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Translation-Factorable Surfaces in 4-dimensional Euclidean Space

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Abstract

In this study we consider translation-factorable (TF-type) surfaces in Euclidean 4-space $\mathbb{E}^4$. We have calculated the Gaussian and mean curvature of the TF-type surfaces. Further, we give some sufficient conditions to become flat or minimal for these surfaces. Finally, we give some examples of TF-type surfaces and plot the projection of the graphics into the Euclidean 3-space.

Keywords: *Factorable surface; Gaussian curvature; Mean curvature; Monge patch; Translation surface.*
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1. Introduction

Classical differential geometry provides a complete local description of smooth surfaces. The first and second fundamental forms of surfaces provide a set of differential-geometric shape descriptors that capture domain-independent surface information. Gaussian curvature is an intrinsic surface property which refers to an isometric invariant of a surface. Both Gaussian and mean curvatures have attractive characteristics of translational and rotational invariance. A depth surface is a range image observed from a single view which can be represented by a digital graph (Monge patch) surface. In [1], Yu. A. Aminov introduced the surface M in $\mathbb{E}^4$ given by $X(u,v) = (u, v, z(u,v), w(u,v))$, where z and w are differentiable functions. This representation is called a Monge patch. Also, in [6], the authors investigated the curvature properties of these type of surfaces. An interesting classes of the surfaces given with the Monge patch are that translation and factorable surfaces. Translation surfaces have been studied from the various viewpoints by many differential geometers in Euclidean and pseudo-Euclidean spaces (see, [2], [4], [8], [14], [15], [17], [18], [19], [21]). The another important classification surfaces given with the Monge patch is factorable surfaces or it is called homothetical surfaces [16]. There has been classification of factorable surfaces in the 3- dimensional Euclidean, Lorentz-Minkowski and pseudo-Galilean space ([3], [5], [18], [22]) and also 4-dimensional Euclidean and pseudo-Euclidean space ([7], [9], [10], [11]).
In this paper, firstly, we give some basic concepts of the surfaces in $\mathbb{E}^4$. Further this section provides some basic properties of surfaces in $\mathbb{E}^4$ and the structure of their curvatures. In the third section, we consider a translation-factorable surface in Euclidean 4-space, which is defined in Euclidean 3-space by Difi et. al. [13]. In [20] the authors gave the classification of flat and minimal translation-homothetical (TH) surfaces in $\mathbb{E}^3$ and $\mathbb{E}^3_1$. In this paper we characterize such surfaces (TF-surfaces) in terms of their Gaussian curvature and mean curvature functions. We give a classification for flat and minimal translation-factorable surfaces in $\mathbb{E}^4$. Finally, we give some examples and plot their graphics.

2. Basic Concepts

Let M be a smooth surface in 4-dimensional Euclidean space $\mathbb{E}^4$ given with the surface patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at a point $p = X(u,v)$ of M is spanned by $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \tag{2.1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^4 = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of the tangent plane T_pM in $\mathbb{E}^4$, that is the normal space of M at p.

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Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent to M and the space of smooth vector fields normal to M, respectively. Given any local vector fields X_i and X_j, $(1 \leq i, j \leq 2)$ tangent to M consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j, \tag{2.2}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{E}^4$, respectively. This map is well-defined, symmetric and bilinear.
For any orthonormal frame field $N_k, (1 \leq k \leq 2)$ of M, the shape operator $A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$;

$$A_{N_k} X_j = -\left(\widetilde{\nabla}_{x_j} N_k\right)^T, \tag{2.3}$$

This operator is bilinear, self-adjoint and satisfies the condition:

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k \quad 1 \leq i, j \leq 2, \quad 1 \leq k \leq 2, \tag{2.4}$$

where c_{ij}^k are the coefficients of the second fundamental form. The coefficients of the second fundamental form of a surface M can be calculated by

$$\begin{aligned} c_{11}^k &= \langle X_{uu}, N_k \rangle, \\ c_{12}^k &= \langle X_{uv}, N_k \rangle, \\ c_{22}^k &= \langle X_{vv}, N_k \rangle. \end{aligned} \tag{2.5}$$

