Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (5) (2020), 1635 – 1648
DOI : 10.15672/hujms.500759

Research Article

Rings for which every cosingular module is discrete

Yahya Talebi[1], Ali Reza Moniri Hamzekolaee*[1], Abdullah Harmanci[2], Burcu Ungor[3]

[1] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
[2] *Department of Mathematics, Faculty of Sciences, Hacettepe University, Ankara, Turkey*
[3] *Department of Mathematics, Faculty of Sciences, Ankara University, Ankara, Turkey*

Abstract

In this paper we introduce the concepts of CD-rings and CD-modules. Let R be a ring and M be an R-module. We call R a CD-ring in case every cosingular R-module is discrete, and M a CD-module if every M-cosingular R-module in $\sigma[M]$ is discrete. If R is a ring such that the class of cosingular R-modules is closed under factor modules, then it is proved that R is a CD-ring if and only if every cosingular R-module is semisimple. The relations of CD-rings are investigated with V-rings, GV-rings, SC-rings, and rings with all cosingular R-modules projective. If R is a semilocal ring, then it is shown that R is right CD if and only if R is left SC with $Soc(_RR)$ essential in $_RR$. Also, being a V-ring and being a CD-ring coincide for local rings. Besides of these, we characterize CD-modules with finite hollow dimension.

Mathematics Subject Classification (2010). 16D60, 16E50, 16D90, 16D99

Keywords. CD-module, CD-ring, cosingular module, discrete module, V-ring, semilocal module, finite hollow dimension

1. Introduction

Throughout this paper, R is always an associative ring with identity and all modules are unitary right R-modules, unless otherwise stated. Let M be an R-module. An R-module N is *generated by* M or M*-generated* if there exists an epimorphism $f\colon M^{(I)} \to N$ for some index set I. An R-module N is said to be *subgenerated by* M if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of R-modules whose objects are all R-modules subgenerated by M (see [18]). A submodule L of M is *essential in* M, denoted by $L \leq_e M$, if for every nonzero submodule K of M, $L \cap K \neq 0$. As a dual concept, a submodule N of a module M is called *small* in M, denoted by $N \ll M$, if for every proper submodule L of M, $N + L \neq M$. A module M is called *hollow* if every proper submodule of M is small in M.

*Corresponding Author.
Email addresses: talebi@umz.ac.ir (Y. Talebi), a.monirih@umz.ac.ir (A.R.M. Hamzekolaee), harmanci@hacettepe.edu.tr (A. Harmanci), bungor@science.ankara.edu.tr (B. Ungor)
Received: 21.12.2018; Accepted: 03.12.2019

$Rad(M)$, $Soc(M)$, and $Z(M)$ denote the radical, the socle, and the singular submodule of M, respectively, and $J(R)$ stands for the Jacobson radical of a ring R. Let M be a module. The notations $N \leq M$ and $N \leq_{\oplus} M$ will denote a submodule and a direct summand of M, respectively.

Let M and N be two modules. Then N is said to be *small* (*M-small*) if there exists a module L ($L \in \sigma[M]$) such that $N \ll L$ (in $\sigma[M]$). It is well-known that a module is small (M-small) if and only if it is small in its injective envelope (in $\sigma[M]$). A submodule N of a module M *lies above a direct summand K of M* if $N/K \ll M/K$. Let N and L be submodules of M. N is called a *supplement of L in M* if it is minimal with respect to the property $M = N + L$, equivalently, $M = N + L$ and $N \cap L \ll N$. The module M is called *supplemented* if for each submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll B$. A submodule N of M has a *weak supplement L in M* if $N + L = M$ and $N \cap L \ll M$, and M is called *weakly supplemented* if every submodule N of M has a weak supplement. Any module M is called *amply supplemented* if for any two submodules A and B with $M = A + B$, A contains a supplement of B in M. Recall that M is called *H-supplemented* provided for every submodule N of M, there exists a direct summand D of M such that $\frac{N+D}{N} \ll \frac{M}{N}$ and $\frac{N+D}{D} \ll \frac{M}{D}$. Also M is called *$\oplus$-supplemented* in case for every $N \leq M$, there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$, and in [17], M is called *principally $\oplus$-supplemented* in case for every $m \in M$, there exists a direct summand K of M such that $M = mR + K$ and $mR \cap K \ll K$.

In [15], Talebi and Vanaja define $\overline{Z}_M(N)$ as a dual of M-singular submodule as follows: $\overline{Z}_M(N) = Rej(N, \mathcal{MS}) = \bigcap\{Kerf \mid f\colon N \to S, S \in \mathcal{MS}\} = \bigcap\{U \leq N \mid N/U \in \mathcal{MS}\}$ where $\mathcal{MS}$ denotes the class of all M-small modules. They call N an *M-cosingular* (*non-M-cosingular*) *module* if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$). Clearly, every M-small module is M-cosingular. We should note that cosingular and non-cosingular concepts mean R-cosingular and non-R-cosingular. Let $\mathcal{S}'$ and $\mathcal{S}$ denote the classes of left and right small modules respectively. Recall from [15], $\overline{Z}({}_RR) = Rej(R, \mathcal{S}') = \bigcap\{Kerf \mid f\colon R \to U, U \in \mathcal{S}'\}$ and $\overline{Z}(R_R) = Rej(R, \mathcal{S}) = \bigcap\{Kerf \mid f\colon R \to U, U \in \mathcal{S}\}$. By [1, Corollary 8.23], $\overline{Z}({}_RR)$ and $\overline{Z}(R_R)$ are two-sided ideals of R. A ring R is said to be *right (left) cosingular* if $\overline{Z}(R_R) = 0$ ($\overline{Z}({}_RR) = 0$).

In [6], Keskin and Tribak introduce and study modules M such that every M-cosingular module in $\sigma[M]$ is projective in $\sigma[M]$. They call such modules $COSP$. They investigate some general properties of $COSP$-modules. $COSP$-modules are also characterized when every injective module in $\sigma[M]$ is amply supplemented. Finally they show that a $COSP$-module is Artinian if and only if every submodule has finite hollow dimension.

In [14], the present authors work on rings for which every (simple) cosingular module is projective. They show that for a ring R, every simple cosingular R-module is projective if and only if R is a GV (GCO) ring. They give some conditions for a ring R to have the property that every cosingular R-module is projective. It is also shown for a right perfect ring R under an assumption that every cosingular R-module is projective if and only if R is a left and right Artinian serial ring with $J(R)^2 = 0$.

