following statements are equivalent:

1. \( I \) is a \( z \)-ideal.
2. For every \( a \in R \) and \( b \in I \), if \( M_a \subseteq M_b \) then \( a \in I \).

Proof. \( 1 \Rightarrow 2 \). Let \( a \in R \) and \( b \in I \). Since \( I \) is a \( z \)-ideal and \( b \in I \), we have \( M_b \subseteq I \). Hence, if \( M_a \subseteq M_b \), then \( a \in M_a \subseteq M_b \subseteq I \), and so \( a \in I \).

\( 2 \Rightarrow 1 \). Let \( I \) be a left ideal and \( a \in I \). For each \( x \in M_a \), we have \( M_x \subseteq M_a \). By hypothesis \( x \in I \), and so \( M_a \subseteq I \). Therefore \( I \) is a \( z \)-ideal. \( \square \)

Let \( R \) be a ring and \( I \) be a left (right) ideal of \( R \). The intersection of all \( z \)-ideals containing \( I \) will be denoted by \( I_z \). For each element \( a \in I_z \) and for every \( z \)-ideal \( J \) of \( R \) containing \( I \), we have \( a \in J \). Then \( M_a \subseteq J \), and so \( M_a \subseteq I_z \). Therefore, we have the following Lemma:

Lemma 3.5. For every left (right) ideal \( I \) of a ring \( R \), the intersection of all \( z \)-ideals containing \( I \), which is denoted by \( I_z \), is a \( z \)-ideal. In particular, \( I_z \) is the smallest \( z \)-ideal containing \( I \).

Lemma 3.6. Let \( R \) be a ring. Then the following statements hold.

1. For every left ideals \( I \) and \( J \) of \( R \), if \( I \subseteq J \), then \( I_z \subseteq J_z \).
2. If \( \{I_\lambda\}_{\lambda \in \Lambda} \) is any family of left ideals of \( R \), then
   \[
   \bigcap_{\lambda \in \Lambda} I_\lambda \subseteq \bigcap_{\lambda \in \Lambda} (I_\lambda)_z.
   \]

Proof. 1. Since every \( z \)-ideal containing \( J \) contains \( I \), we see \( I_z \subseteq J_z \).

2. For every \( \mu \in \Lambda \), we have \( \bigcap_{\lambda \in \Lambda} I_\lambda \subseteq I_\mu \). Hence our claim is true, by part (1). \( \square \)
It is immediate that for every $z$-ideal $I$, we have $I_z = I$. In the next two Propositions, we study the structure of $z$-ideals in right duo rings.

**Proposition 3.7.** Let $R$ be a right duo ring and $I$ be an ideal of $R$. Then $I \subseteq \sqrt{I} \subseteq I_z$.

**Proof.** Clearly, $I \subseteq \sqrt{I}$. Now, we assume that $x \in \sqrt{I}$ and $J$ is a $z$-ideal containing $I$. Thus there is a positive integer $n$ such that $x^n \in I \subseteq J$. Hence $x \in M_x = M_{x^n} \subseteq J$, by Lemma 2.5. Therefore $x \in I_z$, and so $\sqrt{I} \subseteq I_z$. $\square$

Recall that a proper ideal $I$ of a ring $R$ is said to be a semiprime ideal if for every ideal $J$ of $R$, $J^2 \subseteq I$ implies that $J \subseteq I$. As an immediate consequence of Proposition 3.7 and [7, Theorem 10.11], we get the following result

**Corollary 3.8.** Let $R$ be a right duo ring and $I$ be a $z$-ideal of $R$. Then $\sqrt{I} = I$. In particular, $I$ is a semiprime ideal of $R$.

**Proposition 3.9.** Let $R$ be a right duo ring and $I$ be an ideal of $R$. Then the following statements hold.

1. $(\sqrt{I})_z = I_z$.
2. If $I$ is a $z$-ideal, then $(\sqrt{I})_z = I$.
3. $\sqrt{I_z} = (\sqrt{I})_z$.

**Proof.** 1. Every $z$-ideal containing $\sqrt{I}$ also contains $I$. Therefore $I_z \subseteq (\sqrt{I})_z$. Conversely, Proposition 3.7 gives $\sqrt{I} \subseteq I_z$. This means that $I_z$ is a $z$-ideal containing $\sqrt{I}$. Thus $(\sqrt{I})_z \subseteq I_z$, and consequently $(\sqrt{I})_z = I_z$.
2. Since $I$ is a $z$-ideal, we have $I_z = I$. The proof is completed by (1).
3. We know that $I_z$ is a $z$-ideal of $R$. Corollary 3.8 yields $\sqrt{I_z} = I_z$. Therefore $\sqrt{I_z} = (\sqrt{I})_z$, by (1). $\square$

The following Proposition is a generalization of [11, Proposition 3.1] to noncommutative case.

**Proposition 3.10.** Let $R$ be a right duo ring. Then the following statements are equivalent:

1. For any $z$-ideals $I$ and $J$, $I + J$ is a $z$-ideal.
2. For any ideals $I$ and $J$, $(I + J)_z = I_z + J_z$.
3. The sum of any nonempty family of $z$-ideals is a $z$-ideal.
4. For every nonempty family $\{I_\alpha\}_{\alpha \in A}$ of ideals,

$$\left( \sum_{\alpha \in A} I_\alpha \right)_z = \sum_{\alpha \in A} (I_\alpha)_z.$$  

**Proof.** 1 $\Rightarrow$ 2. Since $I_z$ and $J_z$ are $z$-ideals, $I_z + J_z$ is a $z$-ideal containing $I + J$, by hypothesis. Hence $(I + J)_z \subseteq I_z + J_z$. It follows from Lemma 3.6 that $I_z + J_z \subseteq (I + J)_z$. Therefore $(I + J)_z = I_z + J_z$.

