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Geodesics and Torsion Tensor according to g-lift of Riemannian Connection on Cotangent Bundle

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ABSTRACT: In this study, the geodesics and torsion tensor according to g-lift of Riemannian connection ∇ on cotangent bundle T^*M are investigated. Firstly, using the components of g-lift of Riemannian connection ∇ on cotangent bundle T^*M the components of torsion tensor according to g-lift of Riemannian connection ∇ are obtained. So the torsion tensor according to g-lift of Riemannian connection ∇ are determined. Finally, geodesics on cotangent bundle T^*M according to g-lift of Riemannian connection ∇ are studied.

Keywords: Cotangent Bundle, Connection, Torsion Tensor, Geodesic, g-lift.

Kotanjant Demette Riemann Konneksiyonun g-liftine göre Burulma Tensörü ve Geodezikler

ÖZET: Bu çalışmada, T^*M kotanjant demette Riemann konneksiyonun g-liftine göre burulma tensörü ve geodezikler incelenir. İlk olarak, T^*M kotanjant demet üzerindeki Riemann konneksiyonun g-liftinin bileşenleri kullanılarak ∇ Riemann konneksiyonun g-liftine göre burulma tensor bileşenleri elde edilir. Böylece ∇ Riemann konneksiyonun g-liftine göre burulma tensörü belirlenir. Son olarak, ∇ Riemann konneksiyonun g-liftine göre T^*M kotanjant demet üzerindeki geodezikler çalışılır.

Anahtar Kelimeler: Kotanjant Demet, Konneksiyon, Burulma Tensörü, Jeodezik, g-lift.

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INTRODUCTION

Let M be a smooth manifold and T^*M be its cotangent bundle. The various problems of the cotangent bundle are studied by many authors (Mok, 1977; Druta Romaniuc, 2012; Kurek and Mikulski, 2013; Çayır 2019a; Çayır 2019b;). One of basic differential geometric structures on a smooth manifold is geodesic. A geodesic is a curve representing the shortest path between two points in a surface, or more generally in a manifold. The shortest path between two given points in a smooth manifold can defined by using the equation for the length of a curve and then minimizing this length between the points using the calculus variations. The problems of geodesics are studied on the different type tensor bundle of a manifold by some authors. The geodesics in tensor bundle of type $(1,q)$ have been investigated according to the Levi-Civita connection of diagonal lift of g metric (Cengiz and Salimov, 2002). The geodesics in tensor bundle of type (p,q), $p+q>0$, have been investigated according to complete lifts of affine connections (Mağden and Salimov, 2004).

On a smooth manifold endowed with an affine connection, torsion and curvature compose the two basic invariants of the connection. In differential geometry, the notion of torsion is a manner of characterizing a twist or screw of a moving frame around a curve. Torsion can be described concretely as a tensor, or as a vector valued two form on the manifold. If ∇ is an affine connection on a smooth manifold, then the torsion tensor is defined

$$T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y]$$

where X,Y are vector field and $[X,Y]$ is the Lie bracket of vector fields. The torsion tensors are studied on cotangent bundle of a manifold by many authors (Yano and Patterson, 1967; Mok, 1977).

MATERIALS AND METHODS

Let M be $n-$ dimensional differentiable manifold. Let TM be the tangent bundle of M . The local coordinates on TM are $(x^i, x^{\bar{i}})=(x^i, y^i)$ where (x^i) are local coordinates on M and (y^i) are vector space coordinates according to the basis $\partial/\partial x^i$. Let T^*M be the cotangent bundle of M . The local coordinates on T^*M are $(x^i, x^{\bar{i}})=(x^i, p_i)$ where (x^i) are local coordinates on M and (p_i) are vector space coordinates according to the basis dx^i . In this paper manifolds, mappings and connection are assumed to be differentiable of class C^∞ . The indices $i,j,k,...$ have range in $\{1,...,n\}$ and indices $\bar{i},\bar{j},\bar{k},...$ have a range in $\{n+1,...,2n\}$.

