Volume 4, Issue 1, Year 2020, pp. 49-70
E - ISSN: 2587-3008
URL: https://ratingacademy.com.tr/ojs/index.php/jsp
DOİ: https://doi.org/10.26900/jsp.4.005
Research Article

Yüksel SOYKAN *

** Zonguldak Bülent Ecevit University, TURKEY, e-mail: yuksel_soykan@hotmail.com*
ORCID ID: https://orcid.org/0000-0002-1895-211X
Received: 26 October 2019; Accepted: 14 January 2020

ON GENERALIZED SİXTH-ORDER PELL SEQUENCES

Abstract. In this paper, we investigate the generalized sixth order Pell sequences and we deal with, in detail, three special cases which we call them as sixth order Pell, sixth order Pell-Lucas and modified sixth order Pell sequences.

2010 Mathematics Subject Classification. 11B39, 11B83, 05A15.

Keywords. Pell numbers, sixth order Pell numbers, sixth order Pell-Lucas numbers, Hexanacci numbers.

1. Introduction

In this paper, we introduce the generalized sixth order Pell sequences and we investigate, in detail, three special cases which we call them sixth order Pell, sixth order Pell-Lucas and modified sixth order Pell sequences. First we recall the definition of a generalized Hexanacci sequence.

A generalized Hexanacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r_1, r_2, r_3, r_4, r_5, r_6)\}_{n\geq 0}$ is defined by the sixth-order recurrence relations

$$
\begin{aligned}
W_n &= r_1W_{n-1} + r_2W_{n-2} + r_3W_{n-3} + r_4W_{n-4} + r_5W_{n-5} + r_6W_{n-6}, \\
W_0 &= a, W_1 = b, W_2 = c, W_3 = d, W_4 = e, W_5 = f
\end{aligned} \tag{1.1}
$$

where the initial values $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary complex (or real) numbers and $r_1, r_2, r_3, r_4, r_5, r_6$ are real numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_5}{r_6}W_{-(n-1)} - \frac{r_4}{r_6}W_{-(n-2)} - \frac{r_3}{r_6}W_{-(n-3)} - \frac{r_2}{r_6}W_{-(n-4)} - \frac{r_1}{r_6}W_{-(n-5)} + \frac{1}{r_6}W_{-(n-6)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

It is well-known that the Pell sequence (OEIS: A000129, [13]) $\{P_n\}$ is defined recursively by the equation, for $n \geq 0$

$$P_{n+2} = 2P_{n+1} + P_n$$

in which $P_0 = 0$ and $P_1 = 1$. Next, we present the first few values of Pell numbers with positive and negative subscripts:

Table 1. The first few values of the Pell numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P_n	0	1	2	5	12	29	70	169	408	985	2378	5741	13860	33461	80782
P_{-n}	0	1	-2	5	-12	29	-70	169	-408	985	-2378	5741	-13860	33461	-80782

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1,2,3,4,6,8,11,12,19]. For higher order Pell sequences, see [9,10,16,17,18].

In this paper we consider the case $r_1 = 2$, $r_2 = r_3 = r_4 = r_5 = r_6 = 1$ and in this case we write $V_n = W_n$. A generalized sixth order Pell sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4, V_5)\}_{n\geq 0}$ is defined by the sixth-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6} \tag{1.2}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$, $V_3 = c_3$, $V_4 = c_4$, $V_5 = c_5$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - V_{-(n-4)} - 2V_{-(n-5)} + V_{-(n-6)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (1.2) holds for all integer n.

As $\{V_n\}$ is a sixth order recurrence sequence (difference equation), it's characteristic equation is

$$x^6 - 2x^5 - x^4 - x^3 - x^2 - x - 1 = 0. \tag{1.3}$$

The approximate value of the roots $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 of Equation (1.3) are given by

$$\begin{aligned}
\theta_1 &= 2.61436627211445042\,08\\
\theta_2 &= -0.76286141326240044899\\
\theta_3 &= 0.45907924801189877223 - 0.76572377800211887372i\\
\theta_4 &= 0.45907924801189877223 + 0.76572377800211887372i\\
\theta_5 &= -0.38483167743792375813 - 0.69350597836224613636i\\
\theta_6 &= -0.38483167743792375813 + 0.69350597836224613636i
\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}
\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 &= 2,\\
\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6 &= -1.
\end{aligned}$$

