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An Examination of Data Dependence For Jungck-Type Iteration Method

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Abstract: Iteration methods are an important field of study in fixed point theory and have an extensive literature. Different types of iteration methods were defined in many spaces by researchers, and the results such as convergence, rate of convergence, stability and data dependence of these methods were examined. In this study, a new iteration method of Jungck Type was defined and the convergence of this method for a certain mapping class was investigated. Then, using this mapping class for this iteration method, the results of stability and data dependence were obtained. Additionally, the rate of convergence of the newly defined iteration method with Jungck CR iteration method was compared under suitable conditions and an example supporting this result was given.

Jungck-Tipi İterasyon Yöntemi İçin Veri Bağlılığının İncelenmesi

Anahtar Kelimeler
İterasyon Yöntemleri,
Yakınsaklık,
Veri Bağlılığı,
Kararlılık

Özet: İterasyon yöntemleri sabit nokta teorisinde önemli bir çalışma alanı olup geniş bir literatüre sahiptir. Araştırmacılar tarafından birçok uzayda farklı türden iterasyon yöntemleri tanımlanarak, bu yöntemlerin yakınsaklığı, yakınsaklık hızları, kararlılığı ve veri bağlılığı gibi sonuçlar irdelenmiştir. Bu çalışmada, Jungck tipi yeni bir iterasyon yöntemi tanımlanarak bu yöntemin belirli bir dönüşüm sınıfı için yakınsaklığı incelenmiştir. Daha sonra bu iterasyon yöntemi için söz konusu dönüşüm sınıfı kullanılarak kararlılık ve veri bağlılığı sonuçları elde edilmiştir. Ayrıca yeni tanımlanan iterasyon yönteminin uygun şartlar altında yakınsaklık hızı Jungck CR iterasyon yöntemiyle karşılaştırılarak bu sonucu destekleyen örnek verilmiştir.

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1. Introduction

There is a wide range of interest in literature of fixed point theory. In this context, many newly iteration methods have been introduced, and strong convergence, data dependence, stability, and rate of convergence of various iteration methods have been studied by researchers (see [1]-[6]). In 1976, Jungck defined the Jungck iteration method [7]. After that, obtaining fixed point results of some iteration methods for Jungck type has been the concept of research. Such studies have became the subject of various researchs (see [8]-[10]). Below, some of these studies has been given in a chronological order:

In 2005, Jungck-Mann iteration method was defined by Singh et al. [11], in 2008 Jungck-Ishikawa iteration method and Jungck-Noor iteration method were defined by Olatinwo [12], [13], in 2011, Jungck-SP iteration method was defined by Chugh et al. [14], in 2013, Jungck-CR iteration method and Jungck-Sahu iteration method were defined by Hussain et al. [15], Jungck-Agarwal iteration method was defined by Chugh et al. [16], in 2014, Jungck-Khan iteration method was defined Khan et al. [17], in 2016 Jungck-Mann hybrid iteration method was defined by Akewe [18], in 2019 Junck type iteration method was defined by Atalan [19].

2. Material and Method

We need some definitions, lemmas for the new Jungck-type iteration method and main results.

Definition 2.1. Let X be Banach space, Y an arbitrary set and $S,T:Y\to X$ such that $T(Y)\subseteq S(Y)$. For $x_0\in Y$, following iteration method:

$$Sx_{n+1}=f(T,x_n)=Tx_n \tag{1}$$

for all $n\in\mathbb{N}$. If $S=I$ (unit mapping) and $Y=X$ in the above equation, it is easily seen that Picard iteration method is obtained. This iteration method is called Jungck iteration method [7].

Definition 2.2. The pair $S,T:Y\to X$ is called contractive if there exist a real number $\delta\in[0,1)$ and a continuous function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ such that $\phi(0)=0$ and for all $x,y\in Y$, we have

$$\|Tx-Ty\|\le\phi(\|Sx-Tx\|)+\delta\|Sx-Sy\| \tag{2}$$

[12].

Definition 2.3. Let X be a nonempty set and $S,T:X\to X$ be mappings. If $Tx=Sx$, then $x\in X$ is called coincidence point of T and S. If $x=Tx=Sx$, then $x\in X$ is called common fixed point of T and S. If $p=Tx=Sx$ for some $x\in X$, then p is called the point of coincidence of T and S. If $TSx=STx$ for all $x\in X$, then a pair (S,T) is called commuting. If $TSx=STx$ whenever $Tx=Sx$ for some $x\in X$, then a pair (S,T) is called weakly compatible [20].

Lemma 2.4. Let $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality

$$\sigma_{n+1}\le(1-\lambda_n)\sigma_n+\mu_n$$

where $\lambda_n\in(0,1)$ for all $n\ge n_0$, $\sum_{n=0}^{\infty}\lambda_n=\infty$ and $\frac{\mu_n}{\lambda_n}\to0$ as $n\to\infty$. Then $\lim_{n\to\infty}\sigma_n=0$ [21].

