

JOURNAL OF SCIENCE



SAKARYA UNIVERSITY

Sakarya University Journal of Science

ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University |
<http://www.saujs.sakarya.edu.tr/>

Title: Stability Switches Of A Neural Field Model : An Algebraic Study On The Parameters

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Received: 2019-02-03 20:28:21

Accepted: 2019-12-01 20:45:23

Article Type: Research Article

Volume: 24

Issue: 1

Month: February

Year: 2020

Pages: 178-182

How to cite

Berrak Özgür; (2020), Stability Switches Of A Neural Field Model : An Algebraic Study On The Parameters. Sakarya University Journal of Science, 24(1), 178-182, DOI: 10.16984/saufenbilder.521545

Access link

<http://www.saujs.sakarya.edu.tr/tr/issue/49430//521545>

New submission to SAUJS

<http://dergipark.gov.tr/journal/1115/submission/start>

Stability Switches Of A Neural Field Model : An Algebraic Study On The Parameters

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Abstract

In this paper, a special case for a delayed neural field model is considered. After constructing its characteristic equation a stability analysis is made. Using Routh-Hurwitz criterion, some conditions for characteristic equation are given for the stability of the system.

Keywords: delay differential equations, characteristic equation, stability analysis

1. INTRODUCTION

Delay differential systems have an important role in applied mathematics. They are frequently used in lots of scientific field such as biology, engineering, medicine and economics etc. Stability analysis of these models is studied by scientists.

To understand the activity of a large neuron populations in the brain, the scientists use neural field models. By use of integral equations or integro-differential equations, we can make their analysis. The studies by Wilson and Cowan [1] and Amari [2] are very important in this area.

Delay terms in these models have a great importance because some biological phenomena take time to occur. After a delay term is seen in a

model, a stability analysis may be performed to see its effect and also one can investigate the existence and uniqueness of their solutions [3-8].

An integro-differential system including delay term is used to represent the relations between neural populations [7-9]. Using this model, a center manifold result is given in [9], the existence and uniqueness of their solutions is studied in [10]. Stability analysis of the same model for some special cases is made in [11-13] using their characteristic equations. The effect of the delay term on its stability is examined using D-curves. Also there are some papers in which numerical techniques are used to determine their stability [14].

In this study, we consider a neural field model. By obtaining its characteristic equation, we get some

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stability and unstability conditions considering parameters in this model. In Section 2.1, the conditions for the stability is given in case of a zero delay. In Section 2.2, the analysis is made to show under which conditions the sytem becomes unstable.

2. STABILITY ANALYSIS OF THE MODEL

In this study we consider the neural field model [7,8,9] for two neuron populations which is defined on a finite piece of cortex $\Omega \subset R$.

$$\left(\frac{d}{dt} + l_i\right) V_i(t, r) = \sum_{j=1}^2 \int_{\Omega} J_{ij}(r, \bar{r}) S[\sigma_j(V_j(t - \tau_{ij}(r, \bar{r}), \bar{r}) - h_j)] d\bar{r} \quad , \quad t \geq 0 \quad , \quad i = 1,2 \quad (1)$$

$$V_i(t, r) = \phi_i(t, r) \quad , \quad t \in [-T, 0]$$

In this model, the functions $V_1(x, t)$ and $V_2(x, t)$ describe the synaptic inputs for a large group of neurons at position x and time t . The synaptic connectivity functions $J_{ij}(x, y)$ gives the relations between neurons on different populations. The constant delay is considered as $\tau(x, y) = \tau$. We consider $\Omega = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In this research we assume the special case that $J_{12}(x, y) = J_{22}(x, y) = 0$ i.e. , the neurons in the second population inhibit each other and the ones in the first population. Hence we have the following linearized system near $(0,0)$

$$\frac{d}{dt} U_1(x, t) + l_1 U_1(x, t) = \sigma_1 s_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{11}(x, y) U_1(y, t - \tau(x - y)) dy \quad (2)$$

$$\frac{d}{dt} U_2(x, t) + l_2 U_2(x, t) = \sigma_1 s_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x, y) U_1(y, t - \tau(x - y)) dy$$

where the functions $U_1(x, t)$ and $U_2(x, t)$ are used instead of $V_1(x, t)$ and $V_2(x, t)$. In this model we have $U_1(x, t) = e^{ikx} u_1(t)$ and $U_2(x, t) = e^{ikx} u_2(t)$ as in Fourier method. Writing $u_1(t) = c_1 e^{\lambda t}$ and $u_2(t) = c_2 e^{\lambda t}$ we get the following

