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Harmonic k-Uniformly Convex, k-Starlike Mappings and Pascal Distribution Series

Elif YAŞAR*

Abstract
In this paper, connections between various subclasses of harmonic univalent functions by using a convolution operator involving the Pascal distribution series are investigated. Furthermore, an example is provided, illustrating graphically with the help of Maple, to illuminate the convolution operator.

Keywords: *Harmonic functions, Univalent functions, Pascal distribution.*

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**Corresponding author*

1. Introduction

Let $\mathcal{H}$ denote the family of continuous complex valued harmonic functions of the form $f = h + \overline{g}$ defined in the open unit disk $\mathfrak{U} = \{z : |z| < 1\}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.1}$$

are analytic in $\mathfrak{U}$.
A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathfrak{U}$ is that $|h'(z)| > |g'(z)|$ in $\mathfrak{U}$ (see [2],[3]).
Denote by $\mathcal{SH}$ the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ which are harmonic, univalent and sense-preserving in $\mathfrak{U}$ and normalized by $f(0) = f_z(0) - 1 = 0$. One can easily show that the sense-preserving property implies that $|b_1| < 1$. The subclass $\mathcal{SH}^0$ of $\mathcal{SH}$ consist of all functions in $\mathcal{SH}$ which have the additional property $b_1 = 0$. Note that $\mathcal{SH}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions in $\mathfrak{U}$, if the co-analytic part of f is identically zero.
Define $\overline{\mathcal{H}}^i$ $(i = 1, 2)$ be the subclass of $\mathcal{SH}$ consisting of the functions $f = h + \overline{g}$ such that $h(z)$ and $g(z)$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \text{ and } g(z) = (-1)^i \sum_{n=1}^{\infty} |b_n| z^n. \tag{1.2}$$

Let $HUC(k, \alpha)$ be a subclass of the functions $f = h + \overline{g}$ in $\mathcal{SH}$ which satisfy the condition

$$\text{Re}\left\{1 + \left(1 + ke^{i\eta}\right) \frac{z^2h''(z) + \overline{2zg'(z) + z^2g''(z)}}{zh'(z) - \overline{zg'(z)}}\right\} \geq \alpha, \tag{1.3}$$

for some k $(k \geq 0)$, α $(0 \leq \alpha < 1)$ and $z \in \mathfrak{U}$. Define $\overline{HUC}(k, \alpha) := HUC(k, \alpha) \cap \overline{\mathcal{H}}^1$. A mapping in $HUC(k, \alpha)$ or $\overline{HUC}(k, \alpha)$ is called harmonic k-uniformly convex in $\mathfrak{U}$. These classes were studied in [5]. For $g \equiv 0$, $k = 1$ and

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$\alpha=0$, the class $HUC(k,\alpha)$ reduces to the class UC of analytic uniformly convex functions defined by [4].
Let $HS^*(k,\alpha)$ be a subclass of the functions $f=h+\overline{g}$ in $\mathcal{SH}$ which satisfy the condition

$$\text{Re}\left\{\frac{zf'(z)}{z'f(z)}-\alpha\right\}\geq k\left|\frac{zf'(z)}{z'f(z)}-1\right|$$

for some $k\ (k\geq 0)$, $\alpha\ (0\leq\alpha<1)$ and $z\in\mathfrak{U}$. Also define $\overline{HS}^*(k,\alpha):=HS^*(k,\alpha)\cap\overline{\mathcal{H}}^2$. These mappings are called harmonic $k-$ starlike in $\mathfrak{U}$. For $\alpha=0$ these classes were studied in [7]. For $g\equiv 0$, $k=1$ and $\alpha=0$, the class $HS^*(k,\alpha)$ reduces to the class US^* of analytic uniformly starlike functions defined by [6].
The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view (see [8] ,[9], [10],[11], [12], [13]).
Let us consider a non-negative discrete random variable $\mathcal{X}$ with a Pascal probability generating function

$$P(\mathcal{X}=n)=\binom{n+r-1}{r-1}p^n(1-p)^r,\quad n\in\{0,1,2,3,...\}$$

where p,r are called the parameters.
Now we introduce a power series whose coefficients are probabilities of the Pascal distribution, that is

$$P_p^r(z)=z+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}p^{n-1}(1-p)^r z^n.\quad (r\geq 1,\ 0\leq p\leq 1,\ z\in\mathfrak{U}) \tag{1.4}$$