It is known by [9, Theorem 2.3] that a ring R is right perfect if and only if every quasi-projective R-module is discrete. Inspired by [6] and [14], in this paper, we study rings R (resp., modules M) such that every (resp., M-)cosingular R-module (resp., in $\sigma[M]$) is discrete. We call them *CD-rings* (resp., *CD-modules*). The aim of this article is to characterize rings for which every cosingular module is discrete. We investigate basic properties of CD-modules. It is obtained that every small module over a right CD-ring is semisimple. It is proved that a lifting CD-module has an essential socle. We show that every module over a right V-ring is CD, and so every right V-ring is right CD, the converse is true for local rings. By [7, Proposition 2.7], it is known that every module with finite hollow dimension is semilocal. We observe that a semilocal Artinian (or Noetherian) CD-module has finite hollow dimension. We also give a characterization of a CD-module with finite hollow dimension. This characterization reveals that this kind of module is finitely generated. On the other hand, we investigate under what conditions a CD-module with finite hollow dimension is finitely cogenerated. We show that for a semilocal ring R, R is right CD if and only if $\frac{R}{\overline{Z}(R_R)}$ is semisimple. For a right perfect ring R, it is proved that every $\overline{Z}^2$-torsionfree R-module is (quasi-)discrete if and only if R is right CD. We also present some examples to illustrate different concepts.

2. CD-Modules and CD-Rings

In this section, we introduce a new class of modules (resp. rings), namely CD-modules (resp. CD-rings). An R-module M is CD provided that every M-cosingular R-module in $\sigma[M]$ is discrete. The class of CD-modules contains semisimple modules and V-modules. We introduce and study rings for which every cosingular module is discrete, in this case we call them right CD-rings. Every right V-ring is right CD. We also investigate general properties and some characterizations of CD-rings. For a ring R, we show that R is right CD if and only if every cosingular module is semisimple, under the additional standing assumption that the class of cosingular R-modules is closed under taking homomorphic images.

Let us recall some conditions on a module M as follows:

(D_0) For every decomposition $M = M_1 \oplus M_2$ of M, M_1 and M_2 are relatively projective;
(D_1) Every submodule of M lies above a direct summand of M;
(D_2) If $M/A \cong B \leq_{\oplus} M$, then $A \leq_{\oplus} M$;
(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2 \leq_{\oplus} M$.

The module M is called *discrete* if it satisfies (D_1) and (D_2), *quasi-discrete* if it satisfies (D_1) and (D_3), and *lifting* if M satisfies (D_1). We have the following hierarchy:
discrete $\Rightarrow$ quasi-discrete $\Rightarrow$ lifting $\Rightarrow$ H-supplemented $\Rightarrow$ $\oplus$-supplemented $\Rightarrow$ supplemented.

It is not hard to verify that a ring R is right CD if and only if the R-module R_R is CD if and only if every cyclic R-module is CD.

Proposition 2.1. *Any homomorphic images of a CD-module is CD. In particular, any direct summand of a CD-module is CD.*

Proof. Let M be CD and $N \leq M$. Suppose that L is an M/N-cosingular module in $\sigma[M/N]$. Since $\sigma[M/N] \subseteq \sigma[M]$, we conclude that $\overline{Z}_M(L) \subseteq \overline{Z}_{M/N}(L)$. Hence L is M-cosingular in $\sigma[M]$. Therefore, L is discrete. □

As a consequence, every ring homomorphic image of a CD-ring is CD. The next result is an immediate consequence of Proposition 2.1.

Corollary 2.2. *The following are equivalent for a ring R.*

(1) *Every R-module is CD;*
(2) *Every free R-module is CD;*
(3) *Every projective R-module is CD;*
(4) *Every flat R-module is CD;*
(5) *R is right CD and the class of CD-modules is closed under direct sums.*

Corollary 2.3. *Let R be a right CD-ring and M be a module with cyclic radical. Then $Rad(M)$ is CD as both an R-module and an $R/\overline{Z}(R_R)$-module.*

Proof. Since R is right CD and $Rad(M)$ is cyclic, clearly, $Rad(M)$ is CD as an R-module. On the other hand, by [16, Proposition 2.1], $Rad(M)$ is an $R/\overline{Z}(R_R)$-module.

Also, by Proposition 2.1, $R/\overline{Z}(R_R)$ is a right CD-ring. Therefore $Rad(M)$ is CD as an $R/\overline{Z}(R_R)$-module. □

Proposition 2.4. *If a module M is CD as an $R/\overline{Z}(R_R)$-module, then it is CD as an R-module. The converse holds if M is a cosingular R-module.*

Proof. Let $N \in \sigma[M]$ be an M-cosingular R-module. By [16, Proposition 2.1], $N\overline{Z}_R(R_R) \subseteq \overline{Z}_R(N)$. Note that $\overline{Z}_R(N) \subseteq \overline{Z}_M(N)$. Since N is M-cosingular, $N\overline{Z}_R(R_R) = 0$. Hence N has an $R/\overline{Z}(R_R)$-module structure. By hypothesis, N is a discrete $R/\overline{Z}(R_R)$-module, and so it is a discrete R-module. Thus M is CD as an R-module. Assume now that M is a CD cosingular R-module. Since M is cosingular, $M\overline{Z}_R(R_R) \subseteq \overline{Z}_R(M)$ implies that M is $R/\overline{Z}(R_R)$-module. Any M-cosingular $R/\overline{Z}(R_R)$-module N in $\sigma[M]$ is also an M-cosingular R-module. Then N is discrete as an R-module. Hence N is a discrete $R/\overline{Z}(R_R)$-module. This completes the proof. □

Let $\mathcal{A}$ be a class of R-modules. An R-module M is said to be *$\mathcal{A}$-projective* in case M is projective relative to all elements of $\mathcal{A}$.

Theorem 2.5. *Let $\mathcal{A}$ be a class of R-modules and consider the following conditions.*

(1) *Every module in $\mathcal{A}$ is semisimple;*
(2) *Every module in $\mathcal{A}$ is discrete;*
(3) *Every module in $\mathcal{A}$ is quasi-discrete;*
(4) *Every module in $\mathcal{A}$ satisfies (D_0);*
(5) *Every module in $\mathcal{A}$ is $\mathcal{A}$-projective.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. If $\mathcal{A}$ is closed under finite direct sums, then $(4) \Rightarrow (5)$. If $\mathcal{A}$ is closed under homomorphic images, then $(5) \Rightarrow (1)$.

Proof. $(1) \Rightarrow (2)$ It is clear by definitions.
$(2) \Rightarrow (3)$ It follows from [8, Lemma 4.6].
$(3) \Rightarrow (4)$ By [8, Lemma 4.23], every quasi-discrete module satisfies (D_0).
Assume now that $\mathcal{A}$ is closed under finite direct sums. $(4) \Rightarrow (5)$ Let $M_1, M_2 \in \mathcal{A}$ and $M = M_1 \oplus M_2$. By assumption $M \in \mathcal{A}$, and by (4), M satisfies (D_0). Hence M_1 and M_2 are relatively projective.
Let $\mathcal{A}$ be closed under homomorphic images. $(5) \Rightarrow (1)$ Let $M \in \mathcal{A}$ and $L \leq M$. By assumption, $M/L \in \mathcal{A}$, and it is M-projective by (5). It follows that L is a direct summand of M. Therefore M is semisimple. □

If we replace $\mathcal{A}$ with the class of cosingular modules, we have the following result.

