Research Article

Fekete-Szegö Problem for Certain Subclass of Analytic Functions with Complex Order Defined by \( q \)-Analogue of Ruscheweyh Operator

MOHAMED KAMAL AOUF AND TAMER MOHAMED SEOUDY*

ABSTRACT. In this paper, we study Fekete-Szegö problem for certain subclass of analytic functions with complex order in the open unit disk by applying the \( q \)-analogue of Ruscheweyh operator in conjunction with the principle of subordination between analytic functions.

Keywords: Analytic functions, univalent functions, \( q \)-derivative operator, \( q \)-analogue of Ruscheweyh operator, Fekete-Szego problem, subordination.

2010 Mathematics Subject Classification: 30C45.

1. INTRODUCTION

Let \( \mathcal{A} \) denote the class of functions of the form:

\[
 f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \). If \( f \) and \( g \) are analytic in \( \mathbb{U} \), we say that \( f \) is subordinate to \( g \), written as \( f \prec g \) in \( \mathbb{U} \) or \( f(z) \prec g(z) \) (\( z \in \mathbb{U} \)), if there exists a Schwarz function \( \omega \), which (by definition) is analytic in \( \mathbb{U} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) (\( z \in \mathbb{U} \)) such that \( f(z) = g(\omega(z)) \) (\( z \in \mathbb{U} \)). Furthermore, if the function \( g \) is univalent in \( \mathbb{U} \), then we have the following equivalence holds (see [12] and [7]):

\[
 f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).
\]

For function \( f \in \mathcal{A} \) given by (1.1) and \( 0 < q < 1 \), the \( q \)-derivative of a function \( f \) is defined by (see [10, 9] and [6])

\[
 D_q f(z) = \begin{cases} 
 \frac{f(qz)-f(z)}{(q-1)z}, & z \neq 0 \\
 f'(0), & z = 0
\end{cases}
\]

provided that \( f'(0) \) exists and \( D_q^2 f(z) = D_q(D_q f(z)) \). We note from (1.2) that

\[
 \lim_{q \to 1^-} D_q f(z) = f'(z) \quad \text{and} \quad \lim_{q \to 1^-} D_q^2 f(z) = f''(z).
\]

Received: 19.11.2019; Accepted: 31.01.2020; Published Online: 05.01.2020
*Corresponding author: T. M. Seoudy; tms00@fayoum.edu.eg
DOI: 10.33205/cma.648478
It is readily deduced from (1.1) and (1.2) that

(1.3) \[ D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \]

where

(1.4) \[ [k]_q = \frac{q^k - 1}{q - 1}. \]

Aldweby and Darus [1] defined $q$–analogue of Ruscheweyh operator $R^\delta_q : A \rightarrow A$ as follows:

\[ R^\delta_q f(z) = z + \sum_{k=2}^{\infty} \frac{(k + \delta - 1)!}{[\delta]! [k - 1]_q} a_k z^k \quad (\delta \geq -1), \]

where $[i]_q$ is given by

\[ [i]_q = \begin{cases} [i]_q [i-1]_q \ldots [1]_q, & i \in \mathbb{N} = \{1, 2, 3, \ldots\} \\ 1, & i = 0 \end{cases}. \]

We note that

\[ R^0_q f(z) = f(z) \quad \text{and} \quad R^1_q f(z) = z D_q f(z). \]

From the definition of $R^\delta_q$ we observe that if $q \rightarrow 1^-$, we have

\[ \lim_{q \rightarrow 1^-} R^\delta_q f(z) = R^\delta f(z) = z + \sum_{k=2}^{\infty} \frac{(k + \delta - 1)!}{\delta! (k - 1)!} a_k z^k, \]

where $R^\delta$ is Ruscheweyh differential operator defined by Ruscheweyh [16].