Let g be a pseudo Riemannian metric. $g^\sharp : T^*M \to TM$ is the musical isomorphism associated with g pseudo Riemannian metric with inverse given by $g^\flat : TM \to T^*M$. Then the musical isomorphisms $g^\sharp$ and $g^\flat$ are given by

$$g^\sharp : x^M = \left(x^m, x^{\bar{m}}\right) = (x^m, p_m) \to x^J = \left(x^j, x^{\bar{j}}\right) = \left(\delta^j_m x^m, y^j = g^{jm} p_m\right) \qquad (1.1)$$

and

$$g^\flat = x^J = \left(x^j, x^{\bar{j}}\right) = (x^j, y^j) \to x^M = \left(x^m, x^{\bar{m}}\right) = \left(\delta^m_j x^j, p_m = g_{mj} y^j\right) \qquad (1.2)$$

where $g^{ik}g_{kj}=\delta^i_j$ is the Kronecker symbol. The Jacobian matrices of $g^\flat$ and $g^\sharp$ are given, respectively, by (Cakan et. al., 2016)

$$\left(g^\flat_*\right)=A=\left(A^M_J\right)=\begin{pmatrix} A^m_j & A^m_{\bar j} \\ A^{\bar m}_j & A^{\bar m}_{\bar j}\end{pmatrix}=\left(\frac{\partial x^M}{\partial x^J}\right)=\begin{pmatrix} \delta^m_j & 0 \\ y^s\partial_j g_{ms} & g_{mj}\end{pmatrix} \quad (1.3)$$

and

$$\left(g^\sharp_*\right)=A=\left(A^J_M\right)=\begin{pmatrix} A^j_m & A^j_{\bar m} \\ A^{\bar j}_m & A^{\bar j}_{\bar m}\end{pmatrix}=\left(\frac{\partial x^J}{\partial x^M}\right)=\begin{pmatrix} \delta^j_m & 0 \\ p_s\partial_m g^{js} & g^{jm}\end{pmatrix}. \quad (1.4)$$

The complete lift on cotangent bundle has been defined and applied to connection in manifold (Yano and Patterson, 1967). The $g-$ lifts of some tensor fields on the cotangent bundle have been defined via musical isomorphism and the $g-$lifts have been applied to problems of some tensor fields (Salimov and Cakan, 2017). The $g-$ lifts of affine connection and curvature tensor on cotangent bundle have been studied (Cakan and Kemer, 2019). In this paper, we investigate the geodesics and torsion tensor according to the $g-$lift of the Riemannian connection on cotangent bundle T^*M.

RESULTS AND DISCUSSION

Let M be an differentiable manifold and ∇ be symmetric affine connection on M. Let ${}^C\overset{*}{\nabla}$ be complete lift of symmetric affine connection on cotangent bundle T^*M. The non-zero components ${}^C\overset{*}{\Gamma}{}^K_{IJ}$ of ${}^C\overset{*}{\nabla}$ is given

$$\begin{aligned} &{}^C\overset{*}{\Gamma}{}^k_{ij}=\Gamma^k_{ij}, \quad {}^C\overset{*}{\Gamma}{}^{\bar k}_{i\bar j}=-\Gamma^j_{ik}, \quad {}^C\overset{*}{\Gamma}{}^{\bar k}_{\bar i j}=-\Gamma^i_{kj} \\ &{}^C\overset{*}{\Gamma}{}^{\bar k}_{ij}=p_s\left(\partial_k\Gamma^s_{ij}-\partial_i\Gamma^s_{jk}-\partial_j\Gamma^s_{ik}+2\Gamma^s_{kt}\Gamma^t_{ij}\right)\end{aligned} \quad (1.5)$$

according to where $I,J,...=1,...,2n$ (Yano and Ishihara, 1973).

Theorem 1 Let M be a $n-$ dimensional pseudo Riemannian manifold with pseudo Riemannian metric g. Let ${}^C\nabla$ and ${}^C\overset{*}{\nabla}$ be complete lifts of ∇ affine connection to TM and T^*M, respectively. Then the differential of ${}^C\nabla$ by $g^\flat$, i.e. a $g-$ lift ${}^G\overset{*}{\nabla}$ in the cotangent bundle T^*M, coincides with the complete lift ${}^C\overset{*}{\nabla}$ in the cotangent bundle T^*M if ∇ is a Riemannian connection which is a metric connection with vanishing torsion. And the $g-$lift ${}^G\overset{*}{\nabla}$ has components (Cakan and Kemer, 2019)