Corollary 2.6. *If the class of cosingular R-modules is closed under homomorphic images, then the following statements are equivalent.*

(1) *R is right CD;*
(2) *Every cosingular R-module is semisimple;*
(3) *Every cyclic cosingular R-module is semisimple;*
(4) *Every cosingular R-module is quasi-discrete;*
(5) *Every cosingular R-module satisfies (D_0);*
(6) *Every cosingular R-module is N-projective for every cosingular R-module N.*

If any of above statements holds, then every cosingular R-module is quasi-projective.

Proposition 2.7. *Let R be a right perfect ring and M an R-module. Then the following are equivalent.*

(1) *Every direct product of M-projective R-modules is discrete;*
(2) *Every direct product of M-projective R-modules satisfies (D_0).*

In this case, the class of M-projective R-modules is closed under direct products.

Proof. (1) ⇒ (2) It follows from [8, Lemma 4.23].
(2) ⇒ (1) Let $N = \prod_{i\in I} N_i$ be a product of M-projective R-modules. Then, by assumption $N \times N \cong N \oplus N$ satisfies (D_0). Hence N is quasi-projective. Since R is right perfect, by [9, Theorem 2.3], N is discrete.

To prove the last statement, note that R is right perfect, so M has a projective cover $f\colon P \to M$. By assumption, $N \oplus P$ satisfies (D_0) where $N = \prod_{i\in I} N_i$ is a product of M-projective R-modules. Hence N is P-projective. Therefore N is M-projective. □

As a consequence of Proposition 2.7, we give a new characterization of commutative Artinian rings.

Corollary 2.8. *Let R be a commutative perfect ring. Then the following are equivalent.*

(1) *R is Artinian;*
(2) *Every direct product of projective R-modules is discrete;*
(3) *Every direct product of projective R-modules is quasi-discrete;*
(4) *Every direct product of projective R-modules satisfies (D_0).*

Proof. (1) ⇒ (2) By [2, Theorems 3.3 and 3.4], every direct product of projective R-modules is projective and also discrete by [9, Theorem 2.3].
(2) ⇒ (3) Obvious.
(3) ⇒ (4) It follows from [8, Lemma 4.23].
(4) ⇒ (1) Let P be a direct product of projective R-modules and M an arbitrary R-module. There exists a set I and a submodule L of $R^{(I)}$ such that $M \cong R^{(I)}/L$. Let $N = P \oplus R^I$ which is a direct product of projective modules. By (4), N satisfies (D_0). It follows that P is R^I-projective. By [8, Proposition 4.31], P is $R^{(I)}/L$-projective. Hence P is M-projective. Therefore P is a projective R-module. The result follows from [2, Theorem 3.4]. □

Now we can replace $\mathcal{A}$ in Theorem 2.5 with the class of small modules.

Corollary 2.9. *Let R be a ring. Then the following statements are equivalent.*

(1) *Every small R-module is semisimple;*
(2) *Every small R-module is discrete;*
(3) *Every small R-module is quasi-discrete;*
(4) *Every small R-module satisfies (D_0);*
(5) *Every small R-module is N-projective for every small R-module N.*

Let M be a module. In [19], M is called *coatomic* if every proper submodule is contained in a maximal submodule, or equivalently, for a submodule N of M, if $Rad(M/N) = M/N$, then $M = N$. Finitely generated modules and semisimple modules are coatomic. The following result exhibits some basic properties of CD-modules.

Proposition 2.10. *Let M be a CD-module. Then the following hold.*

(1) *Every M-small module is semisimple. In particular, every small submodule of M is semisimple.*
(2) *$Rad(M) \subseteq Soc(M)$.*
(3) *M is coatomic.*
(4) *$Rad(M) \ll M$.*
(5) *Every finitely generated submodule of $Rad(M)$ is Artinian (Noetherian).*

Proof. (1) Every M-small module is M-cosingular, therefore discrete. Since the class of M-small modules is closed under finite direct sums and homomorphic images, by Theorem 2.5, every M-small module is semisimple.
(2) By (1), $Rad(M)$ is semisimple and hence $Rad(M) \subseteq Soc(M)$.
(3) By (2), $Rad(M) \subseteq Soc(M)$. If $Soc(M) = M$, then $Rad(M) = 0$ and if $Soc(M) \neq M$,

then $Rad(M) \neq M$. In both conditions, M has a maximal submodule. Applying the same argument for M/N where $N \subsetneq M$ implies that N is contained in a maximal submodule of M since M/N is a CD-module. Thus M is coatomic.
(4) Assume that $Rad(M)$ is not small in M. Then there exists a proper submodule N of M such that $M = Rad(M) + N$. By (3), N is contained in a maximal submodule K of M. It follows that $K = M$. This contradiction implies $Rad(M) \ll M$.
(5) The result follows from the fact that $Rad(M)$ is semisimple. □

By the above proposition, a CD-module cannot be radical and small right ideals of right CD-rings are semisimple as an R-module.

Corollary 2.11. *Let R be a right CD-ring. Then the following statements hold.*

(1) *Every small R-module is semisimple.*
(2) $J(R) \subseteq Soc(R_R)$.

For an easy reference we note the following result.

Lemma 2.12. *Let M be a module such that $M/\overline{Z}_M(M)$ is semisimple, then $Rad(M) \subseteq \overline{Z}_M(M)$. The converse holds if M is a lifting module.*

Proof. Let M be a module such that $M/\overline{Z}_M(M)$ is semisimple and π denote the natural epimorphism from M onto $M/\overline{Z}_M(M)$ with kernel $\overline{Z}_M(M)$. Since $M/\overline{Z}_M(M)$ is semisimple, $Rad(M/\overline{Z}_M(M)) = 0$. Hence $\pi(Rad(M)) = 0$. Therefore $Rad(M) \subseteq \overline{Z}_M(M)$. Conversely, assume that $Rad(M) \subseteq \overline{Z}_M(M)$. Let $N/\overline{Z}_M(M) \leq M/\overline{Z}_M(M)$. By hypothesis, there exists a submodule $A \leq N$ such that $M = A \oplus B$ with $N \cap B$ small in B. Then $N \cap B \subseteq Rad(M)$ and hence $N \cap B \subseteq \overline{Z}_M(M)$. Since $N \cap (B + \overline{Z}_M(M)) = \overline{Z}_M(M) + N \cap B$, $M/\overline{Z}_M(M) = N/\overline{Z}_M(M) \oplus ((B + \overline{Z}_M(M))/\overline{Z}_M(M))$. This completes the proof. □

Let U be a submodule of a module M. Recall that M is called *U-lifting* if for any submodule N of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \leq U$.