Lemma 2.5. Let $\{\sigma_n\}_{n=0}^{\infty}$ be nonnegative real sequence. Assume that there exists $n_0\in\mathbb{N}$, such that for all the $n\ge n_0$ one has the inequality

$$\sigma_{n+1}\le(1-\lambda_n)\sigma_n+\lambda_n\rho_n$$

where $\lambda_n\in(0,1)$ for all $n\in\mathbb{N}$, $\sum_{n=0}^{\infty}\lambda_n=\infty$, and $\rho_n\ge0$. Then the following inequality holds:

$$0\le\limsup_{n\to\infty}\sigma_n\le\limsup_{n\to\infty}\rho_n$$

[22].

Definiton 2.6. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers with same limit c. If

$$\lim_{n\to\infty}\frac{d(a_n,c)}{d(b_n,c)}=0 \tag{3}$$

then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ to c [23].

Definition 2.7. Let $S,T:Y\to X$, $T(Y)\subseteq S(Y)$ and $p=Tx=Sx$. For any $x_0\in Y$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the iteration method $Sx_{n+1}=f(T,x_n)$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty}\subset X$ be an arbitrary sequence and set

$$\epsilon_n=d\big(Sy_{n+1},f(T,y_n)\big),n=0,1,2,\ldots.$$

Then the iteration method $f(T,x_n)$ will be called (S,T)-stable if and only if $\lim_{n\to\infty}\epsilon_n=0$ implies that $\lim_{n\to\infty}Sy_n=p$ [11].

Definition 2.8. Let $(S,T),(\tilde{S},\tilde{T}):Y\to X$ be nonself-mapping pairs on an arbitrary set Y such that $T(Y)\subseteq S(Y)$ and $\tilde{T}(Y)\subseteq\tilde{S}(Y)$. We say that the pair $(\tilde{S},\tilde{T})$ is an approximate mapping pair of (S,T) if for all $x\in Y$ and for fixed $\epsilon_1\geq 0,\epsilon_2\geq 0$, we have

$\|Tx-\tilde{T}x\|\leq\epsilon_1$, $\|Sx-\tilde{S}x\|\leq\epsilon_2$
[17].

Iteration methods which is given below, are used for results we obtained. Thus, let us go through with Jungck-Type iteration method CR [15] as follows:

$$\begin{cases} Sx_{n+1}=(1-\alpha_n)Sy_n+\alpha_nTz_n \\ Sy_n=(1-\beta_n)Tx_n+\beta_nTz_n \\ Sz_n=(1-\gamma_n)Sx_n+\gamma_nTx_n \end{cases} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty},\{\gamma_n\}_{n=0}^{\infty}\subset[0,1]$.

We have defined a new Jungck type iteration method as follows:

$$\begin{cases} Sx_{n+1}=(1-\alpha_n)Ty_n+\alpha_nTz_n \\ Sy_n=(1-\beta_n)Tx_n+\beta_nTz_n \\ Sz_n=(1-\gamma_n)Sx_n+\gamma_nTx_n \end{cases} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty},\{\gamma_n\}_{n=0}^{\infty}\subset[0,1]$.

3. Results

We use the following assumption in this paper: $C(S,T)$ denotes the set of coincidence points of ▯ and ▯.

Theorem 3.1. Let $(X,\|.\|)$ be a Banach space, $S,T:Y\to X$ satisfies condition (2), where $T(Y)\subseteq S(Y),S(Y)$ is a complete subset of X and assume that there exist a $z\in C(S,T)$ such that $Tz=Sz=p$. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (5) with $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty},\{\gamma_n\}_{n=0}^{\infty}\subset[0,1]$ and $\sum_{n=0}^{\infty}\alpha_n\beta_n\gamma_n=\infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Moreover, p is a unique common fixed point of (S,T) provided that $Y=X$, and S and T are weakly compatible.

Proof. By using iterative sequence (5) and condition (2), we have

$$\begin{aligned} \|Sx_{n+1}-p\| &\leq (1-\alpha_n)\|Ty_n-p\|+\alpha_n\|Tz_n-p\| \\ &\leq (1-\alpha_n)\delta\|Sy_n-Sz\|+\phi(\|Sz-Tz\|) \\ &+\alpha_n\delta\|Sz_n-Sz\|+\phi(\|Sz-Tz\|) \\ &=(1-\alpha_n)\delta\|Sy_n-p\|+\alpha_n\delta\|Sz_n-p\| \end{aligned} \tag{6}$$

and

$$\begin{aligned} \|Sz_n-p\| &\leq (1-\gamma_n)\|Sx_n-p\|+\gamma_n\|Tz_n-p\| \\ &\leq (1-\gamma_n)\|Sx_n-Sz\|+\gamma_n\delta\|Sx_n-Sz\|+\phi(\|Sz-Tz\|) \\ &=[1-\gamma_n(1-\delta)]\|Sx_n-p\| \end{aligned} \tag{7}$$

also

$$\begin{aligned}\|Sy_n - p\| &\le (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\ &\le (1-\beta_n)\delta\|Sx_n - Sz\| + (1-\beta_n)\,\phi(\|Sz - Tz\|) \\ &+\beta_n\delta\|Sz_n - Sz\| + \beta_n\phi(\|Sz - Tz\|) \\ &= (1-\beta_n)\delta\|Sx_n - p\| + \beta_n\delta\|Sz_n - p\| \\ &\le \delta[(1-\beta_n) + \beta_n - \beta_n\gamma_n(1-\delta)]\|Sx_n - p\| \\ &\le \delta[1 - \beta_n\gamma_n(1-\delta)]\|Sx_n - p\|. \end{aligned} \tag{8}$$

