SAKARYA ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ

Sakarya University Journal of Science
SAUJS

e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University |
http://www.saujs.sakarya.edu.tr/en/

Title: The Relation Between Chebyshev Polynomials and Jacobsthal and Jacobsthal Lucas Sequences

Authors: Şükran UYGUN
Recieved: 2020-02-11 12:18:20

Accepted: 2020-08-29 10:08:44

Article Type: Research Article

Volume: 24
Issue: 6
Month: December
Year: 2020
Pages: 1162-1170

How to cite
Şükran UYGUN; (2020), The Relation Between Chebyshev Polynomials and Jacobsthal and Jacobsthal Lucas Sequences . Sakarya University Journal of Science, 24(6), 1162-1170, DOI: https://doi.org/10.16984/saufenbilder.687708
Access link
http://www.saujs.sakarya.edu.tr/en/pub/issue/57766/687708

Sakarya University Journal of Science 24(6), 1162-1170, 2020

The Relation Between Chebyshev Polynomials and Jacobsthal and Jacobsthal Lucas Sequences

Şükran UYGUN*[1]

Abstract

In this paper Jacobsthal, Jacobsthal Lucas and generalized Jacobsthal sequences are denoted by aid of first or second type of Chebyshev polynomials by different equalities. Then using these equalities a relation is obtained between Jacobsthal and generalized Jacobsthal numbers. Moreever, the nth powers of some special matrices are found by using Jacobsthal numbers or Chebyshev polynomials. Some connections among Jacobsthal, Jacobsthal Lucas are revealed by using the determinant of the power of some special matrices. Then, the properties of Jacobsthal, Jacobsthal Lucas numbers are obtained by using the identities of Chebyshev polynomials.

Keywords: Chebhshev polynomials, Jacobsthal Sequences, Jacobsthal Lucas Sequences

1. INTRODUCTION

For any $n \geq 2$ integers, $a, b; p, q$ are integers, Horadam sequence was defined by Horadam in 1965, denoted by $\{W_n\}_{n>0}$, by the following recursive relation

$$W_n = W_n(a, b; p, q) = pW_{n-1} - qW_{n-2},$$

$$W_0 = a, \ \ W_1 = b.$$

where $p^2 - 4q \neq 0$. For special choices of $a, b; p, q,$ special integer sequences are obtained. For example,

$W_n(0,1; 1, -1) = F_n$ classic Fibonacci sequence

$W_n(2,1; 1, -1) = L_n$ classic Lucas sequence

$W_n(0,1; p, -q) = \widetilde{F_n}$ generalized Fibonacci sequence

$W_n(0,1; 1, -2) = j_n$ classic Jacobsthal sequence

$W_n(a, b; 1, -2) = J_n$ generalized Jacobsthal sequence

$W_n(2,1; 1, -2) = c_n$ classic Jacobsthal Lucas sequence

$W_n(0,1; 2, -1) = P_n$ classic Pell sequence

* Corresponding Author: suygun@gantep.edu.tr
[1] Gaziantep University, Department of Mathematics, ORCID: https://orcid.org/0000-0002-7878-2175

$W_n(2,2;2,-1) = Q_n$ classic Pell Lucas sequence

$W_n(1,x;2x,1) = T_n$ first kind Chebyshev polynomials

$W_n(1,2x;2x,1) = U_n$ second kind Chebyshev polynomials.

The humankind encountered special integer sequences with Fibonacci in 1202. The importance of Fibonacci sequence was not understood in that century. But now, because of applications of special sequences, there are too many studies on it. For example, the Golden Ratio, the ratio of two consecutive Fibonacci numbers is used in Physics, Art, Architecture, Engineering. We can also encounter Golden Ratio so many areas in nature, human body. Horadam sequence is very important since we can obtain almost most of other special integer sequences by using Horadam sequence. Horadam sequence was studied by Horadam, Carlitz, Riordan and other some mathematicians. Horadam intended to write the first paper which contains the properties of Horadam sequences in [1,2]. In 1969, the relations between Chebyshev functions and Horadam sequences were investigated in [3]. In [6], Udrea found important relations with Horadam sequence and Chebyshev polynomials. In [7], Mansour found a formula for the generating functions of powers of Horadam sequence. Horzum and Koçer studied the properties of Horadam polynomial sequences in [8]. The authors established identities involving sums of products of binomial coefficients that satisfy the general second order linear recurrence in [9]. In [10], the authors obtained Horadam numbers with positive and negative indices by using determinants of some special tridiagonal matrices. In [11], the authors established formulas for odd and even sums of generalized Fibonacci numbers by matrix methods. In [12], some properties of the generalized Fibonacci sequence were obtained by matrix methods. One of important special integer sequences is Jacobsthal sequence because of its application in computer science. In [4,13,14,15], you can find some properties and generalizations of Jacobsthal sequence.

