Existence of solutions for a coupled system of Caputo-Hadamard type fractional differential equations with Hadamard fractional integral conditions

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Abstract

In this work, we study existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations involving two Caputo-Hadamard-type fractional derivatives. By applying the Banach's fixed point theorem and Shaefer's fixed point theorem, the existence of solutions is obtained.The results obtained in this work are well illustrated with the aid of examples.

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1. Introduction and Preliminaries

The fractional-type differential equations are generalizations of classical integer order differential equations and they are increasingly used to model problems in finance, fluid dynamics, biology and other areas of application. Recently, many studies on fractional differential equations, involving fractional derivative operators such as Riemann-Liouville fractional derivative [11, 20], Caputo fractional derivative [3, 5, 7], Hadamard fractional derivative [10, 18, 19], and Caputo-Hadamard fractional derivative [2, 4, 6], have appeared during

the past several years. Moreover, by applying a variety of fixed point theorems (such as Banach contraction principle, Krasnoselskii's fixed point theorem, Schaefers fixe point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder's degree theory) several scholars presented the existence results for various classes of fractional differential equations, see for example [5, 6, 11, 15, 17] and the references cited therein. The study of coupled systems of fractional differential equations is also very significant as such systems appear in a variety of problems of applied nature, see [1, 8, 13, 14]. Some recent works on coupled systems of fractional differential equations with different fractional derivative operators can be found in [1, 9, 17, 21] and the references cited therein. In this paper, we consider a coupled system of Caputo-Hadamard-type fractional differential equations

$$\begin{cases} D^{\beta_1}\left(D^{\alpha_1}+\lambda_1\right)u\left(t\right)-f\left(t,u\left(t\right),v\left(t\right)\right)=I^{\theta_1}\varphi\left(t,u\left(t\right),v\left(t\right)\right),t\in\left[1,e\right], \\ \qquad 0<\beta_1,\alpha_1\leq 1, 1<\beta_1+\alpha_1\leq 2, \quad \theta_1>0, \\ D^{\beta_2}\left(D^{\alpha_2}+\lambda_2\right)v\left(t\right)-g\left(t,u\left(t\right),v\left(t\right)\right)=I^{\theta_2}\psi\left(t,u\left(t\right),v\left(t\right)\right),t\in\left[1,e\right], \\ \qquad 0<\beta_2,\alpha_2\leq 1, 1<\beta_2+\alpha_2\leq 2, \quad \theta_2>0, \end{cases} \tag{1}$$

supplemented with nonlocal Hadamard fractional integral conditions

$$\begin{cases} aI^{p_1}u\left(\eta_1\right)=\gamma_1, bI^{q_1}x\left(\xi_1\right)=\gamma_2, p_1,q_1>0, 1<\eta_1,\xi_1<e, a,b\in\mathbb{R}, \\ cI^{p_2}v\left(\eta_2\right)=\sigma_1, dI^{q_2}v\left(\xi_2\right)=\sigma_2, p_2,q_2>0, 1<\eta_2,\xi_2<e, c,d\in\mathbb{R}, \end{cases} \tag{2}$$

where f, g, φ and $\psi : [1, e] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions, $I^{\theta_i}, I^{p_i}, I^{q_i}$ are the Hadamard fractional integrals and $D^{\alpha_i}, D^{\beta_i}$ $(i = 1, 2)$ are the Caputo-Hadamard fractional derivatives. The Hadamard fractional integral [1, 12, 16] of order α for a continuous function $\varphi : [1, +\infty) \to \mathbb{R}$ is defined by

$$_H I^\alpha \varphi\left(t\right) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\varphi\left(s\right)}{s} ds, \alpha > 0,$$

with $\Gamma\left(\alpha\right) = \int_0^\infty e^u u^{\alpha-1} du$. The Caputo-Hadamard fractional derivative [2, 12, 16] of order α for a continuous function $\varphi : [1, +\infty) \to \mathbb{R}$ is defined by

$$_H^C D^\alpha \varphi(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n \varphi(s) \frac{ds}{s} = {}_H I^{n-\alpha}\left(\delta^n \varphi\right)(t),$$

where $n = [\alpha] + 1$ and $\delta^n = \left(t \frac{d}{dt}\right)^n$.

Lemma 1.1. *[2, 12, 16] Let $x(t) \in AC^n_\delta[a,b]$ where $0 < a < b < \infty$ and $AC^n_\delta[a,b] = \left\{g : [a,b] \to \mathbb{C}, \ \delta^{n-1}g(t) \in AC[a,$ We define*

$$I^\rho D^\rho x(t) = x(t) + \sum_{i=0}^{n-1} c_i (\log t)^i,$$

where $c_i \in R, \ i = 1, 2, ..., n-1$ and $n = [\rho] + 1$.

Lemma 1.2. *[15, 17] Let S be a Banach space. Assume that $\mathcal{A} : S \to S$ is a completely continuous operator and that the set $\Lambda = \{x \in S : x = \sigma\mathcal{A}(x), 0 < \sigma < 1\}$ is bounded. Then, $\mathcal{A}$ has a fixed point in S.*

We prove the following auxiliary lemma.

Lemma 1.3. *Let $\Pi_1 \neq 0$ and $h_1 \in C([1, e], \mathbb{R})$. Then the solution of the problem*

$$D^{\beta_1}(D^{\alpha_1} + \lambda_1)u(t) = h_1(t), t \in [1, e], 0 < \beta_1, \alpha_1 \leq 1, 1 < \beta_1 + \alpha_1 \leq 2, \tag{3}$$

with the Hadamard fractional integral conditions

$$aI^{p_1}u(\eta_1) = \gamma, bI^{q_1}u(\xi_1) = \gamma, 1 < \eta_1, \xi_1 < e, a, b \in \mathbb{R}, \tag{4}$$

is given by

$$u(t) = I^{\alpha_1+\beta_1} h_1(t) - \lambda_1 I^{\alpha_1} u(t) \tag{5}$$
$$+ \frac{\Lambda_4 (\log t)^{\alpha_1} - \Lambda_3 \Gamma(\alpha_1+1)}{\Pi_1 \Gamma(\alpha_1+1)} \left(\gamma_1 - aI^{p_1+\alpha_1+\beta_1} h_1(\eta_1) + \lambda_1 aI^{p_1+\alpha_1} u(\eta_1) \right)$$
$$- \frac{\Lambda_2 (\log t)^{\alpha_1} - \Lambda_1 \Gamma(\alpha_1+1)}{\Pi_1 \Gamma(\alpha_1+1)} \left(\gamma_2 - bI^{q_1+\alpha_1+\beta_1} h_1(\xi_1) + \lambda_1 bI^{q_1+\alpha_1} u(\xi_1) \right),$$

where

$$\begin{aligned}
\Pi_1 &= \Lambda_1 \Lambda_4 - \Lambda_2 \Lambda_3 \neq 0, \\
\Lambda_1 &= \frac{a(\log \eta_1)^{p_1+\alpha_1}}{\Gamma(p_1+\alpha_1+1)}, \quad \Lambda_2 = \frac{a(\log \eta_1)^{p_1}}{\Gamma(p_1+1)}, \\
\Lambda_3 &= \frac{b(\log \xi_1)^{q_1+\alpha_1}}{\Gamma(q_1+\alpha_1+1)}, \quad \Lambda_4 = \frac{b(\log \xi_1)^{q_1}}{\Gamma(q_1+1)},
\end{aligned}$$