$$\lambda e^{ikx} u_1(t) + l_1 e^{ikx} u_1(t) - K_1 e^{-\lambda \tau} u_1(t) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{11}(x, y) e^{iky} dy = 0 \quad (3)$$

$$\lambda e^{ikx} u_2(t) + l_2 e^{ikx} u_2(t) - K_1 e^{-\lambda \tau} u_1(t) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x, y) e^{iky} dy = 0$$

where $K_1 = \sigma_1 s_1$ [7] and considering that the solutions of the system are the functions $\cos(2nx)$ and $\sin(2nx)$ [7], we have the following system of equations

$$\lambda u_1(t) + l_1 u_1(t) - K_1 e^{-\lambda \tau} u_1(t) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{11}(x, y) e^{iky} dy = 0 \quad (4)$$

$$\lambda u_2(t) + l_2 u_2(t) - K_1 e^{-\lambda \tau} u_1(t) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x, y) e^{iky} dy = 0$$

Assuming that

$$F_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{11}(x, y) e^{iky} dy \quad (5)$$

$$F_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x, y) e^{iky} dy$$

we have the following system

$$\lambda u_1(t) + l_1 u_1(t) - K_1 e^{-\lambda \tau} u_1(t) F_1 = 0 \quad (6)$$

$$\lambda u_2(t) + l_2 u_2(t) - K_1 e^{-\lambda \tau} u_1(t) F_2 = 0$$

To get the solution of the system we need the following equation derived by determinant of the system for $u_1(t)$ and $u_2(t)$

$$\lambda^2 + \lambda l_2 + \lambda l_1 + l_1 l_2 - \lambda K_1 F_1 e^{-\lambda \tau} - K_1 F_1 l_2 e^{-\lambda \tau} = 0 \quad (7)$$

or

$$\lambda^2 + (l_1 + l_2 - K_1 F_1 e^{-\lambda \tau}) \lambda + l_1 l_2 - K_1 F_1 l_2 e^{-\lambda \tau} = 0 \quad (8)$$

The stability analysis is done for this system regarding its characteristic equation with delay

term. To do this, we have to determine if there are some roots with positive real parts. The Routh-Hurwitz criterion is used to find some conditions on the parameters of the system to classify the roots having positive or negative real parts.

Here we make this analysis in two parts.

2.1. Case 1:

Here, in the absence of the delay term, i.e., $\tau = 0$, the characteristic equation (8) becomes

$$\lambda^2 + (l_1 + l_2 - K_1 F_1)\lambda + l_1 l_2 - K_1 F_1 l_2 = 0 \quad (9)$$

Since the leading coefficient in characteristic polynomial is positive, if the coefficient of λ and the constant term are positive then all the roots of (9) have negative real parts. Hence we conclude the result for the stability of the system by the following theorem.

Theorem 1: Consider the system (2) with a zero delay term, i.e., $\tau = 0$. If the conditions

$$l_1 + l_2 - K_1 F_1 > 0 \quad (10)$$

and

$$l_1 l_2 - K_1 F_1 l_2 > 0 \quad (11)$$

are satisfied then all characteristic roots of the characteristic equation have negative real parts. Under these conditions, the system (2) is stable.

Proof: Considering that the leading coefficient in characteristic polynomial of order two is positive, by Routh-Hurwitz criterion, it has roots with negative real parts if the coefficient of λ and the constant term are positive. Hence, if the conditions given above are satisfied then the system is stable.

2.2. Case 2 :

If $\tau \neq 0$ then considering the characteristic equation (8) we have the procedure given in [15].

As the delay term increases, the roots of this equation may change. They may have positive or negative real parts. To talk about this case, we need a polynomial. To get it, we introduce a purely imaginary root for the characteristic equation given below

$$\lambda = i\sigma, \sigma \in R \quad (12)$$

hence we have

$$\lambda^2 = -\sigma^2 \quad (13)$$

Writing this into (8), we get the following

$$\begin{aligned} &-\sigma^2 + i\sigma l_2 + i\sigma l_1 + l_1 l_2 - \\ &i\sigma K_1 F_1 (\cos(\sigma\tau) - i\sin(\sigma\tau)) - \\ &K_1 F_1 l_2 (\cos(\sigma\tau) - i\sin(\sigma\tau)) = 0 \end{aligned} \quad (14)$$

or

$$\begin{aligned} &-\sigma^2 + l_1 l_2 + i\sigma(l_1 + l_2) \\ &- (K_1 F_1 l_2 + i\sigma K_1 F_1) (\cos(\sigma\tau) - i\sin(\sigma\tau)) \\ &= 0 \end{aligned} \quad (15)$$