2. MAIN RESULTS

Definition 1. Let $n \geq 2$ integer, the Jacobsthal $\{j_n\}_{n>0}$, the Jacobsthal Lucas $\{c_n\}_{n>0}$ and generalized Jacobsthal $\{J_n\}_{n>0}$ sequences are defined by

$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 0, \quad j_1 = 1,$$

$$c_n = c_{n-1} + 2c_{n-2}, \quad c_0 = 2, \quad c_1 = 1,$$

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = a, \quad J_1 = b.$$

respectively.

Definition 2. Let $n \geq 2$ integer, the first kind $\{T_n\}_{n>0}$, and second kind $\{U_n\}_{n>0}$, Chebyshev polynomial sequences are defined by the following recurrence relations

$$T_n = 2xT_{n-1} - T_{n-2}, \quad T_0 = 1, \quad T_1 = x,$$

$$U_n = 2xU_{n-1} - U_{n-2}, \quad U_0 = 1, \quad U_1 = 2x,$$

respectively.

The Binet formula for the Horadam sequence is given by

$$W_n = \frac{Xr_1^n - Yr_2^n}{r_1 - r_2},$$

where $X = b - ar_2$, $Y = b - ar_1$; r_1, r_2 being the roots of the associated characteristic equation of the Horadam sequence $\{W_n\}_{n>0}$. It is obtained the quadratic characteristic equation for $\{W_n\}_{n>0}$, as $r^2 - pr + q = 0$, with roots r_1, r_2 defined by

$$r_1 = \frac{p+\sqrt{p^2-4q}}{2}, \ r_2 = \frac{p-\sqrt{p^2-4q}}{2}.$$

The summation, difference and product of the roots are given as

$$r_1 + r_2 = p, \quad r_1 - r_2 = \sqrt{p^2 - 4q}, \quad r_1 r_2 = q.$$

The Binet formulas for the Jacobsthal, Jacobsthal Lucas and generalized Jacobsthal sequences are given by respectively

$$j_n = \frac{2^n - (-1)^n}{3},$$

$c_n = 2^n + (-1)^n$,

$J_n = \frac{A2^n - B(-1)^n}{3}$,

where $A = b + a$, $B = b - 2a$.

We define $E_w = -XY = pab - qa^2 - b^2$ for Horadam sequence $\{W_n\}_{n>0}$. Similarly, for the first kind Chebyshev polynomials $E_T = -XY = pab - qa^2 - b^2 = x^2 - 1$, and for the second kind Chebyshev polynomials $E_U = -1$. For the Jacobsthal, Jacobsthal Lucas, generalized Jacobsthal numbers, we have $E_J = ab + 2a^2 - b^2$, $E_j = -1$, $E_c = 9$.

Chebyshev polynomials are also defined by

$$T_n(cos\varphi) = cosn\varphi,\ \ U_n(cos\varphi) = \frac{sinn\varphi}{sin\varphi},$$

$n \in Z^+, sin\varphi \neq 0$.

Proposition 3. Jacobsthal, Jacobsthal Lucas, generalized Jacobsthal numbers are obtained by using Chebyshev polynomials as

$$j_n = (2i^2)^{\frac{n-1}{2}} U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right),$$

$$c_n = 2(2i^2)^{\frac{n}{2}} T_n\left(\frac{1}{2\sqrt{2}i}\right),$$

$$\begin{aligned} J_n &\\ = a(2i^2)^{\frac{n}{2}} T_n\left(\frac{1}{2\sqrt{2}i}\right) &\\ + \frac{(2b-a)(2i^2)^{\frac{n-1}{2}}}{2} U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right) &\\ = (2i^2)^{\frac{n}{2}}\left[\frac{b}{\sqrt{2}i} U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right) - aU_{n-2}\left(\frac{1}{2\sqrt{2}i}\right)\right]. & \end{aligned}$$

