

On a Fuzzy Problem with Variable Coefficient by Fuzzy Laplace Transform

Hülya GÜLTEKİN ÇİTİL^{1*}

ABSTRACT: This study is on solutions of a fuzzy problem with variable coefficient. Solutions are found by fuzzy Laplace transform. Generalized differentiability is used. It is searched whether the solutions are valid α -level sets or not. Examples are solved on studied problem. Conclusions are given.

Keywords: Fuzzy initial value problem, fuzzy Laplace transform, generalized differentiability, triangular fuzzy number

¹ Hülya GÜLTEKİN ÇİTİL (Orcid ID: 0000-0002-3543-033X), Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun, Turkey

*Sorumlu Yazar/Corresponding Author: Hülya GÜLTEKİN ÇİTİL, e-mail: hulya.citil@giresun.edu.tr

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INTRODUCTION

The theory of fuzzy differential equation is very important topic. Therefore, it is studied by many researchers. For example, in population models (Buckley et al., 2002) and growth model (Mondal et al., 2013).

Zadeh first introduced fuzzy number and fuzzy arithmetic (Zadeh, 1965). There are different approaches to solve the fuzzy differential equation. The first approach is Hukuhara derivative (Puri and Ralescu, 1983; Gültekin and Altınışik, 2014) or generalized Hukuhara derivative (Stefanini and Bede 2008; Ceylan and Altınışik, 2018; Gültekin Çitil, 2018). But in Hukuhara derivative, the solution becomes uncertain as time goes on. Thus, generalized Hukuhara derivative was introduced (Bede et al., 2007). The second approach is extension principle (Zadeh, 1975). The third approach is differential inclusion (Hüllermeier, 1997).

Fuzzy Laplace transform is important topic to solve fuzzy differential equation. Also, it gives solution satisfying the initial values of fuzzy differential equation directly. Firstly, it was introduced by Allahviranloo and Barchordary Ahmadi (2010). They solved first order fuzzy initial value problem by fuzzy Laplace transform. In many areas, many researchers used fuzzy Laplace transform (Ramazannia Tolouti and Barchordary Ahmadi, 2010; Ahmad et al., 2012; Mondal and Roy, 2015).

In this paper, a fuzzy problem with variable coefficient is investigated by fuzzy Laplace transform. The concept of generalized differentiability, the properties of fuzzy Laplace transform, fuzzy arithmetic, Hukuhara difference are used. It is searched whether the solutions are valid fuzzy functions or not. Examples are solved.

In section 2, it is given definitions and theorems that we will use in our study. In section 3, the considered problem is introduced and the problem is solved by fuzzy Laplace transform, several theorems are given and examples are solved. In section 4, conclusion is given.

MATERIALS AND METHODS

Definition 1. A fuzzy number is a mapping $u: \mathbb{R} \rightarrow [0,1]$ where,

$\overline{\{x \in \mathbb{R} | u(x) > 0\}}$ is compact, u is convex fuzzy set, normal, upper semi-continuous on \mathbb{R} (Liu, 2011).

Let's show the set of all fuzzy numbers with \mathbb{R}_F .

Definition 2. $u \in \mathbb{R}_F$, $[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\} = [\underline{u}_\alpha, \bar{u}_\alpha]$, $0 < \alpha \leq 1$ is the α -level set of u .

If $\alpha = 0$, $[u]^0 = \overline{\{supp u\}} = \overline{\{x \in \mathbb{R} | u(x) > 0\}}$ (Khastan and Nieto, 2010).

Definition 3. The parametric form $[\underline{u}_\alpha, \bar{u}_\alpha]$ of A fuzzy number satisfy the following requirements:

The lower part \underline{u}_α is left-continuous bounded non-decreasing on $(0,1]$, also it is right-continuous for $\alpha = 0$,

The upper part \bar{u}_α is left-continuous bounded non-increasing on $(0,1]$, also it is right-continuous for $\alpha = 0$,

$\underline{u}_\alpha \leq \bar{u}_\alpha$, $0 \leq \alpha \leq 1$ (Salahshour and Allahviranloo, 2013).

Definition 4. $[B]^\alpha = \left[\underline{b} + \left(\frac{\bar{b}-\underline{b}}{2} \right) \alpha, \bar{b} - \left(\frac{\bar{b}-\underline{b}}{2} \right) \alpha \right]$ is the α -level set of B , which is a symmetric triangular fuzzy number with support $[\underline{b}, \bar{b}]$ (Liu, 2011).