The first few generalized sixth order Pell numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few generalized sixth order Pell numbers

n	V_n	V_{-n}
0	V_0	V_0
1	V_1	$-V_0 - V_1 - V_2 - V_3 - 2 \times V_4 + V_5$
2	V_2	$-V_5 + 3V_4 - V_3$
3	V_3	$-V_4 + 3V_3 - V_2$
4	V_4	$-V_3 + 3V_2 - V_1$
5	V_5	$-V_2 + 3V_1 - V_0$
6	$2V_5 + V_4 + V_3 + V_2 + V_1 + V_0)$	$-V_5 + 2V_4 + V_3 + V_2 + 4V_0$
7	$5V_5 + 3V_4 + 3V_3 + 3V_2 + 3V_1 + 2V_0$	$4V_5 - 9V_4 - 2V_3 - 3V_2 - 3V_1 - 4V_0$
8	$13V_5 + 8V_4 + 8V_3 + 8V_2 + 7V_1 + 5V_0$	$-4V_5 + 12V_4 - 5V_3 + 2V_2 + V_1 + V_0$
9	$34V_5 + 21V_4 + 21V_3 + 20V_2 + 18V_1 + 13V_0$	$V_5 - 6V_4 + 11V_3 - 6V_2 + V_1$
10	$89V_5 + 55V_4 + 54V_3 + 52V_2 + 47V_1 + 34V_0$	$V_4 - 6V_3 + 11V_2 - 6V_1 + V_0$

Now we define three special case of the sequence $\{V_n\}$. Sixth-order Pell sequence $\{P_n^{(6)}\}_{n\geq 0}$, sixth-order Pell-Lucas sequence $\{Q_n^{(6)}\}_{n\geq 0}$ and modified sixth-order Pell sequence $\{E_n^{(6)}\}_{n\geq 0}$ are defined, respectively, by the sixth-order recurrence relations

(1.4)

$$P_{n+6}^{(6)} = P_{n+5}^{(6)}+2P_{n+4}^{(6)}+P_{n+3}^{(6)}+P_{n+2}^{(6)}+P_{n+1}^{(6)}+P_{n}^{(6)},\ \ P_0^{(6)} = 0, P_1^{(6)} = 1, P_2^{(6)} = 2, P_3^{(6)} = 5, P_4^{(6)} = 13, P_5^{(6)} = 34$$

and

(1.5)

$$Q_{n+6}^{(6)} = Q_{n+5}^{(6)}+2Q_{n+4}^{(6)}+Q_{n+3}^{(6)}+Q_{n+2}^{(6)}+Q_{n+1}^{(6)}+Q_{n}^{(6)}, Q_0^{(6)} = 4, Q_1^{(6)} = 2, Q_2^{(6)} = 6, Q_3^{(6)} = 17, Q_4^{(6)} = 46, Q_5^{(6)} = 122$$

and

(1.6)

$$E_{n+6}^{(6)} = E_{n+5}^{(6)}+2E_{n+4}^{(6)}+E_{n+3}^{(6)}+E_{n+2}^{(6)}+E_{n+1}^{(6)}+E_{n}^{(6)},\ \ E_0^{(6)} = 0, E_1^{(6)} = 1, E_2^{(6)} = 1, E_3^{(6)} = 3, E_4^{(6)} = 8, E_5^{(6)} = 21.$$

The sequences $\{P_n^{(6)}\}_{n\geq 0}$, $\{Q_n^{(6)}\}_{n\geq 0}$ and $\{E_n^{(6)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n}^{(6)} = -P_{-(n-1)}^{(6)} - P_{-(n-2)}^{(6)} - P_{-(n-3)}^{(6)} - P_{-(n-4)}^{(6)} - 2P_{-(n-5)}^{(6)} + P_{-(n-6)}^{(6)} \tag{1.7}$$

and

$$Q_{-n}^{(6)} = -Q_{-(n-1)}^{(6)} - Q_{-(n-2)}^{(6)} - Q_{-(n-3)}^{(6)} - Q_{-(n-4)}^{(6)} - 2Q_{-(n-5)}^{(6)} + Q_{-(n-6)}^{(6)} \tag{1.8}$$

and

$$E_{-n}^{(6)} = -E_{-(n-1)}^{(6)} - E_{-(n-2)}^{(6)} - E_{-(n-3)}^{(6)} - E_{-(n-4)}^{(6)} - 2E_{-(n-5)}^{(6)} + E_{-(n-6)}^{(6)} \tag{1.9}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.7), (1.8) and (1.9) hold for all integer n.