$$\begin{aligned} &{}^G\overset{*}{\Gamma}{}^c_{ab}=\Gamma^c_{ab}, \quad {}^G\overset{*}{\Gamma}{}^c_{a\bar b}=0, \quad {}^G\overset{*}{\Gamma}{}^c_{\bar a b}=0, \quad {}^G\overset{*}{\Gamma}{}^c_{\bar a\bar b}=0 \\ &{}^G\overset{*}{\Gamma}{}^{\bar c}_{ab}=p_t\left(\partial_c\Gamma^t_{ab}-\partial_a\Gamma^t_{bc}-\partial_b\Gamma^t_{ac}+2\Gamma^t_{cr}\Gamma^r_{ab}\right) \\ &{}^G\overset{*}{\Gamma}{}^{\bar c}_{a\bar b}=-\Gamma^b_{ac}, \quad {}^G\overset{*}{\Gamma}{}^{\bar c}_{\bar a b}=-\Gamma^a_{cb}, \quad {}^G\overset{*}{\Gamma}{}^{\bar c}_{\bar a\bar b}=0 \; .\end{aligned} \quad (1.6)$$

Let ∇ be a Riemannian connection and T be torsion tensor of Riemannian connection ∇ on M. Let $\bar{T}$ be torsion tensor of the $g-$lift ${}^G\overset{*}{\nabla}$ on cotangent bundle T^*M. Using the components ${}^G\overset{*}{\Gamma}{}^C_{AB}$ of ${}^G\overset{*}{\nabla}$ in (1.6) the components $\bar{T}^C_{AB}$ of $\bar{T}$ are obtained with the equation

$$\bar{T}^C_{AB}={}^G\overset{*}{\Gamma}{}^C_{AB}-{}^G\overset{*}{\Gamma}{}^C_{BA} \quad (1.7)$$

according to the induced coordinates $\left(x^h, p_h\right)$. So we obtain

$$\bar{T}^{c}_{ab} = {}^{G}\overset{*}{\Gamma}{}^{c}_{ab} - {}^{G}\overset{*}{\Gamma}{}^{c}_{ba} = \Gamma^{c}_{ab} - \Gamma^{c}_{ba} = \Gamma^{c}_{ab} - \Gamma^{c}_{ab} = 0$$
$$\bar{T}^{c}_{\bar{a}b} = {}^{G}\overset{*}{\Gamma}{}^{c}_{\bar{a}b} - {}^{G}\overset{*}{\Gamma}{}^{c}_{b\bar{a}} = 0$$

$$\bar{T}^{c}_{a\bar{b}} = {}^{G}\overset{*}{\Gamma}{}^{c}_{a\bar{b}} - {}^{G}\overset{*}{\Gamma}{}^{c}_{\bar{b}a} = 0$$