2 $\Rightarrow$ 3. Let $\{I_\alpha\}_{\alpha \in A}$ be a family of $z$-ideals and $a \in \sum_{\alpha \in A} I_\alpha$. Then there exists a finite subset $F$ of $A$ such that $a \in \sum_{\alpha \in F} I_\alpha$. Since $I_\alpha$ is a $z$-ideal, we have $(I_\alpha)_z = I_\alpha$, for every $\alpha \in F$. A simple induction argument shows that

$$\left( \sum_{\alpha \in F} I_\alpha \right)_z = \sum_{\alpha \in F} (I_\alpha)_z = \sum_{\alpha \in F} I_\alpha.$$

Consequently, $\sum_{\alpha \in F} I_\alpha$ is a $z$-ideal, and so

$$M_\alpha \subseteq \sum_{\alpha \in A} I_\alpha \subseteq \sum_{\alpha \in A} I_\alpha.$$
3.6 Let $I$ be a family of ideals. Since $I_\beta \subseteq \sum_{\alpha \in A} I_\alpha$, for all $\beta \in A$, we have $(I_\beta)_z \subseteq (\sum_{\alpha \in A} I_\alpha)_z$, for all $\beta \in A$, by Lemma 3.6. Therefore

$$\sum_{\alpha \in A} (I_\alpha)_z \subseteq (\sum_{\alpha \in A} I_\alpha)_z.$$ 

Since $(I_\alpha)_z$ is a z-ideal containing $I_\alpha$, for all $\alpha \in A$, we may conclude from assumption that $\sum_{\alpha \in A} (I_\alpha)_z$ is a z-ideal containing $\sum_{\alpha \in A} I_\alpha$. Hence

$$\sum_{\alpha \in A} I_\alpha = \sum_{\alpha \in A} (I_\alpha)_z.$$ 

Therefore

$$\sum_{\alpha \in A} I_\alpha = \sum_{\alpha \in A} (I_\alpha)_z.$$ 

4 $\Rightarrow$ 1. If $I$ and $J$ are z-ideals, then $I_z = I$ and $J_z = J$. By hypothesis, we have $(I + J)_z = I_z + J_z$. Therefore $(I + J)_z = I + J$, and so $I + J$ is a z-ideal. 

**Lemma 3.11.** Let $R$ be a right duo ring and $P$ be a prime ideal of $R$. Let $n \in \mathbb{N}$, $I_1, \ldots, I_{n-1}$ be ideals and $I_n$ be a left ideal of $R$. Then the following statements are equivalent:

1. $I_j \subseteq P$, for some $1 \leq j \leq n$.
2. $\bigcap_{i=1}^n I_i \subseteq P$.
3. $I_1 I_2 \cdots I_n \subseteq P$.

**Proof.** 1 $\Rightarrow$ 2. $\bigcap_{i=1}^n I_i \subseteq I_j \subseteq P$.

2 $\Rightarrow$ 3. Since $I_n$ is a left ideal of $R$, we have $I_1 I_2 \cdots I_n \subseteq I_n$. On the other hand, $I_i$ is an ideal, for every $1 \leq i \leq n - 1$, and hence $I_1 I_2 \cdots I_n \subseteq I_i$, for all $1 \leq i \leq n$. Thus $I_1 I_2 \cdots I_n \subseteq \bigcap_{i=1}^n I_i \subseteq P$.

3 $\Rightarrow$ 1. Suppose that $I_i \not\subseteq P$ and $x_i \in I_i \setminus P$, for every $1 \leq i \leq n$. Thus

$$x_1 x_2 \cdots x_n \in I_1 I_2 \cdots I_n \subseteq P$$

which yields $x_j \in P$, for some $1 \leq j \leq n$, by Lemma 2.2. This contradicts the choice of $x_j$. 

**Proposition 3.12.** Let $R$ be a right duo ring and $I$ an ideal of $R$. If $I$ is a finite intersection of maximal ideals of $R$, then any minimal prime ideal of $I$ is a z-ideal.

**Proof.** Since $I$ is a finite intersection of maximal ideals, Lemma 3.11 implies that any minimal prime ideal of $I$ is a maximal ideal. Hence each minimal prime ideal of $I$ is a z-ideal. 

Recall from [7, Definition 10.3] that a nonempty set $S$ of a ring $R$ is said to be $m$-system if for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$.

**Theorem 3.13.** Let $R$ be a right duo ring, $I$ a z-ideal of $R$ and $Q$ be a minimal prime ideal of $I$. If for every $a, b \in R$ with $a \notin Q$, there exists $r \in R \setminus Q$ such that $ab = br$, then $Q$ is a z-ideal.

**Proof.** Suppose $Q$ is not z-ideal. Then there exist elements $a \in R \setminus Q$ and $b \in Q$ such that $M_a \not\subseteq M_b$, by Lemma 3.4. We now assume that

$$S = (R \setminus Q) \cup \{ b^n c \mid n \in \mathbb{N}, c \in R \setminus Q \}.$$
We first prove that $r_1r_2 \in S$, for every $r_1, r_2 \in S$. Consider $r_1, r_2 \in S$.