Proof. Using Lemma 1.1 , we can write

$$u(t) = I^{\alpha+\beta} h_1(t) - \lambda_1 I^{\alpha_1} u(t) + e_0 \frac{1}{\Gamma(\alpha_1+1)} (\log t)^{\alpha_1} + e_1, \tag{6}$$

where e_0 and e_1 are arbitrary constants. By taking the Hadamard fractional integral of order p_1 for (6), we have

$$I^{p_1} u(t) = I^{p_1+\alpha_1+\beta_1} h_1(t) - \lambda_1 I^{p_1+\alpha_1} u(t) + e_0 \frac{(\log t)^{p_1+\alpha_1}}{\Gamma(p_1+\alpha_1+1)} + e_1 \frac{(\log t)^{p_1}}{\Gamma(p_1+1)}. \tag{7}$$

Now we multiply (7) by a and in particular, for $t = \eta_1$, we obtain

$$\begin{aligned}
aI^{p_1} x(\eta_1) &= aI^{p_1+\alpha_1+\beta_1} \varphi(\eta_1) - a\lambda_1 I^{p_1+\alpha_1} x(\eta_1) \tag{8} \\
&\quad + ae_0 \frac{(\log t)^{p_1+\alpha_1}}{\Gamma(p_1+\alpha_1+1)} + ae_1 \frac{(\log t)^{p_1}}{\Gamma(p_1+1)}.
\end{aligned}$$

Using the boundary conditions (4), we find that

$$e_0 \Lambda_1 + e_1 \Lambda_2 = \gamma_1 - aI^{p_1+\alpha_1+\beta_1} \varphi(\eta_1) + a\lambda_1 I^{p_1+\alpha_1} u(\eta_1), \tag{9}$$

and

$$e_0 \Lambda_3 + e_1 \Lambda_4 = \gamma_2 - bI^{q_1+\alpha_1+\beta_1} \varphi(\xi_1) + b\lambda_1 I^{q_1+\alpha_1} u(\xi_1). \tag{10}$$

Solving (9) and (10) for e_0, e_1, we get

$$\begin{aligned}
e_0 &= \frac{\Lambda_4}{\Pi_1} \left[\gamma_1 - aI^{p_1+\alpha_1+\beta_1} \varphi(\eta_1) + a\lambda_1 I^{p_1+\alpha_1} u(\eta_1) \right] \\
&\quad - \frac{\Lambda_4}{\Pi_1} \left[\gamma_2 - bI^{q_1+\alpha_1+\beta_1} \varphi(\xi_1) + b\lambda_1 I^{q_1+\alpha_1} u(\xi_1) \right],
\end{aligned}$$

and

$$\begin{aligned}
e_1 &= \frac{\Lambda_3}{\Pi_1} \left[\gamma_2 - bI^{q_1+\alpha_1+\beta_1} \varphi(\xi_1) + b\lambda_1 I^{q_1+\alpha_1} u(\xi_1) \right] \\
&\quad - \frac{\Lambda_1}{\Pi_1} \left[\gamma_1 - aI^{p_1+\alpha_1+\beta_1} \varphi(\eta_1) + a\lambda_1 I^{p_1+\alpha_1} u(\eta_1) \right].
\end{aligned}$$

By substituting the value of e_0 and e_1 in (6), we obtain the solution (5). $\quad\square$

Lemma 1.4. *Let $\Pi_2 \neq 0$ and $h_2 \in C([1,e],\mathbb{R})$. Then the solution of the problem*

$$
\begin{cases}
D^{\beta_2}(D^{\alpha_2} + \lambda_2)v(t) = h_2(t), t \in [1,e], 0 < \beta_2, \alpha_2 \leq 1, 1 < \beta_2 + \alpha_2 \leq 2 \\
\\
cI^{p_2}v(\eta_2) = \sigma_1, dI^{q_2}v(\xi_2) = \sigma_2, 1 < \eta_2, \xi_2 < e, c, d \in \mathbb{R},
\end{cases}
\tag{11}
$$

is given by

$$
\begin{aligned}
v(t) &= I^{\alpha_2+\beta_2}h_2(t) - \lambda_2 I^{\alpha_2}v(t) \\
&+ \frac{\Delta_2(\log t)^{\alpha_2} - \Delta_1\Gamma(\alpha_2+1)}{\Pi_2\Gamma(\alpha_2+1)}\left(\sigma_1 - cI^{p_2+\alpha_2+\beta_2}h_2(\eta_2) + \lambda_2 cI^{p_2+\alpha_2}v(\eta_2)\right) \\
&- \frac{\Delta_4(\log t)^{\alpha_2} - \Delta_3\Gamma(\alpha_2+1)}{\Pi_2\Gamma(\alpha_2+1)}\left(\sigma_2 - dI^{q_2+\alpha_2+\beta_2}h_2(\xi_2) + \lambda_2 dI^{q_2+\alpha_2}v(\xi_2)\right),
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
\Delta_1 &= \frac{c(\log\eta_2)^{p_2+\alpha_2}}{\Gamma(p_2+\alpha_2+1)}, \quad \Delta_2 = \frac{c(\log\eta_2)^{p_2}}{\Gamma(p_2+1)}, \\
\Delta_3 &= \frac{d(\log\xi_2)^{q_2+\alpha_2}}{\Gamma(q_2+\alpha_2+1)}, \quad \Delta_4 = \frac{d(\log\xi_2)^{q_2}}{\Gamma(q_2+1)},
\end{aligned}
$$

and

$$
\Pi_2 = \Delta_1\Delta_4 - \Delta_2\Delta_3, \quad \Pi_2 \neq 0.
$$

Proof. The proof it is similar to that of Lemma 1.4. □

Let us now introduce the space
$W = \{(u,v) : u,v \in C([1,T],\mathbb{R})\}$ endowed with the norm $\|u,v\|_W = \|u\| + \|v\|\}$, where

$$
\|u\| = \sup\{|u(t)|, t \in [1,T]\}, \|v\| = \sup\{|v(t)|, t \in [1,T]\}
$$

It is clear that $(W, \|.\|_W)$ is a Banach space. Throughout this paper, for convenience, the expressions $I^\omega\phi(s,u(s),v(s))(t)$, and $I^\varpi z(s)(t)$ mean

$$
\begin{aligned}
I^\omega\phi(s,u(s),v(s))(t) &= \frac{1}{\Gamma(\omega)}\int_1^t\left(\log\frac{t}{s}\right)^{\omega-1}\phi(s,u(s),u(s))\frac{ds}{s}. \\
I^\omega z(s)(t) &= \frac{1}{\Gamma(\omega)}\int_1^t\left(\log\frac{t}{s}\right)^{\omega-1}z(s)\frac{ds}{s}.
\end{aligned}
$$

2. Existence and uniqueness result

In view of Lemma 1.3 and Lemma 1.4 , we define the operator $H : W \to W$ by:

$$
H(u,v)(t) = \begin{pmatrix} H_1(u,v)(t) \\ H_2(u,v)(t) \end{pmatrix}, \ t \in [1,e],
$$