Definition 5. The metric is defined by

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha),$$

$$d([u]^\alpha, [v]^\alpha) = \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\bar{u}_\alpha - \bar{v}_\alpha|\}$$

on \mathbb{R}_F (Fatullayev et al., 2013).

Definition 6. Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, w is Hukuhara difference of u and v , $w = u \ominus v$ (Puri and Ralescu, 1983).

Definition 7. Let $f: [a, b] \rightarrow \mathbb{R}_F$ and $x_0 \in [a, b]$. If there exists $f'(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), f(x_0) \ominus f(x_0 - h)$ and the limits hold

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

f is Hukuhara differentiable at x_0 (Bede, 2008).

Definition 8. Let $f: [a, b] \rightarrow \mathbb{R}_F$ and $x_0 \in [a, b]$. If there exists $f'(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), f(x_0) \ominus f(x_0 - h)$ and the limits hold

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

f is (1)-differentiable at x_0 . If there exists $f'(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h), f(x_0 - h) \ominus f(x_0)$ and the limits hold

$$\lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

f is (2)-differentiable at x_0 . (Khastan and Nieto, 2010).

Theorem 1. Let $f: [a, b] \rightarrow \mathbb{R}_F$ be a function and denote $[f(t)]^\alpha = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)]$ for each $\alpha \in [0, 1]$.

1. If the function f is (1)-differentiable, the lower function $\underline{f}_\alpha(t)$ and the upper function $\bar{f}_\alpha(t)$ are differentiable,

$$[f'(t)]^\alpha = [\underline{f}'_\alpha(t), \bar{f}'_\alpha(t)].$$

2. If the function f is (2)-differentiable, the lower function $\underline{f}_\alpha(t)$ and the upper function $\bar{f}_\alpha(t)$ are differentiable,

$$[f'(t)]^\alpha = [\bar{f}'_\alpha(t), \underline{f}'_\alpha(t)] \quad (\text{Khastan et al., 2009}).$$

Definition 9. Let $f: [a, b] \rightarrow \mathbb{R}_F$ be a fuzzy function. The fuzzy Laplace transform of f is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) dt = \left[\lim_{\rho \rightarrow \infty} \int_0^\rho e^{-st} \underline{f}(t) dt, \lim_{\rho \rightarrow \infty} \int_0^\rho e^{-st} \bar{f}(t) dt \right].$$

$$F(s, \alpha) = L([f(t)]^\alpha) = \left[L(\underline{f}_\alpha(t)), L(\bar{f}_\alpha(t)) \right],$$

$$L(\underline{f}_\alpha(t)) = \int_0^\infty e^{-st} \underline{f}_\alpha(t) dt = \lim_{\rho \rightarrow \infty} \int_0^\rho e^{-st} \underline{f}_\alpha(t) dt,$$

$$L(\bar{f}_\alpha(t)) = \int_0^\infty e^{-st} \bar{f}_\alpha(t) dt = \lim_{\rho \rightarrow \infty} \int_0^\rho e^{-st} \bar{f}_\alpha(t) dt \quad (\text{Patel and Desai, 2017}).$$

Theorem 2. Let $f'(t)$ be an integrable fuzzy function, $f(t)$ is primitive of $f'(t)$ on $(0, \infty]$. If f is (1)-differentiable,

$$L(f'(t)) = sL(f(t)) \ominus f(0),$$

if f is (2)-differentiable,

$$L(f'(t)) = (-f(0)) \ominus (-sL(f(t))) \quad (\text{Allahviranloo and Barchordari Ahmadi, 2010}).$$

Theorem 3. Let be $f(x), g(x)$ continuous fuzzy functions and c_1, c_2 constants, then

$$L((c_1 \odot f(x)) \oplus (c_2 \odot g(x))) = (c_1 \odot L(f(x))) \oplus (c_2 \odot L(g(x))) \quad (\text{Allahviranloo and Barchordari Ahmadi, 2010}).$$

Theorem 4. Let $f''(t)$ be an integrable fuzzy function and $f(t), f'(t)$ are primitive of $f'(t), f''(t)$ on $(0, \infty]$. Then, if f and f' are (1)-differentiable,

$$L(f''(t)) = s^2 L(f(t)) \ominus sf(0) \ominus f'(0),$$

if f and f' are (2)-differentiable,

$$L(f''(t)) = s^2 L(f(t)) \ominus sf(0) - f'(0),$$

if f is (1)-differentiable and f' is (2)-differentiable,

$$L(f''(t)) = \ominus (-s^2)L(f(t)) - sf(0) - f'(0)$$

if f is (2)-differentiable and f' is (1)-differentiable,

$$L(f''(t)) = \ominus (-s^2)L(f(t)) - sf(0) \ominus f'(0) \quad (\text{Patel and Desai, 2017}).$$