The equation (2.2) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k, \quad 1 \leq i, j \leq 2. \tag{2.6}$$

Then the *Gaussian curvature* K of a regular patch $X(u,v)$ is given by

$$K = \frac{1}{W^2} \sum_{k=1}^{2} (c_{11}^k c_{22}^k - (c_{12}^k)^2). \tag{2.7}$$

Further, the *mean curvature vector* of a regular patch $X(u,v)$ is given by

$$\overrightarrow{H} = \frac{1}{2W^2} \sum_{k=1} (c_{11}^k G + c_{22}^k E - 2c_{12}^k F) N_k. \tag{2.8}$$

The norm of the mean curvature vector$\left\| \overrightarrow{H} \right\|$ is called the *mean curvature* of M. Recall that a surface M is said to be *flat* (respectively *minimal*) if its Gaussian curvature (respectively mean curvature) vanishes identically [12].

3. Translation-Factorable Surfaces in $\mathbb{E}^4$

In this section, we consider translation-factorable surfaces in $\mathbb{E}^4$ which is defined by the means of Monge patch. Firstly, we give definition of the translation and factorable surfaces in $\mathbb{E}^4$.

Definition 3.1. *[2] A surfaces M defined as the sum of two space curves $\alpha(u) = (u, 0, f_1(u), f_2(u))$ and $\beta(v) = (0, v, g_1(v), g_2(v))$ is called a translation surface in $\mathbb{E}^4$. So, a translation surface is defined by a patch*

$$X(u,v) = (u, v, f_1(u) + g_1(v), f_2(u) + g_2(v)). \tag{3.1}$$

Definition 3.2. *[10] Let M be a surface given by an explicit form $z(u,v) = f_1(u)g_1(v)$ and $w(u,v) = f_2(u)g_2(v)$ is called a factorable surface in $\mathbb{E}^4$. The factorable surface is defined by a patch*

$$X(u,v) = (u, v, f_1(u)g_1(v), f_2(u)g_2(v)). \tag{3.2}$$

Definition 3.3. *A surface M is translation-factorable (TF-type) surface if it can be parametrized by*

$$X(u,v) = (u, v, A(f_1(u) + g_1(v)) + Bf_1(u)g_1(v), C(f_2(u) + g_2(v)) + Df_2(u)g_2(v)),$$

where A,B,C and D are non-zero real numbers.

Remark 3.4. *In [2] we have if $A, C \neq 0$ and $B, D = 0$ in, then the surface is a translation surface. In [10] we have if $A, C = 0$ and $B, D \neq 0$, then surface is a factorable surface.*

So, we consider TF-type surface in Euclidean 4-space given with the parametrization

$$M: X(u,v) = (u, v, f_1(u) + g_1(v) + f_1(u)g_1(v), f_2(u) + g_2(v) + f_2(u)g_2(v)). \tag{3.3}$$

The tangent space of M is spanned by the vector fields

$$\begin{aligned} X_u &= (1, 0, f_1'g_1 + f_1', f_2'g_2 + f_2') \\ X_v &= (0, 1, f_1g_1' + g_1', f_2g_2' + g_2'). \end{aligned} \tag{3.4}$$

Hence the coefficients of the first fundamental form of the surfaces are

$$\begin{aligned} E &= \langle X_u, X_u \rangle = 1 + (f_1'g_1 + f_1')^2 + (f_2'g_2 + f_2')^2 \\ F &= \langle X_u, X_v \rangle = (f_1'g_1 + f_1')(f_1g_1' + g_1') + (f_2'g_2 + f_2')(f_2g_2' + g_2') \\ G &= \langle X_v, X_v \rangle = 1 + (f_1g_1' + g_1')^2 + (f_2g_2' + g_2')^2, \end{aligned} \tag{3.5}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^4$. Since the surface M is non-degenerate, $\|X_u \times X_v\| = \sqrt{EG - F^2} \neq 0$. For the later use we define a smooth function W as $W = \|X_u \times X_v\|$.
The second partial derivatives of $X(u,v)$ are given by

$$\begin{aligned} X_{uu} &= (0, 0, f_1''g_1 + f_1'', f_2''g_2 + f_2'') \\ X_{uv} &= (0, 0, f_1'g_1', f_2'g_2') \\ X_{vv} &= (0, 0, f_1g_1'' + g_1'', f_2g_2'' + g_2''). \end{aligned} \tag{3.6}$$