Note that, by using ratio test we conclude that the radius of convergence of the above power series is infinity. Now, for $r,s\geq 1$ and $0\leq p,q\leq 1$, we introduce the operator

$$P_{p,q}^{r,s}(f)(z)=P_p^r(z)*h(z)+\overline{P_q^s(z)*g(z)}=H(z)+\overline{G(z)}$$

where

$$H(z)=z+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}p^{n-1}(1-p)^r a_n z^n \tag{1.5}$$

$$G(z)=b_1 z+\sum_{n=2}^{\infty}\binom{n+s-2}{s-1}q^{n-1}(1-q)^s b_n z^n$$

and "$*$" denotes the convolution (or Hadamard product) of power series.

Example 1.1. Consider the harmonic polynomial $f_1(z)=z+\frac{1}{6}z^2+\frac{1}{6}\overline{z}^4$. If we take $r=7$, $s=7$, $p=0.1$ and $q=0.3$ then from (1.5), we have

$$P_{p,q}^{r,s}(f_1)(z)=z+0.05z^2+0.03\overline{z}^4.$$

Images of concentric circles inside $\mathfrak{U}$ under the functions f_1 and $P_{p,q}^{r,s}(f_1)$ are shown in Figure 1 and Figure 2.

In this paper, we deal mainly with connections between the classes harmonic starlike, harmonic convex, harmonic k-uniformly convex and harmonic k-starlike by using above convolution operator involving the Pascal distribution series.

2. Preliminary Lemmas

To prove our theorems we will use the following lemmas.

Lemma 2.1. *[2] If $f=h+\overline{g}\in\mathcal{KH}^0$ where h and g are given by (1.1) with $b_1=0$, then*

$$|a_n|\leq\frac{n+1}{2},\quad |b_n|\leq\frac{n-1}{2}.$$

Lemma 2.2. *[5] Let $f=h+\overline{g}$ be given by (1.1). If $k\geq 0$, $0\leq\alpha<1$ and*

$$\sum_{n=2}^{\infty}n\left(n(k+1)-(k+\alpha)\right)|a_n|+\sum_{n=1}^{\infty}n\left(n(k+1)+(k+\alpha)\right)|b_n|\leq 1-\alpha, \tag{2.1}$$

then f is harmonic, sense-preserving, univalent in $\mathfrak{U}$ and $f\in HUC(k,\alpha)$.

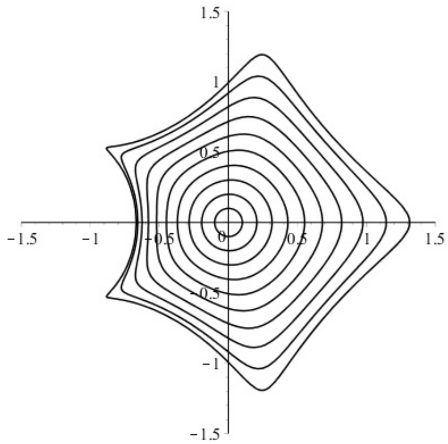

Figure 1. Image of $f_1(\mathfrak{U})$

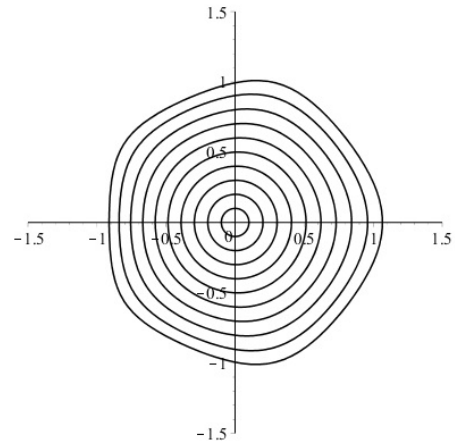

Figure 2. Image of $P_{p,q}^{r,s}(f_1)(\mathfrak{U})$

Lemma 2.3. *[1] Let $f = h + \overline{g} \in T^1$ be given by (1.2). Then $f \in \overline{HUC}(k,\alpha)$ if and only if the coefficient condition (2.1) is satisfied. Also, if $f \in \overline{HUC}(k,\alpha)$, then*