Theorem 5. Let $f(t)$ satisfies the condition of existence theorem of Laplace transform and

$$L(f(t)) = F(s),$$

then

$$L(tf(t)) = -F'(s).$$

If $f'(t)$ satisfies the condition of existence theorem of Laplace transform, then

$$L(tf'(t)) = -sF'(s) - F(s),$$

similarly for $f''(t)$

$$L(tf''(t)) = -s^2F'(s) - 2sF(s) + f(0) \quad (\text{Patel and Desai, 2017}).$$

RESULTS AND DISCUSSION

Consider fuzzy problem with variable coefficient

$$ty'' + \lambda y' = [A]^\alpha, \quad y(0) = [B]^\alpha, \quad t > 0 \quad (1)$$

by fuzzy Laplace transform, where $\lambda < 0$,

$$[A]^\alpha = [\underline{A}_\alpha, \overline{A}_\alpha] = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2} \right) \alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2} \right) \alpha \right],$$

$$[B]^\alpha = [\underline{B}_\alpha, \overline{B}_\alpha] = \left[\underline{b} + \left(\frac{\overline{b} - \underline{b}}{2} \right) \alpha, \overline{b} - \left(\frac{\overline{b} - \underline{b}}{2} \right) \alpha \right]$$

are symmetric triangular fuzzy numbers with the supports $[\underline{a}, \overline{a}]$, $[\underline{b}, \overline{b}]$, respectively.

Firstly, taking the Laplace transform of fuzzy differential equation (1), we have

$$L(ty'' + \lambda y') = L([A]^\alpha). \quad (2)$$

Using the properties of fuzzy Laplace transform,

$$L(ty'') + \lambda L(y') = L([A]^\alpha). \quad (3)$$

Using the Theorem 4 and Theorem 5, we obtain

$$-s^2 Y'(s, \alpha) - 2sY(s, \alpha) - (\ominus y(0, \alpha)) + \lambda(sY(s, \alpha) \ominus y(0, \alpha)) = L([A]^\alpha). \quad (4)$$

Using the Hukuhara difference, fuzzy arithmetic and the properties of fuzzy Laplace transform, the equations

$$-s^2 \underline{Y}'(s, \alpha) - 2s\underline{Y}(s, \alpha) + \underline{y}(0, \alpha) + \lambda s\underline{Y}(s, \alpha) - \lambda \underline{y}(0, \alpha) = \frac{\overline{A}_\alpha}{s}, \quad (5)$$

$$-s^2 \overline{Y}'(s, \alpha) - 2s\overline{Y}(s, \alpha) + \overline{y}(0, \alpha) + \lambda s\overline{Y}(s, \alpha) - \lambda \overline{y}(0, \alpha) = \frac{\underline{A}_\alpha}{s} \quad (6)$$

are obtained. The equation (5) gives the equation

$$\underline{Y}'(s, \alpha) + \frac{2-\lambda}{s} \underline{Y}(s, \alpha) = -\frac{\overline{A}_\alpha}{s^3} + \frac{(1-\lambda)\underline{B}_\alpha}{s^2}. \quad (7)$$

Using the integral multiplier

$$\mu(s) = e^{\int \frac{2-\lambda}{s} ds} = s^{2-\lambda}, \quad (8)$$

$$s^{2-\lambda} \underline{Y}(s, \alpha) = \int \left(-\frac{\overline{A}_\alpha}{s^{1+\lambda}} + \frac{(1-\lambda)\underline{B}_\alpha}{s^\lambda} \right) ds \quad (9)$$

is obtained. From this, we have

$$\underline{Y}(s, \alpha) = \frac{\overline{A}_\alpha}{\lambda} \frac{1}{s^2} + \underline{B}_\alpha \frac{1}{s} + C \frac{1}{s^{2-\lambda}}. \quad (10)$$

Taking the inverse Laplace transform of (10), the lower solution of the problem (1) is obtained as

$$\underline{y}(t, \alpha) = \frac{\overline{A}_\alpha}{\lambda} t + \underline{B}_\alpha + C_* t^{1-\lambda}. \quad (11)$$

Similarly, the upper solution of the problem (1) is obtained as

$$\overline{y}(t, \alpha) = \frac{\underline{A}_\alpha}{\lambda} t + \overline{B}_\alpha + C_* t^{1-\lambda}. \quad (12)$$

Consequently, the solution of the problem (1) is

$$[y(t)]^\alpha = [\underline{y}(t, \alpha), \bar{y}(t, \alpha)] \quad (13)$$

by fuzzy Laplace transform.