Further, the normal space of M is spanned by the orthonormal vector fields

$$\begin{aligned} N_1 &= \frac{1}{\sqrt{\widetilde{E}}}(-f_1'g_1 - f_1', -f_1g_1' - g_1', 1, 0), \\ N_2 &= \frac{1}{\sqrt{\widetilde{E}}\widetilde{W}}(\widetilde{F}(f_1'g_1 + f_1') - \widetilde{E}(f_2'g_2 + f_2'), \widetilde{F}(f_1g_1' + g_1') - \widetilde{E}(f_2g_2' + g_2'), -\widetilde{F}, \widetilde{E}) \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \widetilde{E} &= 1 + (f_1'g_1 + f_1')^2 + (f_1g_1' + g_1')^2 \\ \widetilde{F} &= (f_1'g_1 + f_1')(f_2'g_2 + f_2') + (f_1g_1' + g_1')(f_2g_2' + g_2') \\ \widetilde{G} &= 1 + (f_2'g_2 + f_2')^2 + (f_2g_2' + g_2')^2, \\ \widetilde{W} &= \sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2} = W. \end{aligned} \tag{3.8}$$

Using (3.6) and (3.7) into (2.5), we can calculate the coefficients of the second fundamental form as follows:

$$\begin{aligned} c_{11}^1 &= \frac{f_1''g_1 + f_1''}{\sqrt{\widetilde{E}}},\ c_{12}^1 = \frac{f_1'g_1'}{\sqrt{\widetilde{E}}},\ c_{22}^1 = \frac{f_1g_1'' + g_1''}{\sqrt{\widetilde{E}}} \\ c_{11}^2 &= \frac{1}{\sqrt{\widetilde{E}}\widetilde{W}}(\widetilde{E}(f_2''g_2 + f_2'') - \widetilde{F}(, f_1''g_1 + f_1'')) \\ c_{12}^2 &= \frac{1}{\sqrt{\widetilde{E}}\widetilde{W}}(\widetilde{E}f_2'g_2' - \widetilde{F}f_1'g_1') \\ c_{22}^2 &= \frac{1}{\sqrt{\widetilde{E}}\widetilde{W}}(\widetilde{E}(f_2g_2'' + g_2'') - \widetilde{F}(f_1g_1'' + g_1'')). \end{aligned} \tag{3.9}$$

We obtain the following result.

Proposition 3.5. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. Then the Gaussian curvature of M can be given by*

$$K = \frac{1}{W^4}\left\{ \begin{array}{c} \widetilde{G}\left[f_1''g_1''(g_1+1)(f_1+1) - (f_1'g_1')^2\right] + \widetilde{E}\left[f_2''g_2''(g_2+1)(f_2+1) - (f_2'g_2')^2\right] \\ -\widetilde{F}\left[f_2''g_1''(g_2+1)(f_1+1) + f_1''g_2''(g_1+1)(f_2+1) - 2f_1'g_1'f_2'g_2'\right] \end{array} \right\}. \tag{3.10}$$

Proof. Using the equations (3.5) and (3.9) into (2.7) we obtain (3.10). □

Corollary 3.6. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. Then M has vanishing Gaussian curvature if and only if*

$$\begin{array}{c} \widetilde{G}\left[f_1''g_1''(g_1+1)(f_1+1) - (f_1'g_1')^2\right] + \widetilde{E}\left[f_2''g_2''(g_2+1)(f_2+1) - (f_2'g_2')^2\right] \\ -\widetilde{F}\left[f_2''g_1''(g_2+1)(f_1+1) + f_1''g_2''(g_1+1)(f_2+1) - 2f_1'g_1'f_2'g_2'\right] = 0. \end{array} \tag{3.11}$$