Proposition 2.13. *Consider the following conditions for a module M.*

(1) *M is $\overline{Z}_M(M)$-lifting;*
(2) *$M/\overline{Z}_M(M)$ is semisimple;*

Then (1) $\Rightarrow$ (2)*. The converse holds if M is lifting.*

Proof. (1) $\Rightarrow$ (2) Let N be a submodule of M with $\overline{Z}_M(M) \leq N$. There exists a submodule $A \leq N$ such that $M = A \oplus B$ and $N \cap B \leq \overline{Z}_M(M)$. Then $M/\overline{Z}_M(M) = N/\overline{Z}_M(M) \oplus (B + \overline{Z}_M(M))/\overline{Z}_M(M)$.
Assume that M is lifting. (2) $\Rightarrow$ (1) Let N be any submodule of M. By assumption, N has a submodule A such that $M = A \oplus B$ with $N \cap B$ small in B. Then $N \cap B \subseteq Rad(M)$. By Lemma 2.12, all small submodules of M are contained in $\overline{Z}_M(M)$. Hence $N \cap B \leq \overline{Z}_M(M)$. This completes the proof. □

Theorem 2.14. *Let M be a lifting CD-module. Then $Soc(M)$ is essential in M.*

Proof. Assume that $Soc(M)$ is not essential in M. There exists a nonzero submodule N of M such that it is maximal with respect to the property $Soc(M) \cap N = 0$. Then $Soc(M) \oplus N$ is an essential submodule of M. M being lifting implies that there exists a direct summand A of M such that $A \leq N$, $M = A \oplus B$ with $N \cap B$ small in B and also in M. So $N \cap B$ is semisimple by Lemma 2.10. Then $N \cap B = 0$. Hence $M = N \oplus B$. It follows that N is a lifting CD-module as a direct summand of M. Let X be any submodule of N. There exists a direct summand $Y \leq X$ of N such that $N = Y \oplus Z$ with $X \cap Z$ small in Z and in N and so in M. Again by Lemma 2.10, $X \cap Z$ is semisimple. Hence $X \cap Z = 0$. Thus $N = X \oplus Z$. It follows that N is semisimple. Thus $N = 0$ and $Soc(M)$ is essential in M. □

Corollary 2.15. *Let M be a CD-module having a decomposition $M = Soc(M) \oplus N$ with N lifting. Then M is semisimple.*

Proof. As a direct summand, N is a lifting CD-module. By Theorem 2.14, Soc(N) is essential in N. Hence $N = 0$. So M is semisimple. □

Corollary 2.16. *Let R be a right CD-ring having a decomposition $R = Soc(R_R) \oplus N$ with N lifting as an R-module. Then R is semisimple.*

Recall from [18] that a ring R is a *right V-ring* provided that every simple R-module is injective, equivalently, R is a right V-ring if and only if every R-module has zero radical. Since the only cosingular module over a right V-ring is zero, every right V-ring is right CD. A ring R is *right generalized co-semisimple* (GCO for short) provided that every simple singular R-module is injective, and R is a *right GV-ring* if each simple R-module is either injective or projective. Note that R is right GCO if and only if it is right GV. Observe that a right GV-ring with zero socle is a right V-ring. The next result shows that every module over a right V-ring (equivalently, a right CD local ring) is CD.

Theorem 2.17. *Let R be a ring and consider the following conditions.*

(1) *R is a right V-ring;*
(2) *Every R-module is CD;*
(3) *R is right CD.*

Then (1) ⇒ (2) ⇒ (3). *If R is local, then all of them are equivalent.*

Proof. (1) ⇒ (2) Let R be a right V-ring and M an R-module. For any M-cosingular module $N \in \sigma[M]$, by [16, Proposition 2.10], $\overline{Z}_M(N) = N = 0$. Hence N is discrete, thus M is CD.
(2) ⇒ (3) Obvious.
Assume now that R is a local ring. (3) ⇒ (1) Let $a \in R$. Since R is local, it is principally hollow (see [5]). This implies that aR is small in R. Then for any homomorphism $f\colon R \to S$ with S small, $f(a)R$ is small in S. On the other hand, R being right CD implies that S is semisimple by Corollary 2.11(1). Hence $f(a)R$ is a direct summand of S. Thus $f(a)R = 0$, i.e., $a \in \mathrm{Ker} f$. It follows that $a \in \overline{Z}_R(R_R)$, and so $R = \overline{Z}_R(R_R)$. By [15, Corollary 2.6], R is a right V-ring. □

Proposition 2.18. *Let R be a ring such that every cosingular module is amply supplemented. Then R is right GV if and only if every cosingular R-module is projective. In this case R is right CD and the class of cosingular R-modules is closed under homomorphic images.*

Proof. Assume that R is right GV. Let $0 \neq M$ be a cosingular R-module, $0 \neq x \in M$ and K a maximal submodule of xR. Now the simple module xR/K is either singular or projective (but not both). If xR/K is singular, then it will be noncosingular by [10, Theorem 4.1]. Consider the natural epimorphism $\pi\colon xR \to xR/K$. By assumption, xR is amply supplemented. Now [15, Theorem 3.5] implies that $0 = \pi(\overline{Z}^2(xR)) = \overline{Z}^2(xR/K) = \overline{Z}(xR/K) = xR/K$. which is a contradiction. Then xR/K is projective and so K is a direct summand of xR. Hence xR and, therefore M is semisimple. Let $M = \oplus_{i \in I} M_i$ where each M_i is simple. Then M_i is singular or projective. Assume that it is singular. Then [10, Theorem 4.1] implies that it is noncosingular that contradicts M is cosingular. Hence each M_i is projective and so is M. Conversely, suppose that every cosingular module is projective. In particular every simple cosingular module is projective. Let M be a simple singular module. Then M is either small or injective. If M is small, then M is projective by supposition since every small module is cosingular. The module M being simple singular implies that M cannot be projective. Thus M is injective. It follows that R is right GV. □

A ring R is *right* (resp. *left*) *nonsingular* if $Z_r(R) = \{x \in R \mid xI = 0, I \leq_e R_R\} = 0$ (resp. $Z_l(R) = \{x \in R \mid Ix = 0, I \leq_e {}_RR\} = 0$). A ring R is *right* (resp. *left*) *SI* provided that every singular right (resp. left) R-module is injective. These rings were introduced and fully investigated by Goodearl in [4].