where

$$
\begin{aligned}
&H_1(u,v)(t) \\
&= I^{\alpha_1+\beta_1}f(s,u(s),v(s))(t) + I^{\alpha_1+\beta_1+\theta_1}h_1(s,u(s),v(s))(t) - \lambda_1 I^{\alpha_1}u(s)(t) \\
&+ \frac{\Lambda_2(\log t)^{\alpha_1} - \Lambda_1\Gamma(\alpha_1+1)}{\Pi_1\Gamma(\alpha_1+1)}\Big[\gamma_1 - aI^{p_1+\alpha_1+\beta_1}f(s,x(s),y(s))(\eta_1) \\
&- aI^{p_1+\alpha_1+\beta_1+\theta_1}\varphi(s,u(s),v(s))(\eta_1) + \lambda_1 aI^{p_1+\alpha_1}u(s)(\eta_1)\Big] \\
&- \frac{\Lambda_4(\log t)^{\alpha} - \Lambda_3\Gamma(\alpha_1+1)}{\Pi_1\Gamma(\alpha_1+1)}\Big[\gamma_2 - bI^{q_1+\alpha_1+\beta_1}f(s,u(s),v(s))(\xi_1) \\
&- bI^{q_1+\alpha_1+\beta_1+\theta_1}h_1(s,u(s),v(s))(\xi_1) + \lambda_1 bI^{q_1+\alpha_1}u(s)(\xi_1)\Big],
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
&H_2\left(u,v\right)(t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (14)\\
&= I^{\alpha_2+\beta_2}g\left(s,u\left(s\right),v\left(s\right)\right)(t) + I^{\alpha_2+\beta_2+\theta_2}\psi\left(s,u\left(s\right),v\left(s\right)\right)(t) - \lambda_2 I^{\alpha_2}v\left(s\right)(t)\\
&\quad + \frac{\Delta_2\left(\log t\right)^{\alpha_2}-\Delta_1\Gamma\left(\alpha_2+1\right)}{\Pi_2\Gamma\left(\alpha_2+1\right)}\Big[\sigma_1 - cI^{p_2+\alpha_2+\beta_2}g\left(s,u\left(s\right),v\left(s\right)\right)\left(\eta_2\right)\\
&\quad - cI^{p_2+\alpha_2+\beta_2+\theta_2}\psi\left(s,u\left(s\right),v\left(s\right)\right)\left(\eta_2\right) + \lambda_2 cI^{p_2+\alpha_2}v\left(s\right)\left(\eta_2\right)\Big]\\
&\quad - \frac{\Delta_4\left(\log t\right)^{\alpha_2}-\Delta_3\Gamma\left(\alpha_2+1\right)}{\Pi_2\Gamma\left(\alpha_2+1\right)}\Big[\sigma_2 - dI^{q_2+\alpha_2+\beta_2}g\left(s,u\left(s\right),v\left(s\right)\right)\left(\xi_2\right)\\
&\quad - dI^{q_2+\alpha_2+\beta_2+\theta_2}\psi\left(s,u\left(s\right),v\left(s\right)\right)\left(\xi_2\right) + \lambda_2 dI^{q_2+\alpha_2}v\left(s\right)\left(\xi_2\right)\Big].
\end{aligned}
$$

For the sake of convenience, we impose the following conditions:

$(C_1): f,\varphi:[1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and there exist constants $k_1>0, k_2>0$ shch that for all $t\in[1,e]$ and $u_i,v_i\in\mathbb{R}\,(i=1,2)$,

$$
\left|f\left(t,u_1,v_1\right)-f\left(t,u_2,v_2\right)\right|\le k_1\left|u_1-u_2\right|+\left|v_1-v_2\right|,
$$

and

$$
\left|\varphi\left(t,u_1,v_1\right)-\varphi\left(t,u_2,v_2\right)\right|\le k_2\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right).
$$

$(C_2): g,\psi:[1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and there exist constants $l_1>0, l_2>0$ shch that for all $t\in[1,e]$ and $u_i,v_i\in\mathbb{R}\,\,(i=1,2)$,

$$
\left|g\left(t,u_1,v_1\right)-g\left(t,u_2,v_2\right)\right|\le l_1\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right).
$$

and

$$
\left|\psi\left(t,u_1,v_1\right)-\psi\left(t,u_2,v_2\right)\right|\le l_2\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right).
$$

We also introduce the following quantities:

$$
\begin{aligned}
\nabla_1\; :&\; = \frac{1}{\Gamma\left(\alpha_1+\beta_1+1\right)} + \frac{|a|\left[|\Lambda_2|+|\Lambda_1|\Gamma\left(\alpha_1+1\right)\right]\left(\log\eta_1\right)^{p_1+\alpha_1+\beta_1}}{|\Pi_1|\Gamma\left(\alpha_1+1\right)\Gamma\left(p_1+\alpha_1+\beta_1+1\right)} \qquad (15)\\
&\quad + \frac{|b|\left[|\Lambda_4|+|\Lambda_3|\Gamma\left(\alpha_1+1\right)\right]\left(\log\xi_1\right)^{q_1+\alpha_1+\beta_1}}{|\Pi_1|\Gamma\left(\alpha_1+1\right)\Gamma\left(q_1+\alpha_1+\beta_1+1\right)},\\
\nabla_2\; :&\; = \frac{1}{\Gamma\left(\alpha_1+\beta_1+\theta_1+1\right)} + \frac{|a|\left[|\Lambda_2|+|\Lambda_1|\Gamma\left(\alpha_1+1\right)\right]\left(\log\eta_1\right)^{p_1+\alpha_1+\beta_1+\theta_1}}{|\Pi_1|\Gamma\left(\alpha_1+1\right)\Gamma\left(p_1+\alpha_1+\beta_1+\theta_1+1\right)}\\
&\quad + \frac{|b|\left[|\Lambda_4|+|\Lambda_3|\Gamma\left(\alpha_1+1\right)\right]\left(\log\xi_1\right)^{q_1+\alpha_1+\beta_1+\theta_1}}{|\Pi_1|\Gamma\left(\alpha_1+1\right)\Gamma\left(q_1+\alpha_1+\beta_1+\theta_1+1\right)}\\
\nabla_3\; :&\; = |\lambda_1|\left[\frac{1}{\Gamma\left(\alpha_1+1\right)} + \frac{|a|\left[|\Lambda_2|+|\Lambda_1|\Gamma\left(\alpha_1+1\right)\right]\left(\log\eta_1\right)^{p_1+\alpha_1}}{|\Pi_1|\Gamma\left(\alpha_1+1\right)\Gamma\left(p_1+\alpha_1+1\right)},\right.\\
&\quad \left. + \frac{|b|\left(|\Lambda_4|+|\Lambda_3|\Gamma\left(\alpha_1+1\right)\right)\left(\log\xi_1\right)^{q_1+\alpha_1}}{|\Pi_1|\Gamma\left(\alpha_1+1\right)\Gamma\left(q_1+\alpha_1+1\right)}\right],\\
\nabla_4\; :&\; = \frac{1}{|\Pi_1|\Gamma\left(\alpha_1+1\right)}\left[|\Lambda_2|+|\Lambda_1|\Gamma\left(\alpha_1+1\right)|\gamma_1| + |\Lambda_4|+|\Lambda_3|\Gamma\left(\alpha_1+1\right)|\gamma_2|\right].
\end{aligned}
$$