Considering (15) in the form of $P_1(i\sigma) + P_2(i\sigma) e^{-i\sigma\tau} = 0$ or more clearly

$$\begin{aligned} &R_1(\sigma) + iQ_1(\sigma) \\ &+ (R_2(\sigma) + iQ_2(\sigma)) (\cos(\sigma\tau) - i\sin(\sigma\tau)) = 0 \end{aligned}$$

and separating its real and imaginary parts, we get the followings

$$\begin{aligned} R_1(\sigma) &= -\sigma^2 + l_1 l_2 \\ Q_1(\sigma) &= \sigma(l_1 + l_2) \\ R_2(\sigma) &= -K_1 F_1 l_2 \\ Q_2(\sigma) &= -\sigma K_1 F_1 \end{aligned} \quad (16)$$

Considering them we may write a new polynomial equation given below including no delay term

$$R_1(\sigma)^2 + Q_1(\sigma)^2 = R_2(\sigma)^2 + Q_2(\sigma)^2 \quad (17)$$

i.e.,

$$\sigma^4 - 2\sigma^2 l_1 l_2 + l_1^2 l_2^2 + \sigma^2 (l_1 + l_2)^2 = K_1^2 F_1^2 l_2^2 + \sigma^2 K_1^2 F_1^2 \quad (18)$$

With the new substitution

$$\mu = \sigma^2 \quad (19)$$

we get the following equation

$$\mu^2 - 2\mu l_1 l_2 + l_1^2 l_2^2 + \mu (l_1 + l_2)^2 - K_1^2 F_1^2 l_2^2 - \mu K_1^2 F_1^2 = 0 \quad (20)$$

Rewriting it the following is obtained.

$$\mu^2 + [(l_1 + l_2)^2 - K_1^2 F_1^2 - 2l_1 l_2] \mu + l_1^2 l_2^2 - K_1^2 F_1^2 l_2^2 = 0 \quad (21)$$

Now we can make an analysis for the stability of the system via coefficients of this equation. Here we have

$$A = (l_1 + l_2)^2 - K_1^2 F_1^2 - 2l_1 l_2 \quad (22)$$

$$B = l_1^2 l_2^2 - K_1^2 F_1^2 l_2^2$$

Regarding the Routh-Hurwitz criterion, there are two cases in which the equation (21) has positive real roots.

i) Since the leading coefficient is positive, if $B < 0$ then there is one positive real root.

ii) If $B > 0$ then the roots are given by $\frac{-A \pm \sqrt{A^2 - 4B}}{2}$. In this case, if $A < 0$ and $A^2 - 4B > 0$ are satisfied then there is a positive real root.

Now we give an important theorem about the stability considering the two cases above.

Theorem 2: Consider the system (2) with $\tau \neq 0$. If

$$l_1^2 < K_1^2 F_1^2 \text{ is satisfied}$$

or

in case of $l_1^2 - K_1^2 F_1^2 > 0$, if the conditions

$$(l_1 + l_2)^2 - K_1^2 F_1^2 - 2l_1 l_2 < 0 \text{ and}$$

$$((l_1 + l_2)^2 - K_1^2 F_1^2 - 2l_1 l_2)^2 - 4(l_1^2 l_2^2 - K_1^2 F_1^2 l_2^2) > 0 \text{ are}$$

satisfied then the characteristic equation has positive root. Hence the system becomes unstable.

Proof : For the system (2) we get equation (21) according its characteristic equation (8). Considering Routh-Hurwitz criterion, if the condition $l_1^2 < K_1^2 F_1^2$ is satisfied or in case of $l_1^2 - K_1^2 F_1^2 > 0$, if the conditions

$$(l_1 + l_2)^2 - K_1^2 F_1^2 - 2l_1 l_2 < 0 \text{ and}$$

$$((l_1 + l_2)^2 - K_1^2 F_1^2 - 2l_1 l_2)^2 - 4(l_1^2 l_2^2 - K_1^2 F_1^2 l_2^2) > 0 \text{ are}$$

satisfied then we guarantee that there is a positive real root. Hence the system is unstable.

3. CONCLUSION

One of the important application of delayed models in neuroscience is the neural field model. The characteristic equation is the most important part in the stability analysis of these models. In this study, we consider a special case for the neural field model. The main aim of this study is to give some quick and important results for the stability of the model in an algebraic way. The method used here is to determine the characteristic equation first and then by means of Routh-Hurwitz criterion, to give the relation between the parameters of the system and the stability. As a conclusion, in case of the existence and nonexistence of the delay term, some detailed results are given for the stability of the neural field model considered.

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