In the rest of the paper, for easy writing, we drop the superscripts and write P_n, Q_n and E_n for $P_n^{(6)}, Q_n^{(6)}$ and $E_n^{(6)}$, respectively.

Note that P_n, Q_n and E_n sequences are't in the database of http://oeis.org [13], yet.

Next, we present the first few values of the sixth-order Pell, sixth-order Pell-Lucas and modified sixth-order Pell numbers with positive and negative subscripts:

Table 3. The first few values of the special sixth-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	2	5	13	34	89	233	609	1592	4162	10881	28447	74371
P_{-n}	0	0	0	0	0	1	−1	0	0	0	−1	4	−4	1
Q_n	6	2	6	17	46	122	321	835	2182	5705	14916	38997	1 01953	266541
Q_{-n}	6	−1	−1	−1	−1	−6	17	−8	−1	−1	4	−34	65	−40
E_n	0	1	1	3	8	21	55	144	376	983	2570	6719	17566	45924
E_{-n}	0	0	0	0	−1	2	−1	0	0	1	−5	8	−5	1

2. Generating Functions

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1. *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized sixth-order Pell sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{\begin{array}{c} V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 \\ +(V_4 - 2V_3 - V_2 - V_1 - V_0)x^4 + (V_5 - 2V_4 - V_3 - V_2 - V_1 - V_0)x^5 \end{array}}{1 - 2x - x^2 - x^3 - x^4 - x^5 - x^6}. \tag{2.1}$$

Proof.

Using the definition of generalized sixth-order Pell numbers and substracting $xf(x), x^2 f(x), x^3 f(x), x^4 f(x), x^5 f(x)$ and $x^6 f(x)$ from $f(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
& (1 - 2x - x^2 - x^3 - x^4 - x^5 - x^6) f_{V_n}(x) \\
= & \sum_{n=0}^{\infty} V_n x^n - 2x \sum_{n=0}^{\infty} V_n x^n - x^2 \sum_{n=0}^{\infty} V_n x^n - x^3 \sum_{n=0}^{\infty} V_n x^n - x^4 \sum_{n=0}^{\infty} V_n x^n - x^5 \sum_{n=0}^{\infty} V_n x^n - x^6 \sum_{n=0}^{\infty} V_n x^n \\
= & \sum_{n=0}^{\infty} V_n x^n - 2 \sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+2} - \sum_{n=0}^{\infty} V_n x^{n+3} - \sum_{n=0}^{\infty} V_n x^{n+4} - \sum_{n=0}^{\infty} V_n x^{n+5} - \sum_{n=0}^{\infty} V_n x^{n+6} \\
= & \sum_{n=0}^{\infty} V_n x^n - 2 \sum_{n=1}^{\infty} V_{n-1} x^n - \sum_{n=2}^{\infty} V_{n-2} x^n - \sum_{n=3}^{\infty} V_{n-3} x^n - \sum_{n=4}^{\infty} V_{n-4} x^n - \sum_{n=5}^{\infty} V_{n-5} x^n - \sum_{n=6}^{\infty} V_{n-6} x^n
\end{aligned}$$

and then

$$
\begin{aligned}
& (1-2x-x^2-x^3-x^4-x^5-x^6)f_{V_n}(x) \\
= \; & (V_0+V_1x+V_2x^2+V_3x^3+V_4x^4+V_5x^5)-2(V_0x+V_1x^2+V_2x^3+V_3x^4+V_4x^5) \\
& -(V_0x^2+V_1x^3+V_2x^4+V_3x^5)-(V_0x^3+V_1x^4+V_2x^5)-(V_0x^4+V_1x^5)-V_0x^5 \\
& +\sum_{n=6}^{\infty}(V_n-V_{n-1}-V_{n-2}-V_{n-3}-V_{n-4}-V_{n-5}-V_{n-6})x^n \\
= \; & V_0+(V_1-2V_0)x+(V_2-2V_1-V_0)x^2+(V_3-2V_2-V_1-V_0)x^3 \\
& +(V_4-2V_3-V_2-V_1-V_0)x^4+(V_5-2V_4-V_3-V_2-V_1-V_0)x^5.
\end{aligned}
$$

Rearranging the above equation, we get (2.1).

The previous Lemma gives the following results as particular examples.