$$\bar{T}^{\bar{c}}_{\bar{a}b} = {}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{\bar{a}b} - {}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{b\bar{a}} = -\Gamma^{a}_{cb} - \left(-\Gamma^{a}_{bc}\right) = -\Gamma^{a}_{cb} + \Gamma^{a}_{bc} = -\Gamma^{a}_{bc} + \Gamma^{a}_{bc} = 0$$
$$\bar{T}^{\bar{c}}_{a\bar{b}} = {}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{a\bar{b}} - {}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{\bar{b}a} = -\Gamma^{b}_{ac} - \left(-\Gamma^{b}_{ca}\right) = -\Gamma^{b}_{ac} + \Gamma^{b}_{ca} = -\Gamma^{b}_{ca} + \Gamma^{b}_{ca} = 0$$
$$\bar{T}^{c}_{\bar{a}\bar{b}} = {}^{G}\overset{*}{\Gamma}{}^{c}_{\bar{a}\bar{b}} - {}^{G}\overset{*}{\Gamma}{}^{c}_{\bar{b}\bar{a}} = 0$$
$$\bar{T}^{\bar{c}}_{\bar{a}\bar{b}} = {}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{\bar{a}\bar{b}} - {}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{\bar{b}\bar{a}} = 0$$
$$\begin{aligned}
\bar{T}^{\bar{c}}_{ab} &= {}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{ab} - {}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{ba} \\
&= p_s\left(\partial_c\Gamma^{s}_{ab} - \partial_a\Gamma^{s}_{bc} - \partial_b\Gamma^{s}_{ac} + 2\Gamma^{s}_{ct}\Gamma^{t}_{ab}\right) - p_s\left(\partial_c\Gamma^{s}_{ba} - \partial_b\Gamma^{s}_{ac} - \partial_a\Gamma^{s}_{bc} + 2\Gamma^{s}_{ct}\Gamma^{t}_{ba}\right) \\
&= p_s\partial_c\Gamma^{s}_{ab} - p_s\partial_a\Gamma^{s}_{bc} - p_s\partial_b\Gamma^{s}_{ac} + p_s 2\Gamma^{s}_{ct}\Gamma^{t}_{ab} - p_s\partial_c\Gamma^{s}_{ba} + p_s\partial_b\Gamma^{s}_{ac} + p_s\partial_a\Gamma^{s}_{bc} - p_s 2\Gamma^{s}_{ct}\Gamma^{t}_{ba} \\
&= p_s\partial_c\Gamma^{s}_{ab} - p_s\partial_c\Gamma^{s}_{ba} + p_s 2\Gamma^{s}_{ct}\Gamma^{t}_{ab} - p_s 2\Gamma^{s}_{ct}\Gamma^{t}_{ba} \\
&= p_s\partial_c\Gamma^{s}_{ab} - p_s\partial_c\Gamma^{s}_{ab} + 2p_s\Gamma^{s}_{ct}\Gamma^{t}_{ab} - 2p_s\Gamma^{s}_{ct}\Gamma^{t}_{ab} = 0 \quad .
\end{aligned}$$

Corollary 1 The torsion tensor according to $g-$lift ${}^{G}\overset{*}{\nabla}$ of Riemannian connection ∇ is equal to zero.

Let $C:[0,1]\to T^{*}M$ be a curve on cotangent bundle $T^{*}M$. And we suppose that C is expressed locally by $x^{C} = x^{C}(t)$, i.e., $x^{c} = x^{c}(t)$, $x^{\bar{c}} = x^{\bar{c}}(t) = p_c(t)$ according to induced coordinates $\left(x^{i}, p_i\right)$ on cotangent bundle $T^{*}M$. t is a parameter.

A curve C on cotangent bundle $T^{*}M$ is a geodesic according to $g-$lift ${}^{G}\overset{*}{\nabla}$ of a Riemannian connection ∇, when it satisfies the differential equation

$$\frac{d^2x^{C}}{dt^2} + {}^{G}\overset{*}{\Gamma}{}^{C}_{AB}\frac{dx^{A}}{dt}\frac{dx^{B}}{dt} = 0 \qquad (1.8)$$

according to the induced coordinates $\left(x^{c}, x^{\bar{c}}\right) = \left(x^{c}, p_c\right)$.

Using the components of $g-$lift ${}^{G}\overset{*}{\nabla}$ we obtain following equations from (1.8):

$$\begin{gathered}
\frac{d^2x^{c}}{dt^2} + {}^{G}\overset{*}{\Gamma}{}^{c}_{ab}\frac{dx^{a}}{dt}\frac{dx^{b}}{dt} + {}^{G}\overset{*}{\Gamma}{}^{c}_{\bar{a}b}\frac{dx^{\bar{a}}}{dt}\frac{dx^{b}}{dt} + {}^{G}\overset{*}{\Gamma}{}^{c}_{a\bar{b}}\frac{dx^{a}}{dt}\frac{dx^{\bar{b}}}{dt} + {}^{G}\overset{*}{\Gamma}{}^{c}_{\bar{a}\bar{b}}\frac{dx^{\bar{a}}}{dt}\frac{dx^{\bar{b}}}{dt} = 0 \\
\frac{d^2x^{c}}{dt^2} + \Gamma^{c}_{ab}\frac{dx^{a}}{dt}\frac{dx^{b}}{dt} = 0
\end{gathered} \qquad (1.9)$$

and

$$\frac{d^2x^{\bar{c}}}{dt^2}+{}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}+{}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{\bar{a}b}\frac{dx^{\bar{a}}}{dt}\frac{dx^b}{dt}+{}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{a\bar{b}}\frac{dx^a}{dt}\frac{dx^{\bar{b}}}{dt}+{}^{G}\overset{*}{\Gamma}{}^{\bar{c}}_{\bar{a}\bar{b}}\frac{dx^{\bar{a}}}{dt}\frac{dx^{\bar{b}}}{dt}=0$$