Substituting (7) in (6) and (7), (8) in (6) respectively, we obtain

$$\begin{aligned}\|Sx_{n+1} - p\| &\le (1-\alpha_n)\delta[1 - \beta_n(1-\delta)]\|Sx_n - p\| \\ &+\alpha_n[1 - \gamma_n(1-\delta)]\|Sx_n - p\| \\ &\le \delta\begin{Bmatrix}1 - \alpha_n\beta_n\gamma_n(1-\delta) + \alpha_n\beta_n\gamma_n(1-\delta) \\ -\alpha_n\beta_n\gamma_n(1-\delta)\end{Bmatrix}\|Sx_n - p\| \\ &\le \delta[1 - \alpha_n\beta_n\gamma_n(1-\delta)]\|Sx_n - p\|. \end{aligned} \tag{9}$$

Then

$$\begin{aligned}\|Sx_{n+1} - p\| &\le \delta^{n+1}\textstyle\prod_{i=0}^{n}[1 - \alpha_i\beta_i\gamma_i(1-\delta)]\|Sx_{n+1} - p\| \\ &\le \delta^{n+1}\frac{1}{e^{(1-\delta)\sum_{i=0}^{n}\alpha_i\beta_i\gamma_i}}\|Sx_0 - p\|. \end{aligned} \tag{10}$$

Taking the limit the above inequality, it can be seen that $\lim_{n\to\infty}\|Sx_n - p\| = 0$.
We prove that p is a unique common fixed point of S, T, when $Y = X$.
Assume that there exist another point of coincide q of the pair (S, T). Then there exist $r \in C(S,T)$ such that $Sr = T\text{r} = q$ We get,

$$0 \le \|p - q\| = \|Tz - T\text{r}\| \le \phi(\|Sz - Tz\|) + \delta\|Sz - S\text{r}\| = \delta\|p - \text{q}\|$$

which implies that $p = q$. S, T are weakly compatible and $Sz = Tz = p$, then $Tp = TTz = TSz = STz$ implies $Tp = Sp$. Hence, Tp is a point of coincidence of the pair (S, T) and because point of coincidence is unique, then $Tp = p = Sp$ and thus p is a unique common fixed point of S and T.

Theorem 3.2. Let $(X, \|.\|)$ be a Banach space, $S, T: Y \to X$ satisfies condition (2), where $T(Y) \subseteq S(Y), S(Y)$ is a complete subset of X and assume that there exist a $z \in C(S,T)$ such that $Tz = Sz = p$. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (5) with $0 < \alpha_1 < \alpha_n, 0 < \beta_1 < \beta_n, 0 < \gamma_1 < \gamma_n$ converges to p. Also let $\{Su_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and let $\epsilon_n = d\big(Su_{n+1}, f(T, u_n)\big), n = 0,1,2,\ldots.$ Then iteration method (5) is (S,T)-stable.

Proof. Let $\lim_{n\to\infty}\epsilon_n = 0$ such that $\epsilon_n = \big\|Su_{n+1} - \big((1-\alpha_n)Tv_n + \alpha_n Tw_n\big)\big\|$.

$$\begin{aligned}\|Su_{n+1} - p\| &\le \big\|Su_{n+1} - \big((1-\alpha_n)Tv_n + \alpha_n Tw_n\big)\big\| \\ &+\|(1-\alpha_n)Tv_n + \alpha_n Tw_n - p\| \\ &\le \epsilon_n + (1-\alpha_n)\|Tv_n - Tz\| + \alpha_n\|Tw_n - Tz\| \end{aligned}$$

$$\leq \epsilon_n + \delta(1-\alpha_n)\|Sv_n - Sz\| + (1-\alpha_n)\phi(\|Sz - Tz\|)$$

$$+\delta\alpha_n\|Sw_n - Sz\| + \alpha_n\phi(\|Sz - Tz\|)$$

$$= \epsilon_n + \delta(1-\alpha_n)\|Sv_n - p\| + \delta\alpha_n\|Sw_n - p\| \tag{11}$$

and

$$\|Sw_n - p\| \leq (1-\gamma_n)\|Su_n - p\| + \gamma_n\|Tu_n - p\|$$

$$\leq (1-\gamma_n)\|Su_n - p\| + \gamma_n\delta\|Su_n - p\| + \gamma_n\phi(\|Sz - Tz\|)$$