Proof: The roots of characteristic equation for Horadam sequence are $r_1 = \frac{p+\sqrt{p^2-4q}}{2}$, $r_2 = \frac{p-\sqrt{p^2-4q}}{2}$ are demonstrated by

$$\begin{aligned} r_1, r_2 &= \sqrt{q}\left(\frac{p}{2\sqrt{q}} \pm \sqrt{\left(\frac{p}{2\sqrt{q}}\right)^2 - 1}\right) \\ &= \sqrt{q}(cos\theta \pm isin\theta) \end{aligned}$$

where $cos\theta = \frac{p}{2\sqrt{q}}$. By De Moivre formula it is obtained that

$$r_1^n = q^{\frac{n}{2}}(cosn\theta + isinn\theta),$$

$$r_2^n = q^{\frac{n}{2}}(cosn\theta - isinn\theta).$$

We know that for *p=1, q=-2,* Horadam sequence turns out Jacobsthal and Jacobsthal Lucas sequences. Hence

$$W_n(0,1;1,-2) = j_n = \frac{r_1^n - r_2^n}{r_1 - r_2}$$

$$\begin{aligned} &= \frac{q^{\frac{n}{2}}[(cos\theta + isin\theta)^n - (cos\theta - isin\theta)^n]}{2\sqrt{q}isin\theta} \\ &= \sqrt{2i^2}^{n-1}\frac{sinn\theta}{sin\theta} \\ &= \sqrt{2i^2}^{n-1} U_{n-1}(cos\theta) \\ &= \sqrt{2i^2}^{n-1} U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right). \end{aligned}$$

$$W_n(2,1;1,-2) = c_n = r_1^n + r_2^n$$

$$\begin{aligned} &= \sqrt{2i^2}^n 2cosn\theta = 2\sqrt{2i^2}^n T_n(cos\theta) \\ &= 2\sqrt{2i^2}^n T_n\left(\frac{1}{2\sqrt{2}i}\right). \end{aligned}$$

By $A = b + a$, $B = b - 2a$, we have

$W_n(a,b;1,-2) = J_n = \frac{Ar_1^n - Br_2^n}{r_1 - r_2} == \frac{q^{\frac{n}{2}}[A(cos\theta+isin\theta)^n - B(cos\theta-isin\theta)^n]}{2\sqrt{q}isin\theta} = \frac{q^{\frac{n}{2}}[cosn\theta(A-B)+isinn\theta(A+B)]}{2\sqrt{q}isin\theta} = \frac{q^{\frac{n}{2}}cosn\theta(A-B)}{3} + \frac{q^{\frac{n-1}{2}}sinn\theta(A+B)}{2sin\theta} = a(2i^2)^{\frac{n}{2}}T_n(cos\theta) + \frac{(2b-a)(2i^2)^{\frac{n-1}{2}}U_{n-1}(cos\theta)}{2} = a(2i^2)^{\frac{n}{2}}T_n\left(\frac{1}{2\sqrt{2}i}\right) + \frac{(2b-a)(2i^2)^{\frac{n-1}{2}}}{2}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right).$

By using the well- known property of Chebyshev polynomials as $T_n(x) = xU_{n-1}(x) - U_{n-2}(x)$, it is easily seen that

$$J_n = (2i^2)^{\frac{n}{2}}\left[a\frac{1}{2\sqrt{2}i}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right) - aU_{n-2}\left(\frac{1}{2\sqrt{2}i}\right) + \frac{2b-a}{2\sqrt{2}i}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right)\right] = (2i^2)^{\frac{n}{2}}\left[\frac{b}{\sqrt{2}i}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right) - aU_{n-2}\left(\frac{1}{2\sqrt{2}i}\right)\right].$$

Corollary 4. Jacobsthal, Jacobsthal Lucas, generalized Jacobsthal numbers can also be demonstrated by using Chebyshev polynomials as

$$j_{n+1} = 2^{\frac{n}{2}}i^{3n}U_n\left(\frac{i}{2\sqrt{2}}\right),$$

$$c_n = 2^{\frac{n+2}{2}}i^{3n}T_n\left(\frac{i}{2\sqrt{2}}\right),$$

$$J_n = i^n\left[\frac{a2^{\frac{n}{2}}}{3}T_n\left(\frac{-i}{2\sqrt{2}}\right) + \frac{(2b-a)2^{\frac{n-1}{2}}}{2i}U_{n-1}\left(\frac{-i}{2\sqrt{2}}\right)\right].$$