Theorem 3.7. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. If M is a flat surface then it is parametrized by as follows;*
1) $z(u,v)=c_1g_1(v)+c_1+g_1(v),\ w(u,v)=c_2g_2(v)+c_2+g_2(v)$,
2) $z(u,v)=c_1f_1(u)+c_1+f_1(u),\ w(u,v)=c_2f_2(u)+c_2+f_2(u)$,
3) $z(u,v)=c_1g_1(v)+c_1+g_1(v),\ w(u,v)=c_2f_2(u)+c_2+f_2(u)$,
4) $z(u,v)=c_1f_1(u)+c_1+f_1(u),\ w(u,v)=c_2g_2(v)+c_2+g_2(v)$,
5) $z(u,v)=c_1,\ w(u,v)=c_2g_2(v)+c_2+g_2(v)$,
6) $z(u,v)=c_1,\ w(u,v)=c_2f_2(u)+c_2+f_2(u)$,
7) $z(u,v)=c,\ w(u,v)=de^{c_1u}e^{c_2v}-1$,
8) $z(u,v)=c, w(u,v)=f_2(u)g_2(v)+f_2(u)+g_2(v)$ *satisfying*

$$\begin{aligned} f_2(u) &= ((1-k)(c_3u+c_4))^{\frac{1}{1-k}}-1,\\ g_2(v) &= \left(\frac{(k-1)(c_5v+c_6)}{k}\right)^{\frac{k}{k-1}}-1,\end{aligned}$$

9) $z(u,v)=c_1f_1(u)+c_1+f_1(u),\ w(u,v)=d$,
10) $z(u,v)=c_1g_1(v)+c_1+g_1(v),\ w(u,v)=d$,
11) $z(u,v)=ce^{c_1u}e^{c_2v}-1,\ w(u,v)=d$,
12) $z(u,v)=f_1(u)g_1(v)+f_1(u)+g_1(v),\ w(u,v)=d$, *satisfying*

$$\begin{aligned} f_1(u) &= ((1-k)(c_3u+c_4))^{\frac{1}{1-k}}-1,\\ g_1(v) &= \left(\frac{(k-1)(c_5v+c_6)}{k}\right)^{\frac{k}{k-1}}-1,\end{aligned}$$

13) $z(u,v)=c_1c_5e^{c_2u}e^{c_6v}-1,\ w(u,v)=c_3c_7e^{c_4u}e^{c_8v}-1;\ c_4c_6=c_2c_8$,
14) $z(u,v)=c_1c_5e^{c_2u}e^{c_6v}-1,\ w(u,v)=c_3c_7e^{c_4u}e^{c_8v}-1;\ c_2c_4=-c_6c_8$,
where c,d,k,c_i are real constants, $i=1,..,8$, $k\neq 0,1$.

Proof. Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. If the TF-type surface is flat then the equation (3.11) is satisfied.
i) If we take $f_1'(u)=0$ and $f_2'(u)=0$ or $g_1'(v)=0$ and $g_2'(v)=0$ or $f_1'(u)=0$ and $g_2'(v)=0$ or $f_2'(u)=0$ and $g_1'(v)=0$ in Eq. (3.11), one can obtain $K=0$. So the surface parametrization (1), (2), (3) and (4) is obtained, respectively.
ii) If $f_1'(u)=0$ and $g_1'(v)=0$, then from the Eq. (3.11) we obtain

$$f_2''g_2''(g_2+1)(f_2+1)-(f_2'g_2')^2=0. \tag{3.12}$$

If we take $f_2'(u)=0$ and $g_2'(v)=0$, respectively, then we obtain the surface parametrization (5) and (6).
Furthermore, if $f_2'(u)g_2'(v)\neq 0$, then from (3.12) we get

$$\frac{f_2''(u)(f_2(u)+1)}{(f_2'(u))^2}=\frac{(g_2'(v))^2}{g_2''(v)(g_2(v)+1)}=k, \tag{3.13}$$