Remark 2.19. If for a CD-module M, the class of M-cosingular modules is closed under factor modules, then every M-cosingular M-injective module is zero. So for a right CD-ring R such that the class of cosingular R-modules is closed under homomorphic images (e.g. semiperfect right SI-rings), every cosingular injective R-module is zero. This answers one of the questions posed by Talebi and Vanaja (see [15, Page 1460, Question 3]).

Proposition 2.20. *Let R be a right GV-ring. Then R is right CD if and only if every cyclic cosingular R-module is amply supplemented.*

Proof. Assume that every cyclic cosingular R-module is amply supplemented. Let $0 \neq M$ be a cosingular module. By a similar discussion in the proof of Proposition 2.18, M is semisimple. Clearly M is discrete. Conversely, assume that R is CD and let M be a cyclic cosingular module. By assumption, M is discrete. So M is lifting and obviously amply supplemented. □

Remark 2.21. Let R be a right cosingular right CD-ring. Then by Corollary 2.6, every cosingular R-module is R-projective. In particular, any finitely generated cosingular R-module is projective.

A module M is said to have *finite hollow dimension* in case there exists an epimorphism $f: M \to \prod_{i=1}^{n} H_i$ with all H_i hollow and $Kerf \ll M$. In this case, it is said that the *hollow dimension of* M is n. Recall that a module M is called *semilocal* if $M/Rad(M)$ is semisimple (see [7] for details). A ring R is *semilocal* if the right R-module R is semilocal, i.e., $R/J(R)$ is a semisimple ring. By [7, Proposition 2.7], every module with finite hollow dimension is semilocal. The converse statement holds for finitely generated modules. In particular, for CD modules we have the following result.

Proposition 2.22. *Let M be an Artinian (or Noetherian) and CD-module. Then the following conditions are equivalent.*

(1) *M has finite hollow dimension;*
(2) *M is weakly supplemented;*
(3) *M is semilocal.*

Proof. (1) ⇒ (2) ⇒ (3) By [7, Proposition 2.7].
(3) ⇒ (2) Since M is a CD-module, by Proposition 2.10, $Rad(M)$ is small in M. The rest is clear by [7, Proposition 2.7].
(2) ⇒ (1) The module M being CD implies that $Rad(M) \ll M$, and so the hollow dimensions of M and $M/Rad(M)$ are equal due to [7, Remark 1.4]. On the other hand, since M is weakly supplemented, $M/Rad(M)$ is weakly supplemented. Hence by [7, Corollary 2.3], the hollow dimension and length of $M/Rad(M)$ are equal. The hypothesis and the semisimplicity of $M/Rad(M)$ imply that $M/Rad(M)$ is both Artinian and Noetherian. Thus $M/Rad(M)$ has finite length. Therefore the hollow dimension of M is finite. □

The next result shows that every CD-module with finite hollow dimension is finitely generated.

Theorem 2.23. *The following are equivalent for a CD-module M.*

(1) *M has finite hollow dimension;*
(2) *M is semilocal and finitely generated.*

Proof. In light of [7, Proposition 2.7], it is enough to prove that a CD-module with finite hollow dimension is finitely generated. Let M be a CD-module with finite hollow dimension. By [13, Corollary 1.11], $M/Rad(M)$ is semisimple and Artinian. Hence $M/Rad(M)$

is finitely generated. On the other hand, M being a CD-module implies that $Rad(M)$ is small in M by Proposition 2.10. Therefore M is finitely generated due to [1, Theorem 10.4]. □

We now investigate under what conditions a CD-module with finite hollow dimension is finitely cogenerated.

Proposition 2.24. *The following statements are equivalent for a CD-module M with finite hollow dimension.*

(1) *M is finitely cogenerated;*
(2) *$Rad(M)$ is Artinian;*
(3) *$Soc(M)$ is Artinian;*
(4) *M is Artinian.*

Proof. (1) ⇒ (2) $Rad(M)$ is finitely cogenerated as a submodule of finitely cogenerated M, and by Proposition 2.10, $Rad(M)$ is semisimple. Hence $Rad(M)$ is Artinian.
(2) ⇒ (1) Since M has finite hollow dimension, $M/Rad(M)$ is semisimple Artinian by [13, Corollary 1.11], and so $M/Rad(M)$ is finitely cogenerated. On the other hand, by Proposition 2.10, $Rad(M)$ is semisimple. Hence (2) implies that $Rad(M)$ is finitely cogenerated. Since both of $\mathrm{Rad}(M)$ and $M/Rad(M)$ are finitely cogenerated, M is finitely cogenerated.
(1) ⇒ (3) By [1, Theorem 10.4], $\mathrm{Soc}(M)$ is finitely cogenerated, and so it is Artinian.
(3) ⇒ (1) Since M has finite hollow dimension, Proposition 2.22 implies that M is semilocal, i.e., $M/Rad(M)$ is semisimple. Then $M/Soc(M)$ is semisimple as a homomorphic image of semisimple module $M/Rad(M)$. By [7, Proposition 2.1(c)], M has a decomposition $M = M_1 \oplus M_2$ where M_1 is semisimple and $Soc(M)$ is essential in M_2. Hence $M_1 = 0$, and so $Soc(M)$ is essential in M. Thus M is finitely cogenerated due to [1, Theorem 10.4].
(3) ⇒ (4) By a similar discussion in the proof of (3) ⇒ (1), [13, Corollary 1.11] implies $M/Rad(M)$ is Artinian, and so is $M/Soc(M)$. Since both of $Soc(M)$ and $M/Soc(M)$ are Artinian, M is also Artinian.
(4) ⇒ (3) Obvious. □

Corollary 2.25. *Let R be a right Noetherian ring and M a CD-module with finite hollow dimension. Then the following are equivalent.*

(1) *M is finitely cogenerated;*
(2) *$Soc(M)$ is essential in M.*

Proof. (1) ⇒ (2) It is known by [1, Theorem 10.4].
(2) ⇒ (1) Since M is a CD-module with finite hollow dimension, M is finitely generated by Theorem 2.23. The ring R being right Noetherian implies that $Soc(M)$ is also finitely generated. Therefore [1, Proposition 10.7] completes the proof. □

Proposition 2.26. *Let R be a commutative domain. Then the following are equivalent.*

(1) *R is CD;*
(2) *Every cosingular R-module is projective;*
(3) *R is a field.*

Proof. (1) ⇒ (2) Let R be a CD commutative domain. It is well-known that R_R is a small R-module. So, by Proposition 2.11(1), R is semisimple. Then every R-module is projective, so (2) holds.
(2) ⇒ (3) Let $I \leq R$. Then R/I is cosingular since R is small and homomorphic images of small modules are small. By (2), R/I is projective, therefore I is a direct summand of R. Hence R is simple and so a field.
(3) ⇒ (1) Clear. □