Proof:

$$j_n = (2i^2)^{\frac{n-1}{2}}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right) = 2^{\frac{n-1}{2}}i^{n-1}U_{n-1}\left(\frac{-i}{2\sqrt{2}}\right) = 2^{\frac{n-1}{2}}i^{3n-3}U_{n-1}\left(\frac{i}{2\sqrt{2}}\right)$$

$$c_n = (2i^2)^{\frac{n}{2}}T_n\left(\frac{1}{2\sqrt{2}i}\right) = 2^{\frac{n+2}{2}}i^nT_n\left(\frac{-i}{2\sqrt{2}}\right) = 2^{\frac{n+2}{2}}i^{3n}T_n\left(\frac{i}{2\sqrt{2}}\right)$$

$$J_n = i^n\left[\frac{a2^{\frac{n}{2}}}{3}T_n\left(\frac{1}{2\sqrt{2}i}\right) + \frac{(2b-a)2^{\frac{n-1}{2}}}{2i}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right)\right] = i^n\left[\frac{a2^{\frac{n}{2}}(-1)^n}{3}T_n\left(\frac{i}{2\sqrt{2}}\right) + \frac{(2b-a)2^{\frac{n-1}{2}}(-1)^n}{2i}U_{n-1}\left(\frac{i}{2\sqrt{2}}\right)\right] = i^n\left[\frac{a2^{\frac{n}{2}}}{3}T_n\left(\frac{-i}{2\sqrt{2}}\right) + \frac{(2b-a)2^{\frac{n-1}{2}}}{2i}U_{n-1}\left(\frac{-i}{2\sqrt{2}}\right)\right].$$

Theorem 5. Generalized Jacobsthal numbers are denoted by using the first kind Chebyshev polynomials as

$$J_n = \frac{2\sqrt{E_J}(2i^2)^{\frac{n}{2}}}{3}T_n\left(\cos\left(\theta - \frac{\varphi}{n}\right)\right),$$

where $\cos\varphi = \frac{X-Y}{2\sqrt{E}.}$.

Proof: It is easily seen that $\sqrt{(X-Y)^2 + (i(X+Y))^2} = 2\sqrt{E_J}.$ By using this equality and the third part of the proof of Proposition 3, it is obtained that

$$J_n = \frac{(2i^2)^{\frac{n}{2}}}{r_1 - r_2}.[(X-Y)\cos n\theta + i(X+Y)\sin n\theta] = \frac{(2i^2)^{\frac{n}{2}}}{3}.\left[\frac{(X-Y)\cos n\theta}{\sqrt{(X-Y)^2 + \left(i(X+Y)\right)^2}} + \frac{i(X+Y)\sin n\theta}{\sqrt{(X-Y)^2 + \left(i(X+Y)\right)^2}}\right] \cdot \sqrt{(X-Y)^2 + \left(i(X+Y)\right)^2}$$

$$= \frac{(2i^2)^{\frac{n}{2}} 2\sqrt{E}}{3} . \left[\frac{(X-Y)cosn\theta}{2\sqrt{E}} + \frac{i(X+Y)sinn\theta}{2\sqrt{E}} \right] = \frac{(2i^2)^{\frac{n}{2}} 2\sqrt{E}}{3} . [cos\varphi cosn\theta + i(X+Y)sin\varphi sinn\theta] = \frac{(2i^2)^{\frac{n}{2}} 2\sqrt{E}}{3} \cos(n\theta - \varphi).$$

Theorem 6. Let $n, r, s \in N$. The following relation between generalized Jacobsthal numbers and Jacobsthal numbers is satisfied

$$J_n J_{n+r+s} - J_{n+r} J_{n+s} = -(-2)^{2n} j_r j_s.$$

Proof: By using Theorem 5, it is obtained that

$$J_n J_{n+r+s} = \frac{2\sqrt{E_J}(2i^2)^{\frac{n}{2}}}{3} \cos(n\theta - \varphi) \frac{2\sqrt{E_J}(2i^2)^{\frac{n+r+s}{2}}}{3} \cos((n+r+s)\theta - \varphi)$$

$$J_{n+r} J_{n+s} = \frac{4E_J(2i^2)^{\frac{2n+r+s}{2}}}{9} \cos((n+r)\theta - \varphi) \cos((n+s)\theta - \varphi)$$