$$\leq [1-\gamma_n(1-\delta)]\|Su_n - p\| \tag{12}$$

and moreover

$$\|Sv_n - p\| \leq (1-\beta_n)\|Tu_n - p\| + \beta_n\|Tw_n - p\|$$

$$\leq (1-\beta_n)\delta\|Su_n - Sz\| + (1-\beta_n)\ \phi(\|Sz - Tz\|)$$

$$+\beta_n\phi(\|Sz - Tz\|) + \beta_n\delta\|Sw_n - Sz\|$$

$$= (1-\beta_n)\delta\|Su_n - Sz\| + \beta_n\delta\|Sw_n - p\| +$$

$$\leq \delta[(1-\beta_n) + \beta_n(1-\gamma_n(1-\delta))]\|Su_n - p\|$$

$$\leq \delta[(1-\beta_n) + \beta_n - \beta_n\gamma_n(1-\delta)]\|Su_n - p\|$$

$$\leq \delta[1-\beta_n\gamma_n(1-\delta)]\|Su_n - p\|. \tag{13}$$

If we combine (11) in (12) and (13) then, we have

$$\|Su_{n+1} - p\| \leq \epsilon_n + \delta[1-\alpha_n\beta_n\gamma_n(1-\delta)]\|Su_n - p\|. \tag{14}$$

Hence $0 < \alpha_1 < \alpha_n, 0 < \beta_1 < \beta_n, 0 < \gamma_1 < \gamma_n$ and $\delta \in [0,1)$, we get

$$\left[1-\alpha_n\beta_n\gamma_n(1-\delta))\right] \leq \left[1-\alpha_1\beta_n\gamma_1(1-\delta)\right] < 1.$$

Then we obtain $\lim_{n\to\infty} u_n = p$.
Now, we suppose that $\lim_{n\to\infty} u_n = p$. It will be shown that $\lim_{n\to\infty} \epsilon_n = 0$:

$$\epsilon_n = \left\|Su_{n+1} - \left((1-\alpha_n)Tv_n + \alpha_n Tw_n\right) - p + p\right\|$$

$$\leq \|Su_{n+1} - p\| + (1-\alpha_n)\|Tv_n - Tz\| + \alpha_n\|Tw_n - Tz\|$$

$$\leq \|Su_{n+1} - p\| + \delta(1-\alpha_n)\|Sz - Sv_n\| + (1-\alpha_n)\phi(\|Sz - Tz\|)$$

$$+\alpha_n\|Sw_n - Sz\| + \alpha_n\phi(\|Sz - Tz\|)$$

$$\leq \|Su_{n+1} - p\| + \delta(1-\alpha_n)\|Sv_n - p\| + \alpha_n\|Sw_n - p\|.$$

From (13) it can be seen easily $\|Sv_n - p\| \leq \delta[1-\beta_n\gamma_n(1-\delta)]\|Su_n - p\|$. Then we get

$$\epsilon_n \leq \|Su_{n+1} - p\| + \delta[1-\alpha_n(1-\delta)]\|Sw_n - p\|$$

Also, from (12), $\|Sw_n - p\| \leq [1-\gamma_n(1-\delta)]\|Su_n - p\|$. We obtain

$$\epsilon_n \leq \|Su_{n+1} - p\| + \delta[1-\alpha_n\beta_n\gamma_n(1-\delta)]\|Su_n - p\|.$$

Taking the limit in the above inequality, it can be seen that $\lim_{n\to\infty} \epsilon_n = 0$.

Theorem 3.3. Let $(X, \|.\|)$ be a Banach space, $(S,T), (\tilde{S},\tilde{T}): Y \to X$ satisfies condition (2), where $T(Y) \subseteq S(Y)$ and $\tilde{T}(Y) \subseteq \tilde{S}(Y)$. $\tilde{S}(Y)$ is a complete subset of X and assume that there exist a $z \in C(S,T)$ and a $\tilde{z} \in C(\tilde{S},\tilde{T})$ such that $Tz = Sz = p$ and $\tilde{T}\tilde{z} = \tilde{S}\tilde{z} = \tilde{p}$ Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (5) with $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n = \infty$ and $\{\tilde{S}u_{n+1}\}_{n=0}^{\infty}$ a sequence defined by

$$\begin{cases} \tilde{S}u_{n+1} = (1-\alpha_n)\tilde{T}v_n + \alpha_n\tilde{T}w_n \\ \tilde{S}v_n = (1-\beta_n)\tilde{T}u_n + \beta_n\tilde{T}w_n \\ \tilde{S}w_n = (1-\gamma_n)\tilde{S}u_n + \gamma_n\tilde{T}u_n \end{cases} \tag{15}$$