$$|a_n| \leq \frac{1-\alpha}{n\left(n\left(k+1\right)-\left(k+\alpha\right)\right)}, \quad n \geq 2, \quad |b_n| \leq \frac{1-\alpha}{n\left(n\left(k+1\right)+\left(k+\alpha\right)\right)}, \quad n \geq 1.$$

Lemma 2.4. *[1] Let $f = h + \overline{g}$ be given by (1.1). If $k \geq 0$, $0 \leq \alpha < 1$ and*

$$\sum_{n=2}^{\infty}\left(n\left(k+1\right)-\left(k+\alpha\right)\right)|a_n| + \sum_{n=1}^{\infty}\left(n\left(k+1\right)+\left(k+\alpha\right)\right)|b_n| \leq 1-\alpha, \tag{2.2}$$

then f is harmonic, sense-preserving, univalent in $\mathfrak{U}$ and $f \in HS^*(k,\alpha)$.

Lemma 2.5. *[1] Let $f = h + \overline{g} \in T^2$ be given by (1.2). Then $f \in \overline{HS}^*(k,\alpha)$ if and only if the coefficient condition (2.2) is satisfied. Also, if $f \in \overline{HS}^*(k,\alpha)$, then*

$$|a_n| \leq \frac{1-\alpha}{n\left(k+1\right)-\left(k+\alpha\right)}, \quad n \geq 2, \quad |b_n| \leq \frac{1-\alpha}{n\left(k+1\right)+\left(k+\alpha\right)}, \quad n \geq 1. \tag{2.3}$$

Lemma 2.6. *[2] If $f = h + \overline{g} \in SH^{*,0}$ where h and g are given by (1.1) with $b_1 = 0$, then*

$$|a_n| \leq \frac{\left(2n+1\right)\left(n+1\right)}{6}, \quad |b_n| \leq \frac{\left(2n-1\right)\left(n-1\right)}{6}, \; n \geq 2.$$

3. Main Results

From now, throughout the main results, we will consider $0 \leq \alpha < 1$, $k \geq 0$, $r, s \geq 1$, and $0 \leq p, q < 1$.

Theorem 3.1. *If the inequality*

$$\begin{aligned} & \frac{(k+1)r\left(r+1\right)\left(r+2\right)p^3}{\left(1-p\right)^3} + \frac{\left(4k+5-\alpha\right)r\left(r+1\right)p^2}{\left(1-p\right)^2} + \frac{\left(2k+4-2\alpha\right)rp}{1-p} \\ & + \frac{(k+1)s\left(s+1\right)\left(s+2\right)q^3}{\left(1-q\right)^3} + \frac{\left(6k+5+\alpha\right)s\left(s+1\right)q^2}{\left(1-q\right)^2} + \frac{\left(6k+4+2\alpha\right)sq}{1-q} \\ \leq \; & 2(1-\alpha)(1-p)^r \end{aligned} \tag{3.1}$$

is hold, then $P_{p,q}^{r,s}\left(\mathcal{K}H^0\right) \subset HUC(k,\alpha)$.