where k is a constant.
a) If $k=1$, from (3.13) we obtain the differential equations $f_2''(u)(f_2(u)+1)=(f_2'(u))^2$ and $(g_2'(v))^2=g_2''(v)(g_2(v)+1)$. Solving these differential equations we have $f_2(u)=c_3e^{c_4u}-1$ and $g_2(v)=c_5e^{c_6v}-1$. Then we obtain the parametrization (7) of TF-type surface.
b) If $k\neq 1$, the solutions of the differential equations (3.13) are $f_2(u)=((1-k)(c_3u+c_4))^{\frac{1}{1-k}}-1$ and $g_2(v)=\left(\frac{(k-1)(c_5v+c_6)}{k}\right)^{\frac{k}{k-1}}-1$. So, we obtain the parametrization (8) of TF-type surface.
iii) If $f_2'(u)=0$ and $g_2'(v)=0$, then from the Eq. (3.11) we obtain

$$f_1''g_1''(g_1+1)(f_1+1)-(f_1'g_1')^2=0. \tag{3.14}$$

So, in a similar way for the Case ii) we obtain TF-type surfaces given with the parametrization (9)-(12).
iv) Let in the Eq. (3.11)

$$\begin{aligned} f_1''g_1''(g_1+1)(f_1+1)-(f_1'g_1')^2 &= 0,\\ f_2''g_2''(g_2+1)(f_2+1)-(f_2'g_2')^2 &= 0\end{aligned} \tag{3.15}$$

with

$$f_2''g_1''(g_2+1)(f_1+1)+f_1''g_2''(g_1+1)(f_2+1)-2f_1'g_1'f_2'g_2'=0 \tag{3.16}$$

or

$$\widetilde{F}=f_1'f_2'(g_1+1)(g_2+1)+g_1'g_2'(f_1+1)(f_2+1)=0. \tag{3.17}$$

So, the Eq.(3.15) is equivalent to (3.12) and (3.14). Hence, the Eq. (3.15) is arranged with respect to u and v we obtain

$$\frac{f_i''(f_i+1)}{(f_i')^2}=\frac{(g_i')^2}{g_i''(g_i+1)}=m, \quad i=1,2 \tag{3.18}$$

where m is a constant. In that case, solution of the differential equations (3.18) we get,

$$\begin{aligned} f_1(u) &= c_1e^{c_2u}-1,\ f_2(u)=c_3e^{c_4u}-1, \\ g_1(v) &= c_5e^{c_6v}-1,\ g_2(v)=c_7e^{c_8v}-1. \end{aligned} \tag{3.19}$$

If we use the functions in Eq. (3.19) together with (3.16) and (3.17) we get $c_4c_6=c_2c_8$ and $c_2c_4=-c_6c_8$ respectively. So, we obtain the parametrization (13) and (14) of TF-type surface. This completes the proof of the theorem. □

Proposition 3.8. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. Then the mean curvature vector field of M is given by*

$$\begin{aligned} \overrightarrow{H} &= \frac{1}{2\sqrt{\widetilde{E}}W^2}[Eg_1''(f_1+1)+Gf_1''(g_1+1)-2Ff_1'g_1']N_1 \\ &+\frac{1}{2\sqrt{\widetilde{E}}W^3}\left\{\begin{array}{l} \widetilde{E}\left[Eg_2''(f_2+1)+Gf_2''(g_2+1)-2Ff_2'g_2'\right] \\ -\widetilde{F}\left[Eg_1''(f_1+1)+Gf_1''(g_1+1)-2Ff_1'g_1'\right] \end{array}\right\}N_2. \end{aligned} \tag{3.20}$$

Proof. Using the equations (3.5) and (3.9) into (2.8) we obtain (3.20). □

Corollary 3.9. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. Then M has vanishing mean curvature if and only if*

$$\begin{aligned} &\widetilde{G}\left(Eg_1''(f_1+1)+Gf_1''(g_1+1)-2Ff_1'g_1'\right)^2+\widetilde{E}\left(Eg_2''(f_2+1)+Gf_2''(g_2+1)-2Ff_2'g_2'\right)^2 \\ &-2\widetilde{F}\left(Eg_1''(f_1+1)+Gf_1''(g_1+1)-2Ff_1'g_1'\right)\left(Eg_2''(f_2+1)+Gf_2''(g_2+1)-2Ff_2'g_2'\right)=0. \end{aligned}$$

Corollary 3.10. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. Then M is minimal if and only if*

$$Eg_i''(f_i+1)+Gf_i''(g_i+1)-2Ff_i'g_i'=0, \quad i=1,2, \tag{3.21}$$

holds.