It is easy to check that

(1.5) \[ z D_q \left( R^\delta_q f(z) \right) = \left( 1 + \frac{[\delta]_q}{q^\delta} \right) R^{1+\delta}_q f(z) - \frac{[\delta]_q}{q^\delta} R^\delta_q f(z). \]

If $q \rightarrow 1^-$, the equality (1.5) implies

\[ z \left( R^{\delta} f(z) \right)' = (1 + \delta) R^{\delta+1} f(z) - \delta R^{\delta} f(z) \]

which is the well known recurrence formula for Ruscheweyh differential operator.

By making use of the $q$–analogue of Ruscheweyh operator $R^\delta_q$ and the principle of subordination, we now introduce the following subclass of analytic functions of complex order.

**Definition 1.1.** Let $P$ be the class of all functions $\phi$ which are analytic and univalent in $U$ and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $R\phi(z) > 0$ for $z \in U$. A function $f \in A$ is said to be in the class $K^0_{q,b}(\gamma, \phi)$ if it satisfies the following subordination condition:

(1.6) \[ 1 + \frac{1}{b} \left[ \frac{(1 - \gamma) z D_q R^\delta_q f(z) + \gamma z D_q (z D_q R^\delta_q f(z))}{(1 - \gamma) R^\delta_q f(z) + \gamma z D_q R^\delta_q f(z)} - 1 \right] < \phi(z) \quad (b \in \mathbb{C}^*). \]

We note that:

(i) $\lim_{q \rightarrow 1^-} K^0_{q,b}(\gamma, \phi) = K_b(\gamma, \phi) \quad (b \in \mathbb{C}^*)$

\[ = \left\{ f \in A : 1 + \frac{1}{b} \left[ \frac{zf'(z) + \gamma z^2 f''(z)}{(1 - \gamma) f(z) + \gamma zf'(z)} - 1 \right] < \phi(z) \right\}. \]
(ii) $K_{q,(1-\alpha)}^0 e^{-i \theta \cos \theta} (0, \phi) = S_q^0 (\alpha; \phi) (|\theta| \leq \frac{\pi}{2}, 0 \leq \alpha < 1)$

$$= \left\{ f \in A : \frac{e^{i \theta D_q f(z)}}{f(z)} - \frac{\alpha \cos \theta - i \sin \theta}{(1 - \alpha) \cos \theta} \prec \phi(z) \right\},$$

(iii) $K_{q,(1-\alpha)}^0 e^{-i \theta \cos \theta} (1, \phi) = C_q^0 (\alpha; \phi) (|\theta| \leq \frac{\pi}{2}, 0 \leq \alpha < 1)$

$$= \left\{ f \in A : \frac{e^{i \theta D_q (1 + (1 - 2\alpha) z)}}{D_q f(z)} - \frac{\alpha \cos \theta - i \sin \theta}{(1 - \alpha) \cos \theta} \prec \phi(z) \right\}.$$

(iv) $K_{q,1}^0 (0, \phi) = S_q^0 (\phi)$ and $K_{q,1}^0 (1, \phi) = C_q^0 (\phi)$ (Alweby and Darus [3]),

(v) $K_{q,b}^0 (0, \phi) = S_{q,b} (\phi)$ and $K_{q,b}^0 (1, \phi) = C_{q,b} (\phi)$ (Seoudy and Aouf [18]),

(vi) $K_{q,1}^0 (0, \phi) = S_q (\phi)$ and $K_{q,1}^0 (1, \phi) = C_q (\phi)$ (Alweby and Darus [2]),

(vii) $\lim_{q \to 1^+} K_{q,b}^0 (0, \phi) = S_b (\phi)$ and $\lim_{q \to 1^-} K_{q,b}^0 (1, \phi) = C_b (\phi)$ (Ravichandran et al. [15]),

(viii) $\lim_{q \to 1^-} K_{q,1}^0 (0, \phi) = S^* (\phi)$ and $\lim_{q \to 1^-} K_{q,1}^0 (1, \phi) = C (\phi)$ (Ma and Minda [11]),