Proof. Suppose $f = h + \overline{g} \in \mathcal{K}H^0$ where h and g are given by (1.1) with $b_1 = 0$. We need to show that $P_{p,q}^{r,s}(f) = H + \overline{G} \in HUC(k,\alpha)$ where H and G are given by (1.5) with $b_1 = 0$. By Lemma 2.2, we need to establish that

$Q_1 \leq 1-\alpha$, where

$$
\begin{aligned}
Q_1 &= \sum_{n=2}^{\infty} n\left(n\left(k+1\right)-\left(k+\alpha\right)\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1}\left|a_n\right| \\
&\quad +\sum_{n=2}^{\infty} n\left(n\left(k+1\right)+\left(k+\alpha\right)\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1}\left|b_n\right|.
\end{aligned}
$$

Using Lemma 2.2, we obtain

$$
\begin{aligned}
Q_1 &\leq \frac{1}{2}\left\{\sum_{n=2}^{\infty} n\left(n+1\right)\left(n\left(k+1\right)-\left(k+\alpha\right)\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1}\right. \\
&\quad \left. +\sum_{n=2}^{\infty} n\left(n-1\right)\left(n\left(k+1\right)+\left(k+\alpha\right)\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1}\right\} \\
&= \frac{1}{2}\left\{\left(k+1\right)\sum_{n=2}^{\infty}\left(n-1\right)\left(n-2\right)\left(n-3\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1}\right. \\
&\quad +(6k+7-\alpha)\sum_{n=2}^{\infty}\left(n-1\right)\left(n-2\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1} \\
&\quad +(6k+10-4\alpha)\sum_{n=2}^{\infty}\left(n-1\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1} \\
&\quad +2(1-\alpha)\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1} \\
&\quad +(k+1)\sum_{n=2}^{\infty}\left(n-1\right)\left(n-2\right)\left(n-3\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1} \\
&\quad +(6k+5+\alpha)\sum_{n=2}^{\infty}\left(n-1\right)\left(n-2\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1} \\
&\quad \left. +(6k+4+2\alpha)\sum_{n=2}^{\infty}\left(n-1\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1}\right\} \\
&= \frac{1}{2}\left\{(k+1)r\left(r+1\right)\left(r+2\right)p^3\left(1-p\right)^r\sum_{n=4}^{\infty}\binom{n+r-2}{r+2}p^{n-4}\right. \\
&\quad +(4k+5-\alpha)r\left(r+1\right)p^2\left(1-p\right)^r\sum_{n=3}^{\infty}\binom{n+r-2}{r+1}p^{n-3} \\
&\quad +(2k+4-2\alpha)rp\left(1-p\right)^r\sum_{n=2}^{\infty}\binom{n+r-2}{r}p^{n-2} \\
&\quad +2(1-\alpha)(1-p)^r\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}p^{n-1} \\
&\quad +(k+1)s\left(s+1\right)\left(s+2\right)q^3\left(1-q\right)^s\sum_{n=4}^{\infty}\binom{n+s-2}{s+2}q^{n-4} \\
&\quad +(6k+5+\alpha)s\left(s+1\right)q^2\left(1-q\right)^s\sum_{n=3}^{\infty}\binom{n+s-2}{s+1}q^{n-3} \\
&\quad \left. +(6k+4+2\alpha)sq\left(1-q\right)^s\sum_{n=2}^{\infty}\binom{n+s-2}{s}q^{n-2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\Bigg\{(k+1)r\,(r+1)\,(r+2)p^3\,(1-p)^r\sum_{n=0}^{\infty}\binom{n+r+2}{r+2}p^n \\
&+ (4k+5-\alpha)\,r\,(r+1)\,p^2\,(1-p)^r\sum_{n=0}^{\infty}\binom{n+r+1}{r+1}p^n \\
&+ (2k+4-2\alpha)\,rp\,(1-p)^r\sum_{n=0}^{\infty}\binom{n+r}{r}p^n \\
&+2(1-\alpha)(1-p)^r\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}p^n-2(1-\alpha)(1-p)^r \\
&+(k+1)s\,(s+1)\,(s+2)q^3\,(1-q)^s\sum_{n=0}^{\infty}\binom{n+s+2}{s+2}q^n \\
&+(6k+5+\alpha)s\,(s+1)\,q^2\,(1-q)^s\sum_{n=0}^{\infty}\binom{n+s+1}{s+1}q^n \\
&+ (6k+4+2\alpha)\,sq\,(1-q)^s\sum_{n=0}^{\infty}\binom{n+s}{s}q^n\Bigg\} \\
&= \frac{1}{2}\Bigg\{\frac{(k+1)r\,(r+1)\,(r+2)p^3}{(1-p)^3}+\frac{(4k+5-\alpha)\,r\,(r+1)\,p^2}{(1-p)^2}+\frac{(2k+4-2\alpha)\,rp}{1-p} \\
&+\frac{(k+1)s\,(s+1)\,(s+2)q^3}{(1-q)^3}+\frac{(6k+5+\alpha)s\,(s+1)\,q^2}{(1-q)^2}+\frac{(6k+4+2\alpha)\,sq}{1-q} \\
&+2(1-\alpha)-2(1-\alpha)(1-p)^r\Big\}.
\end{aligned}
$$