Theorem 3.11. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. If M is a minimal surface then it is parametrized by as follows;*
1) $z(u,v)=c_1v+c_2,\ w(u,v)=c_3v+c_4,$
2) $z(u,v)=c_1u+c_2,\ w(u,v)=c_3u+c_4,$
3) $z(u,v)=c_1u+c_2,\ w(u,v)=c_3v+c_4,$
4) $z(u,v)=c_1v+c_2,\ w(u,v)=c_3u+c_4,$
5) $z(u,v)=c,\ w(u,v)=(u+d)\tan(av+b)-1,$
6) $z(u,v)=c,\ w(u,v)=(v+d)\tan(au+b)-1,$
7) $z(u,v)=(u+d)\tan(av+b)-1,\ w(u,v)=(u+d)\tan(av+b)-1,$
8) $z(u,v)=(v+d)\tan(au+b)-1,\ w(u,v)=(v+d)\tan(au+b)-1,$
9) $z(u,v)=f_1(u)g_1(v)+f_1(u)+g_1(v), w(u,v)=f_2(u)g_2(v)+f_2(u)+g_2(v)$
where the functions f_i and g_i satisfying

$$\begin{aligned} u &= \pm\int\frac{df_i(u)}{\sqrt{2a\ln(f_i(u)+1)-2ac_1}}, \\ v &= \pm\int\frac{dg_i(v)}{\sqrt{c_2(g_i(v)+1)^4-\frac{b}{2}}}, \end{aligned} \tag{3.22}$$

or

$$\begin{aligned} u &= \pm\int\frac{df_i(u)}{\sqrt{c_1(f_i(u)+1)^4-\frac{a}{2}}}, \\ v &= \pm\int\frac{dg_i(v)}{\sqrt{2b\ln(g_i(v)+1)-2bc_2}}, \end{aligned} \tag{3.23}$$

or

$$\begin{aligned} u &= \pm\int\frac{df_i(u)}{\sqrt{c_1(f_i(u)+1)^{2(1+k)}-c_2}}, \\ v &= \pm\int\frac{dg_i(v)}{\sqrt{c_3(g_i(v)+1)^{2(1-k)}+c_4}}. \end{aligned} \tag{3.24}$$

Proof. Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. If M is minimal then $H=0$, so using Eq. (3.21) and (3.5) we obtain

$$\begin{aligned} 0 &= g_1''(f_1+1)(1+(f_1')^2(g_1+1)^2+(f_2')^2(g_2+1)^2) \\ &+f_1''(g_1+1)(1+(g_1')^2(f_1+1)^2+(g_2')^2(f_2+1)^2) \\ &-2f_1'g_1'(f_1'g_1'(g_1+1)(f_1+1)+f_2'g_2'(g_2+1)(f_2+1)), \end{aligned} \tag{3.25}$$

and

$$0 = g_2''(f_2+1)(1+(f_1')^2(g_1+1)^2+(f_2')^2(g_2+1)^2) \\ +f_2''(g_2+1)(1+(g_1')^2(f_1+1)^2+(g_2')^2(f_2+1)^2) \\ -2f_2'g_2'(f_1'g_1'(g_1+1)(f_1+1)+f_2'g_2'(g_2+1)(f_2+1)). \tag{3.26}$$

i) We obtain the surface parametrization of TF-type (1), (2), (3) and (4) in Theorem 3.7, if we take $f_1'(u)=0$ and $f_2'(u)=0$ or $g_1'(v)=0$ and $g_2'(v)=0$ or $f_2'(u)=0$ and $g_1'(v)=0$ or $f_1'(u)=0$ and $g_2'(v)=0$ in Eq. (3.25) and Eq. (3.26), respectively.
ii) If $f_1'(u)=0$ and $g_1'(v)=0$ then Eq. (3.25) holds, so from the Eq. (3.26) we obtain

$$\frac{g_2''}{g_2+1}+\frac{f_2''}{f_2+1}+(f_2')^2(g_2''(g_2+1)-(g_2')^2)+(g_2')^2(f_2''(f_2+1)-(f_2')^2)=0. \tag{3.27}$$