By substracting the equalities,

$$J_n J_{n+r+s} - J_{n+r} J_{n+s} = \frac{4E_J(2i^2)^{\frac{2n+r+s}{2}}}{9} .$$

$$\left[\frac{cos\big((2n+r+s)\theta - 2\varphi\big) + \cos(r+s)\theta - \cos((2n+r+s)\theta - 2\varphi) - \cos(r-s)\theta}{2} \right]$$

$$= \frac{4E_J(2i^2)^{\frac{2n+r+s}{2}}}{9} \left[\frac{\cos(r+s)\theta - \cos(r-s)\theta}{2} \right] = -\frac{4E_J(2i^2)^{\frac{2n+r+s}{2}}}{9} [\sin r\,\theta \sin s\theta] = -sin^2\theta \frac{4E_J(2i^2)^{\frac{2n+r+s}{2}}}{9} \left[\frac{\sin r\,\theta \sin s\theta}{\sin\theta \sin\theta} \right]$$

$$= -sin^2\theta \frac{4E_J(2i^2)^{\frac{2n+r+s}{2}}}{9} \frac{\sin r\,\theta \sin s\theta}{\sin\theta \sin\theta} = = (cos^2\theta - 1) \frac{4E_J(2i^2)^{\frac{2n+r+s}{2}}}{9} U_{r-1}(cos\theta) U_{s-1}(cos\theta) = \left[\left(\frac{1}{2\sqrt{2}i} \right)^2 - 1 \right] \frac{4E_J(2i^2)^{\frac{2n+r+s}{2}}}{9} U_{r-1}\left(\frac{1}{2\sqrt{2}i} \right) U_{s-1}\left(\frac{1}{2\sqrt{2}i} \right) = E_J(2i^2)^{\frac{2n+r+s-2}{2}} U_{r-1}\left(\frac{1}{2\sqrt{2}i} \right) U_{s-1}\left(\frac{1}{2\sqrt{2}i} \right).$$

For the other side of the equality, from Proposition 3, it is obtained that

$$E_J(2i^2)^{2n} j_r j_s = E_J(2i^2)^{2n} (2i^2)^{\frac{r-1}{2}} U_{r-1}\left(\frac{1}{2\sqrt{2}i} \right) \cdot (2i^2)^{\frac{s-1}{2}} U_{s-1}\left(\frac{1}{2\sqrt{2}i} \right).$$

The equality of the results is proved the theorem.

The applications of Theorem 6 for Jacobsthal sequence

$$j_n j_{n+r+s} - j_{n+r} j_{n+s} = -(-2)^{\frac{2n+r+s-2}{2}} U_{r-1}\left(\frac{1}{2\sqrt{2}i} \right) U_{s-1}\left(\frac{1}{2\sqrt{2}i} \right) = -(-)^{2n} j_r j_s$$

The applications of theorem for Jacobsthal Lucas sequence

$$c_n c_{n+r+s} - c_{n+r} c_{n+s} = 9(-2)^{\frac{2n+r+s-2}{2}} U_{r-1}\left(\frac{1}{2\sqrt{2}i} \right) U_{s-1}\left(\frac{1}{2\sqrt{2}i} \right) = 9(-)^{2n} j_r j_s$$

Lemma 7. It is well-known that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(C), \text{ then}$$

$$A^n = \begin{cases} \dfrac{x_1^n - x_2^n}{x_1 - x_2} A - \det(A) \dfrac{x_1^{n-1} - x_2^{n-1}}{x_1 - x_2} I_2, & x_1 \neq x_2 \\ n x_1^{n-1} A - (n-1)\det(A) x_1^{n-2} I_2, & x_1 = x_2 \end{cases}$$

x_1 and x_2 being the roots of the associated characteristic equation of the matrix A:

$$x^2-(a+d)x+\det(A)=0.$$

Corollary 8. If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in M_2(C)$ whose trace is $a+d=1$ and determinant is $\det(A)=-2$, then

$$A^n=j_nA-j_{n-1}I_2 .$$

Proof: We know that the quadratic characteristic equation for the Jacobsthal sequence is r^2-r-$2=0$ with roots $x_1=2,\ x_2=-1$. If a $2x2$ square matrix is chosen whose trace is $a+d=1$ and determinant is $\det(A)=-2$, then we will get

$$\begin{aligned} A^n=\frac{x_1^n-x_2^n}{x_1-x_2}A-\det(A)\frac{x_1^{n-1}-x_2^{n-1}}{x_1-x_2}I_2 \\ =j_nA-j_{n-1}I_2 \end{aligned}$$

Because the determinant of the matrix is equal the product of the eigenvalues of the matrix. The trace is equal the sum of the eigenvalues.