(ix) $\lim_{q \to 1^-} K_{q,b}^0 \left( 0, \frac{1 + (1 - 2\alpha) z}{1 - z} \right) = S^*_b (\alpha)$ and $\lim_{q \to 1^-} K_{q,b}^0 \left( 1, \frac{1 + (1 - 2\alpha) z}{1 - z} \right) = C_\alpha (b)$ $(0 \leq \alpha < 1)$ (Frasin [8]),

(x) $\lim_{q \to 1^-} K_{q,b}^0 \left( 0, \frac{1 + z}{1 - z} \right) = S^*_b (b)$ (Nasr and Aouf [14]),

(xi) $\lim_{q \to 1^-} K_{q,b}^0 \left( 1, \frac{1 + z}{1 - z} \right) = C (b) \in C^*$ (Nasr and Aouf [13] and Wiatrowski [19]),

(xii) $\lim_{q \to 1^-} K_{q,1-\alpha}^0 \left( 0, \frac{1 + z}{1 - z} \right) = S^* (\alpha)$ and $\lim_{q \to 1^-} K_{q,1-\alpha}^0 \left( 1, \frac{1 + z}{1 - z} \right) = C (\alpha) \left( 0 \leq \alpha < 1 \right)$ (Robertson [17]),

(xiii) $\lim_{q \to 1^-} K_{q,be^{-i \theta \cos \theta}}^0 \left( 0, \frac{1 + z}{1 - z} \right) = S^0 (b)$ and $\lim_{q \to 1^-} K_{q,be^{-i \theta \cos \theta}}^0 \left( 1, \frac{1 + z}{1 - z} \right) = C^0 (b)$ $(|\theta| < \frac{\pi}{2})$ (Al-Oboudi and Haidan [4] and Aouf et al. [5]).

In order to establish our main results, we need the following lemma.

**Lemma 1.1.** [11] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $\mathbb{U}$ and $\mu$ is a complex number, then

$$|c_2 - \mu c_1^2| \leq 2 \max \{ 1; |2\mu - 1| \}.$$ 

The result is sharp for the functions given by

$$p(z) = \frac{1 + z}{1 - z^2} \text{ and } p(z) = \frac{1 + z}{1 - z}.$$

**Lemma 1.2.** [11] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with a positive real part in $\mathbb{U}$, then

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2 & \text{if } \nu \leq 0, \\ 2 & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2 & \text{if } \nu \geq 1 \end{cases},$$

when $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z)$ is $(1 + z)/(1 - z)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z)$ is $(1 + z^2)/(1 - z^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(z) = \left( \frac{1 + \lambda}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1 - \lambda}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \lambda \leq 1).$$
or one of its rotations. If \( \nu = 1 \), the equality holds if and only if \( p \) is the reciprocal of one of the functions such that equality holds in the case of \( \nu = 0 \).

Also the above upper bound is sharp, and it can be improved as follows when \( 0 < \nu < 1 \):

\[
|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2 \quad \left( 0 \leq \nu \leq \frac{1}{2} \right)
\]

and

\[
|c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2 \quad \left( \frac{1}{2} \leq \nu \leq 1 \right).
\]

In the present paper, we obtain the Fekete-Szegö inequalities for the class \( \mathcal{K}_{q,b} (\gamma, \phi) \). The motivation of this paper is to generalize previously results. Unless otherwise mentioned, we assume throughout this paper that the function \( 0 < q < 1, \ b \in \mathbb{C}^*, \ 0 \leq \gamma \leq 1, \ \phi \in \mathcal{P}, \ [k]_q \) is given by (1.4) and \( z \in \mathbb{U} \).

**Theorem 1.1.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( \mathcal{K}_{q,b} (\gamma, \phi) \), then

\[
|a_3 - \mu a_2^2| \leq \frac{|b B_1|}{q[1+\gamma q(q+1)][\delta+2]q[\delta+1]_q} \max \left\{ 1; \left| \frac{B_2}{B_1} + \left( 1 - \frac{1+\gamma q(q+1)}{(1+\gamma q)^2 [\delta+1]_q} \mu \right) \frac{B_1 b}{q} \right| \right\}.
\]

The result is sharp.