Now let's investigate whether the solution is a valid fuzzy function or not.

Theorem 1. The solution $[y(t)]^\alpha = [\underline{y}(t, \alpha), \bar{y}(t, \alpha)]$ of the problem (1) is a valid α -level set.

Proof. As

$$\frac{\partial y(t, \alpha)}{\partial \alpha} > 0, \quad \frac{\partial \bar{y}(t, \alpha)}{\partial \alpha} < 0, \quad \underline{y}(t, \alpha) < \bar{y}(t, \alpha),$$

the solution of the problem (1) is a valid α -level set. According to this, it must be

$$-\frac{t}{\lambda} \left(\frac{\bar{a}-a}{2} \right) + \left(\frac{\bar{b}-b}{2} \right) > 0.$$

That is,

$$t > \frac{\lambda(\bar{b}-b)}{(\bar{a}-a)}.$$

Since $\bar{b} - b > 0$, $\bar{a} - a > 0$ and $\lambda < 0$, this equality is provided. The proof is completed.

Theorem 2. The solution $[y(t)]^\alpha = [\underline{y}(t, \alpha), \bar{y}(t, \alpha)]$ of the problem (1) is symmetric triangular fuzzy number for any $t > 0$ time.

Proof. Since

$$\underline{y}(t, 1) = \frac{t}{\lambda} \left(\frac{\bar{a}+a}{2} \right) + \left(\frac{\bar{b}+b}{2} \right) + C_* t^{1-\lambda} = \bar{y}(t, 1)$$

and

$$\underline{y}(t, 1) - \underline{y}(t, \alpha) = (1 - \alpha) \left(\left(\frac{\bar{b}-b}{2} \right) - \frac{t}{\lambda} \left(\frac{\bar{a}-a}{2} \right) \right) = \bar{y}(t, \alpha) - \bar{y}(t, 1),$$

the solution of the problem (1) is symmetric triangular fuzzy number for any $t > 0$ time .

Example 1. We consider the fuzzy problem

$$ty'' - y' = [0]^\alpha, \quad y(0) = [1]^\alpha \quad (14)$$

by the fuzzy Laplace transform, where $[0]^\alpha = [-1 + \alpha, 1 - \alpha]$, $[1]^\alpha = [\alpha, 2 - \alpha]$.

Solving the fuzzy problem (14), the solution is obtained as

$$\underline{y}(t, \alpha) = \alpha + (\alpha - 1)t + C_* t^2, \quad (15)$$

$$\bar{y}(t, \alpha) = (2 - \alpha) + (1 - \alpha)t + C_* t^2, \quad (16)$$

$$[y(t)]^\alpha = [\underline{y}(t, \alpha), \bar{y}(t, \alpha)]. \quad (17)$$

The solution (15)-(17) of the problem (14) is a valid α -level set since $t > -1$ and the solution is symmetric triangular fuzzy number for any $t > 0$ time.

Example 2. Consider the problem

$$ty'' - 2y' = [1]^\alpha, y(0) = [2]^\alpha \quad (18)$$

where $[1]^\alpha = [\alpha, 2 - \alpha]$, $[2]^\alpha = [1 + \alpha, 3 - \alpha]$.

Solving the problem (18) by fuzzy Laplace transform, we obtain

$$\underline{y}(t, \alpha) = (1 + \alpha) + \left(\frac{\alpha}{2} - 1\right)t + C_* t^2, \quad (19)$$

$$\bar{y}(t, \alpha) = (3 - \alpha) - \frac{\alpha}{2}t + C_* t^2, \quad (20)$$

$$[y(t)]^\alpha = \left[\underline{y}(t, \alpha), \bar{y}(t, \alpha) \right]. \quad (21)$$

The solution (19)-(21) of the problem (18) is a valid α -level set since $t > -2$ and the solution is symmetric triangular fuzzy number for any $t > 0$ time.

CONCLUSION

In this paper, a fuzzy problem with variable coefficient and symmetric triangular fuzzy number initial value is searched by fuzzy Laplace transform. It is used the concept of generalized differentiability. It is found that the solution is a valid α -level set. Also, it is found that the solution is symmetric triangular fuzzy number for any $t > 0$ time. Different problems can be investigated and different conclusions can be found. This topic is important and it will be useful for other mathematicians.

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