$$
\begin{aligned}
\Phi_1 \quad : \quad &= \frac{1}{\Gamma\left(\alpha_2+\beta_2+1\right)} + \frac{|c|\left[|\Delta_2|+|\Delta_1|\,\Gamma\left(\alpha_2+1\right)\right]\left(\log\eta_2\right)^{p_2+\alpha_2+\beta_2}}{|\Pi_2|\,\Gamma\left(\alpha_2+1\right)\Gamma\left(p_2+\alpha_2+\beta_2+1\right)} \\
&\quad + \frac{|d|\left[|\Delta_4|+|\Delta_3|\,\Gamma\left(\alpha_2+1\right)\right]\left(\log\xi_2\right)^{q_2+\alpha_2+\beta_2}}{|\Pi_2|\,\Gamma\left(\alpha_2+1\right)\Gamma\left(q_2+\alpha_2+\beta_2+1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_2 \quad : \quad &= \frac{1}{\Gamma\left(\alpha_2+\beta_2+\theta_2+1\right)} + \frac{|c|\left[|\Delta_2|+|\Delta_1|\,\Gamma\left(\alpha_2+1\right)\right]\left(\log\eta_2\right)^{p_2+\alpha_2+\beta_2+\theta_2}}{|\Pi_2|\,\Gamma\left(\alpha_2+1\right)\Gamma\left(p_2+\alpha_2+\beta_2+\theta_2+1\right)} \\
&\quad + \frac{|d|\left[|\Delta_4|+|\Delta_3|\,\Gamma\left(\alpha_2+1\right)\right]\left(\log\xi_2\right)^{q_2+\alpha_2+\beta_2+\theta_2}}{|\Pi_2|\,\Gamma\left(\alpha_2+1\right)\Gamma\left(q_2+\alpha_2+\beta_2+\theta_2+1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3 \quad : \quad &= |\lambda_2|\left[\frac{1}{\Gamma\left(\alpha_2+1\right)} + \frac{|c|\left[|\Delta_2|+|\Delta_1|\,\Gamma\left(\alpha_2+1\right)\right]\left(\log\eta_1\right)^{p_2+\alpha_2}}{|\Pi_2|\,\Gamma\left(\alpha_2+1\right)\Gamma\left(p_2+\alpha_2+1\right)},\right. \\
&\quad \left. + \frac{|d|\left(|\Delta_4|+|\Delta_3|\,\Gamma\left(\alpha_2+1\right)\right)\left(\log\xi_2\right)^{q_2+\alpha_2}}{|\Pi_2|\,\Gamma\left(\alpha_2+1\right)\Gamma\left(q_2+\alpha_2+1\right)}\right],
\end{aligned}
$$

$$
\Phi_4 \quad : \quad = \frac{1}{|\Pi_1|\,\Gamma\left(\alpha_2+1\right)}\left[|\Delta_2|+|\Delta_1|\,\Gamma\left(\alpha_2+1\right)|\sigma_1|+|\Delta_4|+|\Delta_3|\,\Gamma\left(\alpha_2+1\right)|\sigma_2|\right].
$$

(16)

Theorem 2.1. *Assume that conditions (C_1) and (C_1) hold and suppose that $\Pi_1 \neq 0$ and $\Pi_2 \neq 0$. If these inequalities*

$$
k_1\nabla_1 + k_2\nabla_2 + \nabla_3 < \frac{1}{2}, \quad l_1\Phi_1 + l_2\Phi_2 + \Phi_3 < \frac{1}{2},
\tag{17}
$$

are valid, then system (1)-(2) has a unique solution.

Proof. Let us fix $\sup_{t\in[1,e]}|f\left(t,0,0\right)| = L < \infty$, $\sup_{t\in[1,e]}|\varphi\left(t,0,0\right)| = M < \infty$ and define

$$
\max\left[\frac{k_1\nabla_1 + k_2\nabla_2 + \nabla_4}{\frac{1}{2} - \left(L\nabla_1 + M\nabla_2 + \nabla_3\right)}, \frac{l_1\Phi_1 + l_2\Phi_2 + \Phi_4}{\frac{1}{2} - \left(L'\Phi_1 + M'\Phi_2 + \Phi_3\right)}\right] \leq r.
$$

We show that $HB_r \subset B_r$, where $B_r = \{(u,v) \in X \times Y : \|(u,v)\| \leq r\}$. For $(u,v) \in B_r$ and by (H_1), we have

$$
\begin{aligned}
|f\left(t,u,v\right)| \quad &\leq \quad |f\left(t,u,v\right)-f\left(t,0,0\right)|+|f\left(t,0,0\right)| \leq k_1\|u\|+k_1\|v\|+L \\
&\leq \quad k_1\|u,v\|_W + L \leq k_1r + L,
\end{aligned}
$$

$$
\begin{aligned}
|\varphi\left(t,u,v\right)| \quad &\leq \quad |\varphi\left(t,u,v\right)-\varphi\left(t,0,0\right)|+|\varphi\left(t,0,0\right)| \leq k_2\|u\|+k_2\|v\|+M \\
&\leq \quad k_2\|u,v\|_W + M \leq k_2r + M,
\end{aligned}
$$

(18)

Similarly, we have

$$
|g\left(t,u,v\right)| \quad \leq \quad l_1\|u\|+l_1\|v\|+L' \leq l_1\|u,v\|_W + L' \leq l_1r + L',
$$

$$
|\psi\left(t,u,v\right)| \quad \leq \quad l_2\|u\|+l_2\|v\|+M' \leq l_2\|u,v\|_W + M' \leq l_2r + M',
$$

(19)

By (18), we can write

$$\|H_1(u,v)\|$$
$$\leq \sup_{t\in[1,e]} \left\{ I^{\alpha_1+\beta_1} |f(s,u(s),v(s))|(t) + I^{\alpha_1+\beta_1+\theta_1} |\psi(s,u(s),v(s))|(t) + |\lambda_1| I^{\alpha_1} |v(s)|(t) \right.$$

$$+ \frac{|\Lambda_2|(\log t)^{\alpha_1} + |\Lambda_1|\Gamma(\alpha_1+1)}{|\Pi_1|\Gamma(\alpha_1+1)} \left(|\gamma_1| + |a| I^{p_1+\alpha_1+\beta_1} |f(s,u(s),v(s))|(\eta_1) \right.$$

$$+ |a| I^{p_1+\alpha_1+\beta_1+\theta_1} |\varphi(s,u(s),v(s))|(\eta_1) + |\lambda_1||a_1| I^{p_1+\alpha_1} |u(s)|(\eta_1) \Big)$$

$$+ \frac{|\Lambda_4|(\log t)^{\alpha_1} + |\Lambda_3|\Gamma(\alpha_1+1)}{|\Pi_1|\Gamma(\alpha_1+1)} \left(|\gamma_2| + |b| I^{q_1+\alpha_1+\beta_1} |f_1(s,u(s),v(s))|(\xi_1) \right.$$

$$\left. + |b| I^{q_1+\alpha_1+\beta_1+\theta_1} |\varphi(s,u(s),v(s))|(\xi_1) + |\lambda_1||b| I^{q_1+\alpha_1} |u(s)|(\xi_1) \Big) \right\}$$

$$\leq \left[\frac{1}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1|\Gamma(\alpha_1+1))(\log \eta_1)^{p_1+\alpha_1+\beta_1}}{|\Pi_1|\Gamma(\alpha_1+1)\Gamma(p_1+\alpha_1+\beta_1+1)} \right.$$

$$\left. + \frac{|b|(|\Lambda_4| + |\Lambda_3|\Gamma(\alpha_1+1))(\log \xi_1)^{q_1+\alpha_1+\beta_1}}{|\Pi_1|\Gamma(\alpha_1+1)\Gamma(q_1+\alpha_1+\beta_1+1)} \right] (Lr+k_1)$$

$$+ \left[\frac{1}{\Gamma(\alpha_1+\beta_1+\theta_1+1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1|\Gamma(\alpha_1+1))(\log \eta_1)^{p_1+\alpha_1+\beta_1+\theta_1}}{|\Pi_1|\Gamma(\alpha_1+1)\Gamma(p_1+\alpha_1+\beta_1+\theta_1+1)} \right.$$

$$\left. + \frac{|b|(|\Lambda_4| + |\Lambda_3|\Gamma(\alpha_1+1))(\log \xi_1)^{q_1+\alpha_1+\beta_1+\theta_1}}{|\Pi_1|\Gamma(\alpha_1+1)\Gamma(q_1+\alpha_1+\beta_1+\theta_1+1)} \right] (Lr+k_2)$$

$$+ |\lambda_1| \left[\frac{1}{\Gamma(\alpha_1+1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1|\Gamma(\alpha_1+1))(\log \eta_1)^{p_1+\alpha_1}}{|\Pi_1|\Gamma(\alpha_1+1)\Gamma(p_1+\alpha_1+1)} \right.$$

$$\left. + \frac{|b_1|(|\Lambda_4| + |\Lambda_3|\Gamma(\alpha_1+1))(\log \xi_1)^{q_1+\alpha_1}}{|\Pi_1|\Gamma(\alpha_1+1)\Gamma(q_1+\alpha_1+1)} \right] r$$

$$+ \frac{1}{|\Pi_1|\Gamma(\alpha_1+1)} [|\Delta_2| + |\Delta_1|\Gamma(\alpha_1+1)|\gamma_1| + |\Delta_4| + |\Delta_3|\Gamma(\alpha_1+1)|\gamma_2|]$$

$$= (Lr+k_1)\nabla_1 + (Mr+k_2)\nabla_2 + \nabla_3 r + \nabla_4,$$

which implies that

$$\|H_1(u,v)\| \leq (L\nabla_1 + M\nabla_2 + \nabla_3)r + k_1\nabla_1 + k_2\nabla_2 + \nabla_4 \leq \frac{r}{2}.$$