- If $r_1, r_2 \in R \setminus Q$, then $r_1r_2 \notin Q$, by Lemma 2.2, and so $r_1r_2 \in S$.
- If there are $c_1, c_2 \in R \setminus Q$ such that $r_1 = b^n c_1$ and $r_2 = b^m c_2$, for some positive integers $n$ and $m$, then $c_1 b^m = b^n r$, for some $r \in R \setminus Q$, by hypothesis. Therefore
  \[ r_1r_2 = b^n c_1 b^m c_2 = b^{n+m} r c_2 \in S \]
  because $r, c_2 \in R \setminus Q$, and so $rc_2 \notin Q$, by Lemma 2.2.
- If $r_1 \in R \setminus Q$ and $r_2 = b^n c$, for some $n \in \mathbb{N}$ and $c \in R \setminus Q$, then there exists an element $r \in R \setminus Q$ such that $r_1 b^n = b^n r$, by hypothesis. This yields
  \[ r_1r_2 = r_1 b^n c = b^n rc. \]
  Since $rc \notin Q$, we have $r_1r_2 \in S$. Also, we see that $r_2 r_1 = b^n c r_1 \in S$. Hence for every $r_1, r_2 \in S$ we have $r_1 r_2 \in S$. Therefore $S$ is an $m$-system of $R$.

Now, we show that $I \cap S = \emptyset$. If $x \in I \cap S$, then $x \in I \subseteq Q$ and $x \in S$. Hence $x = b^n c$, for some $n \in \mathbb{N}$ and $c \in R \setminus Q$. From Lemma 2.5 we see that
  \[ ac \in M_{ac} = M_a \cap M_c \subseteq M_b \cap M_c = M_{b^n} \cap M_c = M_x \subseteq I \]
because $M_a \subseteq M_b$ and $I$ is a $z$-ideal. This yields $ac \in Q$. Hence $a \in Q$ or $c \in Q$, by Lemma 2.2. This contradicts the choice of $a$ and $c$. Therefore $I \cap S = \emptyset$. By Zorn’s Lemma, there exists an ideal $I \subseteq P$ which is maximal with respect to being disjoint from $S$. From [7, Proposition 10.5] it follows that $P$ is a prime ideal. Since $S \cap P = \emptyset$ and $b \in S$, we have $I \subseteq P \subseteq Q$. However, this contradicts our assumption that $Q$ is a minimal prime ideal of $I$. Therefore $Q$ is a $z$-ideal.

We know that the Jacobson radical of a ring $R$, which denoted by $\text{rad}(R)$, is the intersection of all maximal right (or left) ideals of $R$. Now, if $R$ is a right (or left) duo ring, then every maximal right (or left) ideal is a maximal ideal. Therefore, if $R$ is a right (or left) duo ring, then we can say that $\text{rad}(R)$ is the intersection of all maximal ideals of $R$.

**Example 3.14.** Let $D$ be a division ring and $\mathbb{C}$ be the field of complex numbers. Let $R = D \times \mathbb{C}[x]$. We know that $R$ is a duo ring. If $f \in \text{rad}(\mathbb{C}[x])$, then $1 - xf$ is a unit of $\mathbb{C}[x]$, by [7, Lemma 4.1], which yields $f = 0$. Hence $\text{rad}(\mathbb{C}[x]) = 0$. This implies that $\text{rad}(R) = \text{rad}(D) \times \text{rad}(\mathbb{C}[x]) = 0$, and so $I = \{0\}$ is a $z$-ideal of $R$. If $P$ is a prime ideal of $R$, then $P = 0 \times \mathbb{C}[x]$ or $P = D \times 0$, where $Q$ is a prime ideal of $\mathbb{C}[x]$. Obviously, $D \times 0$ and $0 \times \mathbb{C}$ are minimal prime ideals of $I$ which $0 \times \mathbb{C}$ is maximal, and so is a $z$-ideal. Consider $(a, f), (c, g) \in R$ such that $(a, f) \notin D \times 0$. It is clear that $ac = cr$, for some $r \in D$. Thus
  \[ (a, f)(c, g) = (ac, fg) = (cr, gf) = (c, g)(r, f). \]
Since $(a, f) \notin D \times 0$, we have $f \neq 0$, and so $(r, f) \notin D \times 0$. Therefore $D \times 0$ is a $z$-ideal, by Theorem 3.13.

**Proposition 3.15.** Let $R$ be a right duo ring and $I$ be a left ideal of $R$. Then $(I^n)_z = I_z$, for every $n \in \mathbb{N}$.

**Proof.** Clearly, $(I^n)_z \subseteq I_z$. For every $x \in I$, we have $x^n \in I^n \subseteq (I^n)_z$, and so $M_x \subseteq (I^n)_z$. From Lemma 2.5, we see that $x \in M_x = M_x^n \subseteq (I^n)_z$. Hence $I_z \subseteq (I^n)_z$.

By Lemma 3.5, $(I^n)_z$ is a $z$-ideal, and so $I_z \subseteq (I^n)_z$. Therefore $(I^n)_z = I_z$. \[ \square \]

Recall that a ring $R$ is said to be a von Neumann regular ring if for any $a \in R$, there exists an element $r \in R$ such that $a = ar$. Furthermore, for any ideal $I$ of a von Neumann regular ring $R$, it is clear that $R/I$ is also a von Neumann regular ring. Therefore, we have

**Proposition 3.16.** Let $R$ be a right (or left) duo ring. If $R$ is a von Neumann regular ring, then every ideal of $R$ is a $z$-ideal.
Proof. Let $I$ be a proper ideal of $R$. Since $\frac{R}{I}$ is a von Neumann regular ring, we have $rad(\frac{R}{I}) = 0$, by [7, Corollary 4.24]. On the other hand, $\frac{R}{I}$ is also a right (or left) duo ring. Thus $rad(\frac{R}{I})$ is the intersection of all maximal ideals of $\frac{R}{I}$. Hence $I$ is the intersection of all maximal ideal of $R$ containing $I$, and so $I$ is a z-ideal.

Proposition 3.17. Let $R$ be a right duo ring. If every left ideal of $R$ is a z-ideal, then $R$ is a von Neumann regular ring.