Assume that $\{Sx_{n+1}\}_{n=0}^{\infty}$ and $\{\tilde{S}u_{n+1}\}_{n=0}^{\infty}$ convergence to p and $\tilde{p}$. Then we have

$$\|p-\tilde{p}\| = \frac{8\epsilon}{(1-\delta)}.$$

Proof. By using iterative sequence (5), iterative sequence (15), condition (2), and Definition 2.8

$$\begin{aligned} \|Sx_{n+1} - \tilde{S}u_{n+1}\| &\le (1-\alpha_n)\|Ty_n - \tilde{T}v_n\| + \alpha_n\|Tz_n - \tilde{T}w_n\| \\ &\le (1-\alpha_n)\|Ty_n - Tv_n\| + (1-\alpha_n)\|Tv_n - \tilde{T}v_n\| \\ &+\alpha_n\|Tz_n - Tw_n\| + \alpha_n\|Tw_n - \tilde{T}w_n\| \\ &\le (1-\alpha_n)\delta\|Sy_n - Sv_n\| + (1-\alpha_n)\phi(\|Sy_n - Ty_n\|) \\ &+\alpha_n\delta\|Sz_n - Sw_n\| + \alpha_n\phi(\|Sz_n - Tz_n\|) + \epsilon_1 \\ &\le (1-\alpha_n)\delta\|Sy_n - \tilde{S}v_n\| + (1-\alpha_n)\delta\|\tilde{S}v_n - Sv_n\| \\ &+ (1-\alpha_n)\phi(\|Sy_n - Ty_n\|) + \alpha_n\delta\|Sz_n - \tilde{S}w_n\| \\ &+\alpha_n\delta\|\tilde{S}w_n - Sw_n\| + \alpha_n\phi(\|Sz_n - Tz_n\|) + \epsilon_1 \\ &\le (1-\alpha_n)\delta\|Sy_n - \tilde{S}v_n\| + (1-\alpha_n)\phi(\|Sy_n - Ty_n\|) \\ &+\alpha_n\delta\|Sz_n - \tilde{S}w_n\| + \alpha_n\phi(\|Sz_n - Tz_n\|) + \epsilon_1 + \delta\epsilon_2 \end{aligned} \tag{16}$$

and

$$\begin{aligned} \|Sz_n - \tilde{S}w_n\| &\le (1-\gamma_n)\|Sx_n - \tilde{S}u_n\| + \gamma_n\|Tx_n - \tilde{T}u_n\| \\ &\le (1-\gamma_n)\|Sx_n - \tilde{S}u_n\| + \gamma_n\|Tx_n - Tu_n^{(1)}\| + \gamma_n\|Tu_n - \tilde{T}u_n\| \\ &\le (1-\gamma_n)\|Sx_n - \tilde{S}u_n\| + \gamma_n\epsilon_1 + \gamma_n\phi(\|Sx_n - Tx_n\|) \\ &+\delta\gamma_n\|Sx_n - \tilde{S}u_n\| + \delta\gamma_n\|Su_n - \tilde{S}u_n\| \\ &\le (1-(1-\delta)\gamma_n)\|Sx_n - \tilde{S}u_n\| + \gamma_n\phi(\|Sx_n - Tx_n\|) + \gamma_n\epsilon_1 + \delta\gamma_n\epsilon_2 \end{aligned} \tag{17}$$

and

$$\begin{aligned} \|Sy_n - \tilde{S}v_n\| &\le (1-\beta_n)\|Tx_n - \tilde{T}u_n\| + \beta_n\|Tz_n - \tilde{T}w_n\| \\ &\le (1-\beta_n)\|Tx_n - Tu_n\| + (1-\beta_n)\|Tu_n - \tilde{T}u_n\| \\ &+\beta_n\|Tz_n - Tw_n\| + \beta_n\|Tw_n - \tilde{T}w_n\| \end{aligned}$$

$$\leq (1-\beta_n)\delta\|Sx_n - Su_n\| + (1-\beta_n)\phi(\|Sx_n - Tx_n\|)$$
$$+\beta_n\delta\|Sz_n - Sw_n\| + \beta_n\phi(\|Sz_n - Tz_n\|) + \epsilon_1$$
$$\leq (1-\beta_n)\delta\|Sx_n - Su_n\| + (1-\beta_n)\phi(\|Sx_n - Tx_n\|)$$
$$+\beta_n\delta\epsilon_2 + \beta_n\phi(\|Sz_n - Tz_n\|) + \epsilon_1 + \beta_n\delta\delta\gamma_n\epsilon_2 + \beta_n\delta\gamma_n\epsilon_1$$
$$+\beta_n\delta(1-(1-\delta)\gamma_n)\left\|Sx_n - \tilde{S}u_n\right\| + \beta_n\delta\gamma_n\phi(\|Sx_n - Tx_n\|)$$
$$\leq \left(1-\beta_n\gamma_n(1-\delta)\right)\left\|Sx_n - \tilde{S}u_n\right\| + (1-\beta_n)\delta\epsilon_2$$
$$+\beta_n\delta\epsilon_2 + \beta_n\phi(\|Sz_n - Tz_n\|) + \epsilon_1 + \beta_n\delta\delta\gamma_n\epsilon_2 + \beta_n\delta\gamma_n\epsilon_1$$
$$+\beta_n\delta\gamma_n\phi(\|Sx_n - Tx_n\|) + (1-\beta_n)\phi(\|Sx_n - Tx_n\|). \tag{18}$$