Proposition 2.27. *Let R be a ring such that the class of cosingular R-modules is closed under factor modules. Then the following statements are equivalent.*

(1) *R is right CD;*
(2) *Every cosingular R-module is semisimple;*
(3) *The ring $R/\overline{Z}(R_R)$ is semisimple.*

Proof. (1) $\Leftrightarrow$ (2) It follows from Corollary 2.6.
(2) $\Leftrightarrow$ (3) This follows from [16, Proposition 2.1(2)] and the fact that $R/\overline{Z}(R_R)$ is a cosingular R-module. □

Proposition 2.28. *Let R be a ring such that every cosingular R-module is semisimple. If for every R-module M, $\overline{Z}(M) \leq_\oplus M$, then every cosingular R-module is projective.*

Proof. Let N be an R-module. Then $N = \overline{Z}(N) \oplus T$, where $\overline{Z}(N)$ is non-cosingular and L is cosingular and hence semisimple. We show that every cosingular R-module is projective. Let M be a cosingular R-module and $f\colon N \longrightarrow M$ an epimorphism with N a free module. Now, $f(\overline{Z}(N)) \subseteq \overline{Z}(M) = 0$. Hence $\overline{Z}(N) \subseteq Kerf$. It follows that $Kerf = \overline{Z}(N) \oplus (T \cap Kerf)$. Since T is semisimple, $T = (T \cap Kerf) \oplus S$ for some submodule S of T. It is easy to check that $N = Kerf \oplus S$. Therefore M is projective. □

Corollary 2.29. *Every cosingular R-module is projective in each of the following cases:*

(1) *R is a right CD-ring such that the class of cosingular R-modules is closed under factor modules and for every R-module M, $\overline{Z}(M) \leq_\oplus M$.*
(2) *Every R-module is a direct sum of a non-cosingular R-module and a semisimple R-module. (Clearly in this case R is also right CD).*

Proof. (1) It follows from Corollary 2.6 and Proposition 2.28.
(2) By [15, Corollary 3.9]. □

3. Applications to some classes of modules and rings

In this section, we study the CD-property for some classes of modules and rings, and present some examples. We show that for a semilocal ring, being a right CD-ring implies being a left CD-ring. By a similar argument to [16, Corollary 2.7], for a semilocal ring R, we have $\overline{Z}(R_R) = Soc({}_RR)$ and $\overline{Z}({}_RR) = Soc(R_R)$.

Lemma 3.1. *Let R be a semilocal ring. Then there exists a decomposition $R = R_1 \oplus R_2$ with R_1 semisimple, $J(R)$ essential in R_2, $R_2/J(R)$ semisimple and $Soc(R_R) \subseteq R_1 \oplus J(R)$. If $J(R) \subseteq Soc(R_R)$, then $Soc(R_R) = R_1 \oplus J(R)$.*

Proof. By [7, Theorem 3.5], R has a decomposition $R = R_1 \oplus R_2$ with R_1 semisimple, $J(R)$ essential in R_2 and $R_2/J(R)$ semisimple. $J(R)$ being essential in R_2 implies that $Soc(R_R) \subseteq R_1 \oplus J(R)$. If $J(R) \subseteq Soc(R_R)$, then clearly, $R_1 \oplus J(R) \subseteq Soc(R_R)$. □

The following result introduces a large class of two-sided CD-rings. It is known by Corollary 2.11 that if a ring R is right CD, then $J(R) \subseteq Soc(R_R)$, and so $J(R)^2 = 0$. The next result also exhibits that the converse of this statement holds for semilocal rings.

Proposition 3.2. *Let R be a semilocal ring with $J(R) \subseteq Soc(R_R)$ (resp., $J(R) \subseteq Soc({}_RR)$). Then R is left (resp., right) CD. In particular, every semilocal ring with $J(R)^2 = 0$ is left and right CD.*

Proof. Let R be a semilocal ring with $J(R) \subseteq Soc(R_R)$. It follows that $\frac{R}{Soc(R_R)} = \frac{R}{\overline{Z}({}_RR)}$ is semisimple, since $\frac{R}{\overline{Z}({}_RR)}$ is a homomorphic image of $R/J(R)$. Hence every cosingular left R-module is semisimple by [16, Proposition 2.1(2)] and therefore R is left CD. To prove the last part, let R be semilocal with $J(R)^2 = 0$. By [16, Proposition 2.6 and Corollary 2.7], $Soc({}_RR) = Ann_r(J(R))$ and $Soc(R_R) = Ann_l(J(R))$. Since $J(R)^2 = 0$, we have $J(R) \subseteq Soc(_RR)$ and $J(R) \subseteq Soc(R_R)$. Hence by the first part, R is left and right CD. □

We now present a right (left) cosingular semilocal ring which is not right (left) CD.

Example 3.3. Let D be a commutative local integral domain with field of fractions Q (for example, we might take D the localization of the integers $\mathbb{Z}$ by a prime number p, i.e., D is the subring of the field of rational numbers consisting of fractions a/b such that b is not divisible by p). Let $R = \begin{pmatrix} D & Q \\ 0 & Q \end{pmatrix}$. The operations are given by the ordinary matrix operations. Since D is local it has a unique maximal ideal, say m and the Jacobson radical of R is $J(R) = \begin{pmatrix} m & Q \\ 0 & 0 \end{pmatrix}$. Then $R/J(R) \cong (D/m) \times Q$. Thus R is semilocal. On the other hand, if we suppose that D has zero socle, then R has zero left socle and so $\overline{Z}(R_R) = Soc(_RR) = 0$. Hence R is right cosingular. But R has non-zero right socle, namely, $\overline{Z}(_RR) = Soc(R_R) = \begin{pmatrix} 0 & Q \\ 0 & Q \end{pmatrix}$. It follows that R is right cosingular but not left cosingular. Since $J(R) \nsubseteq Soc(R_R)$ and $J(R) \nsubseteq Soc(_RR)$, R is neither right CD nor left CD by Corollary 2.11.

The following example shows that the class of CD-rings contains properly the class of V-rings.

Example 3.4. Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ the ring of 2×2 upper triangular matrices over F. It is well-known that R is a right and left (SI) GV-ring which is neither a right nor a left V-ring because of $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Since R is left and right Artinian serial with $J(R)^2 = 0$, by Proposition 3.2, R is left and right CD.

Recall that a ring R is called *right Harada* (a *right H-ring* for short) provided that every injective right R-module is lifting. It is well-known that R is a right H-ring if and only if every right R-module is decomposed to a small module and an injective module.