Proof. Let $a \in R$ and $I = Ra$. By hypothesis, $I$ is a z-ideal, and so $Iz = I$. Hence $(I^2)z = Iz = I$, by Proposition 3.15. On the other hand, from Lemma 2.1, we see that $I^2 = RaRa = aRa$. Since $I^2$ is a left ideal, $I^2$ is a z-ideal, and so $(I^2)z = I^2$. Hence $I^2 = (I^2)z = Iz = I$. Then we may conclude from $a \in I = I^2 = aRa$ that there exists an element $r \in R$ such that $a = ara$. Therefore $R$ is a von Neumann regular ring.

The following result, which is a generalization of [10, Theorem 1.2] to noncommutative case, follows immediately from Proposition 3.16 and Proposition 3.17.

Corollary 3.18. Let $R$ be a duo ring. Then $R$ is a von Neumann regular ring if and only if every ideal of $R$ is a z-ideal.

Recall that if $I$ and $J$ are two left ideals of a ring $R$, then the subset $\{ x \in R \mid xI \subseteq J \}$ is denoted by $(J : I)$. It is easily seen that $(J : I)$ is an ideal of $R$. In particular, for each left ideal $I$, the subset $(0 : I)$, which will be denote by $Ann_l(I)$, is also an ideal of $R$. We call it the left annihilator of $I$.

Proposition 3.19. Let $I$ and $J$ be two left ideals of a right duo ring $R$. If $J$ is a z-ideal, then $(J : I)$ is a z-ideal of $R$.

Proof. By Lemma 3.4, it is sufficient to show that for every $a \in R$ and $b \in (J : I)$, if $M_a \subseteq M_b$, then $a \in (J : I)$. Now, we assume that $a \in R$, $b \in (J : I)$ and $M_a \subseteq M_b$. Thus for every $x \in I$, we have $bx \in J$. Moreover

$$M_ax = M_a \cap M_x \subseteq M_b \cap M_x = M_{bx}$$

by Lemma 2.5. Since $bx \in J$ and $J$ is a z-ideal, $M_{bx} \subseteq J$, and so $ax \in M_{ax} \subseteq M_{bx} \subseteq J$. Therefore $a \in (J : I)$.

Lemma 3.20. If $e$ is an idempotent element of a right duo ring $R$, then $Re = Ann_l(R(1 - e))$.

Proof. For every $r \in R$, we have $reR(1 - e) \subseteq re(1 - e)R = 0$, by Lemma 2.1. Hence $Re \subseteq Ann_l(R(1 - e))$. We now assume that $r \in Ann_l(R(1 - e))$. Thus $r - re = r(1 - e) = 0$, and so $r = re \in Re$.

From [7, Theorem 2.5], it follows that every right ideal of a ring $R$ is a direct summand of $R$ if and only if every left ideal of $R$ is a direct summand of $R$. A ring satisfying these equivalent conditions is called a semisimple ring.

Proposition 3.21. Let $R$ be a semisimple right duo ring. Then every one sided ideal of $R$ is an ideal.

Proof. Since $R$ is a right duo ring, every right ideal of $R$ is an ideal. By [7, Theorem 4.25], semisimple rings are exactly the left Noetherian von Neumann regular rings. Let $I$ be a left ideal of $R$. Since $R$ is left Noetherian, every left ideal of $R$ is finitely generated, and so $I = Re$, for an idempotent element $e$ of $R$, by using the characterization (3) of [7, Theorem 4.23]. Hence $I = Ann_l(R(1 - e))$, by Lemma 3.20, and consequently $I$ is an ideal of $R$.
As observed in the proof of Proposition 3.21, every ideal of a semisimple right duo ring is an annihilator of a left ideal. On the other hand, we know from [7, Theorem 4.25] that \( rad(R) = 0 \), for every semisimple ring \( R \), and hence the zero ideal of \( R \) is a \( z \)-ideal. Now, we may by using Proposition 3.19 conclude that the following result.

**Corollary 3.22.** Every ideal of a semisimple right duo ring is a \( z \)-ideal.

4. Relative \( z \)-ideals in a right duo ring

The main goal of this section is to introduce left relative \( z \)-ideals. We define the concept of \( p \)-right duo rings to obtain equivalent condition to minimal prime ideals of an ideal, and then study left relative \( z \)-ideals of their rings. Finally, we prove that if every proper ideal of a \( p \)-right duo ring \( R \) is a left relative \( z \)-ideal, then every ideal of \( R \) is a \( z \)-ideal.

**Definition 4.1.** Let \( J \) be a left ideal of a ring \( R \). A left ideal \( I \) of \( R \) is said to be a left \( z \)-ideal if \( M_a \cap J \subseteq I \), for every \( a \in I \). Whenever, for a left ideal \( I \), there exists a left ideal \( J \) such that \( J \not\subseteq I \) and \( I \) is a left \( z \)-ideal, we say that \( I \) is a left relative \( z \)-ideal and \( J \) is called a \( z \)-factor of \( I \).

Recall that a ring \( R \) is said to be a reduced ring if \( R \) has no nonzero nilpotent element. In the following, we introduce a class of left relative \( z \)-ideals in a right duo ring. Before giving it, let us state the following Lemma which follows immediately from Lemma 2.2.

**Lemma 4.2.** For each right duo ring \( R \), if \( rad(R) = 0 \), then \( R \) is a reduced ring.

**Proposition 4.3.** Let \( R \) be a right duo ring with \( rad(R) = 0 \). If \( I \) is a left ideal of \( R \) such that \( Ann_l(I) \neq 0 \), then \( I \) is a left relative \( z \)-ideal.