*Proof.* If \( f \in \mathcal{K}_{q,b} (\gamma, \phi) \), then there is a Schwarz function \( \omega \), analytic in \( \mathbb{U} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) in \( \mathbb{U} \) such that

\[
1 + \frac{1}{b} \left[ \frac{(1-\gamma) D_q R_q^\delta f(z) + \gamma z D_q(z D_q R_q^\delta f(z))}{(1-\gamma) R_q^\delta f(z) + \gamma z D_q R_q^\delta f(z)} - 1 \right] = \phi(\omega(z)).
\]

Define the function \( p(z) \) by

\[
p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + \ldots.
\]

Since \( \omega \) is a Schwarz function, we see that \( \Re p(z) > 0 \) and \( p(0) = 1 \). Therefore,

\[
\phi(\omega(z)) = \phi \left( \frac{p(z) - 1}{p(z) + 1} \right)
\]

\[
= \phi \left( \frac{1}{2} \left[ c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) z^3 + \ldots \right] \right)
\]

\[
= 1 + \frac{B_1 c_1}{2} z + \left[ \frac{B_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} \right] z^2 + \ldots.
\]

Now, by substituting (1.10) in (1.8), we have

\[
1 + \frac{1}{b} \left[ \frac{(1-\gamma) D_q R_q^\delta f(z) + \gamma z D_q(z D_q R_q^\delta f(z))}{(1-\gamma) R_q^\delta f(z) + \gamma z D_q R_q^\delta f(z)} - 1 \right]
\]

\[
= 1 + \frac{B_1 c_1}{2} z + \left[ \frac{B_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} \right] z^2 + \ldots.
\]

From the above equation, we obtain

\[
\frac{1}{b} q (1 + \gamma q) [\delta + 1]_q a_2 = \frac{B_1 c_1}{2}
\]
and

\[
\frac{q}{b} \left( [1 + \gamma q (q + 1)] \left[ \delta + 2 \right]_q [\delta + 1]_q a_3 - (1 + \gamma q)^2 \left( [\delta + 1]_q \right)^2 a_2 \right) = \frac{B_1 c_2}{2} - \frac{B_1 c_1}{4} + \frac{B_2 c_1^2}{4}
\]
or, equivalently,

\[
a_2 = \frac{B_1 c_1 b}{2q (1 + \gamma q) [\delta + 1]_q}
\]

and

\[
a_3 = \frac{b B_1}{2 [1 + \gamma q (q + 1)] q [\delta + 2]_q [\delta + 1]_q} \left\{ c_2 - \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} - \frac{B_1 b}{q} \right] c_1^2 \right\}.
\]

Therefore, we have

\[
a_3 - \mu a_2^2 = \frac{b B_1}{2q [1 + \gamma q (q + 1)] [\delta + 2]_q [\delta + 1]_q} \left\{ c_2 - \nu c_1^2 \right\},
\]

where

\[
\nu = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} - \frac{B_1 b}{q} \left( 1 - \frac{[1 + \gamma q (q + 1)] [\delta + 2]_q \mu}{(1 + \gamma q)^2 [\delta + 1]_q} \right) \right].
\]

Our result now follows from Lemma 1.1. The result is sharp for the functions

\[
1 + \frac{1}{b} \left[ \frac{(1 - \gamma) z D_q R_q^\delta f(z) + \gamma z D_q (z D_q R_q^\delta f(z))}{(1 - \gamma) R_q^\delta f(z) + \gamma z D_q R_q^\delta f(z)} - 1 \right] = \phi \left( \frac{z}{2} \right)
\]

and

\[
1 + \frac{1}{b} \left[ \frac{(1 - \gamma) z D_q R_q^\delta f(z) + \gamma z D_q (z D_q R_q^\delta f(z))}{(1 - \gamma) R_q^\delta f(z) + \gamma z D_q R_q^\delta f(z)} - 1 \right] = \phi \left( \frac{z}{4} \right).
\]

This completes the proof of Theorem 1.1. \( \square \)

Taking \( \gamma = 0 \) and \( b = 1 \) in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 6].

