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g-Steiner, co-Steiner and Normal Points of Bounded Euclidean Submanifolds

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ABSTRACT

The original "Steiner point" introduced in 1838 by the Swiss mathematician Jakob Steiner (1796-1863), also known as the "Steiner curvature centroid", is the geometric centroid of the system obtained by placing a mass equal to the magnitude of the exterior angle at each vertex of a triangle. Steiner points have been studied and applied to networks, combinatorics, computational geometry and even in game theory.
In this article, we extend the notion of Steiner points to the notion of g-Steiner points for bounded Euclidean submanifolds with arbitrary codimension. In this article, we also introduce the notions of co-Steiner and normal points for bounded Euclidean submanifolds. We prove several fundamental properties for such points. Furthermore, we establish some links between g-Steiner, co-Steiner and normal points.

Keywords: *Steiner point, g-Steiner point, co-Steiner point, normal point, G-total curvature, Gauss-Kronecker curvature, Lipschitz-Killing curvature.*
AMS Subject Classification (2020): *Primary: 52A20, 53C40; Secondary: 52A39.*

1. Introduction

The name of "Steiner point" was named after Swiss mathematician Jakob Steiner (1796-1863). The Steiner point, also known as the Steiner curvature centroid, is originally defined to be the geometric centroid of the system obtained by placing a mass equal to the magnitude of the exterior angle at each vertex of a triangle (cf. [19, 20]). Since then Steiner points have been studied and applied to networks, combinatorics, computational geometry and game theory (cf. e.g., [12, 13, 15, 16, 17]).

Throughout this article, by a *bounded manifold* we mean a compact manifold with or without smooth boundary. By a *closed manifold* we mean a bounded manifold without boundary.

For an even-dimensional convex closed hypersurface M^n in a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$, H. Flanders proved in [11] that the Steiner point of M^n can be defined as

$$s(M^n) = \frac{1}{c_n}\int_{p\in M^n} \mathbf{x}K(p)dv, \tag{1.1}$$

where $\mathbf{x}$ denotes the position vector field of M^n in $\mathbb{E}^{n+1}$, dv is the volume element of M^n, and $K(p)$ denotes the Gauss-Kronecker curvature of M^n at a point $p \in M^n$.

It is known that the Steiner point defined by (1.1) satisfies the following properties (cf. [17, 18]):

- $s(aM^n) = as(M^n)$ for any similar transformation a;
- $s(M^n + c) = s(M^n) + c$ for a constant vector $c \in \mathbb{E}^{n+1}$;
- $s(M^n)$ is a continuous function of M^n;
- If $\dim M^n$ is positive, then $s(M^n)$ is a relative interior point of M^n.

In this article, we extend the notion of Steiner points to the notion of g-Steiner points for bounded Euclidean submanifolds with arbitrary codimension. We also introduce the notions of co-Steiner and normal points for

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bounded Euclidean submanifolds via the notion of G-total curvatures introduced in [4, 7]. In this article, we also prove several fundamental properties for such points. Furthermore, we establish some links between g-Steiner, co-Steiner and normal points.

2. Preliminaries

In this article, we follow the notations from [7, 8, 9, 10].

Let M^n be a bounded manifold of dimension n and let $\varphi : M^n \to \mathbb{E}^m$ be an immersion of M^n into an oriented Euclidean space $\mathbb{E}^m$ of dimension m. By a frame $\{p, e_1, \ldots, e_m\}$ in the space $\mathbb{E}^m$, we mean a point $p \in \mathbb{E}^m$ together with an ordered set of mutually perpendicular unit vectors $e_1, \ldots, e_m$ whose orientation is coherent with that of the Euclidean space $\mathbb{E}^m$. In the following, we shall identify e_i with its image $\varphi_*(e_i)$ under the differential map φ_* of the immersion φ.

Let $\mathcal{F}(\mathbb{E}^m)$ be the set of all frames on the Euclidean m-space space $\mathbb{E}^m$. We let $\mathcal{F}(M^n)$ denote the set of all orthonormal frames in M^n (with respect to the induced metric on M^n) such that $e_1, \ldots, e_n$ are tangent to M^n and hence $e_{n+1}, \ldots, e_m$ are normal to M^n.

Let us denote by $B_1(\varphi)$ the bundle space of unit normal vectors of $\varphi(M^n)$ so that a point in $B_1(\varphi)$ is a pair (p, e), where e is a unit normal vector of M^n at $\varphi(p)$. Then $B_1(\varphi)$ forms a principal bundle of $(m-n-1)$-dimensional unit-spheres S_p^{m-n-1} at $p \in M^n$. Clearly, $B_1(\varphi)$ is a manifold of dimension $m-1$. Let $B(\varphi)$ denote the set consists of all $b = (p, e_1, \ldots, e_m)$ such that $(p, e_1, \ldots, e_m) \in \mathcal{F}(M^n)$ and $(\varphi(p), e_1, \ldots, e_m) \in \mathcal{F}(\mathbb{E}^m)$. Then the natural projection $B(\varphi) \to M^n$ can be regarded as a principal bundle with fiber $O(n) \times SO(m-n)$, and $\tilde{\varphi} : B(\varphi) \to \mathcal{F}(\mathbb{E}^m)$ is naturally defined by $\tilde{\varphi}(b) = (\varphi(p), e_1, \ldots, e_m)$.

To avoid confusion, we shall use Einstein's convention on summation and also use the following ranges of indices:

$$1 \leq i, j, k, \ldots \leq n; \quad n+1 \leq r, s, t, \ldots \leq m; \quad A, B, C \ldots, \ldots \leq m$$

throughout this article, unless otherwise stated.