Theorem 9. If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in M_2(C)$ whose trace is $a+d=1$ and determinant is $\det(A)=-2$, then another relation with Jacobsthal sequence and Chebyshev polynomials is established by using the matrix of A as

$$\begin{aligned} A^n=(2i^2)^{\frac{n-1}{2}}\Big[U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right)A \\ -\frac{1}{\sqrt{2}i}U_{n-2}\left(\frac{1}{2\sqrt{2}i}\right)I_2\Big] \end{aligned}$$

$$\begin{aligned} A^n=(2i^2)^{\frac{n-1}{2}}\Big[U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right)\left(A-\frac{1}{\sqrt{2}i}I_2\right) \\ +\frac{1}{\sqrt{2}i}T_n\left(\frac{1}{2\sqrt{2}i}\right)I_2\Big]. \end{aligned}$$

Proof: We know that

$$\begin{aligned} W_n(0,1;1,-2)=j_n=\frac{r_1^n-r_2^n}{r_1-r_2} \\ =(2i^2)^{\frac{n-1}{2}}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right). \end{aligned}$$

By Corollary 8,

$$A^n=j_nA-j_{n-1}I_2=(2i^2)^{\frac{n-1}{2}}\Big[U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right)A-\frac{1}{\sqrt{2}i}U_{n-2}\left(\frac{1}{2\sqrt{2}i}\right)I_2\Big].$$

By using the property between Chebyshev polynomials

$T_n(x)=xU_{n-1}(x)-U_{n-2}(x)$, it is obtained that

$$\begin{aligned} A^n=(2i^2)^{\frac{n-1}{2}}\Big[U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right)A \\ -\frac{1}{\sqrt{2}i}\Big(\frac{1}{2\sqrt{2}i}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right)I_2 \\ -T_n\left(\frac{1}{2\sqrt{2}i}\right)I_2\Big)\Big] \end{aligned}$$

$$\begin{aligned} A^n=(2i^2)^{\frac{n-1}{2}}\Big[U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right)\left(A-\frac{1}{\sqrt{2}i}I_2\right) \\ +\frac{1}{\sqrt{2}i}T_n\left(\frac{1}{2\sqrt{2}i}\right)I_2\Big]. \end{aligned}$$

Example 10. Let $A=\begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}$, then

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}^n=j_n\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}-j_{n-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}= \begin{bmatrix} \frac{j_n}{2}-j_{n-1} & \frac{3j_n}{2} \\ \frac{3j_n}{2} & \frac{j_n}{2}-j_{n-1} \end{bmatrix}$$

By the equality of the determinant of matrices, we get a property of Jacobsthal sequence

$$(-2)^n=-2j_n^2+j_{n-1}^2-j_nj_{n-1}.$$

Example 11. Let $A=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, then

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n=j_n\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}-j_{n-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ =\begin{bmatrix} -j_{n-1} & j_n \\ 2j_n & 2j_{n-2} \end{bmatrix}. \end{aligned}$$

By the equality of the determinant of matrices, we get

$$(-2)^{n-1}=j_n^2+j_{n-2}j_{n-1}.$$

Example 12. Let $A=\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}$, then

$$\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}^n = j_n\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix} - j_{n-1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}2j_{n-2} & j_n\\ 2j_n & -j_{n-1}\end{bmatrix}.$$

By the equality of the determinant of matrices, we get the same result with the previous example.

Theorem 13. By using the properties of Chebyshev polynomials in [16], we get some properties of Jacobsthal and Jacobsthal Lucas sequences as

a) $c_{m+n}+(-2)^n c_{m-n}=c_m c_n$,

b) $j_{n+1}j_{n+2r+1}+(-2)^{n+1}j_r^2=j_{n+r+1}^2$

c) $c_n c_{n+2r}=(-2)^r[2c_{n+r}^2-9j_r^2]$,

d) $\frac{j_{nk}}{j_k(-2)^{k(n-1)/2}}=\frac{sinn\left(cos^{-1}\left(\frac{c_n}{(-2)^{n/2}}\right)\right)}{sin\left(cos^{-1}\left(\frac{c_n}{(-2)^{n/2}}\right)\right)}$,

e) $c_n^2=2(-2)^n+c_{2n}$,

f) $c_n^2-c_{n-1}c_{n+1}=-9(-2)^{n-1}$,

g) $c_n^2-9j_r^2=(-2)^{n+2}$.