**Corollary 1.1.** Let \( \phi (z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( S_q^\delta (\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \frac{|B_1|}{q|\delta + 2|_q|\delta + 1|_q} \max \left\{ 1; \frac{B_2}{B_1} + \left( 1 - \frac{[\delta + 2]_q \mu}{[\delta + 1]_q} \right) \frac{B_1}{q} \right\}.
\]

The result is sharp.

Taking \( \gamma = b = 1 \) in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 7].

**Corollary 1.2.** Let \( \phi (z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( K_q^\delta (\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \frac{|B_1|}{q^{1+q+1} [\delta + 2]_q [\delta + 1]_q} \max \left\{ 1; \frac{B_2}{B_1} + \left( 1 - \frac{[1+q(1+q)] [\delta + 2]_q \mu}{[\delta + 1]_q (1+q)^2 \mu} \right) \frac{B_1 b}{q} \right\}.
\]

The result is sharp.

Taking \( \gamma = \delta = 0 \) and \( b = 1 \) in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [2, Theorem 2.1].
Corollary 1.3. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( S_q(\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \frac{|B_1|}{q(q+1)} \max \left\{ \frac{1}{2}, \frac{B_2}{B_1} + (1 - (q + 1) \mu) \frac{B_1}{q} \right\}.
\]

The result is sharp.

Taking \( \gamma = b = 1 \) and \( \delta = 0 \) in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [2, Theorem 2.2].

Corollary 1.4. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( K_q(\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \frac{|B_1|}{q(q+1)} \frac{|B_1|}{|1 + q(q+1)|} \max \left\{ \frac{1}{2}, \frac{B_2}{B_1} + \left(1 - \frac{[1+q(q+1)]}{[1+q(q+1)]^2} \mu \right) \frac{B_1}{q} \right\}.
\]

The result is sharp.

Taking \( \gamma = \delta = 0 \) and \( q \to 1^- \) in Theorem 1.1, we obtain the following corollary which improves the result of Ravichandran et al. [15, Theorem 4.1].

Corollary 1.5. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( S_b(\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \frac{|B_1 b|}{2} \max \left\{ \frac{1}{2}, \frac{B_2}{B_1} + (1 - 2\mu) \frac{B_1 b}{1} \right\}.
\]

The result is sharp.

Theorem 1.2. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let

\[
\begin{align*}
\sigma_1 &= \frac{(1 + \gamma q)^2 [\delta + 1]_q}{[1 + \gamma q (q + 1)] [\delta + 2]_q b B_1^2 + q (B_2 - B_1)} , \\
\sigma_2 &= \frac{(1 + \gamma q)^2 [\delta + 1]_q [b B_1^2 + q (B_2 + B_1)]}{[1 + \gamma q (q + 1)] [\delta + 2]_q b B_1^2} , \\
\sigma_3 &= \frac{(1 + \gamma q)^2 [\delta + 1]_q (b B_1^2 + q B_2)}{[1 + \gamma q (q + 1)] [\delta + 2]_q b B_1^2} .
\end{align*}
\]

If \( f \) given by (1.1) belongs to the class \( K^\delta_{q,b}(\gamma, \phi) \) with \( b > 0 \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b}{[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q} \left[ B_2 + \frac{B_1^2 b}{q} \left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q} \mu \right) \right] , & \mu \leq \sigma_1 \\
\frac{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}{[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q} \left[-B_2 - \frac{B_1^2 b}{q} \left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q} \mu \right) \right] , & \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{b}{[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q} \left[ B_1 - B_2 - \frac{B_1^2 b}{q} \left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q} \mu \right) \right] , & \mu \geq \sigma_2
\end{cases}
\]

Further, if \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[
|a_3 - \mu a_2^2| \leq \frac{b B_1}{q [1 + \gamma q (q + 1)] [\delta + 2]_q [\delta + 1]_q}.
\]

If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_3 - \mu a_2^2| \leq \frac{q[1+\gamma q(q+1)][\delta+2]_q}{[1+\gamma q(q+1)][\delta+2]_q B_1^2} \left[ B_1 + B_2 + \frac{B_1^2 b}{q} \left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q} \mu \right) \right] |a_2|^2
\]
Corollary 1.6. \( \text{the result of Aldweby and Darus} [3, \text{Theorem 10}] \).