The last expression is bounded above by $(1-\alpha)$ by the given condition (3.1). Thus the proof of Theorem 3.1 is complete. □

Theorem 3.2. *If the inequality*

$$
\frac{(k+1)r\,(r+1)\,p^2}{(1-p)^2}+\frac{(3k+4-\alpha)\,rp}{1-p}+\frac{(k+1)s\,(s+1)\,q^2}{(1-q)^2}+\frac{(3k+2+\alpha)\,sq}{1-q}\le 2(1-\alpha)\,(1-p)^r \tag{3.2}
$$

is hold, then $P^{r,s}_{p,q}\left(\mathcal{K}H^0\right)\subset HS^*(k,\alpha)$.

Proof. Suppose that $f=h+\overline{g}\in\mathcal{KH}^0$ where h and g are given by (1.1) with $b_1=0$. It suffices to show that $P^{r,s}_{p,q}(f)=H+\overline{G}\in HS^*(k,\alpha)$ where H and G are given by (1.5) with $b_1=0$ in $\mathfrak{U}$. Using Lemma 2.4, we need to show that $Q_2\le 1-\alpha$, where

$$
\begin{aligned}
Q_2 &= \sum_{n=2}^{\infty}(n\,(k+1)-(k+\alpha))\binom{n+r-2}{r-1}(1-p)^r\,p^{n-1}\,|a_n| \\
&+\sum_{n=2}^{\infty}(n\,(k+1)+(k+\alpha))\binom{n+s-2}{s-1}(1-q)^s\,q^{n-1}\,|b_n|\,.
\end{aligned}
$$

Using Lemma 2.2, we obtain

$$
\begin{aligned}
Q_2 &\le \frac{1}{2}\Bigg\{\sum_{n=2}^{\infty}(n+1)\,(n\,(k+1)-(k+\alpha))\binom{n+r-2}{r-1}(1-p)^r\,p^{n-1} \\
&+\sum_{n=2}^{\infty}(n-1)\,(n\,(k+1)+(k+\alpha))\binom{n+s-2}{s-1}(1-q)^s\,q^{n-1}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{2}\left\{(k+1)\sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+r-2}{r-1}(1-p)^{r}p^{n-1}\right. \\
& +(3k+4-\alpha)\sum_{n=2}^{\infty}(n-1)\binom{n+r-2}{r-1}(1-p)^{r}p^{n-1} \\
& +2(1-\alpha)\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}(1-p)^{r}p^{n-1} \\
& +(k+1)\sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+s-2}{s-1}(1-q)^{s}q^{n-1} \\
& \left.+(3k+2+\alpha)\sum_{n=2}^{\infty}(n-1)\binom{n+s-2}{s-1}(1-q)^{s}q^{n-1}\right\} \\
= & \frac{1}{2}\left\{(k+1)r(r+1)p^{2}(1-p)^{r}\sum_{n=3}^{\infty}\binom{n+r-2}{r+1}p^{n-3}\right. \\
& +(3k+4-\alpha)rp(1-p)^{r}\sum_{n=2}^{\infty}\binom{n+r-2}{r}p^{n-2} \\
& +2(1-\alpha)(1-p)^{r}\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}p^{n-1} \\
& +(k+1)s(s+1)q^{2}(1-q)^{s}\sum_{n=3}^{\infty}\binom{n+s-2}{s+1}q^{n-3} \\
& \left.+(3k+2+\alpha)sq(1-q)^{s}\sum_{n=2}^{\infty}\binom{n+s-2}{s}q^{n-2}\right\} \\
= & \frac{1}{2}\left\{(k+1)r(r+1)p^{2}(1-p)^{r}\sum_{n=0}^{\infty}\binom{n+r+1}{r+1}p^{n}\right. \\
& +(3k+4-\alpha)rp(1-p)^{r}\sum_{n=0}^{\infty}\binom{n+r}{r}p^{n} \\
& +2(1-\alpha)(1-p)^{r}\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}p^{n}-2(1-\alpha)(1-p)^{r} \\
& +(k+1)s(s+1)q^{2}(1-q)^{s}\sum_{n=0}^{\infty}\binom{n+s+1}{s+1}q^{n} \\
& \left.+(2k+2+\alpha)sq(1-q)^{s}\sum_{n=0}^{\infty}\binom{n+s}{s}q^{n}\right\} \\
= & \frac{1}{2}\left\{\frac{(k+1)r(r+1)p^{2}}{(1-p)^{2}}+\frac{(3k+4-\alpha)rp}{1-p}+2(1-\alpha)-2(1-\alpha)(1-p)^{r}\right. \\
& \left.+\frac{(k+1)s(s+1)q^{2}}{(1-q)^{2}}+\frac{(3k+2+\alpha)sq}{1-q}\right\}.
\end{aligned}
$$