On $\mathcal{F}(\mathbb{E}^m)$, we introduce the 1-forms θ^A, θ_A^B defined by

$$dp = \sum \theta^A e_A, \quad de_A = \sum \theta_A^B e_B, \quad \theta_A^B = -\theta_B^A. \tag{2.1}$$

Since $d^2 = 0$, it follows from (2.1) that

$$d\theta^A = \sum \theta^B \wedge \theta_B^A, \quad d\theta_B^A = -\sum \theta_C^A \wedge \theta_B^C, \tag{2.2}$$

where $\wedge$ denotes the exterior product.

Let ω^A and ω_B^A denote the restrictions of the 1-forms θ^A and θ_A^A to M^n via the immersion φ. Then we have $\omega^r = 0$. Thus, we find from (2.1) and (2.2) that

$$d\varphi = \sum \theta^i e_i, \quad de_m = -\sum_{A=1}^{m-1} \omega_A^m e_A, \tag{2.3}$$

$$d\omega^i = \sum \omega^j \wedge \omega_j^i, \quad d\omega_j^i = -\sum \omega_k^i \wedge \omega_j^k + \sum_r \omega_i^r \wedge \omega_j^r, \tag{2.4}$$

It was well-known that $\omega^1, \ldots, \omega^n$ are linearly independent 1-forms on M^n and the volume element of M^n is given by $dv = \omega_1 \wedge \ldots \wedge \omega_n$.

Since $0 = d\omega^r = \sum \omega^i \wedge \omega_i^s$, Cartan's lemma implies that

$$\omega_i^r = \sum h_{ij}^r \omega^j, \quad h_{ij}^r = h_{ji}^r, \tag{2.5}$$

where h_{ij}^r are called the coefficients of the second fundamental form h. The eigenvalues $\kappa_1(p, e_r), \ldots, \kappa_n(p, e_r)$ of the symmetric matrix (h_{ij}^r) via the second fundamental form are called the *principal curvatures* of M^n at a point $(p, e_r) \in B_1(\varphi)$.

From (2.4) and (2.5) we obtain

$$d\omega_j^i = \sum \omega_i^k \wedge \omega_j^k + \sum_{r,k,\ell} (h_{ik}^r h_{j\ell}^r - h_{i\ell}^r h_{jk}^r) \omega^k \wedge \omega^\ell, \tag{2.6}$$

The ℓ-th mean curvature $K_\ell(p,e_r)$ at (p,e_r) is defined by the elementary symmetric functions such that

$$\binom{n}{\ell} K_\ell(p,e_r) = \sum \kappa_1(p,e_r)\cdots\kappa_\ell(p,e_r), \quad \ell = 1,\ldots,n, \tag{2.7}$$

where $\binom{n}{\ell} = n!/(\ell!(n-\ell)!)$ denotes the binomial coefficients (cf. [1, 8, 9, 10, 14]). In particular, the n-th mean curvature $K_n(p,e)$ is well-known as the *Lipschitz-Killing curvature* at (p,e).

Consider the principal bundle $B_1(\varphi)\to M^n$ with fiber S_p^{m-n-1}. Then

$$d\sigma = \omega_{n+1}^m \wedge\cdots\wedge \omega_{m-1}^m$$

is a differential form of degree $(m-n-1)$ on $B_1(\varphi)$ such that its restriction to the fiber S_p^{m-n-1} of $B_1(\varphi)$ at a point $p\in M^n$ is the volume element of S_p^{m-n-1}. Therefore, $dv\wedge d\sigma$ is the volume element of the principal bundle $B_1(\varphi)$ (cf. [5, 10]).

Also, it follows from (2.5) and (2.7) that we have

$$\omega_1^m\wedge\cdots\wedge\omega_n^m = K_n(p,e_m)dv. \tag{2.8}$$

Note that, for a hypersurface M^n in $\mathbb{E}^{n+1}$, the fiber S_p^0 at a point $p\in M^n$ consists of two unit vectors at p; namely the unit outward normal vector and unit inner normal vector at $\varphi(p)$. Also, note that the ℓ-th mean curvature satisfies

$$K_\ell(p,-e) = (-1)^\ell K_\ell(p,e) \tag{2.9}$$

for $\ell = 0,1,\ldots,n$.

3. G-total curvatures, g-Steiner, co-Steiner and normal points

Now, we recall the notion of G-total curvatures from [4, 7]. For a given immersion $\varphi: M^n\to\mathbb{E}^m$ of a bounded n-manifold M^n into $\mathbb{E}^m$, let $\eta: B_1(\varphi)\to\mathbb{E}^r$ be a $\mathbb{E}^r$-valued function on the principal bundle $B_1(\varphi)$. In particular, if $r=1$, then η is nothing but a real-value function on $B_1(\varphi)$.

For $\ell\in\{0,1,2,\ldots\}$ and $k\geq 0$, the integral

$$G_\ell(\varphi,p,\eta,k) = \int_{e\in S_p^{m-n-1}} \eta(p,e)(K_\ell(p,e))^k d\sigma, \tag{3.1}$$

is called the *ℓ-th G-total curvature of rank k at the point $p\in M^n$ with respect to the function η* if the right-hand-side of (3.1) exists (cf. [4, 7]).

An immersion $\varphi: M^n\to\mathbb{E}^m$ is called *pseudo-flat* if $G_n(\varphi,p,1,1)=0$ for all $p\in M^n$ (cf. [4]). It follows from (2.9) that every immersion $\varphi: M^n\to\mathbb{E}^m$ of M^n is always pseudo-flat whenever $n=\dim M^n$ is odd.