If \( \sigma \)

Proof. Applying Lemma 1.2 to (1.11) and (1.12), we can obtain our results asserted by Theorem 1.2.

Taking \( \gamma = 0 \) and \( b = 1 \) in Theorem 1.2, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 10].

Corollary 1.6. Let \( \phi(z) = 1 + B_1z + B_2z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let

\[
\chi_1 = \frac{[\delta + 1]_q \left[ B_1^2 + q(B_2 - B_1) \right]}{[\delta + 2]_q B_1^2} , \\
\chi_2 = \frac{[\delta + 1]_q \left[ B_1^2 + q(B_2 + B_1) \right]}{[\delta + 2]_q B_1^2} , \\
\chi_3 = \frac{[\delta + 1]_q \left( B_1^2 + qB_2 \right)}{[\delta + 2]_q B_1^2} .
\]

If \( f \) given by (1.1) belongs to the class \( S^\delta_q(\phi) \), then

\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{l}
\frac{1}{q[\delta + 2]_q [\delta + 1]_q} \left[ B_2 + B_1^2 \left( 1 - \frac{[\delta + 2]_q}{[\delta + 1]_q} \mu \right) \right] , \mu \leq \chi_1
\\
\frac{1}{q[\delta + 2]_q [\delta + 1]_q} \left[ B_1 - B_2 - \frac{B_1^2}{q} \left( 1 - \frac{[\delta + 2]_q}{[\delta + 1]_q} \mu \right) \right] , \chi_1 \leq \mu \leq \chi_2
\\
\frac{1}{q[\delta + 2]_q [\delta + 1]_q} \left[ B_1 + B_2 + \frac{B_1^2}{q} \left( 1 - \frac{[\delta + 2]_q}{[\delta + 1]_q} \mu \right) \right] , \mu \geq \chi_2
\end{array} \right. ,
\]

Further, if \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[ |a_3 - \mu a_2^2| + \frac{q[\delta + 1]_q}{[\delta + 2]_q B_1^2} \left[ B_1 - B_2 - \frac{B_1^2}{q} \left( 1 - \frac{[\delta + 2]_q}{[\delta + 1]_q} \mu \right) \right] |a_2|^2 \leq \frac{B_1}{q[\delta + 2]_q [\delta + 1]_q}.
\]

If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[ |a_3 - \mu a_2^2| + \frac{q[\delta + 1]_q}{[\delta + 2]_q B_1^2} \left[ B_1 + B_2 + \frac{B_1^2}{q} \left( 1 - \frac{[\delta + 2]_q}{[\delta + 1]_q} \mu \right) \right] |a_2|^2 \leq \frac{B_1}{q[\delta + 2]_q [\delta + 1]_q}.
\]

The result is sharp.

Taking \( \gamma = b = 1 \) in Theorem 1.2, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 11].

Corollary 1.7. Let \( \phi(z) = 1 + B_1z + B_2z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let

\[
\kappa_1 = \frac{[2]_q \left[ 2 \delta + 1 \right]_q \left[ B_1^2 + q(B_2 - B_1) \right]}{[3]_q [\delta + 2]_q B_1^2} , \\
\kappa_2 = \frac{[2]_q \left[ 2 \delta + 1 \right]_q \left[ B_1^2 + q(B_2 + B_1) \right]}{[3]_q [\delta + 2]_q B_1^2} , \\
\kappa_3 = \frac{[2]_q \left[ 2 \delta + 1 \right]_q \left( B_1^2 + qB_2 \right)}{[3]_q [\delta + 2]_q B_1^2} .
\]