If we take $f_2''(u)=0$ and $g_2''(v)=0$ in the Eq. (3.27), respectively, then we get the functions $f_2(u)=\frac{\tan(au+b)}{c_2}-1$ and $g_2(v)=\frac{\tan(cv+d)}{c_1}-1$. So we obtain the surface parametrization (5) and (6) of TF-type.
If $f_2''(u)g_2''(v)\neq 0$, then differentiating (3.27) with respect to u and v, respectively, we have

$$\frac{\left(f_2''(f_2+1)-(f_2')^2\right)'}{\left((f_2')^2\right)'}=-\frac{\left(g_2''(g_2+1)-(g_2')^2\right)'}{\left((g_2')^2\right)'}=k \tag{3.28}$$

where k is constant. Thus, we integrate the both sides of the Eq. (3.28) with respect to u and v we obtain,

$$f_2''(f_2+1)-(1+k)(f_2')^2 = a, \\ g_2''(g_2+1)-(1-k)(g_2')^2 = b, \tag{3.29}$$

where a and b nonzero real integrand constants.
If we take $k=-1$ in Eq. (3.29) we obtain

$$f_2''(f_2+1) = a, \\ g_2''(g_2+1)-2(g_2')^2 = b. \tag{3.30}$$

Solving the differential equations (3.30) we get (3.22).
Therefore, if $k=1$ we get

$$f_2''(f_2+1)-2(f_2')^2 = a, \\ g_2''(g_2+1) = b. \tag{3.31}$$

So, solution of the (3.31) we obtain (3.23).
If $k\neq\pm1$, then solving the differential equations (3.29) with respect to the constant k we have (3.24). So we obtain the surface parametrization (9).
If we take $f_2'(u)=0$ and $g_2'(v)=0$, then the similar classification is obtained for the functions $f_1(u)$ and $g_1(v)$.
iii) If $f_1(u)=f_2(u)$ and $g_1(v)=g_2(v)$, then the equations (3.25) and (3.26) is equivalent. So we obtain the similar solutions with respect to the previous cases for all f_i and g_i functions. We can obtain the surface parametrizations (7)-(9). This completes the proof of the theorem. □

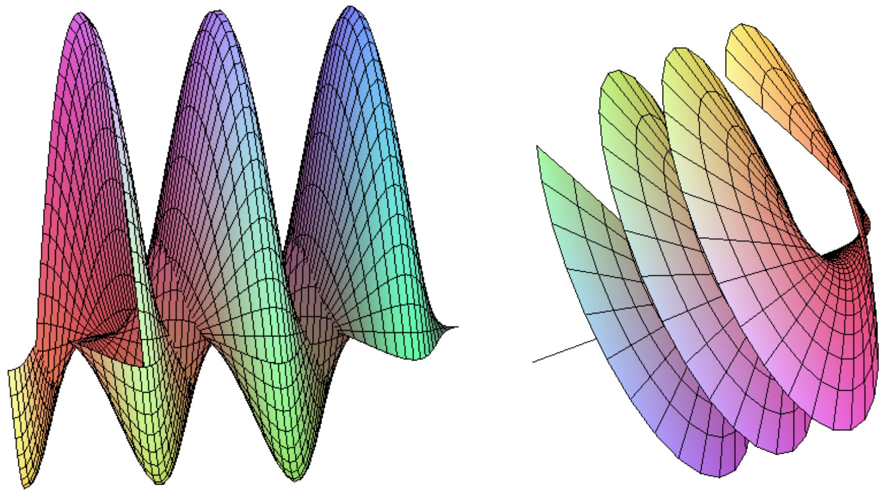
4. Visualization

In this part, we give a geometric model of the TF-type surfaces in Euclidean 4-space. We plot the graph of the projection of these surfaces in $\mathbb{E}^3$ by the use of following plotting command;

$$plot3d([x_1(u,v),x_2(u,v),x_3(u,v)+x_4(u,v)],u=a..b,v=c..d]).$$

Example 4.1. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. For the given functions we plot the projection of the TF surfaces in $\mathbb{E}^3$ as follows,*