In a similar manner, using (19), we obtain

$$\|H_2(u,v)\| \leq \left(L'\Phi_1 + M'\Phi_2 + \Phi_3 \right)r + l_1\Phi_1 + l_2\Phi_2 + \Phi_4 \leq \frac{r}{2}.$$

From the definition of $\|.\|_W$, we can write

$$\|H(u,v)\|_W = \|H_1(u,v)\| + \|H_2(u,v)\| \leq r.$$

Now, for $u_i, v_i \in B_r, i = 1, 2$, we have

$$|H_1(u_1, v_1) - H_1(u_2, v_2)(t)|$$

$$\leq \sup_{t \in [1,e]} \left\{ I^{\alpha_1 + \beta_1} |f(s, u_1(s), v_1(s)) - f(s, u_2(s), v_2(s))|(t) \right.$$

$$+ |\lambda_1| I^{\alpha_1} |u_1(s) - u_2(s)|(t)$$

$$+ I^{\alpha_1 + \beta_1 + \theta_1} |\varphi(s, u_1(s), v_1(s)) - \varphi(s, u_2(s), v_2(s))|(t)$$

$$+ \frac{|\Lambda_2|(\log t)^{\alpha_1} + |\Lambda_1| \Gamma(\alpha_1 + 1)}{|\Pi_1| \Gamma(\alpha_1 + 1)}$$

$$\times \left[|a| I^{p_1 + \alpha_1 + \beta_1} |f(s, u_1(s), v_1(s)) - f(s, u_2(s), v_2(s))|(\eta_1) \right.$$

$$+ |a| I^{p_1 + \alpha_1 + \beta_1 + \theta_1} |\varphi(s, u_1(s), v_1(s)) - \varphi(s, u_2(s), v_2(s))|(\eta_1)$$

$$+ |\lambda_1| |a| I^{p_1 + \alpha_1} |u_1(s) - u_2(s)|(\eta_1) \Big]$$

$$+ \frac{|\Lambda_4|(\log t)^{\alpha_1} + |\Lambda_3| \Gamma(\alpha_1 + 1)}{|\Pi_1| \Gamma(\alpha_1 + 1)}$$

$$\times \left[|b| I^{q_1 + \alpha_1 + \beta_1} |f(s, u_1(s), v_1(s)) - f(s, u_2(s), v_2(s))|(\xi_1) \right.$$

$$+ |b| I^{q_1 + \alpha_1 + \beta_1 + \theta_1} |\varphi(s, u_1(s), v_1(s)) - \varphi(s, u_2(s), v_2(s))|(\xi_1)$$

$$+ |\lambda_1| |b| I^{q_1 + \alpha_1} |u_1(s) - u_2(s)|(\xi_1) \Big] \right\}$$

$$\leq \left[\frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1| \Gamma(\alpha_1 + 1))(\log \eta_1)^{p_1 + \alpha_1 + \beta_1}}{|\Pi_1| \Gamma(\alpha_1 + 1) \Gamma(p_1 + \alpha_1 + \beta_1 + 1)} \right.$$

$$+ \frac{|b|(|\Lambda_4| + |\Lambda_4| \Gamma(\alpha_1 + 1))(\log \xi_1)^{q_1 + \alpha_1 + \beta_1}}{|\Pi_1| \Gamma(\alpha_1 + 1) \Gamma(q_1 + \alpha_1 + \beta_1 + 1)} \right] k_1 (\|u_1 - u_2\| + \|v_1 - v_2\|)$$

$$+ \left[\frac{1}{\Gamma(\alpha_1 + \beta_1 + \theta_1 + 1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1| \Gamma(\alpha_1 + 1))(\log \eta_1)^{p_1 + \alpha_1 + \beta_1 + \theta_1}}{|\Pi_1| \Gamma(\alpha_1 + 1) \Gamma(p_1 + \alpha_1 + \beta_1 + \theta_1 + 1)} \right.$$

$$+ \frac{|b|(|\Lambda_4| + |\Lambda_3| \Gamma(\alpha_1 + 1))(\log \xi_1)^{q_1 + \alpha_1 + \beta_1 + \theta_1}}{|\Pi_1| \Gamma(\alpha_1 + 1) \Gamma(q_1 + \alpha_1 + \beta_1 + \theta_1 + 1)} \right] k_2 (\|u_1 - u_2\| + \|v_1 - v_2\|)$$

$$+ |\lambda_1| \left[\frac{1}{\Gamma(\alpha_1 + 1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1| \Gamma(\alpha_1 + 1))(\log \eta_1)^{p_1 + \alpha_1}}{|\Pi_1| \Gamma(\alpha_1 + 1) \Gamma(p_1 + \alpha_1 + 1)} \right.$$

$$+ \frac{|b|(|\Lambda_4| + |\Lambda_3| \Gamma(\alpha_1 + 1))(\log \xi_1)^{q_1 + \alpha_1}}{|\Pi_1| \Gamma(\alpha_1 + 1) \Gamma(q_1 + \alpha_1 + 1)} \right] \|u_1 - u_2\|$$

$$= [k_1 \nabla_1 + k_2 \nabla_2 + \nabla_3] (\|u_1 - u_2\| + \|v_1 - v_2\|),$$

and, consequently, we obtain

$$\|H_1(u_1, v_1) - H_1(u_2, v_2)\| \leq [k_1 \nabla_1 + k_2 \nabla_2 + \nabla_3] (\|u_1 - u_2\| + \|v_1 - v_2\|).$$

Similarly, we can have

$$\|H_2(u_1, v_1) - H_2(u_2, v_2)\| \leq [l_1 \Phi_1 + l_2 \Phi_2 + \Phi_3] (\|u_1 - u_2\| + \|v_1 - v_2\|).$$

Consequently, we obtain

$$\|H(u_1, v_1) - H(u_2, v_2)\|_W$$
$$\leq [k_1 \nabla_1 + k_2 \nabla_2 + \nabla_3 + l_1 \Phi_1 + l_2 \Phi_2 + \Phi_3] (\|u_1 - u_2\| + \|v_1 - v_2\|).$$

Since $k_1 \nabla_1 + k_2 \nabla_2 + \nabla_3 + l_1 \Phi_1 + l_2 \Phi_2 + \Phi_3 < 1$, therefore, H is a contraction. So, by Banach's fixed point theorem, the operator H has a unique fixed point, which is the unique solution of system (1)-(2). This completes the proof. $\square$