The result is sharp.
If \( f \) given by (1.1) belongs to the class \( \mathcal{K}_q^\delta (\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{q[3]_q[\delta+2]_q[\delta+1]_q} \left[ B_2 + \frac{B_1^2}{q} \left( 1 - \frac{[3]_q[\delta+2]_q[\delta+1]_q}{[2]_q[\delta+1]_q} \mu \right) \right] , & \mu \leq \kappa_1 \\
\frac{[3]_q[\delta+2]_q[\delta+1]_q}{q[3]_q[\delta+2]_q[\delta+1]_q} \left[ B_2 - \frac{B_1^2}{q} \left( 1 - \frac{[3]_q[\delta+2]_q[\delta+1]_q}{[2]_q[\delta+1]_q} \mu \right) \right] , & \kappa_1 \leq \mu \leq \kappa_2 \\
\frac{B_1}{q[3]_q[\delta+2]_q[\delta+1]_q} , & \mu \geq \kappa_2
\end{cases}
\]
Further, if \( \kappa_1 \leq \mu \leq \kappa_3 \), then
\[
|a_3 - \mu a_2^2| + \frac{q[2]_q[\delta+1]_q}{[3]_q[\delta+2]_q B_1^2} \left[ B_1 - B_2 - \frac{B_1^2}{q} \left( 1 - \frac{[3]_q[\delta+2]_q[\delta+1]_q}{[2]_q[\delta+1]_q} \mu \right) \right] |a_2|^2 \leq \frac{B_3}{q[3]_q[\delta+2]_q[\delta+1]_q}.
\]
If \( \kappa_3 \leq \mu \leq \kappa_2 \), then
\[
|a_3 - \mu a_2^2| + \frac{q[2]_q[\delta+1]_q}{[3]_q[\delta+2]_q B_1^2} \left[ B_1 + B_2 + \frac{B_1^2}{q} \left( 1 - \frac{[3]_q[\delta+2]_q[\delta+1]_q}{[2]_q[\delta+1]_q} \mu \right) \right] |a_2|^2 \leq \frac{B_3}{q[3]_q[\delta+2]_q[\delta+1]_q}.
\]
The result is sharp.

**Remark 1.1.** Putting \( \delta = \gamma = 0 \) in Theorems 1.1 and 1.2, respectively, we deduce the corresponding results derived by Seoudy and Aouf \[18\], Theorems 1 and 3, respectively.

**Remark 1.2.** Putting \( \delta = 0 \) and \( \gamma = 1 \) in Theorems 1.1 and 1.2, respectively, we deduce the corresponding results derived by Seoudy and Aouf \[18\], Theorems 2 and 4, respectively.

**Remark 1.3.** For different choices of the parameters \( b, \delta, q, \gamma \) and \( \phi \) in Theorems 1.1 and 1.2, we can deduce some results for the classes \( \mathcal{K}_b (\gamma, \phi), \mathcal{S}_q^\alpha (\alpha; \phi), \mathcal{C}_q^\alpha (\alpha; \phi), \mathcal{S}_q^\alpha (\phi), \mathcal{C}_q (\phi), \mathcal{S}_b (\phi), \mathcal{C}_b (\phi), \mathcal{S}_b^* (\phi), \mathcal{C} (\phi), \mathcal{S}_\alpha^* (b), \mathcal{C}_\alpha (b), \mathcal{S}^* (b), \mathcal{C} (b), \mathcal{S}^* (\alpha), \mathcal{C} (\alpha), \mathcal{S}^0 (b) \) and \( \mathcal{C}^0 (b) \) which are defined in Section 1.

**REFERENCES**


MANSOURA UNIVERSITY
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
MANSOURA 35516, EGYPT
E-mail address: mkaouf127@yahoo.com

FAYOUm UNIVERSITY
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
FAYOUm 63514, EGYPT
UMM AL-QURA UNIVERSITY
DEPARTMENT OF MATHEMATICS, JAMOUm UNIVERSITY COLLEGE,
MAKKAH, SAUDI ARABIA
ORCID: 0000-0001-6427-6960
E-mail address: tms00@fayoum.edu.eg, tmsaman@uqu.edu.sa