The last expression is bounded above by $(1-\alpha)$ by the condition (3.2). Thus the proof of Theorem 3.2 is complete. □

Theorem 3.3. *If the inequality*

$$(1-p)^{r}+(1-q)^{s}\geq 1+\frac{(2k+1+\alpha)}{1-\alpha}|b_{1}| \tag{3.3}$$

is hold, then $P_{p,q}^{r,s}\left(\overline{HUC}(k,\alpha)\right)\subset HUC(k,\alpha)$.

Proof. Suppose $f = h + \overline{g} \in \overline{HUC}(k,\alpha)$ where h and g are given by (1.2) with $i = 1$. We need to establish that the operator

$$\begin{aligned} P_{p,q}^{r,s}(f)(z) &= z - \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} p^{n-1} (1-p)^r a_n z^n \\ &\quad - |b_1|\,\overline{z} - \sum_{n=2}^{\infty} \binom{n+s-2}{s-1} q^{n-1} (1-q)^s |b_n|\,\overline{z}^n \end{aligned}$$

is in $HUC(k,\alpha)$ if and only if $Q_3 \le 1-\alpha$, where

$$\begin{aligned} Q_3 &= \sum_{n=2}^{\infty} n\left(n(k+1) - (k+\alpha)\right) \binom{n+r-2}{r-1} (1-p)^r p^{n-1} |a_n| \\ &\quad + (2k+1+\alpha)|b_1| + \sum_{n=2}^{\infty} n\left(n(k+1) + (k+\alpha)\right) \binom{n+s-2}{s-1} (1-q)^s q^{n-1} |b_n|. \end{aligned}$$

Using Lemma 2.4, we have

$$\begin{aligned} Q_3 &\le (1-\alpha)\left\{ \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} (1-p)^r p^{n-1} \right. \\ &\quad \left. + \sum_{n=2}^{\infty} \binom{n+s-2}{s-1} (1-q)^s q^{n-1} \right\} + (2k+1+\alpha)|b_1| \\ &= (1-\alpha)\left\{ (1-p)^r \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} p^n - (1-p)^r \right. \\ &\quad \left. + (1-q)^s \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} q^n - (1-q)^s \right\} + (2k+1+\alpha)|b_1| \\ &= (1-\alpha)\{2 - (1-p)^r - (1-q)^s\} + (2k+1+\alpha)|b_1| \le 1-\alpha. \end{aligned}$$