Example 2.1. *Consider the following system*

$$
\begin{cases}
D^{\frac{22}{31}} \left(D^{\frac{10}{13}} + \frac{2}{49} \right) u(t) - \frac{1}{(25\pi^2 + t)} \left(\sin u(t) - \cos v(t) \right) \\
\quad = I^{\frac{3}{5}} \left[\frac{1}{55t^2 + 1} \left(\frac{|u(t)|}{|u(t)| + 1} + \frac{|v(t)|}{|v(t)| + 1} \right) \right], t \in [1, e], \\
\\
D^{\frac{11}{14}} \left(D^{\frac{8}{9}} + \frac{7}{80} \right) v(t) - \frac{1 - e^{-t}}{57(t+1)^2} \left(\frac{|u(t)|}{2|u(t)| + 1} + \frac{|v(t)|}{3|v(t)| + 2} \right) \\
\quad = I^{\frac{19}{22}} \left[\frac{1}{49(t^2 + 2)} \left(\cos u(t) + \cos v(t) \right) \right], t \in [1, e], \\
\\
I^{\frac{2}{3}} u \left(\frac{8}{5} \right) = (\log 8 - \log 5)^{-\frac{1}{3}}, I^{\frac{57}{60}} u \left(\frac{7}{4} \right) = (\log 7 - \log 4)^{-\frac{1}{3}}, \\
\frac{1}{20} I^{\frac{3}{4}} v \left(\frac{e}{2} \right) = (1 - \log 2)^{-\frac{1}{4}}, \frac{2}{35} I^{\frac{42}{51}} v \left(\sqrt{2} \right) = \frac{2}{5} (\log 2)^{-\frac{9}{51}}.
\end{cases} \tag{20}
$$

For this example, we have

$$
\begin{aligned}
f(t, u, v) &= \frac{1}{(25\pi^2 + t)} (\sin u - \cos v), \\
g(t, u, v) &= \frac{1 - e^{-t}}{57(t+1)^2} \left(\frac{|u(t)|}{2|u| + 1} + \frac{|v|}{3|v| + 2} \right), \\
\varphi(t, u, v) &= \frac{1}{55t^2 + 1} \left(\frac{|u|}{|u| + 1} + \frac{|v|}{|v| + 1} \right), \\
\psi(t, u, v) &= \frac{1}{49(t^2 + 2)} (\cos u + \cos v).
\end{aligned}
$$

With the given data, we find that

$$
\begin{aligned}
\Lambda_1 &\simeq -0.26587, \Lambda_2 \simeq -0.66963, \Lambda_3 \simeq 0.37963, \Lambda_4 \simeq 0.58793, \\
\Pi_1 &= \Lambda \Lambda_4 - \Lambda_2 \Lambda_3 \simeq 0.097899, \\
\nabla_1 &\simeq 2.588\,2, \nabla_2 = 0.914\,63, \nabla_3 \simeq 0.25392,
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_1 &\simeq -4.8983 \times 10^{-3}, \Delta_2 \simeq -2.243 \times 10^{-2}, \\
\Delta_3 &\simeq -1.177 \times 10^{-2}, \Delta_4 \simeq -3.4429 \times 10^{-2}, \\
\Pi_2 &= \Delta_1 \Delta_4 - \Delta_2 \Delta_3 \simeq -9.535\,8 \times 10^{-5}, \\
\Phi_1 &\simeq 2.143\,2, \Phi_2 = 0.407\,70, \Phi_3 \simeq 0.59613.
\end{aligned}
$$

So, for $t \in [1, e]$ and $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$, we have

$$
\begin{aligned}
|f(t, u_1, v_1) - f(t, u_2, v_2)| &\leq \frac{1}{(25\pi^2 + t)} (|u_1 - u_2| + |v_1 - v_2|), \\
|g(t, u_1, v_1) - g(t, u_2, v_2)| &\leq \frac{1}{57(t+1)^2} (|u_1 - u_2| + |v_1 - v_2|), \\
|\varphi(t, u_1, v_1) - \varphi(t, u_2, v_2)| &\leq \frac{1}{55t^2 + 1} (|u_1 - u_2| + |v_1 - v_2|), \\
|g_2(t, u_1, v_1) - g_2(t, u_2, v_2)| &\leq \frac{1}{49(t^2 + 2)} (|u_1 - u_2| + |v_1 - v_2|).
\end{aligned}
$$

It follows that

$$k_1 = \frac{1}{(25\pi^2 + e)}, k_2 = \frac{1}{140}, l_1 = \frac{1}{55e^2 + 1}, l_2 = \frac{1}{49(e^2 + 2)}.$$

Then,

$$k_1\nabla_1 + k_2\nabla_2 + \nabla_3 + l_1\Phi_1 + l_2\Phi_2 + \Phi_3 \approx 0.976\,76 < 1.$$

By Theorem 2.1 , we conclude that the system (20). has a unique solution on $[1, e]$.

3. Existence result

We show the existence of solutions for the system (1)-(2) by applying Lemma 1.2.

For the forthcoming result, we impose the following conditions:

$(C_3) : f, \varphi : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, are continuous functions and there exist real constants $\omega_1 > 0, \varpi_1 > 0$ such that for any $t \in [1, e]$ and $u, v \in \mathbb{R}$, we have

$$|f(t, u(t), v(t))| \le \omega_1, \quad |\varphi(t, u(t), v(t))| \le \omega_2.$$

$(C_4) : g, \psi : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, are continuous functions and there exist real constants $\omega_2 > 0, \varpi_2 > 0$ such that for all $u, v \in \mathbb{R}$ and $t \in [1, e]$, we have

$$|g(t, u(t), v(t))| \le \varpi_1, \quad |\psi(t, u(t), v(t))| \le \varpi_2.$$

Theorem 3.1. *Assume that hypotheses* (C_3) *and* (C_4) *hold. Furthermore, assume that* $\Pi_1 \ne 0$ *and* $\Pi_2 \ne 0$. *Then, system (1)-(2) has at least one solution.*

Proof. By continuity of functions of f, g φ and ψ, the operator H is continuous.

Now, we show that the operator H is completely continuous.

(I) First, we show that H maps bounded sets of W into bounded sets of W. Let us take $\varepsilon > 0$ and $B_\varepsilon = \{(u, v) \in W : \|u, v\|_W \le \varepsilon\}$. Then for $(u, v) \in B_\varepsilon$, we have

$$\|H_1(u, v)\|$$
$$\le \sup_{t \in [1, e]} \left\{ I^{\alpha_1 + \beta_1} |f(s, u(s), v(s))|(t) + I^{\alpha_1 + \beta_1 + \theta_1} |\psi(s, u(s), v(s))|(t) + |\lambda_1| I^{\alpha_1} |v(s)|(t) \right.$$
$$+ \frac{|\Lambda_2|(\log t)^{\alpha_1} + |\Lambda_1|\Gamma(\alpha_1 + 1)}{|\Pi_1|\Gamma(\alpha_1 + 1)} \left(|\gamma_1| + |a| I^{p_1 + \alpha_1 + \beta_1} |f(s, u(s), v(s))|(\eta_1)\right.$$
$$+ |a| I^{p_1 + \alpha_1 + \beta_1 + \theta_1} |\varphi(s, u(s), v(s))|(\eta_1) + |\lambda_1| |a_1| I^{p_1 + \alpha_1} |u(s)|(\eta_1)\Big)$$
$$+ \frac{|\Lambda_4|(\log t)^{\alpha_1} + |\Lambda_3|\Gamma(\alpha_1 + 1)}{|\Pi_1|\Gamma(\alpha_1 + 1)} \left(|\gamma_2| + |b| I^{q_1 + \alpha_1 + \beta_1} |f_1(s, u(s), v(s))|(\xi_1)\right.$$
$$\left. + |b| I^{q_1 + \alpha_1 + \beta_1 + \theta_1} |\varphi(s, u(s), v(s))|(\xi_1) + |\lambda_1| |b| I^{q_1 + \alpha_1} |u(s)|(\xi_1)\Big) \right\}$$