The integral

$$T_\ell(\varphi,\eta,k) = \frac{1}{c_{m-1}}\int_{p\in M^n} G_\ell(\varphi,p,\eta,k)dv = \int_{(p,e)\in B(\varphi)} \eta(p,e)(K_\ell(p,e))^k dv\wedge d\sigma \tag{3.2}$$

is called the *ℓ-th G-total curvature with resect to η* if the right-hand-side of (3.2) exists (cf. [4, 7]).

For some special cases of G-total curvatures, we have the following.

- $T_\ell(\varphi,\eta,0)$ is the ordinary $\mathbb{E}^m$-valued integral over $B_1(\varphi)$;
- $T_\ell(\varphi,1,0)$ is the volume of $B_1(\varphi)$, where 1 denotes the constant function 1 on $B_1(\varphi)$;
- $T_0(\varphi,\varphi,0)$ is the center of mass of M^n in $\mathbb{E}^m$;
- If M^n is a closed manifold, then $T_n(\varphi,1,1)=\mathcal{X}(M^n)$, where $\mathcal{X}(M^n)$ denotes the Euler characteristic of M^n (see [7, Proposition 6.4]);
- If $m=n+1$ and n is even, then $\frac{1}{2}G_n(\varphi,p,1,1)$ is the Gauss-Kronecker curvature of the hypersurface M^n in $\mathbb{E}^{n+1}$ (cf. [2, 6]).

For simplicity, we put

$$T(\varphi) = T_n(\varphi,1,1) \quad\text{and}\quad T(\varphi,\eta) = T_n(\varphi,\eta,1). \tag{3.3}$$

For an immersion $\varphi: M^n\to\mathbb{E}^m$ and for $(p,e)\in B_1(\varphi)$, we define *g-Steiner points, co-Steiner points and normal points* of φ via (3.2) and (3.3) as follows:

- $gs(\varphi) = T(\varphi, \varphi)$ is called the *g-Steiner point* of φ;
- $cs(\varphi) = T(\varphi, \langle \varphi, e \rangle e)$ is called the *co-Steiner point* of φ;
- $\mathbf{n}(\varphi) = T(\varphi, e)$ is called the *normal point* of φ,

where $\langle \varphi, e \rangle$ denotes the inner product for the $\mathbb{E}^m$-valued functions φ and the unit normal vector e on M^n.
Now, we prove the following two lemmas.

Lemma 3.1. *If $\varphi : M^n \to \mathbb{E}^m$ is an immersion of a bounded manifold M^n in $\mathbb{E}^m$, then the g-Steiner point $gs(\varphi)$ of φ is given by*

$$gs(\varphi) = \frac{1}{c_{m-1}} \int_{(p,e_m)\in B_1(\varphi)} \varphi \omega_1^m \wedge \cdots \wedge \omega_{m-1}^m. \tag{3.4}$$

Proof. Follows easily from the definition of g-Steiner point and Eqs. (2.8), (3.1) and (3.2). □

Lemma 3.2. *For an immersion $\varphi : M^n \to \mathbb{E}^{n+1}$ of an even-dimensional closed convex hypersurface M^n in $\mathbb{E}^{n+1}$, the g-Steiner point $gs(\varphi)$ of φ and the Steiner point $s(M^n)$ of M^n defined by (1.1) are related by $gs(\varphi) = 2s(M^n)$.*

Proof. Let $\varphi : M^n \to \mathbb{E}^{n+1}$ be the immersion of a closed convex hypersurface. Denote by $\mathbf{x}$ the position vector field of M^n in $\mathbb{E}^{n+1}$. Then it follows from Eqs. (3.1), (3.2) and the definition of the g-Steiner point $gs(\varphi)$ of φ that

$$\begin{aligned} gs(\varphi) &= \frac{1}{c_n} \int_{(p,e_{n+1})\in B_1(\varphi)} \varphi \omega_1^{n+1} \wedge \cdots \wedge \omega_n^{n+1} \\ &= \frac{1}{c_n} \int_{p\in M^n} \varphi \left\{ \int_{e\in S_p^0} K_n(p,e)d\sigma \right\} dv \\ &= \frac{2}{c_n} \int_{p\in M^n} \mathbf{x}K(p)dv \end{aligned} \tag{3.5}$$

where $K(p)$ is the Gauss-Kronecker curvature of M^n at a point $p \in M^n$, $K_n(p,e)$ is the Lipschitz-Killing curvature at $(p,e) \in B_1(\varphi)$ and $\mathbf{x}$ is the position vector field of $\varphi : M^n \to \mathbb{E}^{n+1}$. Therefore, after comparing (3.5) with (1.1), we obtain $gs(\varphi) = 2s(M^n)$. □

The next lemma is direct to verify.

Lemma 3.3. *If $\varphi : M^n \to \mathbb{E}^m$ is an immersion of a bounded manifold M^n in $\mathbb{E}^m$ and $\psi : \mathbb{E}^m \hookrightarrow \mathbb{E}^{\bar{m}}$ is an inclusion map, then we have $gs(\varphi) = gs(\bar{\varphi})$ with $\bar{\varphi} = \psi \circ \varphi$.*

4. Some properties of g-Steiner and co-Steiner points

For g-Steiner points, we have the following.