**Proof.** First, we show that \( M_a \cap Ann_l(I) = 0 \), for every \( a \in I \). Suppose that \( x \in M_a \cap Ann_l(I) \). Then \( M_x \subseteq M_a \) and \( xa = xI = 0 \), for every \( a \in I \). From Lemma 2.5, it thus follows that \( x \in M_x = M_x \cap M_a = M_{xa} = M_0 = rad(R) = 0 \).

Hence \( M_a \cap Ann_l(I) = 0 \). We now put \( J = Ann_l(I) \), and show that \( J \not\subseteq I \). If \( J \subseteq I \), then \( J^2 \subseteq JI = Ann_l(I)I = 0 \). Thus \( J^2 = 0 \), and so \( J = 0 \), because \( R \) is a reduced ring, by Lemma 4.2. But this contradicts the assumption that \( J = Ann_l(I) \neq 0 \). Therefore \( J \not\subseteq I \), and so \( I \) is a left relative \( z \)-ideal.

**Definition 4.4.** A right duo ring \( R \) is called a \( p \)-right duo ring if for every prime ideal \( P \) of \( R \) and every elements \( a, b \in R \), which \( a \notin P \), there exists \( r \in R \setminus P \) such that \( ab = br \).

In the following, we give some examples of \( p \)-right duo rings.

**Proposition 4.5.** Let \( R \) be a prime right duo ring. If \( R \) has a unique nonzero prime ideal, then \( R \) is a \( p \)-right duo ring.

**Proof.** Let \( P \) be the unique nonzero prime ideal of \( R \) and \( a, b \in R \) such that \( a \notin P \). If \( b = 0 \), then \( ab = b1 \). Now, we assume that \( b \neq 0 \). Since \( R \) is a right duo ring, \( ab = br \), for some \( r \in R \). On the other hand, \( P \) is the unique nonzero prime ideal of \( R \) and \( a \notin P \). Thus \( a \) is a unit element of \( R \) which yields \( b = a^{-1}br = br' r \), for some \( r' \in R \). It follows \( b(1 - r'r) = 0 \). Since \( R \) is a prime right duo ring and \( b \neq 0 \), we have \( r'r = 1 \). Therefore \( r \notin P \). \( \square \)

**Example 4.6.** Let \( D \) be a division ring and \( R = D \times Z \). We show that \( R \) is a \( p \)-right duo ring. It is easily seen that \( R \) is a right duo ring. If \( P \) is a prime ideal of \( R \), then \( P = D \times 0 \), or \( P = 0 \times Z \) or \( P = D \times pZ \), for some prime number \( p \). We assume that \( (a, b), (c, d) \in R \) and \( (a, b) \notin P \). It is clear that \( ac = cr \), for some \( r \in D \). Thus

\[(a, b)(c, d) = (ac, bd) = (cr, db) = (c, d)(r, b).\]
We consider the following three cases:
1. If $P = D \times 0$, then $b \neq 0$, because $(a, b) \notin P$, and so $(r, b) \notin P$.
2. If $P = 0 \times \mathbb{Z}$, then $a \neq 0$, because $(a, b) \notin P$. Now, if $c \neq 0$, then $r \neq 0$, and so $(r, b) \notin P$, and if $c = 0$, we have
   \[(a, b)(c, d) = (0, bd) = (0, db) = (c, d)(1, b)\]
   which $(1, b) \notin P$.
3. If $P = D \times p\mathbb{Z}$, for some prime number $p$, then $p \nmid b$, because $(a, b) \notin P$, and so $(r, b) \notin P$.

**Proposition 4.7.** Let $R$ be a $p$-right duo ring with $\text{rad}(R) = 0$ and $P$ be a prime ideal of $R$. Let $\Gamma$ be the set of all $z$-ideals of $R$ contained in $P$. Then $\Gamma$ (partially ordered by inclusion) has a maximal element. Furthermore, every maximal element of $\Gamma$ is a prime $z$-ideal of $R$.

**Proof.** Since $\text{rad}(R) = 0$, the zero ideal of $R$ is a $z$-ideal, and so $\Gamma \neq \emptyset$. If $P$ is a $z$-ideal, then clearly $P$ is the only maximal element of $\Gamma$. We now assume that $P$ is not $z$-ideal. If $\Sigma$ is a chain in $\Gamma$, then it is quite obvious that $\bigcup I_a \in \Sigma$ is a $z$-ideal contained in $P$. Therefore $\Gamma$ has a maximal element $J$, by Zorn’s Lemma. Hence $J \subseteq P$, because $P$ is not $z$-ideal. Suppose that $Q$ is a minimal prime ideal of $J$ such that $J \subseteq Q \subseteq P$. Theorem 3.13 implies that $Q$ is a $z$-ideal, because $R$ is a $p$-right duo ring, and so $Q \in \Gamma$. Since $J$ is a maximal element in $\Gamma$, we have $J = Q$. Therefore $J$ is a prime $z$-ideal. \[\square\]

**Proposition 4.8.** Let $R$ be a ring and $I$ be a proper ideal of $R$. Let
\[\Gamma = \{ S \subseteq R \mid S \text{ is an } m \text{-system and } S \cap I = \emptyset \}.\]
If $P$ is a prime ideal of $R$, then $P \in \text{Min}(I)$ if and only if $R \smallsetminus P$ is a maximal element of $\Gamma$.