Using $1-\alpha_n\beta_n\gamma_n \leq \alpha_n\beta_n\gamma_n$ and combining (16), (17) and (18)

$$\left\|Sx_{n+1} - \tilde{S}u_{n+1}\right\| \leq \left(1-\alpha_n\beta_n\gamma_n(1-\delta)\right)\left\|Sx_n - \tilde{S}u_n\right\| + 2\alpha_n\gamma_n\beta_n\phi(\|Sx_n - Tx_n\|)$$
$$+2\alpha_n\gamma_n\beta_n\phi(\|Sy_n - Ty_n\|) + 2\alpha_n\gamma_n\beta_n\phi(\|Sz_n - Tz_n\|) + 4(\epsilon_1 + \epsilon_2)$$

and

$$\|Sz_n - Tz_n\| \leq \|Sz_n - p\| + \|Tz_n - Tz\|$$
$$\leq \|Sz_n - p\| + \delta\|Sz_n - Sz\| + \phi(\|Sz - Tz\|)$$
$$\leq (1+\delta)\|(1-\gamma_n)Sx_n + \gamma_n Tx_n - p\| + \phi(\|Sz - Tz\|)$$
$$\leq (1+\delta)(1-\gamma_n)\|Sx_n - p\| + \gamma_n\|Tx_n - Tz\| + \phi(\|Sz - Tz\|)$$
$$\leq \left((1+\delta)(1-\gamma_n) + \gamma_n\right)\|Sx_n - p\| + (1+\gamma_n)\phi(\|Sz - Tz\|) \tag{19}$$

and

$$\|Sx_n - Tx_n\| \leq \|Sx_n - p\| + \|Tx_n - Tz\|$$
$$\leq (1+\delta)\|Sx_n - p\| + \phi(\|Sz - Tz\|) \tag{20}$$

and

$$\|Sz_n - p\| \leq (1-\gamma_n)\|Sx_n - p\| + \gamma_n\|Tx_n - p\|$$
$$\leq (1-\gamma_n)\|Sx_n - p\| + \gamma_n\delta\|Sx_n - p\| + \gamma_n\phi(\|Sz - Tz\|)$$
$$= [1-\gamma_n(1-\delta)]\|Sx_n - p\| + \gamma_n\phi(\|Sz - Tz\|) \tag{21}$$

and

$$\|Sy_n - Ty_n\| \leq \|Sy_n - p\| + \|Ty_n - Tz\|$$
$$\leq (1+\delta)\|Sy_n - p\| + \phi(\|Sz - Tz\|)$$
$$\leq (1+\delta)\|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| + \phi(\|Sz - Tz\|)$$

$$\leq (1+\delta)(1-\beta_n)\|Tx_n - p\| + (1+\delta)\beta_n\|Tz_n - p\| + \phi(\|Sz - Tz\|)$$

$$\leq (1+\delta)(1-\beta_n)\|Sx_n - p\| + (1+\delta)\beta_n\|Sz_n - p\| + \phi(\|Sz - Tz\|)$$

$$\leq (1+\delta)\big((1-\beta_n) + \beta_n[1-\gamma_n(1-\delta)]\big)\|Sx_n - p\|$$

$$+\gamma_n\phi(\|Sz - Tz\|) + \phi(\|Sz - Tz\|) \tag{22}$$

$$\sigma_n = \|Sx_n - \tilde{S}u_n\|$$

$$\lambda_n = \alpha_n\beta_n\gamma_n(1-\delta)$$

$$\rho_n = \frac{2\phi(\|Sx_n - Tx_n\|) + 2\phi(\|Sy_n - Ty_n\|) + 2\phi(\|Sz_n - Tz_n\|) + 8(\epsilon_1 + \delta\epsilon_2)}{(1-\delta)}$$

Taking the limit in both sides of the (19), (20), (21) and (22) inequalities, it can be seen that $\lim_{n\to\infty}\phi(\|Sx_n - Tx_n\|) = \lim_{n\to\infty}\phi(\|Sy_n - Ty_n\|) = \lim_{n\to\infty}\phi(\|Sz_n - Tz_n\|) = 0$.

Hence an application of Lemma 2.5 and we obtain

$$\|p - \tilde{p}\| = \frac{8\epsilon}{(1-\delta)}.$$

Theorem 3.4. Let $(X, \|.\|)$ be a Banach space, $S, T: Y \to X$ satisfies condition (2), where $T(Y) \subseteq S(Y), S(Y)$ is a complete subset of X and assume that there exist a $z \in C(S,T)$ such that $Tz = Sz = p$. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (5) with $\sum_{n=0}^{\infty}\alpha_n\beta_n\gamma_n = \infty$ and $lim_{n\to\infty}\alpha_n = 0$. Let $\{Su_n\}_{n=0}^{\infty}$ be iterative sequence (4) with $lim_{n\to\infty}\beta_n = 0$. Assume that p is the unique common fixed point of the pair (S,T). Then $\{Sx_n\}_{n=0}^{\infty}$ converges to p faster than $\{Su_n\}_{n=0}^{\infty}$ for $x_0 = u_0 \in Y$.