Theorem 4.1. *For a given immersion $\varphi : M^n \to \mathbb{E}^m$ of a bounded n-manifold M^n into $\mathbb{E}^m$, the g-Steiner point $gs(\varphi)$ of φ satisfies the following properties:*

(a) *If a is a positive number and $\bar{\varphi} = a\varphi$ is the similarity transformation of φ given by $(a\varphi)(p) = a(\varphi(p))$, $p \in M^n$, then we have $gs(a\varphi) = a(gs(\varphi))$;*
(b) *For a constant vector $c \in \mathbb{E}^m$, we have*

$$gs(\varphi_c) = gs(\varphi) + T(\varphi)c,$$

where $\varphi_c = \varphi + c$ is the parallel translation of φ given by $\varphi_c(p) = \varphi(p) + c$, and $T(\varphi)$ is the total Lipschitz-Killing curvature of M^n defined by (3.3), i.e.,

$$T(\varphi) = \frac{1}{c_{m-1}} \int_{(p,e_m)\in B_1(\varphi)} \omega_1^m \wedge \cdots \wedge \omega_{m-1}^m; \tag{4.1}$$

(c) *If $\varphi : M^n \to \mathbb{E}^m$ is a pseudo-flat immersion, then $gs(\varphi) = 0$. In particular, we have $gs(\varphi) = 0$ whenever M^n is of odd dimension;*

(d) *If $\varphi : M^n \to \mathbb{E}^m$ and $\bar\varphi : \bar M^{\bar n} \to \mathbb{E}^{\bar m}$ are immersions of even-dimensional bounded manifolds M^n and $\bar M^{\bar n}$ into $\mathbb{E}^m$ and $\mathbb{E}^{\bar m}$ respectively, then we have*

$$gs(\varphi\times\bar\varphi) = \big(T(\bar\varphi)gs(\varphi), T(\varphi)gs(\bar\varphi)\big), \tag{4.2}$$

where $\varphi\times\bar\varphi$ is the product immersion of φ and $\bar\varphi$ given by $M^n\times\bar M^{\bar n}\ni(p,\bar p)\mapsto(\varphi(p),\bar\varphi(\bar p))\in\mathbb{E}^m\times\mathbb{E}^{\bar m}$.

Proof. (a) Let $a\varphi$ be a similarity transformation of φ. Then, the definition of $B_1(\varphi)$ yields $B_1(a\varphi) = B_1(\varphi)$. For a given point $(p, e_1, \ldots, e_m) \in B(a\varphi)$, we put

$$d(a\varphi) = \sum \bar\omega^i e_i, \quad \bar\omega_A^m = \langle de_m, e_A\rangle,$$

and let ω^i, ω_A^m denote the corresponding forms on $B(\varphi)$. Then we obtain

$$\bar\omega_1^m\wedge\cdots\wedge\bar\omega_{m-1}^m = \omega_1^m\wedge\cdots\wedge\omega_{m-1}^m. \tag{4.3}$$

Consequently, we obtain from the definition of g-Steiner points and Lemma 3.1 that

$$\begin{aligned} gs(a\varphi) &= \frac{1}{c_{m-1}}\int_{(p,e_m)\in B_1(a\varphi)}(a\varphi)\bar\omega_1^m\wedge\cdots\wedge\bar\omega_{m-1}^m \\ &= \frac{a}{c_{m-1}}\int_{(p,e_m)\in B_1(\varphi)}\varphi\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m \\ &= a(gs(\varphi)). \end{aligned}$$

(b) For a given vector $c\in\mathbb{E}^m$, we have $B_1(\varphi) = B_1(\varphi_c)$. Thus, if we denote by $\bar\omega$ the form for φ_c corresponding to a form ω for φ, then we obtain (4.3) as well. Therefore

$$\begin{aligned} gs(\varphi_c) &= \frac{1}{c_{m-1}}\int_{(p,e_m)\in B_1(\varphi+c)}(\varphi+c)\bar\omega_1^m\wedge\cdots\wedge\bar\omega_{m-1}^m \\ &= \frac{1}{c_{m-1}}\int_{(p,e_m)\in B_1(\varphi)}\varphi\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m + \frac{c}{c_{m-1}}\int_{(p,e_m)\in B_1(\varphi)}\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m \\ &= gs(\varphi) + \frac{c}{c_{m-1}}\int_{(p,e_m)\in B_1(\varphi)}\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m \\ &= gs(\varphi) + T(\varphi)c, \end{aligned}$$

where $T(\varphi)$ is defined by (4.1).

(c) The first part of statement (c) follows from Eq. (2.8) and Lemma 3.2. And the second part of statement (c) follows from (2.9).

(d) Let $\varphi : M^n \to \mathbb{E}^m$ and $\varphi' : M^{n'} \to \mathbb{E}^{m'}$ be immersions of bounded manifolds M^n and $M^{n'}$ into $\mathbb{E}^m$ and $\mathbb{E}^{m'}$ respectively. According to Lemma 3.3, to prove statement (d), without loss of generality, we may assume m and $\bar m$ are both odd.