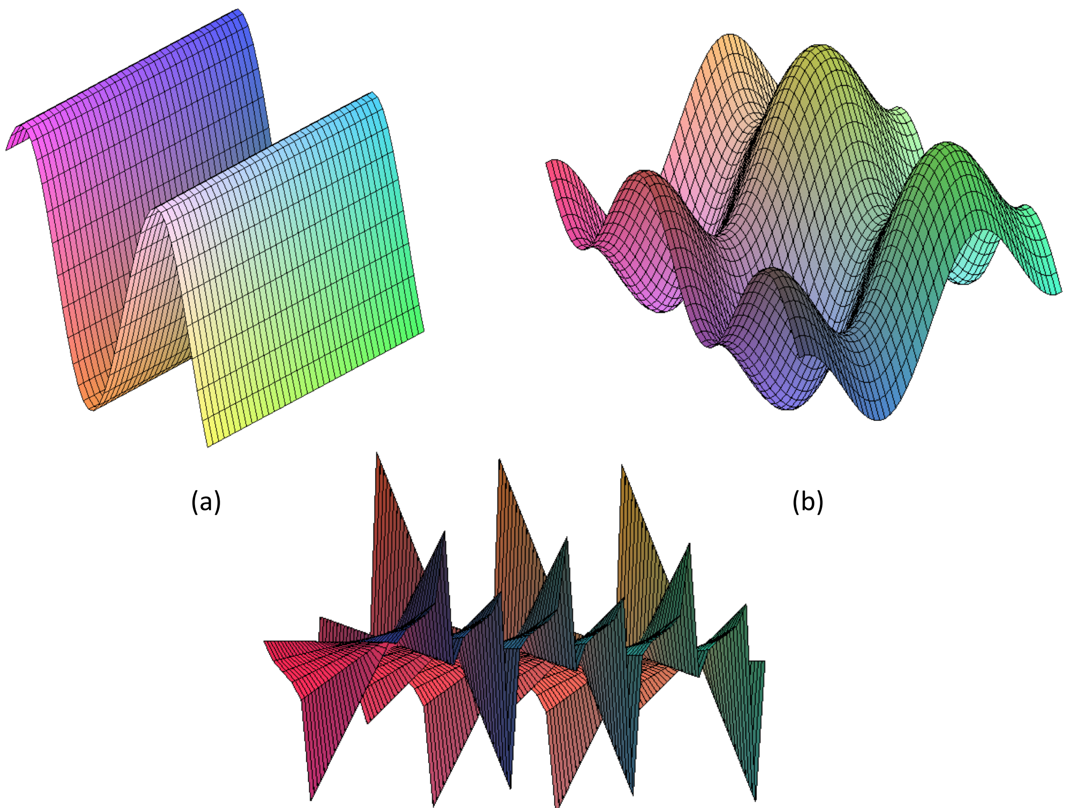
$$a) f_1(u) = f_2(u)=\cos(u);\ g_1(v)=\cos(v), g_2(v)=\sin(v), \\ b) f_1(u) = f_2(u)=\exp(u);\ g_1(v)=\cos(v), g_2(v)=\sin(v).$$



(a) (b)

Example 4.2. *Let M be TF-type surface given with the parametrization (3.3) in $\mathbb{E}^4$. For the given functions we plot the projection of the flat (a-b) and minimal (c) TF surfaces in $\mathbb{E}^3$,*

$$\begin{aligned} a)\ f_1(u) &= f_2(u) = 1;\ g_1(v) = \cos(v), g_2(v) = \sin(v), \\ b)\ f_1(u) &= g_2(v) = 1;\ g_1(v) = \cos(v), f_2(u) = \sin(u), \\ c)\ z(u,v) &= w(u,v) = (2u+1)\tan(3v-1) - 1. \end{aligned}$$



(a) (b) (c)

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