Proof: a) Let $x=\frac{1}{2\sqrt{2}i}$. By using this property

$$j_n=(2i^2)^{\frac{n-1}{2}}U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right),$$

$$c_n=2(2i^2)^{\frac{n}{2}}T_n\left(\frac{1}{2\sqrt{2}i}\right),$$

it is obtained that

$$T_{m+n}+T_{m-n}=2T_mT_n$$

$$\frac{c_{m+n}}{2(2i^2)^{\frac{m+n}{2}}}+\frac{c_{m-n}}{2(2i^2)^{\frac{m-n}{2}}}=2\frac{c_m}{2(2i^2)^{\frac{m}{2}}}\frac{c_n}{2(2i^2)^{\frac{n}{2}}}$$

$$\frac{(2i^2)^{\frac{m-n}{2}}c_{m+n}+(2i^2)^{\frac{m+n}{2}}c_{m-n}}{(2i^2)^m}=\frac{c_mc_n}{(2i^2)^{\frac{m+n}{2}}}$$

$$c_{m+n}+(-2)^n c_{m-n}=c_mc_n$$

b) Similarly

$$U_nU_{n+2r}+U_{r-1}^2=U_{n+r}^2$$

$$\frac{j_{n+1}}{(-2)^{n/2}}\frac{j_{n+2r+1}}{(-2)^{(n+2r)/2}}+\frac{j_r^2}{(-2)^{(r-1)}}=\frac{j_{n+r+1}^2}{(-2)^{n+r}}$$

$$j_{n+1}j_{n+2r+1}+(-2)^{n+1}j_r^2=j_{n+r+1}^2$$

c) $T_nT_{n+2r}-(x^2-1)U_{r-1}^2=T_{n+r}^2$

$$T_n\left(\frac{1}{2\sqrt{2}i}\right)T_{n+2r}\left(\frac{1}{2\sqrt{2}i}\right)-(\frac{-1}{8}-1)U_{r-1}^2\left(\frac{1}{2\sqrt{2}i}\right)=T_{n+r}^2\left(\frac{1}{2\sqrt{2}i}\right)$$

$$\frac{c_nc_{n+2r}}{2(2i^2)^{\frac{n}{2}}2(2i^2)^{\frac{n+2r}{2}}}+\frac{9j_r^2}{8(2i^2)^{n-1}}=\frac{c_{n+r}^2}{2(2i^2)^n}$$

$$\frac{c_nc_{n+2r}}{4(2i^2)^r}+\frac{9j_r^2}{4}=\frac{c_{n+r}^2}{2}$$

d) By using this property $U_{n-1}(T_k(x))=\frac{U_{nk-1}(x)}{U_{k-1}(x)}$ and the equality of the results we prove the statement. For the first part of the equality, we get

$$U_{n-1}(T_k(x))=U_{n-1}\left(\frac{c_n}{2(-2)^{n/2}}\right)=\frac{sinn\left(cos^{-1}\left(\frac{c_n}{(-2)^{n/2}}\right)\right)}{sin\left(cos^{-1}\left(\frac{c_n}{(-2)^{n/2}}\right)\right)}$$

And for the second part of the equality, we get

$$\frac{U_{nk-1}(x)}{U_{k-1}(x)}=\frac{\left(\frac{j_{nk}}{(-2)^{\frac{nk-1}{2}}}\right)}{\left(\frac{j_k}{(-2)^{\frac{k-1}{2}}}\right)}=\frac{j_{nk}}{j_k(-2)^{\frac{k(n-1)}{2}}}.$$

e) $2T_n^2=1+T_{2n}$

$$2\left(\frac{c_n}{2(-2)^{n/2}}\right)^2=1+\frac{c_{2n}}{2(-2)^n}$$

$$c_n^2=2(-2)^n+c_{2n}$$

f) $T_n^2-T_{n+1}T_{n-1}=1-x^2$

$$\left(\frac{c_n}{2(-2)^{n/2}}\right)^2-\frac{c_{n+1}}{2(-2)^{\frac{n+1}{2}}}\frac{c_{n-1}}{2(-2)^{\frac{n-1}{2}}}=1-\left(-\frac{1}{8}\right)$$