Let $B(\varphi\times\bar\varphi)$ be the bundle space over $M^n\times\bar M^{\bar n}$ consisting of

$$((p,\bar p), e_1,\ldots,e_n,\bar e_1,\ldots,\bar e_{\bar n}, e_{n+1},\ldots,e_m,\bar e_{\bar n+1},\ldots,\bar e_{\bar m})$$

such that $(p, e_1, \ldots, e_m) \in B(\varphi)$ and $(\bar p, \bar e_1, \ldots, \bar e_{\bar m}) \in B(\bar\varphi)$. Let us consider the following two unit normal vector fields of $M^n\times\bar M^{\bar n}$ in $\mathbb{E}^m\times\mathbb{E}^{\bar m}$ given by

$$\tilde e_{m+\bar m-1} = -\sin\theta e_m + \cos\theta\bar e_{\bar m}, \quad \tilde e_{m+\bar m} = \cos\theta e_m + \sin\theta\bar e_{\bar m} \tag{4.4}$$

over $M^n\times\bar M^{\bar n}$. Then we have

$$d\tilde e_{m+\bar m} = \cos\theta de_m + \sin\theta d\bar e_{\bar m} + \tilde e_{m+\bar m-1}d\theta. \tag{4.5}$$

If $\tilde\omega_\alpha^\beta$ $(\alpha,\beta = 1,\ldots,m+\bar m)$ denote the connection forms associated with the frame

$$((p,\bar p), e_1,\ldots,e_n,\bar e_1,\ldots,\bar e_{\bar n}, e_{n+1},\ldots,e_{m-1}, e_{m+\bar m-1},\bar e_{\bar n+1},\ldots,\bar e_{\bar m-1},\bar e_{m+\bar m}),$$

then we find

$$\tilde{\omega}_i^{m+\bar{m}} = \cos\theta\omega_i^m, \quad \tilde{\omega}_{\bar{i}}^{m+\bar{m}} = \sin\theta\omega_{\bar{i}}^{\bar{m}}, \quad i = 1,\dots,n; \;\; \bar{i} = 1,\dots,\bar{n} \tag{4.6}$$

$$\tilde{\omega}_r^{m+\bar{m}} = \cos\theta\omega_r^m, \quad \tilde{\omega}_{\bar{r}}^{m+\bar{m}} = \cos\theta\omega_{\bar{r}}^{\bar{m}}, \quad r = n+1,\dots,m-1; \;\; \bar{r} = \bar{n}+1,\dots,\bar{m}-1; \tag{4.7}$$

$$\tilde{\omega}_{m+\bar{m}-1}^{m+\bar{m}} = d\theta. \tag{4.8}$$

Now, it follows from (4.6), (4.7) and (4.8) that

$$\begin{aligned} d\tilde{v}\wedge d\tilde{\sigma} &= \tilde{\omega}_1^{m+\bar{m}}\wedge\cdots\wedge\tilde{\omega}_{m+\bar{m}-1}^{m+\bar{m}} \\ &= (\cos\theta)^{m-1}(\sin\theta)^{\bar{m}-1}K_n(p,e_m)\bar{K}_{\bar{n}}(\bar{p},\bar{e}_{\bar{m}}) \\ &\quad \times dv\wedge d\bar{v}\wedge\omega_{n+1}^m\wedge\cdots\wedge\omega_{m-1}^m\wedge\omega_{\bar{n}+1}^{\bar{m}}\wedge\cdots\wedge\omega_{\bar{m}-1}^{\bar{m}}\wedge d\theta. \end{aligned} \tag{4.9}$$

Consequently,

$$\begin{aligned} gs(\varphi\times\bar{\varphi}) &= \frac{1}{c_{m+\bar{m}-1}}\int_{e_{m+\bar{m}}\in B_1(\varphi\times\bar{\varphi})}(\varphi\times\bar{\varphi})\,\tilde{\omega}_1^{m+\bar{m}}\wedge\cdots\wedge\tilde{\omega}_{m+\bar{m}-1}^{m+\bar{m}} \\ &= \frac{1}{c_{m+\bar{m}-1}}\int_{e_{m+\bar{m}}\in B_1(\varphi\times\bar{\varphi})}(\varphi\times\bar{\varphi})(\cos^{m-1}\theta)(\sin^{\bar{m}-1}\theta) \\ &\qquad \times K_n(p,e_m)\bar{K}_{\bar{n}}(\bar{p},\bar{e}_{\bar{m}})dv\wedge d\bar{v}\wedge d\sigma\wedge d\bar{\sigma}\wedge d\theta \\ &= \big(T(\bar{\varphi})gs(\varphi), T(\varphi)gs(\bar{\varphi})\big), \end{aligned} \tag{4.10}$$

where we have used (cf. Formula (4.6) of [9, page 135])

$$\int_{S^1}(\cos^p\theta\sin^q\theta)d\theta = \frac{2\Gamma((1+p)/2)\Gamma((1+q)/2)}{\Gamma((2+p+q)/2))}$$

for even integers $p, q \geq 0$ and

$$c_n = \frac{(n+1)\pi^{(n+1)/2}}{\Gamma((n+3)/2)}.$$

□

The next three corollaries follow from Theorem 4.1 and the fact that the total Gauss-Kronecker curvature of φ satisfies $T(\varphi) = c_{m-1}\mathcal{X}(M^n)$ (cf. [7, Proposition 6.4]).

Corollary 4.1. *Let $\varphi : M^n \to \mathbb{E}^m$ be an immersion of an even-dimensional closed oriented manifold M^n into $\mathbb{E}^m$. Then the Euler characteristic of M^n satisfies $\mathcal{X}(M^n) = 0$ if and only if the g-Steiner point of φ is invariant under translations.*

Corollary 4.2. *Let $\varphi : M^n \to \mathbb{E}^m$ and $\bar{\varphi} : \bar{M}^{\bar{n}} \to \mathbb{E}^{\bar{m}}$ be immersions of even-dimensional closed oriented manifolds M^n and $\bar{M}^{\bar{n}}$ into $\mathbb{E}^m$ and $\mathbb{E}^{\bar{m}}$ respectively. If $\mathcal{X}(M^n) = \mathcal{X}(\bar{M}^{\bar{n}}) = 0$, then $gs(\varphi\times\bar{\varphi}) = 0$.*

Corollary 4.3. *Let M^n be an oriented closed manifold with $\mathcal{X}(M^n) \neq 0$. If M^n is immersible into $\mathbb{E}^m$, then for any $c \in \mathbb{E}^m$, there exists an immersion of $\varphi : M^n \to \mathbb{E}^m$ such that $gs(\varphi) = c$.*

We have the following properties for co-Steiner points.