By (C_3), we obtain

$$\|H_1(u,v)\|$$
$$\leq \left[\frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1|\Gamma(\alpha_1 + 1))(\log \eta_1)^{p_1 + \alpha_1 + \beta_1}}{|\Pi_1|\Gamma(\alpha_1 + 1)\Gamma(p_1 + \alpha_1 + \beta_1 + 1)} \right.$$
$$\left. + \frac{|b|(|\Lambda_4| + |\Lambda_3|\Gamma(\alpha_1 + 1))(\log \xi_1)^{q_1 + \alpha_1 + \beta_1}}{|\Pi_1|\Gamma(\alpha_1 + 1)\Gamma(q_1 + \alpha_1 + \beta_1 + 1)} \right] \omega_1$$
$$+ \left[\frac{1}{\Gamma(\alpha_1 + \beta_1 + \theta_1 + 1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1|\Gamma(\alpha_1 + 1))(\log \eta_1)^{p_1 + \alpha_1 + \beta_1 + \theta_1}}{|\Pi_1|\Gamma(\alpha_1 + 1)\Gamma(p_1 + \alpha_1 + \beta_1 + \theta_1 + 1)} \right.$$
$$\left. + \frac{|b|(|\Lambda_4| + |\Lambda_3|\Gamma(\alpha_1 + 1))(\log \xi_1)^{q_1 + \alpha_1 + \beta_1 + \theta_1}}{|\Pi_1|\Gamma(\alpha_1 + 1)\Gamma(q_1 + \alpha_1 + \beta_1 + \theta_1 + 1)} \right] \omega_2$$
$$+ |\lambda_1| \left[\frac{1}{\Gamma(\alpha_1 + 1)} + \frac{|a|(|\Lambda_2| + |\Lambda_1|\Gamma(\alpha_1 + 1))(\log \eta_1)^{p_1 + \alpha_1}}{|\Pi_1|\Gamma(\alpha_1 + 1)\Gamma(p_1 + \alpha_1 + 1)} \right.$$
$$\left. + \frac{|b_1|(|\Lambda_4| + |\Lambda_3|\Gamma(\alpha_1 + 1))(\log \xi_1)^{q_1 + \alpha_1}}{|\Pi_1|\Gamma(\alpha_1 + 1)\Gamma(q_1 + \alpha_1 + 1)} \right] \varepsilon$$
$$+ \frac{1}{|\Pi_1|\Gamma(\alpha_1 + 1)} \left[|\Delta_2| + |\Delta_1|\Gamma(\alpha_1 + 1)|\gamma_1| + |\Delta_4| + |\Delta_3|\Gamma(\alpha_1 + 1)|\gamma_2| \right]$$
$$= \nabla_1 \omega_1 + \nabla_2 \omega_2 + \nabla_3 \varepsilon + \nabla_4,$$

which implies that

$$\|H_1(u,v)\| \leq \nabla_1 \omega_1 + \nabla_2 \omega_2 + \nabla_3 \varepsilon + \nabla_4. \tag{21}$$

As before, it can be shown that

$$\|H_2(u,v)\| \leq \Phi_1 \varpi_1 + \Phi_2 \varpi_2 + \Phi_3 \varepsilon + \Phi_4. \tag{22}$$

It follows from (21) and (22) that

$$\|H(u,v)\|_W$$
$$\leq \nabla_1 \omega_1 + \nabla_2 \omega_2 + \nabla_3 \varepsilon + \nabla_4 + \Phi_1 \varpi_1 + \Phi_2 \varpi_2 + \Phi_3 \varepsilon + \Phi_4 < \infty.$$

(II) Next, we show that H is equicontinuous. Let $(u,v) \in B_\varepsilon$ and $t_1, t_2 \in [1, e]$ with $t_1 < t_2$. Then we have

$$|H_2(u,v)(t_2) - H_2(u,v)(t_1)|$$
$$\leq \frac{\omega_1}{\Gamma(\alpha_1 + \beta_1)} \left(\left| (\log t_2)^{\alpha_1 + \beta_1} - (\log t_1)^{\alpha_1 + \beta_1} \right| + \left| (\log t_2 - \log t_1)^{\alpha_1 + \beta_1} \right| \right)$$
$$+ \frac{\omega_2}{\Gamma(\alpha_1 + \beta_1 + \theta_1)} \left(\left| (\log t_2)^{\alpha_1 + \beta_1 + \theta_1} - (\log t_1)^{\alpha_1 + \beta_1 + \theta_1} \right| + \left| (\log t_2 - \log t_1)^{\alpha_1 + \beta_1 + \theta_1} \right| \right)$$
$$+ \frac{\varepsilon |\lambda_1|}{\Gamma(\alpha_1)} \left(|(\log t_2)^{\alpha_1} - (\log t_1)^{\alpha_1}| + |(\log t_2 - \log t_1)^{\alpha_1}| \right)$$
$$+ \frac{|\Lambda_2||(\log t_2)^{\alpha_1} - (\log t_1)^{\alpha_1}|}{|\Pi_1|\Gamma(\alpha_1 + 1)} \left(|\gamma_1| + \frac{\omega_1 |a|(\log \eta_1)^{p_1 + \alpha_1 + \beta_1}}{\Gamma(p_1 + \alpha_1 + \beta_1)} \right.$$
$$\left. + \frac{\omega_2 |a|(\log \eta_1)^{p_1 + \alpha_1 + \beta_1 + \theta_1}}{\Gamma(p_1 + \alpha_1 + \beta_1 + \theta_1)} + \frac{\varepsilon |\lambda_1| |a|(\log \eta_1)^{p_1 + \alpha_1}}{\Gamma(p_1 + \alpha_1)} \right)$$
$$+ \frac{|\Lambda_4||(\log t_2)^{\alpha_1} - (\log t_1)^{\alpha_1}|}{|\Pi_1|\Gamma(\alpha_1 + 1)} \left(|\gamma_2| + \frac{\omega_1 |b|(\log \xi_1)^{q_1 + \alpha_1 + \beta_1}}{\Gamma(q_1 + \alpha_1 + \beta_1)} \right.$$
$$\left. + \frac{\omega_2 |b|(\log \xi_1)^{q_1 + \alpha_1 + \beta_1 + \theta_1}}{\Gamma(q_1 + \alpha_1 + \beta_1 + \theta_1)} + \frac{\varepsilon |\lambda_1| |b|(\log \xi_1)^{q_1 + \alpha_1}}{\Gamma(q_1 + \alpha_1)} \right),$$

which imply that $\|H_1(u,v)(t_2) - H_1(u,v)(t_1)\| \to 0$ independent of u and v as $t_2 \to t_1$. Analogously, we obtain