Proof. From inequality (16) in Theorem 3.1., we have

$$\|Sx_{n+1} - p\| \leq \delta^{n+1}\prod_{i=0}^{n}[1-\alpha_i\beta_i\gamma_i(1-\delta)]\|Sx_0 - p\|. \tag{23}$$

By using iteration method (4) and condition (2), we have

$$\|Sw_n - p\| \leq (1-\gamma_n)\|Su_n - Sz\| + \gamma_n\|Tu_n - Tz\|$$

$$\leq (1-\gamma_n)\|Su_n - Sz\| + \gamma_n\delta\|Su_n - Sz\| + \phi(\|Sz - Tz\|)$$

$$= [1-\gamma_n(1-\delta)]\|Su_n - p\|$$

and

$$\|Sv_n - p\| \leq (1-\beta_n)\|Tu_n - p\| + \beta_n\|Tw_n - p\|$$

$$\leq (1-\beta_n)\delta\|Su_n - Sz\| + (1-\beta_n)\,\phi(\|Sz - Tz\|)$$

$$+\beta_n\delta\|Sw_n - Sz\| + \beta_n\phi(\|Sz - Tz\|)$$

$$\leq \delta[(1-\beta_n) + \beta_n - \beta_n\gamma_n(1-\delta)]\|Su_n - p\|$$

$$\leq \delta[1-\beta_n\gamma_n(1-\delta)]\|Su_n - p\|$$

and

$$\|Su_{n+1} - p\| \geq (1-\alpha_n)\|Sv_n - p\| - \alpha_n\|Tw_n - p\|$$

$$\geq (1-\alpha_n)\|Sv_n - Sz\| + \phi(\|Sz - Tz\|) - \alpha_n\delta\|Sw_n - Sz\| - \phi(\|Sz - Tz\|)$$

$$\geq [1-\alpha_n - \beta_n\gamma_n(1-\delta) + \alpha_n\beta_n\gamma_n(1-\delta) - \delta\alpha_n + \alpha_n\gamma_n(1-\delta)]\|Su_n - p\|$$

$$\geq [1-\alpha_n(1-\delta) - \beta_n\gamma_n(1-\delta)]\|Su_n - p\|$$

$$\geq [1-(1-\delta)(\alpha_n + \beta_n\gamma_n)]\|Su_n - p\|$$

Also, we have

$$\left\|\frac{Sx_{n+1} - p}{Su_{n+1} - p}\right\| \leq \delta^{(n+1)} \frac{\prod_{i=0}^{n}[1-\alpha_i\beta_i\gamma_i(1-\delta)]\|Sx_0 - p\|}{\prod_{i=0}^{n}[1-(\alpha_i + \beta_i\gamma_i)(1-\delta)]\|Su_0 - p\|}$$

Define

$$\phi_n = \delta^{(n+1)} \frac{\prod_{i=0}^{n}[1-\alpha_i\beta_i\gamma_i(1-\delta)]\|Sx_0 - p\|}{\prod_{i=0}^{n}[1-(\alpha_i + \beta_i\gamma_i)(1-\delta)]\|Su_0 - p\|}.$$

We get $\lim_{n\to\infty} \frac{\phi_{n+1}}{\phi_n} < 1$ That is $\lim_{n\to\infty} \phi_n = 0$ which implies that $\{Sx_n\}_{n=0}^{\infty}$ converges to p faster than $\{Su_n\}_{n=0}^{\infty}$.

Example 3.5. Let $Y = [0,1] \subset \mathbb{R}$ be endowed with usual metric. Define operators $T, S: [0,1] \to [0,17]$ with a coincidence point $p = 17$ by $Tx = 24x - 2x^5 - 5$ and $Sx = 17x^2$. It is clear that $T([0,1]) \subseteq S([0,1])$ and $S([0,1])$ is a complete subset of $[0,17]$. Let $\delta \in (0,1)$, $x_0 = 0.6$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{3}$. The convergence result for two Jungck-type iteration methods (5), and (4) to $p = 17 = S1 = T1$ are listed in the following tables:

Table 1: Convergence of iteration Method (5) with initial point $x_0 = 0.6$

Iteration Steps	Iteration Method (5)	Sx_n	Tx_n
1.	0,60000000000000	6,12000000000000	9,24448000000000
2.	0,83405675327201	11,82606135053700	14,21011215711840
3.	0,95226046870888	15,41560000451620	16,28818988746500
⋮	⋮	⋮	⋮
12.	0,99999994528445	16,99999813967140	16,99999923398220
⋮	⋮	⋮	⋮
22.	0,99999999999999	16,99999999999970	16,99999999999990
23.	1,00000000000000	17,00000000000000	17,00000000000000
⋮	⋮	⋮	⋮
33.	1,00000000000000	17,00000000000000	17,00000000000000
34.	1,00000000000000	17,00000000000000	17,00000000000000