Theorem 4.2. *For a given immersion $\varphi : M^n \to \mathbb{E}^m$ of a bounded n-manifold M^n into $\mathbb{E}^m$, the co-Steiner point $cs(\varphi)$ of φ satisfies the following properties:*

(a) *If a is a positive number, then we have $cs(a\varphi) = a(cs(\varphi))$;*
(b) *For a constant vector $c \in \mathbb{E}^m$, the translation φ_c of φ satisfies*

$$cs(\varphi_c) = cs(\varphi) + T(\varphi, \langle c, e\rangle e), \tag{4.11}$$

where $T(\varphi, c)$ is the n-th G-total curvature of M^n defined by

$$T(\varphi, \langle c, e\rangle e) = \frac{1}{c_{m-1}}\int_{(p,e)\in B_1(\varphi)}\langle c, e\rangle e\,\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m, \quad e = e_m; \tag{4.12}$$

(c) *If $\dim M^n$ is odd, then $cs(\varphi) = 0$.*

Since this theorem can be proved in similar way as Theorem 4.1, so we omit its proof.

5. Some properties of normal points

Let $v_1, \ldots, v_{m-1}, v$ be m vector in $\mathbb{E}^m$ and let $v_1 \times \cdots \times v_{m-1}$ denote the vector product of $v_1, \ldots, v_{m-1}$. Then we get

$$v \cdot (v_1 \times \cdots \times v_{m-1}) = (-1)^{m-1} |v, v_1, \cdots, v_{m-1}|, \tag{5.1}$$

where $|v, v_1, \cdots, v_{m-1}|$ denotes the determinant of $v, v_1, \ldots, v_{m-1}$. From (5.1) we find

$$e_1 \times \cdots \times \widehat{e}_\alpha \times \cdots \times e_m = (-1)^{m+\alpha} e_\alpha, \tag{5.2}$$

where the roof $\widehat{\cdot}$ means the omitted term.

Let $\varphi : M^n \to \mathbb{E}^m$ be an immersion of a bounded n-manifold M^n into $\mathbb{E}^m$. As before, we denote by $B(\varphi)$ the bundle space consists of all $(p, e_1, \ldots, e_m)$ over M^n defined as in Section 2. Now, we define a $\mathbb{E}^m$-valued $(m-1)$-form Ω given by

$$\Omega = \frac{1}{(m-1)c_{m-1}} \sum_{\alpha=1}^{m-1} (-1)^\alpha (\omega_1^m \wedge \cdots \wedge \widehat{\omega}_\alpha^m \wedge \cdots \wedge \omega_{m-1}^m) e_\alpha. \tag{5.3}$$

Proposition 5.1. *For a given immersion $\varphi : M^n \to \mathbb{E}^m$ of a bounded n-manifold M^n into $\mathbb{E}^m$, the normal point $\mathbf{n}(\varphi)$ of φ satisfies*

$$\mathbf{n}(\varphi) = \int_{\partial B_1(\varphi)} \Omega, \tag{5.4}$$

where $\partial B_1(\varphi)$ denotes the boundary of $B_1(\varphi)$.

Proof. For a given immersion $\varphi : M^n \to \mathbb{E}^m$ of a bounded n-manifold M^n into $\mathbb{E}^m$, let $[\cdot, \ldots, \cdot]$ denote the combining operation of the vector product of $\mathbb{E}^m$ with the exterior product.

If we denote e_m by e, then we have

$$\begin{aligned}
[\overbrace{de, \ldots, de}^{m-2 \text{ times}}, e] &= (-1)^m \left[\sum \omega_{\alpha_1}^m e_{\alpha_1}, \ldots, \sum \omega_{\alpha_{m-2}}^m e_{\alpha_{m-2}}, e\right] \\
&= (-1)^m \sum \omega_{\alpha_1}^m \wedge \cdots \wedge \omega_{\alpha_{m-2}}^m [e_{\alpha_1}, \ldots, e_{\alpha_{m-2}}, e] \\
&= (-1)^m (m-2)! \sum_{\alpha=1}^{m-1} \omega_1^m \wedge \cdots \wedge \widehat{\omega}_\alpha^m \wedge \ldots \wedge \omega_{m-1}^m [e_1, \ldots, \widehat{e}_\alpha, \ldots, e_{m-1}, e] \\
&= (m-2)! \sum_{\alpha=1}^{m-1} (-1)^\alpha (\omega_1^m \wedge \cdots \wedge \widehat{\omega}_\alpha^m \wedge \cdots \wedge \omega_{m-1}^m) e_\alpha,
\end{aligned}$$

Combining this with (5.3) gives

$$\Omega = \frac{1}{(m-1)! c_{m-1}} [\overbrace{de, \ldots, de}^{m-2 \text{ times}}, e]. \tag{5.5}$$