Corollary 2. *Generated functions of sixth-order Pell, Pell-Lucas and modified Pell numbers are*

$$\sum_{n=0}^{\infty}P_nx^n=\frac{x}{1-2x-x^2-x^3-x^4-x^5-x^6},$$

and

$$\sum_{n=0}^{\infty}Q_nx^n=\frac{6-10x-4x^2-3x^3-2x^4-x^5}{1-2x-x^2-x^3-x^4-x^5-x^6},$$

and

$$\sum_{n=0}^{\infty}E_nx^n=\frac{x-x^2}{1-2x-x^2-x^3-x^4-x^5-x^6},$$

respectively.

3. Obtaning Binet Formula From Generating Function

We next find Binet formula of generalized sixth order Pell numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3. *(Binet formula of generalized sixth order Pell numbers)*

$$V_n=\sum_{k=1}^{6}\frac{d_k\theta_k^n}{\prod\limits_{\substack{j=1\\k\neq j}}^{6}(\theta_k-\theta_j)} \tag{3.1}$$

where

$$
\begin{aligned}
d_k \; = \; & V_0\theta_k^5+(V_1-2V_0)\theta_k^4+(V_2-2V_1-V_0)\theta_k^3+(V_3-2V_2-V_1-V_0)\theta_k^2 \\
& +(V_4-2V_3-V_2-V_1-V_0)\theta_k+(V_5-2V_4-V_3-V_2-V_1-V_0)
\end{aligned}
$$

for each $1\leq k\leq 6$.

Proof. Let

$$h(x) = 1 - 2x - x^2 - x^3 - x^4 - x^5 - x^6.$$

Then for some $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 we write

$$h(x) = (1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x)$$

i.e.,

$$1 - 2x - x^2 - x^3 - x^4 - x^5 - x^6 = (1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x) \tag{3.2}$$

Hence $\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_3}, \frac{1}{\theta_4}, \frac{1}{\theta_5}$ ve $\frac{1}{\theta_6}$ are the roots of $h(x)$. This gives $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 as the roots of

$$h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5} - \frac{1}{x^6} = 0.$$

This implies $x^6 - x^5 - x^4 - x^3 - 2x^2 - x - 1 = 0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{\begin{array}{c} V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 \\ +(V_4 - 2V_3 - V_2 - V_1 - V_0)x^4 + (V_5 - 2V_4 - V_3 - V_2 - V_1 - V_0)x^5 \end{array}}{(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x)}.$$

Then we write

$$\sum_{n=0}^{\infty} V_n x^n = \frac{A_1}{(1-\theta_1 x)} + \frac{A_2}{(1-\theta_2 x)} + \frac{A_3}{(1-\theta_3 x)} + \frac{A_4}{(1-\theta_4 x)} + \frac{A_5}{(1-\theta_5 x)} + \frac{A_6}{(1-\theta_6 x)}. \tag{3.3}$$

So

$$\begin{aligned} & V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 \\ & +(V_4 - 2V_3 - V_2 - V_1 - V_0)x^4 + (V_5 - 2V_4 - V_3 - V_2 - V_1 - V_0)x^5 \\ = \; & A_1(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x) + A_2(1-\theta_1 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x) \\ & +A_3(1-\theta_1 x)(1-\theta_2 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x) + A_4(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_5 x)(1-\theta_6 x) \\ & +A_5(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_6 x) + A_6(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x). \end{aligned}$$

If we consider $x = \frac{1}{\theta_1}$, we get

$$\begin{aligned} & V_0 + (V_1 - 2V_0)\frac{1}{\theta_1} + (V_2 - 2V_1 - V_0)\frac{1}{\theta_1^2} + (V_3 - 2V_2 - V_1 - V_0)\frac{1}{\theta_1^3} \\ & +(V_4 - 2V_3 - V_2 - V_1 - V_0)\frac{1}{\theta_1^4} + (V_5 - 2V_4 - V_3 - V_2 - V_1 - V_0)\frac{1}{\theta_1^5} \\ = \; & A_1(1 - \frac{\theta_2}{\theta_1})(1 - \frac{\theta_3}{\theta_1})(1 - \frac{\theta_4}{\theta_1})(1 - \frac{\theta_5}{\theta_1})(1 - \frac{\theta_6}{\theta_1}). \end{aligned}$$