**Proof.** We know from [7, Corollary 10.4] that an ideal $P$ of $R$ is prime if and only if $R \smallsetminus P$ is an $m$-system. Therefore, if $T = R \smallsetminus P$ is a maximal element of $\Gamma$, then $P$ is a prime ideal of $R$. Also, $T \cap I = \emptyset$ implies that $I \subseteq P$. Now, we assume that there exists a prime ideal $Q$ of $R$ such that $I \subseteq Q \subseteq P$. It follows that $(R \smallsetminus Q) \cap I = \emptyset$ and $R \smallsetminus Q$ is an $m$-system, by [7, Corollary 10.4]. Thus $R \smallsetminus Q \in \Gamma$. Since $T \subseteq R \smallsetminus Q$ and $T$ is a maximal element of $\Gamma$, we have $R \smallsetminus Q = T$, and so $P = Q$. Therefore $P \in \text{Min}(I)$.

Conversely, if $P \in \text{Min}(I)$, then $T = R \smallsetminus P$ is an $m$-system, by [7, Corollary 10.4]. Furthermore, $T \cap I = \emptyset$. Thus $T \in \Gamma$. Suppose that there exists $S \in \Gamma$ such that $T \subseteq S$. Hence $S$ is an $m$-system and $S \cap I = \emptyset$. By Zorn’s Lemma, there exists an ideal $I \subseteq Q$ which is maximal with respect to being disjoint from $S$. From [7, Proposition 10.5], it follows that $Q$ is a prime ideal. Since $S \cap Q = \emptyset$ and $T \subseteq S$, we have $Q \cap T = \emptyset$. Hence $I \subseteq Q \subseteq P$, and consequently $Q = P$, because $P \in \text{Min}(I)$. It follows from $P \cap S = \emptyset$ that $S \subseteq R \smallsetminus P = T$, and so $T = S$. Therefore, $T$ is a maximal element of $\Gamma$. \[\square\]

It is well known that, if $I$ is an ideal of a commutative ring $R$, then $P \in \text{Min}(I)$ if and only if for each $a \in P$, there exist $c \in R \smallsetminus P$ and $n \in \mathbb{N}$ such that $(ac)^n \in I$. We need generalization of this conclusion for the noncommutative rings. In the following Lemma, we will generalize it to right duo rings.

**Proposition 4.9.** Let $R$ be a $p$-right duo ring and $I$ be a proper ideal of $R$. If $P$ is a nonzero prime ideal of $R$ containing $I$ and $T = R \smallsetminus P$, then the following statements are equivalent:

1. $P \in \text{Min}(I)$.
2. For every $x \in P$, there exist $y, z \in T$ and $n \in \mathbb{N}$ such that $yx^n z \in I$.

**Proof.** 1 $\Rightarrow$ 2. Let $P \in \text{Min}(I)$ and $0 \neq x \in P$. If
\[\Gamma = \{ S \subseteq R \mid S \text{ is an } m \text{-system and } S \cap I = \emptyset \},\]
Then $T$ is a maximal element of $\Gamma$, by Proposition 4.8. Now, we assume that

$$T' = \{ yx^n z \mid y, z \in T, n \in \mathbb{N} \cup \{0\} \}.$$ 

Let $y_1x^m z_1, y_2x^n z_2 \in T'$. From Lemma 2.2, it is clear that $z_1y_2 \in T$, and hence there is an element $r \in T$ such that $z_1y_2x^n = x^nr$, because $R$ is a $p$-right duo ring. Thus

$$y_1x^m z_1y_2x^n z_2 = y_1x^{m+n} z_2 \in T'.$$

Therefore $T'$ is an $m$-system. Obviously, $x \in T' \setminus T$, consequently $T \subsetneq T'$. Hence $T' \notin \Gamma$, by the maximality of $T$. However, this yields $T' \cap I \neq \emptyset$. Therefore, there exist $y, z \in T$ and $n \in \mathbb{N}$ such that $yx^n z \in I$.

2 \Rightarrow 1. Let $Q$ be a prime ideal of $R$ such that $I \subseteq Q \subseteq P$. For each $x \in P$, there exist $n \in \mathbb{N}$ and $y, z \in T$ such that $yx^n z \in I \subseteq Q$, by hypothesis. Since $Q$ is a prime ideal and $y, z \notin Q$, we have $x \in Q$, by Lemma 2.2. Therefore $Q = P$, and so $P \in \text{Min}(I)$.

In the following Proposition, which is an analogue of [1, Lemma 2.1], we give conditions that, whenever $J$ is a left ideal, then every minimal prime ideal of a left $z_J$-ideal, is also a left $z_J$-ideal.

**Proposition 4.10.** Let $R$ be a $p$-right duo ring, $I$ be an ideal and $J$ be a left ideal of $R$. If $I$ is a left $z_J$-ideal, then every minimal prime ideal of $I$ is a left $z_J$-ideal.

**Proof.** Let $P \in \text{Min}(I)$. For every element $a \in P$, there exist elements $b, c \in R \setminus P$ and $n \in \mathbb{N}$ such that $ba^n c \in I$, by Proposition 4.9. Since $I$ is a left $z_J$-ideal, we have $M_{ba^n c} \cap J \subseteq I$. Now, it follows from Lemma 2.5 that

$$M_b \cap M_a \cap M_c \cap J = M_b \cap M_a^n \cap M_c \cap J = M_{ba^n c} \cap J \subseteq I \subseteq P.$$ 

Obviously, $M_b, M_c \subseteq P$, because $b, c \notin P$. It follows that $M_a \cap J \subseteq P$, by Lemma 3.11, and so $P$ is a left $z_J$-ideal of $R$. 

**Proposition 4.11.** Let $R$ be a right duo ring and $J$ be a left ideal of $R$. If $P$ is a prime ideal of $R$ for which $J \not\subseteq P$, then $P$ is a left $z_J$-ideal if and only if $P$ is a $z$-ideal.