Table 2: Convergence of iteration Method (4) with initial point $x_0 = 0.6$

Iteration Steps	Iteration Method (4)	Sx_n	Tx_n
1.	0,60000000000000	6,12000000000000	9,24448000000000
2.	0,76145692681088	9,85688307360059	12,76298187191080
3.	0,88034295744818	13,17506328638630	15,07070884153580
⋮	⋮	⋮	⋮
12.	0,99998051894755	16,99933765066840	16,99972725767560
⋮	⋮	⋮	⋮
22.	0,99999999913855	16,99999997071070	16,99999998793970
23.	0,99999999968392	16,99999998925330	16,99999999557490
⋮	⋮	⋮	⋮
33.	0,99999999999999	16,9999999999970	16,9999999999990
34.	1,00000000000000	17,00000000000000	17,00000000000000

4. Discussion and Conclusion

It is showed that in Theorem 3.1, the newly defined iteration has convergence. In Theorem 3.2, it is indicated that stability results can be obtained by using the new iteration method (5) for certain mapping classes. Moreover, Theorem 3.3 has mentioned results of data dependence. In Theorem 3.4, the rate of convergence for two iterations are compared. Applications provided from Theorem 3.4 that is shown in Table 1 and Table 2, has investigated the new iteration method which is faster than the other in the literature. This research also contributes to the literature.

References

[1] Mann, W.R. 1953. Mean value methods in iteration. Proceedings of the American Mathematical Society, 4, 506–510.

[2] Ishikawa, S. 1974. Fixed points by a new iteration method. Proceedings of the American Mathematical Society, 44(1), 147–150.

[3] Noor, M.A. 2000. New approximation schemes for general variational inequalities. Journal of Mathematical Analysis and Applications, 251(1), 217–229.

[4] Gürsoy F, Karakaya V, Rhoades B.E. 2013. Data dependence results of new multistep and S-iterative schemes for contractive-like operators. Fixed Point Theory and Applications, 2013: 1-12.

[5] Dogan, K. 2019. A Comparative Study On Some Recent Iterative Schemes. Journal Of Nonlinear And Convex Analysis, 20(11), 2411-2423.

[6] Atalan, Y. 2019. On a New Fixed Point Iterative Algorithm For General Variational Inequalities. Journal Of Nonlınear And Convex Analysis, 20 (11), 2371-2386.

[7] Jungck, G. 1976. Commuting mappings and fixed points. American Mathematical Monthly, 83(4), 261-263.

[8] Razani, A., Bagherboum, M. 2013. Convergence and stability of Jungck-type iterative procedures in convex b-metric spaces. Fixed Point Theory and Applications, 2013(1), 1-17.

[9] Okeke, G. A., & Kim, J. K. 2016. Convergence and (S, T)-stability almost surely for random Jungck-type iteration processes with applications. Cogent Mathematics, 3(1), 1258768.

[10] Khan, A. R., Gürsoy, F., Karakaya, V., 2016. Jungck-Khan iterative scheme and higher convergence rate. International Journal of Computer Mathematics, 93(12), 1029-2105.

[11] Singh, S.L., Bhatnagar, C. Mishra, S.N. 2005. Stability of Jungck-type iterative procedures. International Journal of Mathematics and Mathematical Sciences, 19, 3035–3043.

[12] Olatinwo, M.O. 2008. Some stability and strong convergence results for the Jungck-Ishikawa iteration process. Creative Mathematics and Informatics, 17, 33–42.

[13] Olatinwo, M.O. 2008. A generalization of some convergence results using a Jungck-Noor three-step iteration process in arbitrary Banach space. Polytechnica Posnaniensis, 40, 37–43.

[14] Chugh, R., Kumar, V. 2011. Strong Convergence and Stability results for Jungck-SP iterative scheme. International Journal of Computer Applications, 36(12), 40-46.

[15] Hussain, N., Kumar, V. Kutbi, M. A. 2013. On rate of convergence of Jungck-type iterative schemes, Abstract and Applied Analysis, 2013, 1-15.

[16] Agarwal R. P., O'Regan D., Sahu D. R., 2007, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, Journal of Nonlinear and Convex Analysis, vol. 8, no. 1, pp. 61–79.

[17] Khan, A. R., Kumar, V., Hussain, N. 2014. Analytical and numerical treatment of Jungck-type iterative schemes. Applied Mathematics and Computation, 231, 521-535.

[18] Akewe, H. 2016. The stability of a modified Jungck-Mann hybrid fixed point iteration procedure. International Journal of Mathematical Analysis and Optimization: Theory and Applications, (2016), 95-104.

[19] Atalan, Y., Karakaya, V. 2019. Obtaining New Fixed Point Theorems By Using Generalized Banach-Contraction Principle. Journal of Institue Of Science and Technology, 35(3). 34-45.

[20] Jungck, G., Hussain, N. 2007. Compatible Maps and Invariant approximations. J. Math. Anal. Appl., 325 (2), 1003-1012.

[21] Weng X, 1991. Fixed Point Iteration for Local Strictly Pseudo-Contractive Mapping. Proceedings of the American Mathematical Society, 113 (3): 727-731.

[22] Soltuz S.M, Grosan T. 2008. Data dependence for Ishikawa iteration when dealing with contractive like operators. Fixed Point Theory. (2008), 1-7.

[23] Knopp K, 1931. Theory and Application of Infinite Series. Berlin.