Since

$$d([\overbrace{de, \ldots, de}^{m-2 \text{ times}}, e]) = (-1)^m [\overbrace{de, \ldots, de}^{m-1 \text{ times}}],$$

it follows from (5.5) that

$$\begin{aligned}
d\Omega &= \frac{(-1)^m}{(m-1)! c_{m-1}} [\overbrace{de, \ldots, de}^{m-1 \text{ times}}] \\
&= -\frac{1}{(m-1)! c_{m-1}} \left[\sum \omega_{\alpha_1}^m e_{\alpha_1}, \ldots, \sum \omega_{\alpha_{m-1}}^m e_{\alpha_{m-1}}\right] \\
&= -\frac{1}{(m-1)! c_{m-1}} \sum \omega_{\alpha_1}^m \wedge \cdots \wedge \omega_{\alpha_{m-1}}^m \left[e_{\alpha_1}, \ldots, e_{\alpha_{m-1}}\right] \\
&= \frac{1}{c_{m-1}} (e \omega_1^m \wedge \cdots \wedge \omega_{m-1}^m).
\end{aligned} \tag{5.6}$$

Consequently, after applying Stokes' theorem we obtain

$$\begin{aligned}\mathbf{n}(\varphi) &= \frac{1}{c_{m-1}}T(\varphi,e)\\ &= \frac{1}{c_{m-1}}\int_{(p,e)\in B_1(\varphi)} eK_n(p,e)dv\wedge d\sigma\\ &= \frac{1}{c_{m-1}}\int_{(p,e)\in B_1(\varphi)} e\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m\\ &= \int_{B_1(\varphi)} d\Omega = \int_{\partial B_1(\varphi)}\Omega,\end{aligned}\tag{5.7}$$

which proves the Proposition. □

For the normal point $\mathbf{n}(\varphi)$ of φ, we have the following.

Theorem 5.1. *For a given immersion $\varphi : M^n \to \mathbb{E}^m$ of a bounded n-manifold M^n into $\mathbb{E}^m$, the normal point of φ satisfies the following three properties:*

(a) *The normal point is invariant under similarity transformations;*
(b) *The normal point is invariant under translations;*
(c) *If M^n is of even dimension or M^n is a closed manifold, then we have $\mathbf{n}(\varphi) = 0$.*

Proof. Since statements (a) and (b) can be proved in similar ways as the proofs of statements (a) and (b) of Theorem 4.1, so we omit them.

(c) First, if $\dim M^n$ is even, then the Lipschitz-Killing curvature satisfies $K(p,e) = K(p,-e)$, Thus, by the definition of the normal point $\mathbf{n}(\varphi)$ and the symmetry of the fiber S_p^{m-1} over p, we easily see that the G-total curvature $G_n(\varphi,p,e,1) = 0$ for each $p \in M^n$. Hence we get $\mathbf{n}(\varphi) = 0$.

Second, if M^n is a closed manifold, then statement (c) follows from Proposition 5.1. □

6. A link between g-Steiner and co-Steiner points

The next theorem provides a link between g-Steiner and co-Steiner points.

Theorem 6.1. *Let $\varphi : M^n \to \mathbb{E}^m$ be an immersion of a closed manifold M^n into $\mathbb{E}^m$. Then the g-Steiner and co-Steiner points of φ are related by $gs(\varphi) = m(cs(\varphi))$.*

Proof. For an immersion $\varphi : M^n \to \mathbb{E}^m$ of a closed manifold M^n into $\mathbb{E}^m$, let Ω be defined as (5.3). Then it follows from (2.3) and (5.5) that

$$\begin{aligned} d(\langle\varphi,e_m\rangle\,\Omega) &= (d\,\langle\varphi,e_m\rangle)\wedge\Omega + \langle\varphi,e_m\rangle\,d\Omega\\ &= (\langle\varphi,de_m\rangle)\wedge\Omega + \langle\varphi,e_m\rangle\,d\Omega\\ &= -\frac{1}{(m-1)c_{m-1}}\sum_{\alpha}^{m-1}\langle\varphi,e_\alpha\rangle\,\omega_\alpha^m\wedge\sum_{\beta=1}^{m-1}(-1)^\beta(\omega_1^m\wedge\cdots\wedge\widehat{\omega}_\beta^m\wedge\cdots\wedge\omega_{m-1}^m)e_\beta\\ &\quad + \langle\varphi,e_m\rangle\,d\Omega\\ &= -\frac{1}{(m-1)c_{m-1}}\sum_{\alpha}^{m-1}\langle\varphi,e_\alpha\rangle\,e_\alpha(\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m) + \langle\varphi,e_m\rangle\,d\Omega\\ &= -\frac{1}{(m-1)c_{m-1}}(\varphi-\langle\varphi,e_m\rangle\,e_m)(\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m) + \langle\varphi,e_m\rangle\,d\Omega.\end{aligned}\tag{6.1}$$

After combining (6.1) with (5.6) we find

$$\begin{aligned} d(\langle\varphi,e\rangle\,\Omega) &= -\frac{1}{(m-1)c_{m-1}}(\varphi-\langle\varphi,e\rangle\,e)(\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m) + \frac{1}{c_{m-1}}(\langle\varphi,e\rangle\,e)\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m\\ &= \frac{1}{(m-1)c_{m-1}}\left\{m(\langle\varphi,e\rangle\,e)\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m - \varphi(\omega_1^m\wedge\cdots\wedge\omega_{m-1}^m)\right\},\end{aligned}\tag{6.2}$$

with $e = e_m$. Consequently, after applying Stokes' theorem we obtain $sg(\varphi) = m(cs(\varphi))$ since M is a closed manifold. □

Corollary 6.1. *Let $\varphi : M^n \to \mathbb{E}^m$ and $\bar{\varphi} : \bar{M}^{\bar{n}} \to \mathbb{E}^{\bar{m}}$ be two immersions of two even-dimensional closed manifolds M^n and $\bar{M}^{\bar{n}}$ into $\mathbb{E}^m$ and $\mathbb{E}^{\bar{m}}$, respectively. Then we have*

$$cs(\varphi \times \bar{\varphi}) = \frac{1}{m+\bar{m}} \big(T(\bar{\varphi})gs(\varphi), T(\varphi)gs(\bar{\varphi})\big). \tag{6.3}$$

where $\varphi \times \bar{\varphi}$ is the product immersion of φ and $\bar{\varphi}$

Proof. Follows from Theorem 6.1 and statement (d) of Theorem 4.1. □

7. g-Steiner and co-Steiner points of higher order

Let $\varphi : M^n \to \mathbb{E}^m$ be an immersion of a bounded n-manifold M^n into $\mathbb{E}^m$. For $k = 1, 2, \ldots, n$, let us introduce the following notion of *Steiner point of order k* of φ.