Then inequality (3.3) completes the proof. □

Theorem 3.4. *If the inequality*

$$\begin{aligned} &\frac{2(k+1)r(r+1)(r+2)p^3}{(1-p)^3} + \frac{(13k+15-2\alpha)\,r(r+1)\,p^2}{(1-p)^2} + \frac{(15k+24-9\alpha)\,rp}{1-p} \\ &+ \frac{2(k+1)s(s+1)(s+2)q^3}{(1-q)^3} + \frac{(11k+9+2\alpha)\,s(s+1)\,q^2}{(1-q)^2} + \frac{(9k+6+3\alpha)\,sq}{1-q} \\ \le\;& 6(1-\alpha)(1-p)^r \end{aligned} \tag{3.4}$$

is hold, then $P_{p,q}^{r,s}\left(SH^{*,0}\right) \subset HS^*(k,\alpha)$.

Proof. Suppose $f = h + \overline{g} \in SH^{*,0}$ where h and g are given by (1.1) with $b_1 = 0$. We need to prove that $P_{p,q}^{r,s}(f) = H + \overline{G} \in HS^*(k,\alpha)$. In view of Lemma 2.4, we need to prove that $Q_4 \le 1-\alpha$, where

$$\begin{aligned} Q_4 &:= \sum_{n=2}^{\infty} \left(n(k+1) - (k+\alpha)\right) \binom{n+r-2}{r-1} (1-p)^r p^{n-1} |a_n| \\ &\quad + \sum_{n=2}^{\infty} \left(n(k+1) + (k+\alpha)\right) \binom{n+s-2}{s-1} (1-q)^s q^{n-1} |b_n|. \end{aligned}$$

Referring Lemma 2.6, we observe

$$
\begin{aligned}
Q_4 \leq\ & \frac{1}{6}\left\{\sum_{n=2}^{\infty}(2n+1)(n+1)\left(n\left(k+1\right)-\left(k+\alpha\right)\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1}\right. \\
& \left.+\sum_{n=2}^{\infty}(2n-1)(n-1)\left(n\left(k+1\right)+\left(k+\alpha\right)\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1}\right\} \\
=\ & \frac{1}{6}\left\{2(k+1)\sum_{n=2}^{\infty}\left(n-1\right)\left(n-2\right)\left(n-3\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1}\right. \\
& +(13k+15-2\alpha)\sum_{n=2}^{\infty}\left(n-1\right)\left(n-2\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1} \\
& +(15k+24-9\alpha)\sum_{n=2}^{\infty}\left(n-1\right)\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1} \\
& +6(1-\alpha)\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left(1-p\right)^r p^{n-1} \\
& +2(k+1)\sum_{n=2}^{\infty}\left(n-1\right)\left(n-2\right)\left(n-3\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1} \\
& +(11k+9+2\alpha)\sum_{n=2}^{\infty}\left(n-1\right)\left(n-2\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1} \\
& \left.+(9k+6+3\alpha)\sum_{n=2}^{\infty}\left(n-1\right)\binom{n+s-2}{s-1}\left(1-q\right)^s q^{n-1}\right\} \\
=\ & \frac{1}{6}\left\{2(k+1)r\left(r+1\right)\left(r+2\right)p^3\left(1-p\right)^r\sum_{n=0}^{\infty}\binom{n+r+2}{r+2}p^n\right. \\
& +(13k+15-2\alpha)r\left(r+1\right)p^2\left(1-p\right)^r\sum_{n=0}^{\infty}\binom{n+r+1}{r+1}p^n \\
& +(15k+24-9\alpha)rp\left(1-p\right)^r\sum_{n=0}^{\infty}\binom{n+r}{r}p^n \\
& +6(1-\alpha)\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}\left(1-p\right)^r p^n-6(1-\alpha)\left(1-p\right)^r \\
& +2(k+1)s\left(s+1\right)\left(s+2\right)q^3\left(1-q\right)^s\sum_{n=0}^{\infty}\binom{n+s+2}{s+2}q^n \\
& +(11k+9+2\alpha)s\left(s+1\right)q^2\left(1-q\right)^s\sum_{n=0}^{\infty}\binom{n+s+1}{s+1}q^n \\
& \left.+(9k+6+3\alpha)sq\left(1-q\right)^s\sum_{n=2}^{\infty}\binom{n+s}{s}q^n\right\} \\
=\ & \frac{1}{6}\left\{\frac{2(k+1)r\left(r+1\right)\left(r+2\right)p^3}{\left(1-p\right)^3}+\frac{\left(13k+15-2\alpha\right)r\left(r+1\right)p^2}{\left(1-p\right)^2}\right. \\
& +\frac{\left(15k+24-9\alpha\right)rp}{1-p}+6(1-\alpha)-6(1-\alpha)\left(1-p\right)^r \\
& +\frac{2(k+1)s\left(s+1\right)\left(s+2\right)q^3}{\left(1-q\right)^3}+\frac{\left(11k+9+2\alpha\right)s\left(s+1\right)q^2}{\left(1-q\right)^2} \\
& \left.+\frac{\left(9k+6+3\alpha\right)sq}{1-q}\right\}.
\end{aligned}
$$