$$c_n^2-c_{n-1}c_{n+1}=-9(-2)^{n-1}$$

g) $T_n^2-(x^2-1)U_{n-1}^2=1$

$$\left(\frac{c_n}{2(-2)^{n/2}}\right)^2-\left(-\frac{1}{8}-1\right)\frac{j_n^2}{(-2)^{n-1}}=1$$

$$c_n^2-9j_r^2=(-2)^{n+2}.$$

3. CONCLUSION

In this study, it is aimed to develop some properties of Jacobsthal and Jacobsthal Lucas sequences by using Chebyshev polynomials. It is denoted that the entries of nth power of some special matrices are the elements of Jacobsthal numbers. Jacobsthal, Jacobsthal Lucas, generalized Jacobsthal numbers are obtained by using Chebyshev polynomials.

Acknowledgements

The author would like to thank the anonymous referees for necessary comments which have improved the presentation of the paper.

Funding

The author received no financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest

No conflict of interest or common interest has been declared by the author.

The Declaration of Ethics Committee Approval

The author declares that this document does not require an ethics committee approval or any special permission.

The Declaration of Research and Publication Ethics

The author of the paper declares that she complies with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that she does not make any falsification on the data collected. In addition, she declares that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

REFERENCES

[1] A. F. Horadam, “Basic Properties of a certain generalized Squence of Numbers”, Fibonacci Quarterly, pp. 161-176, 1965.

[2] A. F. Horadam, “Special Properties of the Sequence {Wn(a,b;p,q)}”, Fibonacci Quarterly, vol. 5, pp. 424-434, 1967.

[3] A. F. Horadam, “Tschebyscheff and Other Functions Associated with the Sequence”, Fibonacci Quarterly, vol. 7, no. 1, pp. 14-22, 1969.

[4] A. F. Horadam, “Jacobsthal representation numbers”, The Fibonacci Quarterly, vol. 37, no. 2, pp. 40-54, 1996.

[5] T. Koshy, “Fibonacci and Lucas Numbers with Applications”, John Wiley and Sons Inc., NY 2001.

[6] G. Udrea, “A note on Sequence of A. F. Horadam,” Portugalia Mathematica, vol. 53, no. 24, pp. 143-144, 1996.

[7] T. Mansour, “A formula for the generating functions of powers of Horadam sequence”, Australasian Journal of Combinatorics, vol. 30, pp, 207-212, 2004.

[8] T. Horzum and E. G. Kocer, “On Some Properties of Horadam Polynomials”, Int

math. Forum, vol. 4, no. 25-28, pp. 1243-1252, 2009.

[9] E. Kilic and E Tan, "On Binomial Sums for the General Second Order Linear Recurrence", Integers Electronic Journal of Combnatorial Number Theory, vol. 10, pp. 801-806, 2010.

[10] N. Taskara, K.Uslu, Y. Yazlık and N. Yılmaz "The Construction of Horadam Numbers in Terms of the Determinant of Tridioganal Matrices", Numerical Analysis and Applied Mathematics, AIP Conference Proceedings, vol. 1389, pp. 367-370, 2011.

[11] C. K. Ho and C. Y. Chong, "Odd and even sums of generalized Fibonacci numbers by matrix methods". Am. Inst. Phys. Conf. Ser., vol. 1602, pp. 1026-1032, 2014.

[12] S. P. Jun and K. H. Choi, "Some properties of the Generalized Fibonacci Sequence by Matrix Methods", Korean J. Math, vol. 24, no. 4, pp. 681-691, 2016.

[13] S. Uygun, "The (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences", Applied Mathematical Sciences, vol. 9, no. 7, pp. 3467-3476, 2015.

[14] S. Uygun, "The Combinatorial Representation of Jacobsthal and Jacobsthal Lucas Matrix Sequences", Ars Combinatoria, vol. 135, pp. 83-92, 2017.

[15] S. Uygun, "A New Generalization for Jacobsthal and Jacobsthal Lucas Sequences", Asian Journal of Mathematics and Physics, vol. 2, no. 1, pp. 14-21, 2018.

[16] G. Udrea, "A Problem of Diophantos-Fermat and Chebyshev polynomials of the second kind", Portugalia Mathematica, vol. 52, pp. 301-304, 1995.