Proposition 3.5. *Let R be a right CD right H-ring. Then R is an (left and right) Artinian serial ring with $J(R)^2 = 0$.*

Proof. Let R be a right CD right H-ring. By [3, 28.10], for every R-module M, there exists a direct decomposition $M = S \oplus E$ where S is small and E is an injective R-module. Since R is right CD, S is semisimple by Corollary 2.11(1). It follows that R is Artinian serial with $J(R)^2 = 0$ by [3, 29.10]. □

Remark 3.6. Note that a semilocal non-semisimple ring with $Soc(_RR)$ right semisimple cannot have the property that all cosingular right R-modules and all cosingular left R-modules are projective. For if, assume that R is a semilocal ring such that all cosingular right R-modules and all cosingular left R-modules are projective. Then $J(R) \subseteq Soc(_RR) = \overline{Z}(R_R) \leq_\oplus R$. Since $J(R) \ll R$ and $\overline{Z}(R_R) \leq_\oplus R$, we have $J(R) \ll \overline{Z}(R_R)$. Since $\overline{Z}(R_R)$ is a right semisimple R-module, it follows that $J(R) = 0$. Hence R is semisimple. The ring $R = \frac{\mathbb{Z}}{4\mathbb{Z}}$ is a local CD-ring but does not have the property that every cosingular R-module is projective. Also R is not GV.

An R-module M is called an *SI-module* provided that every M-singular R-module is M-injective. A generalization of SI-rings is SC-rings. In [12], Sanh defined and investigated SC-modules. An R-module M is called an *SC-module* if every M-singular R-module is continuous. A ring R is a *right SC-ring* if the right R-module R is an SC-module, that is, every singular right R-module is continuous. Left SC-rings are defined similarly. SC-rings generalizes SI-rings and SC-rings were introduced and studied by Rizvi and Yousif [11]. Note that every semiperfect right SI-ring is a right CD-ring by Proposition 2.18.

Lemma 3.7 ([12, Corollary 8]). *For a module M, the following conditions are equivalent.*
(1) *M is an SC-module with essential $Soc(M)$;*
(2) *$M/Soc(M)$ is semisimple.*

In what follows, we show that being a CD-ring is left-right symmetric for semilocal rings.

Theorem 3.8. *Let R be a semilocal ring. Then the following statements are equivalent.*
(1) *R is a left SC-ring with $Soc(_RR)$ essential as a left ideal in R;*
(2) *R is right CD;*
(3) *The ring $R/\overline{Z}(R_R)$ is semisimple;*
(4) *The ring $R/Soc(_RR)$ is semisimple.*
If R satisfies one of these conditions, then R is a left CD-ring.

Proof. (1) $\Leftrightarrow$ (4) It follows from Lemma 3.7.
(3) $\Leftrightarrow$ (4) It is clear from the fact that R is semilocal and so $\overline{Z}(R_R) = Soc(_RR)$.
(2) $\Rightarrow$ (3) It is well-known that $R/\overline{Z}(R_R)$ is a subdirect product of small R-modules. Since R is right CD, all small right R-modules are semisimple by Corollary 2.11 (1). Also since R is semilocal, every direct product of semisimple R-modules is semisimple. Hence $R/\overline{Z}(R_R)$ is semisimple.
(3) $\Rightarrow$ (2) In this case every cosingular right R-module is semisimple and every semisimple module is discrete. Therefore R is right CD.
For the last statement, since R is right CD, by Corollary 2.11(2), $J(R)^2 = 0$. So R is left CD by Proposition 3.2. ☐

Corollary 3.9. *Let R be a commutative semilocal ring. Then R is CD if and only if R is SC.*

Proof. It follows from [11, Theorem 3.8] and Theorem 3.8. ☐

Remark 3.10. Every non-trivial ideal of a local right CD-ring R (or a ring R with all cosingular right R-modules projective) is semisimple. However, R need not be semisimple. For instance, $R = \mathbb{Z}/4\mathbb{Z}$ is a local CD-ring by Theorem 3.8, and its only non-trivial ideal is simple and R is not semisimple.

Lemma 3.11. *A ring R is left nonsingular, semilocal with $R/\overline{Z}(R_R)$ semisimple if and only if R is semisimple.*

Proof. One direction is clear. For the other direction, assume that R is a semilocal, left nonsingular ring with $R/\overline{Z}(R_R)$ semisimple. Then $R/J(R)$ is semisimple. To complete the proof we show $J(R) = 0$. For the semilocal ring R, $R/\overline{Z}(R_R)$ being semisimple implies that $R/Soc(_RR)$ is semisimple. By [7, Proposition 2.1(c)], $Soc(_RR)$ is essential in R as a left ideal and so $J(R)$ is singular as a left R-module. By assumption, $J(R) = 0$. Thus R is semisimple. ☐

The following example shows that a right CD-ring need not be SI or GV or have the property that every cosingular R-module is projective.

Example 3.12. Let p and q be two distinct prime numbers. Then for $m, n \in \{0, 1, 2\}$, the ring $R = \frac{\mathbb{Z}}{p^m q^n \mathbb{Z}}$ is a CD-ring but does not have the property that every cosingular R-module is projective (m and n cannot both be zero and also cannot both be one).

Proof. It is clear that R is semilocal. Let $m = 2$ and $n = 1$. Then $Soc(R) = \frac{pq\mathbb{Z}}{p^2q\mathbb{Z}} + \frac{p^2\mathbb{Z}}{p^2q\mathbb{Z}}$. Since $\mid Soc(R) \mid= pq$, we have $\frac{R}{\overline{Z}(R)} = \frac{R}{Soc(R)}$ is a field. Now by Theorem 3.8, R is SC and CD. Let I_1, I_2 and I_3 be non-trivial ideals of R with $\mid I_1 \mid= p$, $\mid I_2 \mid= p^2$ and $\mid I_3 \mid= pq$. We also have $\overline{Z}(R) = Soc(R) = I_3$. Since $Soc(I_2) \neq 0$, it follows that $\overline{Z}(R)$ is not a direct summand of R. Therefore, every cosingular R-module is not projective (see Remark 3.6). Now, let $m = 2$ and $n = 2$. Suppose that $I_1, \ldots, I_7$ be non-trivial ideals of R such that $| I_1 |= p$, $| I_2 = q$, $| I_3 |= pq$, $| I_4 |= p^2q$, $| I_5 |= pq^2$, $| I_6 |= p^2$ and $| I_7 |= q^2$. Then $Soc(R) = \overline{Z}(R) = I_1 + I_2 = I_3$, which implies $| \frac{R}{\overline{Z}(R)} |= pq$, hence it is semisimple. It is not hard to verify that $\overline{Z}(R)$ is not a direct summand of R. So not every cosingular R-module is projective. Similar arguments hold for the case $m = 2$ or $n = 2$. Since the rings as above are not semisimple, by Lemma 3.11, R is not nonsingular. Now, by [11, Lemma 3.1] R is not SI. Also R is a perfect ring, so that R is not a GV-ring by Proposition 2.18. We conclude that the class of cosingular R-modules is not closed under homomorphic images. □

Recall that for a module M, $\overline{Z}^2(M)$ is defined as $\overline{Z}(\overline{Z}(M))$.