The last expression bounded above by $(1-\alpha)$ by the given condition (3.4). □

The proof of the following theorem is similar to those of the previous theorems so we state only the result.

Theorem 3.5. *If the inequality* $(1-p)^r + (1-q)^s \geq 1 + |b_1|$ *is hold, then* $P_{p,q}^{r,s}\left(\overline{HS}^*(k,\alpha)\right) \subset HS^*(k,\alpha)$.

References

[1] Ahuja, O.P., Aghalary, R., Joshi, S.B.: *Harmonic univalent functions associated with k-uniformly starlike functions*. Math. Sci. Res. J. **9** (1), 9-17 (2005).

[2] Clunie, J., Sheil-Small, T.: *Harmonic univalent functions*. Ann. Acad. Sci. Fenn. Ser. A I Math. **9**, 3-25 (1984).

[3] Duren, P.: Harmonic Mappings in the Plane. Cambridge University Press. Cambridge (2004).

[4] Goodman, A.W.: *On uniformly convex functions*. Ann. Polon. Math. **56** (1), 87-92 (1991).

[5] Kim, Y.C., Jahangiri, J.M., Choi, J.H.: *Certain convex harmonic functions*. Int. J. Math. Sci. **29** (8), 459-465 (2002).

[6] Rønning, F.: *Uniformly convex functions and a corresponding class of starlike functions*. Proc. Amer. Math. Soc. **118**, 189-196 (1993).

[7] Rosy, T., Stephen, B.A., Subramanian, K.G., Jahangiri, J.M.: *Goodman-Rønning-type harmonic univalent functions*. Kyungpook Math. J. **41**(1) 45-54 (2001).

[8] Altınkaya, Ş., Yalçın, S.: *Poisson Distribution Series for Certain Subclasses of Starlike Functions with Negative Coefficients*. Annals of Oradea University Mathematics Fascicola 2. **24** (2), 5-8 (2017).

[9] Porwal, S., Kumar, M.: *A unified study on starlike and convex functions associated with Poisson distribution series*. Afrika Matematika. **27** (5-6), 1021-1027 (2016).

[10] Porwal, S.: *An application of a Poisson distribution series on certain analytic functions*. J. Complex Anal. Article ID 984135, 1-3 (2014).

[11] Nazeer, W., Mehmood, Q., Kang, S.M., Haq, A.U.: *An application of Binomial distribution series on certain analytic functions*. Journal of Computational Analysis and Applications. **26** (1), 11-17 (2019).

[12] Çakmak, S., Yalçin, S., Altınkaya, Ş.: *Some connections between various classes of analytic functions associated with the power series distribution*. Sakarya Univ. Jour. Science. **23** (5), 982-985 (2019).

[13] El-Deeb, M.S., Bulboaca, T., Dziok, J.: *Pascal Distribution Series connected with certain subclasses of univalent functions*. Kyunpook Math. J. **59**, 301-314 (2019).

Affiliations

ELIF YAŞAR
ADDRESS: Bursa Uludag University, Faculty of Arts and Science, Dept. of Mathematics, 16059, Bursa-Turkey.
E-MAIL: elifyasar@yahoo.com
ORCID ID: 0000-0003-0176-4961