**Proof.** It is clear that if $P$ is a $z$-ideal, then $P$ is also a left $z_J$-ideal. Conversely, if $P$ is a left $z_J$-ideal, then $M_a \cap J \subseteq P$, for each $a \in P$. Since $J \not\subseteq P$, Lemma 3.11 yields $M_a \subseteq P$, for each $a \in P$. Hence $P$ is a $z$-ideal of $R$. 

Let $R$ be a $p$-right duo ring and $I$ be an ideal of $R$. From Lemma 2.2, it is clear that $\sqrt{I} \subseteq P$, for each $P \in \text{Min}(I)$. On the other hand, if $P \in \text{Min}(I)$ and $x \in R \setminus P$, we conclude from Proposition 4.9 that $x^n \notin I$, for every $n \in \mathbb{N}$, and so $x \notin \sqrt{I}$. Therefore, we have

$$\sqrt{I} = \bigcap_{P \in \text{Min}(I)} P.$$ 

The following Lemma corresponds to [1, Lemma 2.2].

**Lemma 4.12.** Let $R$ be a $p$-right duo ring, $I$ be an ideal and $J$ be a left ideal of $R$. If $I$ is a left $z_J$-ideal, then $I_z \cap J \subseteq I_z$.

**Proof.** As we have seen in the preceding paragraph

$$\sqrt{I} = \bigcap_{P \in \text{Min}(I)} P.$$ 

From Proposition 3.9, it follows that

$$I_z \cap J = (\sqrt{I})_z \cap J = \left( \bigcap_{P \in \text{Min}(I)} P \right)_z \cap J.$$
Moreover, Lemma 3.6 yields
\[
\left( \bigcap_{P \in \text{Min}(I)} P \right)_z \subseteq \bigcap_{P \in \text{Min}(I)} P_z.
\]

Thus
\[
I_z \cap J \subseteq \left( \bigcap_{P \in \text{Min}(I)} P_z \right) \cap J.
\]

Since \( I \) is a left \( z_J \)-ideal, from Proposition 4.10, we see that \( P \) is a left \( z_J \)-ideal, for every \( P \in \text{Min}(I) \). However, we conclude from Proposition 4.11 that \( J \subseteq P \) or \( P \) is a \( z \)-ideal, for every \( P \in \text{Min}(I) \).

We now assume that \( P \in \text{Min}(I) \). If \( P \) is a \( z \)-ideal, then \( P_z = P \), and so \( P_z \cap J = P \cap J \). If \( J \subseteq P \), then we also have \( P_z \cap J = P \cap J \). Therefore
\[
\left( \bigcap_{P \in \text{Min}(R)} P_z \right) \cap J = \left( \bigcap_{P \in \text{Min}(R)} P \right) \cap J = \sqrt{I} \cap J
\]
and from (4.1) we get
\[
I_z \cap J \subseteq \sqrt{I} \cap J.
\]

Let us finally prove that \( \sqrt{I} \cap J \subseteq I \). If \( x \in \sqrt{I} \cap J \), then there is a positive integer \( n \) such that \( x^n \in I \). Since \( I \) is a left \( z_J \)-ideal, we have \( M_{x^n} \cap J \subseteq I \). Hence, from Lemma 2.5, it follows that
\[
x \in M_x \cap J = M_{x^n} \cap J \subseteq I.
\]
Thus \( \sqrt{I} \cap J \subseteq I \), and consequently \( I_z \cap J \subseteq \sqrt{I} \cap J \subseteq I \).

**Lemma 4.13.** Let \( R \) be a \( p \)-right duo ring and \( J \) be a left ideal of \( R \). If \( I \) is an ideal of \( R \), then \( I \) is a left \( z_J \)-ideal if and only if \( I \) is a left \( z_{I+J} \)-ideal.

**Proof.** If \( I \) is a left \( z_{I+J} \)-ideal, then clearly \( I \) is a left \( z_J \)-ideal. Conversely, let \( I \) be a left \( z_J \)-ideal. Since \( I \subseteq I_z \), from Lemma 4.12 and modular law follow that
\[
I_z \cap (I + J) = I \cap (I_z \cap J) \subseteq I.
\]
For every \( a \in I \), we have \( M_a \subseteq I_z \), because \( I \subseteq I_z \) and \( I_z \) is a \( z \)-ideal. Hence
\[
M_a \cap (I + J) \subseteq I_z \cap (I + J) \subseteq I.
\]
Therefore, \( I \) is a left \( z_{I+J} \)-ideal.

**Lemma 4.14.** Let \( R \) be a right duo ring. If \( I \) and \( J \) are two left ideals of \( R \) such that at least one of them is ideal, then \( I \cap J \) is a left \( z_J \)-ideal if and only if \( I \) is a left \( z_J \)-ideal.

**Proof.** We first assume that \( J \) is an ideal. If \( I \) is a left \( z_J \)-ideal, then for every \( a \in I \cap J \), we have \( M_a \cap J \subseteq I \), and so \( M_a \cap J \subseteq I \cap J \). Hence \( I \cap J \) is a left \( z_J \)-ideal.

Conversely, let \( I \cap J \) be a left \( z_J \)-ideal and \( a \in I \). We must show that \( M_a \cap J \subseteq I \). We now assume that \( x \in M_a \cap J \). Thus \( xa \in I \cap J \), because \( J \) is an ideal. Since \( I \cap J \) is a left \( z_J \)-ideal, \( M_{xa} \cap J \subseteq I \cap J \). From Lemma 2.5, we see that
\[
x \in M_x \cap M_a \cap J = M_{xa} \cap J \subseteq I \cap J \subseteq I.
\]
Therefore \( M_a \cap J \subseteq I \), and so \( I \) is a left \( z_J \)-ideal.

Now, if \( I \) is an ideal, then we can prove this Lemma by a similar argument.