Definition 3.13. A module M is called $\overline{Z}^2$-*torsionfree* in case $\overline{Z}^2(M) = 0$.

It is easy to see that every cosingular module is $\overline{Z}^2$-torsionfree. The class of $\overline{Z}^2$-torsionfree modules is closed under submodules, direct sums and direct products (see [15, Proposition 2.1]). By [8, Theorem 4.41] and [15, Proposition 2.1 and Theorem 3.5], it also follows that for a perfect ring R, the class of $\overline{Z}^2$-torsionfree R-modules is closed under factor modules.

Theorem 3.14. *Let R be a right perfect ring. Consider the following conditions.*

(1) *Every $\overline{Z}^2$-torsionfree R-module is discrete;*
(2) *Every $\overline{Z}^2$-torsionfree R-module is quasi-discrete;*
(3) *Every $\overline{Z}^2$-torsionfree R-module is semisimple;*
(4) *R is right CD.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *If R is right GV, then* (4) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Clear by definitions.
(2) $\Rightarrow$ (3) Let $M = yR$ be a cyclic $\overline{Z}^2$-torsionfree R-module and $x \in yR$. Let K be a maximal submodule of xR. Since R is right perfect and yR is $\overline{Z}^2$-torsionfree, xR/K is $\overline{Z}^2$-torsionfree. So $xR/K \oplus xR$ is $\overline{Z}^2$-torsionfree. Now, by assumption, $xR/K \oplus xR$ is quasi-discrete and hence satisfies (D_0)-condition by [8, Lemma 4.23]. It follows that xR/K is xR-projective. This implies that $K \leq_\oplus xR$. Hence, xR and finally yR are semisimple. It follows that every $\overline{Z}^2$-torsionfree R-module is semisimple.
(3) $\Rightarrow$ (4) By the fact that every cosingular module is $\overline{Z}^2$-torsionfree, (3) implies that every cosingular R-module is semisimple. Thus R is right CD.
Assume now that R is right GV. (4) $\Rightarrow$ (1) Let R be a right CD ring. Since R is right perfect, every cosingular R-module is projective by Proposition 2.18. Let M be a $\overline{Z}^2$-torsionfree R-module. Then $\overline{Z}(M)$ is cosingular. Since $M/\overline{Z}(M)$ is cosingular, it is projective, and so $\overline{Z}(M)$ is a direct summand of M. Hence $M = \overline{Z}(M) \oplus N$ for some cosingular N. It follows that $\overline{Z}(M) = 0$, i.e., M is cosingular. The assumption of (4) now shows that M is discrete. □

Let R be a ring such that every cyclic cosingular R-module is discrete. Then R need not be a CD-ring as the following example shows.

Example 3.15. The ring $R = \mathbb{Z}_8$ is a local ring such that $\frac{R}{\overline{Z}(R)} = \frac{R}{Soc(R)}$ is not semisimple. So by Theorem 3.8, R is not a CD-ring. Let M be a nonzero cyclic R-module. Then M is isomorphic to $M_1 = \frac{R}{(2)} = \frac{R}{J(R)}$ or $M_2 = \frac{R}{(4)} = \frac{R}{Soc(R)}$ or $M_3 = R$. The module M_1 is simple. The module M_2 is an indecomposable local R-module and M_3 is discrete since R is semiperfect. Hence all cyclic (cosingular) R-modules are discrete.

Acknowledgment. The authors would like to thank the referee for his/her helpful suggestions to improve the presentation of this paper.

References

[1] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1992.

[2] U.S. Chase, *Direct product of modules*, Trans. Amer. Math. Soc. **97**, 457-473, 1960.

[3] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules, Supplements and Projectivity in Module Theory*, Frontiers in Math., Boston, Birkhäuser, 2006.

[4] K.R. Goodearl, *Singular Torsion and the Splitting Properties*, Mem. Amer. Math. Soc., No. 124, 1972.

[5] M.A. Kamal and A. Yousef, *On principally lifting modules*, Int. Electron. J. Algebra **2**, 127-137, 2007.

[6] D. Keskin and R. Tribak, *When M-cosingular modules are projective*, Vietnam J. Math. **33** (2), 214–221, 2005.

[7] C. Lomp, *On semilocal modules and rings*, Comm. Algebra **27** (4), 1921-1935, 1999.

[8] S.H. Mohamed and B.J. Müller, *Continuous and Discrete Modules*, in: London Math. Soc. Lecture Notes Series **147**, Cambridge, University Press, 1990.

[9] S. Mohamed and S. Singh, *Generalizations of decomposition theorems known over perfect rings*, J. Austral. Math. Soc. Ser. A **24**, 496–510, 1977.

[10] A.C. Ozcan, *The torsion theory cogenerated by δ-M-small modules and GCO-modules*, Comm. Algebra **35** (2), 623–633, 2007.

[11] S.T. Rizvi and M.F. Yousif, *On continuous and singular modules*, in: Non-Commutative Ring Theory, Lecture Notes in Mathematics Vol. **1448**, 116-124, Springer, Berlin, Heidelberg, 1990.

[12] N.V. Sanh, *On SC-modules*, Bull. Aust. Math. Soc. **48**, 251-255, 1993.

[13] B. Sarath and K. Varadarajan, *Dual Goldie dimension - II*, Comm. Algebra **7** (17), 1885-1899, 1979.

[14] Y. Talebi, A.R.M. Hamzekolaee, M. Hosseinpour, A. Harmanci, and B. Ungor, *Rings for which every cosingular module is projective*, Hacet. J. Math. Stat. **48** (4), 973-984, 2019.

[15] Y. Talebi and N. Vanaja, *The torsion theory cogenerated by M-small modules*, Comm. Algebra **30** (3), 1449-1460, 2002.

[16] R. Tribak and D. Keskin, *On $\overline{Z}_M$-semiperfect modules*, East-West J. Math. **8** (2), 193-203, 2006.

[17] B. Ungor, S. Halicioglu, and A. Harmanci, *On a class of $\oplus$-supplemented modules*, in: Ring Theory and Its Applications, Contemp. Math. **609**, 123–136, Amer. Math. Soc., Providence, RI, 2014.

[18] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

[19] H. Zöschinger, *Koatomare moduln*, Math. Z. **170**, 221-232, 1980.