$$\frac{d^2p_c}{dt^2}+p_s\left(\partial_c\Gamma^s_{ab}-\partial_a\Gamma^s_{bc}-\partial_b\Gamma^s_{ac}+2\Gamma^s_{ct}\Gamma^t_{ab}\right)\frac{dx^a}{dt}\frac{dx^b}{dt}-\Gamma^a_{cb}\frac{dp_a}{dt}\frac{dx^b}{dt}$$

$$-\Gamma^b_{ac}\frac{dx^a}{dt}\frac{dp_b}{dt}=0$$

$$\frac{d^2p_c}{dt^2}+p_s\partial_c\Gamma^s_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}-p_s\partial_a\Gamma^s_{bc}\frac{dx^a}{dt}\frac{dx^b}{dt}-p_s\partial_b\Gamma^s_{ac}\frac{dx^a}{dt}\frac{dx^b}{dt}$$

$$+2\Gamma^s_{ct}\Gamma^t_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}-\Gamma^a_{cb}\frac{dp_a}{dt}\frac{dx^b}{dt}-\Gamma^b_{ac}\frac{dx^a}{dt}\frac{dp_b}{dt}=0$$

$$\frac{d^2p_c}{dt^2}-\frac{d}{dt}\left(\Gamma^s_{ac}p_s\frac{dx^a}{dt}\right)-\Gamma^s_{ac}\frac{dp_s}{dt}\frac{dx^a}{dt}+\Gamma^s_{ac}\Gamma^m_{bs}p_m\frac{dx^a}{dt}\frac{dx^b}{dt}$$

$$+p_s\partial_c\Gamma^s_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}-p_s\partial_a\Gamma^s_{cb}\frac{dx^a}{dt}\frac{dx^b}{dt}+p_s\Gamma^s_{cm}\Gamma^m_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}$$

$$-p_s\Gamma^s_{am}\Gamma^m_{cb}\frac{dx^a}{dt}\frac{dx^b}{dt}=0$$

$$\frac{d}{dt}\left(\frac{dp_c}{dt}-\Gamma^s_{ac}p_s\frac{dx^a}{dt}\right)-\Gamma^s_{ac}\left(\frac{dp_s}{dt}-\Gamma^m_{bs}p_m\frac{dx^b}{dt}\right)\frac{dx^a}{dt}$$

$$+p_sR^s_{cab}\frac{dx^a}{dt}\frac{dx^b}{dt}=0$$

$$\frac{d}{dt}\left(\frac{\delta p_c}{dt}\right)-\Gamma^s_{ac}\frac{\delta p_s}{dt}\frac{dx^a}{dt}+p_sR^s_{cab}\frac{dx^a}{dt}\frac{dx^b}{dt}=0$$

(1.10)

$$\frac{\delta^2p_c}{dt^2}+p_sR^s_{cab}\frac{dx^a}{dt}\frac{dx^b}{dt}=0$$

where $\delta p_c=dp_c-\Gamma^s_{ac}p_sdx^a$. After expressions (1.9) and (1.10) we have

Theorem 2 Let C be a geodesic on cotangent bundle T^*M according to the $g-$lift ${}^{G}\overset{*}{\nabla}$ of Riemannian connection ∇ on M. The geodesic C has equations

$$\frac{d^2x^c}{dt^2}+\Gamma^c_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}=0,$$

$$\frac{\delta^2p_c}{dt^2}+p_sR^s_{cab}\frac{dx^a}{dt}\frac{dx^b}{dt}=0$$

according to the induced coordinates $\left(x^c,p_c\right)$ on cotangent bundle T^*M.

CONCLUSION

In this paper, The torsion tensor and geodesic are studied according to $g-$lift of Riemannian connection to the cotangent bundle T^*M. The torsion tensor components and geodesic equations are obtained by using components of $g-$lift of Riemannian connection ∇ . So the torsion tensor and geodesic according to $g-$ lift of Riemannian connection ∇ are determined on cotangent bundle $T^{*}M$

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