$$
\begin{aligned}
&|H_1(u,v)(t_2) - H_1(u,v)(t_1)| \\
\leq\ & \frac{\varpi_1}{\Gamma(\alpha_2+\beta_2)} \left(\left|(\log t_2)^{\alpha_2+\beta_2} - (\log t_1)^{\alpha_2+\beta_2}\right| + \left|(\log t_2 - \log t_1)^{\alpha_2+\beta_2}\right| \right) \\
& + \frac{\varpi_2}{\Gamma(\alpha_2+\beta_2+\theta_2)} \left(\left|(\log t_2)^{\alpha_2+\beta_2+\theta_2} - (\log t_1)^{\alpha_2+\beta_2+\theta_2}\right| + \left|(\log t_2 - \log t_1)^{\alpha_2+\beta_2+\theta_2}\right| \right) \\
& + \frac{\varepsilon|\lambda_2|}{\Gamma(\alpha_2)} \left(|(\log t_2)^{\alpha_2} - (\log t_1)^{\alpha_2}| + |(\log t_2 - \log t_1)^{\alpha_2}| \right) \\
& + \frac{|\Delta_2|\,|(\log t_2)^{\alpha_2} - (\log t_1)^{\alpha_2}|}{|\Pi_2|\,\Gamma(\alpha_2+1)} \left(|\sigma_1| + \frac{\varpi_1|c|\,(\log \eta_2)^{p_2+\alpha_2+\beta_2}}{\Gamma(p_2+\alpha_2+\beta_2)} \right. \\
& \left. + \frac{\varpi_2|c|\,(\log \eta_2)^{p_2+\alpha_2+\beta_2+\theta_2}}{\Gamma(p_2+\alpha_2+\beta_2+\theta_2)} + \frac{\varepsilon|\lambda_2|\,|c|\,(\log \eta_2)^{p_2+\alpha_2}}{\Gamma(p_2+\alpha_2)} \right) \\
& + \frac{|\Delta_4|\,|(\log t_2)^{\alpha_2} - (\log t_1)^{\alpha_2}|}{|\Pi_2|\,\Gamma(\alpha_2+1)} \left(|\sigma_2| + \frac{\varpi_1|d|\,(\log \xi_2)^{q_2+\alpha_2+\beta_2}}{\Gamma(q_2+\alpha_2+\beta_2)} \right. \\
& \left. + \frac{\varpi_2|d|\,(\log \xi_2)^{q_2+\alpha_2+\beta_2+\theta_2}}{\Gamma(q_2+\alpha_2+\beta_2+\theta_2)} + \frac{\varepsilon|\lambda_2|\,|d|\,(\log \xi_2)^{q_2+\alpha_2}}{\Gamma(q_2+\alpha_2)} \right),
\end{aligned}
$$

which imply that $\|H_2(u,v)(t_2) - H_2(u,v)(t_1)\| \to 0$ independent of u and v as $t_2 \to t_1$. Therefore, the operator $H(u,v)$ is equicontinuous. Combining (I) and (II) and using the Arzelà-Ascoli theorem, we conclude that H is completely continuous operator.

(III) Finally, it will be verified that the $\Omega = \{(u,v) \in W, (u,v) = \mu H(u,v), 0 < \mu < 1\}$, is bounded. Let $(u,v) \in \Omega$, then $(u,v) = \mu H(u,v)$, for some $0 < \mu < 1$. Hence, for $t \in [1, e]$, we have

$$ u(t) = \mu H_1(u,v)(t), \ v(t) = \mu H_2(u,v)(t). $$

By (C_3) and (C_4), we have

$$ \|u\| \leq \mu \left(\nabla_1 \omega_1 + \nabla_2 \omega_2 + \nabla_3 \varepsilon + \nabla_4 \right), \tag{23} $$

and

$$ \|v\| \leq \mu \left(\Phi_1 \varpi_1 + \Phi_2 \varpi_2 + \Phi_3 \varepsilon + \Phi_4 \right). \tag{24} $$

It follows from (23) and (24), that

$$
\begin{aligned}
& \|(u,v)\|_W \\
\leq\ & \mu \left[\nabla_1 \omega_1 + \nabla_2 \omega_2 + \nabla_3 \varepsilon + \nabla_4, + \Phi_1 \varpi_1 + \Phi_2 \varpi_2 + \Phi_3 \varepsilon + \Phi_4 \right] \\
\leq\ & \nabla_1 \omega_1 + \nabla_2 \omega_2 + \nabla_3 \varepsilon + \nabla_4, + \Phi_1 \varpi_1 + \Phi_2 \varpi_2 + \Phi_3 \varepsilon + \Phi_4.
\end{aligned}
$$

Consequently,

$$ \|(u,v)\|_W < \infty. $$

This shows that the set is bounded.

Thanks to $(I), (II)$ and (III), and by Lemma 1.2, we deduce that H has at least one fixed point, which is a solution of system (1)-(2). $\qquad\square$

Example 3.1. *For* $t \in [1, e]$*, consider the following system*

$$
\begin{cases}
D^{\frac{27}{44}}\left(D^{\frac{16}{25}} + \frac{1}{49}\right)u\left(t\right) - \frac{2\ln(t)|\sin u(t)|}{t^2+1} - \frac{|v(t)|(3t-e)}{4e^t(|y(t)|+1)} = I^{\frac{1}{2}}\left(\frac{|u(t)+v(t)|(3-t)^2}{\sqrt{t+8}(|u(t)|+|v(t)|+1)}\right), \\[2mm]
D^{\frac{13}{20}}\left(D^{\frac{8}{11}} - \frac{1}{33}\right)y\left(t\right) - \frac{e^{-|u(t)-v(t)|}+7}{(t+3)^3} = I^{\frac{21}{40}}\left(\frac{|\cos u(t)|+e^{t-e}}{5t^2+\ln t-2} + \frac{|\sin v(t)|}{3t^4-1}\right), \\[2mm]
4I^{\frac{17}{50}}u\left(\frac{38}{25}\right) = \left(\log 8 - \log 5\right)^{-\frac{1}{2}}, 3I^{\frac{9}{10}}u\left(\frac{8}{3}\right) = 6\left(\log 8 - \log 3\right)^{-\frac{1}{10}}, \\[2mm]
\frac{3}{2}I^{\frac{1}{3}}v\left(\frac{8}{5}\right) = 2\left(\log 8 - \log 5\right)^{-\frac{2}{3}}, \frac{7}{2}I^{\frac{23}{25}}v\left(\frac{5}{2}\right) = \frac{21}{2}\left(\log 5 - \log 2\right)^{-\frac{2}{25}}.
\end{cases}
\tag{25}
$$

We have

$$
\begin{aligned}
f\left(t, u, v\right) &= \frac{2\ln\left(t\right)|\sin u|}{t^2+1} + \frac{\left(3t-e\right)|v|}{4e^t\left(|y\left(t\right)|+1\right)}, \varphi\left(t, u, v\right) = \frac{|u+v|\left(3-t\right)^2}{\sqrt{t+8}\left(|u|+|v|+1\right)}, \\
g\left(t, u, v\right) &= \frac{e^{-|u-v|}+7}{\left(t+3\right)^3}, \psi\left(t, u, v\right) = \frac{|\cos u|+e^{t-e}}{5t^2+\ln t-2} + \frac{|\sin v|}{3t^4-1}.
\end{aligned}
$$

and

$$
\begin{aligned}
\Lambda_1 &\simeq -1.7185, \Lambda_2 \simeq -3.3346, \Lambda_3 \simeq -2.1289, \Lambda_4 \simeq -3.0654, \\
\Delta_1 &\simeq 0.317\,90, \Delta_2 \simeq 0.633\,22, \Delta_3 \simeq 2.044\,8, \Delta_4 \simeq 3.333\,6,
\end{aligned}
$$

We remark that

$$
\begin{aligned}
\Pi_1 &= \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3 \simeq -1.8311, \\
\Pi_2 &= \Delta_1\Delta_4 - \Delta_2\Delta_3 \simeq -0.235\,06,
\end{aligned}
$$

and

$$
f\left(t, u, v\right) \leq \frac{3}{2}, \varphi\left(t, u, v\right) \leq \frac{\left(e-3\right)^2}{3}, g\left(t, u, v\right) \leq \frac{1}{8}, \psi\left(t, u, v\right) \leq \frac{7}{6}.
$$

So, by Theorem 3.1, system (25) has at least one solution on $[1, e]$.

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