Definition 7.1. The point $gs^k(\varphi) = T(\varphi, \langle \varphi, e\rangle^{k-1} \varphi)$ is called the *g-Steiner point of order k*.

Similarly, we make the following.

Definition 7.2. The point $cs^k(\varphi) = T(\varphi, \langle \varphi, e\rangle^{k-1} e)$ is called the *co-Steiner point of order k*.

Clearly, the g-Steiner (respectively, co-Steiner) point of order 1 is nothing but the g-Steiner (respectively, co-Steiner) point defined in Section 3.

The g-Steiner and co-Steiner points of higher order are related by the following.

Theorem 7.1. *Let $\varphi : M^n \to \mathbb{E}^m$ be an immersion of a bounded n-manifold M^n into $\mathbb{E}^m$. Then we have:*

$$k(gs^k(\varphi)) = (m+k-1)cs^k(\varphi) + \frac{1}{c_{m-1}} \int_{\partial B(\varphi)} \langle \varphi, e\rangle^k \,\Omega. \tag{7.1}$$

Proof. For an immersion $\varphi : M^n \to \mathbb{E}^m$ of a bounded manifold M^n into $\mathbb{E}^m$, let Ω be defined as in (5.3). Then it follows from (2.3) and (5.5) that

$$\begin{aligned} d(\langle \varphi, e_m\rangle^k \,\Omega) &= (d\,\langle \varphi, e_m\rangle^k) \wedge \Omega + \langle \varphi, e_m\rangle^k \,d\Omega \\ &= k(\langle \varphi, de_m\rangle^{k-1}) \wedge \Omega + \langle \varphi, e_m\rangle^k \,d\Omega \\ &= -\frac{k\,\langle \varphi, e_m\rangle^{k-1}}{(m-1)c_{m-1}} \sum_{\alpha}^{m-1} \langle \varphi, e_\alpha\rangle\, \omega_\alpha^m \wedge \sum_{\beta=1}^{m-1} (-1)^\beta (\omega_1^m \wedge \cdots \wedge \widehat{\omega}_\beta^m \wedge \cdots \wedge \omega_{m-1}^m) e_\beta \\ &\qquad + \langle \varphi, e_m\rangle^k \,d\Omega \\ &= -\frac{k\,\langle \varphi, e_m\rangle^{k-1}}{(m-1)c_{m-1}} \sum_{\alpha}^{m-1} \langle \varphi, e_\alpha\rangle\, e_\alpha(\omega_1^m \wedge \cdots \wedge \omega_{m-1}^m) + \langle \varphi, e_m\rangle^k \,d\Omega \\ &= -\frac{k\,\langle \varphi, e_m\rangle^{k-1}}{(m-1)c_{m-1}} (\varphi - \langle \varphi, e_m\rangle\, e_m)(\omega_1^m \wedge \cdots \wedge \omega_{m-1}^m) + \langle \varphi, e_m\rangle^k \,d\Omega. \end{aligned} \tag{7.2}$$

After combining (7.2) with (5.6) we obtain

$$\begin{aligned} d(\langle \varphi, e\rangle^k \,\Omega) &= -\frac{k\,\langle \varphi, e_m\rangle^{k-1}}{(m-1)c_{m-1}} (\varphi - \langle \varphi, e\rangle\, e)(\omega_1^m \wedge \cdots \wedge \omega_{m-1}^m) + \langle \varphi, e\rangle^k \,d\Omega. \\ &= -\frac{k\,\langle \varphi, e_m\rangle^{k-1}}{(m-1)c_{m-1}} (\varphi - \langle \varphi, e\rangle\, e)(\omega_1^m \wedge \cdots \wedge \omega_{m-1}^m) + \frac{1}{c_{m-1}} (\langle \varphi, e\rangle^k\, e)\omega_1^m \wedge \cdots \wedge \omega_{m-1}^m \\ &= \frac{\langle \varphi, e_m\rangle^{k-1}}{(m-1)c_{m-1}} \left\{ (m+k-1)(\langle \varphi, e\rangle\, e)\omega_1^m \wedge \cdots \wedge \omega_{m-1}^m - k\varphi(\omega_1^m \wedge \cdots \wedge \omega_{m-1}^m) \right\}, \end{aligned} \tag{7.3}$$

with $e = e_m$. Consequently, after applying Stokes' theorem to (7.3) we obtain (7.1). □

The next corollary is an immediate consequence of Theorem 7.1.

Corollary 7.1. *Let $\varphi : M^n \to \mathbb{E}^m$ be an immersion of a closed manifold M^n into $\mathbb{E}^m$. Then the g-Steiner and co-Steiner points of order k are related by*

$$gs^k(\varphi) = \frac{m+k-1}{k} cs^k(\varphi)$$

for $k = 2, \ldots, n$.

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