The following result is an analogue of [1, Proposition 2.5].

**Proposition 4.15.** Let \( R \) be a \( p \)-right duo ring and \( M \) be a maximal ideal of \( R \). If \( I \) is an ideal of \( R \), then \( I \) is a \( z \)-ideal if and only if \( I \cap M \) is a \( z \)-ideal.
Proof. If $I$ is a $z$-ideal, then clearly $I \cap M$ is a $z$-ideal. We now assume that $I \cap M$ is a $z$-ideal of $R$. If $I \subseteq M$, then $I = I \cap M$, and so $I$ is a $z$-ideal. If $I \nsubseteq M$, then $M_a \subseteq I \cap M$, for every $a \in I \cap M$, and so $M_a \cap M \subseteq I \cap M$. Thus $I \cap M$ is a $z_{M}$-ideal. It follows from Lemma 4.14 that $I$ is a left $z_{M}$-ideal. Now, Lemma 4.13 implies that $I$ is a left $z_{I}$-ideal, because $I + M = R$. However, $I$ is a $z$-ideal of $R$. □

The following Proposition is an analogue of [1, Proposition 2.6] and [2, Proposition 2.2].

Proposition 4.16. Let $R$ be a $p$-right duo ring and $J$ be an ideal of $R$ with $J \nsubseteq \text{rad}(R)$. If $J$ is not a $z$-ideal, then there exists an ideal $I$ of $R$ such that $I \nsubseteq J$ and $I$ is a left $z_{J}$-ideal which is not a $z$-ideal.

Proof. Since $J \nsubseteq \text{rad}(R)$, there is a maximal ideal $M$ of $R$ such that $J \nsubseteq M$. Thus $I = J \cap M$ is an ideal of $R$ and $I \nsubseteq J$. Obviously, for every $a \in I$, $M_a \cap J \subseteq M \cap J = I$, and so $I$ is a left $z_{J}$-ideal of $R$. From Proposition 4.15, it follows that $J$ is a $z$-ideal if and only if $I$ is a $z$-ideal. Therefore, the desired conclusion trivially holds. □

Lemma 4.17. Let $R$ be a $p$-right duo ring and $I$ be an ideal of $R$. If $I$ is a left relative $z$-ideal of $R$, then the set

$$\Gamma = \{ J \mid J \text{ is a } z\text{-factor of } I \}$$

has a maximal member with respect to inclusion. Furthermore, every maximal element of $\Gamma$ properly contains $I$.

Proof. Obviously, $\Gamma \neq \emptyset$. If $\Sigma$ is a non-empty totally ordered subset of $\Gamma$, then clearly $L = \bigcup_{J \in \Sigma} J$ is a left ideal which $L \nsubseteq I$. We will show that $I$ is a $z_{L}$-ideal. For every $a \in I$, we have

$$M_a \cap L = M_a \cap \left( \bigcup_{J \in \Sigma} J \right) = \bigcup_{J \in \Sigma} (M_a \cap J) \subseteq I,$$

because $J$ is a $z$-factor of $I$, for all $J \in \Sigma$, and so $M_a \cap L \subseteq I$. Hence $I$ is a left relative $z_{L}$-ideal, and consequently $I$ is an upper bound for $\Sigma$ in $\Gamma$. From Zorn’s Lemma, we see that $\Gamma$ has a maximal element.

Now, we show that every maximal element of $\Gamma$ properly contains $I$. If $J$ is a maximal element of $\Gamma$, then $I$ is a left $z_{J}$-ideal and $J \nsubseteq I$. Hence $I$ is a left $z_{I + J}$-ideal, by Lemma 4.13. Since $I + J \nsubseteq I$, $I + J$ is a $z$-factor of $I$, and so $I + J \in \Gamma$. Therefore, by the maximality of $J$, we deduce that $J = I + J$, and consequently $I \nsubseteq J$. □

Lemma 4.18. Let $R$ be a right duo ring and $I, J$ and $L$ be left ideals of $R$ such that $I \subseteq J$. If $I$ is a left $z_{J}$-ideal and $J$ is a left $z_{L}$-ideal, then $I$ is a left $z_{L}$-ideal.

Proof. Since $I$ is a left $z_{J}$-ideal, we have $M_a \cap J \subseteq I$, for every $a \in I$. Moreover, $M_a \cap L \subseteq J$, for every $a \in I$, because $I \subseteq J$ and $J$ is a left $z_{J}$-ideal. Thus $M_a \cap L \subseteq M_a \cap J \subseteq I$, for every $a \in I$. Therefore $I$ is a left $z_{L}$-ideal. □

Theorem 4.19. Let $R$ be a duo ring such that every proper ideal of $R$ is a left relative $z$-ideal. If $R$ is a $p$-right duo ring, then every ideal of $R$ is a $z$-ideal.

Proof. It is clear that $R$ is a $z$-ideal. Let $I$ be a proper ideal of $R$. Then $I$ is a left relative $z$-ideal, by hypothesis. It follows from Lemma 4.17 that there exists a maximal $z$-factor $J$ of $I$ such that $I \nsubseteq J$. We claim that $J = R$. If $J \neq R$, then $J$ is also a left relative $z$-ideal, and hence we can assume that $L$ is a $z$-factor of $J$ such that $J \nsubseteq L$, by Lemma 4.17. It follows that $I$ is a left $z_{J}$-ideal and $J$ is a left $z_{L}$-ideal. From Lemma 4.18, we may conclude that $I$ is a left $z_{L}$-ideal. Since $I \nsubseteq J \nsubseteq L$, $L$ is a $z$-factor of $I$, which contradicts the maximality of $J$. Therefore, $J = R$, and so $I$ is a $z$-ideal. □

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