This gives

$$
\begin{aligned}
A_1 &= \frac{\begin{array}{c}\theta_1^5(V_0+(V_1-2V_0)\frac{1}{\theta_1}+(V_2-2V_1-V_0)\frac{1}{\theta_1^2}+(V_3-2V_2-V_1-V_0)\frac{1}{\theta_1^3}\\ +(V_4-2V_3-V_2-V_1-V_0)\frac{1}{\theta_1^4}+(V_5-2V_4-V_3-V_2-V_1-V_0)\frac{1}{\theta_1^5})\end{array}}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)(\theta_1-\theta_6)} \\
&= \frac{d_1}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)(\theta_1-\theta_6)}
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
A_2 &= \frac{d_2}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)(\theta_2-\theta_6)}, \\
A_3 &= \frac{d_3}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)(\theta_3-\theta_6)} \\
A_4 &= \frac{d_4}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)(\theta_4-\theta_6)} \\
A_5 &= \frac{d_5}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)(\theta_5-\theta_6)} \\
A_5 &= \frac{d_6}{(\theta_6-\theta_1)(\theta_6-\theta_2)(\theta_6-\theta_3)(\theta_6-\theta_4)(\theta_6-\theta_5)}.
\end{aligned}
$$

Thus (3.3) can be written as

$$
\sum_{n=0}^{\infty} V_n x^n = A_1(1-\theta_1 x)^{-1}+A_2(1-\theta_2 x)^{-1}+A_3(1-\theta_3 x)^{-1}+A_4(1-\theta_4 x)^{-1}+A_5(1-\theta_5 x)^{-1}+A_6(1-\theta_6 x)^{-1}.
$$

This gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} V_n x^n &= A_1\sum_{n=0}^{\infty}\theta_1^n x^n + A_2\sum_{n=0}^{\infty}\theta_2^n x^n + A_3\sum_{n=0}^{\infty}\theta_3^n x^n + A_4\sum_{n=0}^{\infty}\theta_4^n x^n + A_5\sum_{n=0}^{\infty}\theta_5^n x^n + A_6\sum_{n=0}^{\infty}\theta_6^n x^n \\
&= \sum_{n=0}^{\infty}(A_1\theta_1^n + A_2\theta_2^n + A_3\theta_3^n + A_4\theta_4^n + A_5\theta_5^n + A_6\theta_6^n)x^n.
\end{aligned}
$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$
V_n = A_1\theta_1^n + A_2\theta_2^n + A_3\theta_3^n + A_4\theta_4^n + A_5\theta_5^n + A_6\theta_6^n
$$

and then we get (3.1).

Next, using Theorem 3, we present the Binet formulas of sixth-order Pell, Pell-Lucas and modified Pell sequences.

Corollary 4. *Binet formulas of sixth-order Pell, Pell-Lucas and modified Pell sequences are*

$$
P_n = \sum_{k=1}^{6} \frac{\theta_k^{n+4}}{\prod\limits_{\substack{j=1\\ k\neq j}}^{6}(\theta_k-\theta_j)}
$$

and

$$
Q_n = \sum_{k=1}^{6}\theta_k^n = \theta_1^n+\theta_2^n+\theta_3^n+\theta_4^n+\theta_5^n+\theta_6^n,
$$

and

$$E_n = \sum_{k=1}^{6} \frac{(\theta_k - 1)\theta_k^{n+3}}{\prod\limits_{\substack{j=1 \\ k \neq j}}^{6} (\theta_k - \theta_j)}.$$

respectively.

Note that Binet formula of generalized sixth order Pell numbers can be represented as

$$V_n = \sum_{k=1}^{6} \frac{\theta_k d_k \theta_k^n}{(2\theta_k^5 + 2\theta_k^4 + 3\theta_k^3 + 4\theta_k^2 + 5\theta_k + 6)} \tag{3.4}$$

which can be derived from a result ((4.20) in page 25) of Hanusa [5]. When we compare (3.1) and (3.4), we see the following identities:

$$\begin{aligned}
\frac{1}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)(\theta_1-\theta_6)} &= \frac{\theta_1}{2\theta_1^5 + 2\theta_1^4 + 3\theta_1^3 + 4\theta_1^2 + 5\theta_1 + 6} \\
\frac{1}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)(\theta_2-\theta_6)} &= \frac{\theta_2}{2\theta_2^5 + 2\theta_2^4 + 3\theta_2^3 + 4\theta_2^2 + 5\theta_2 + 6} \\
\frac{1}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)(\theta_3-\theta_6)} &= \frac{\theta_3}{2\theta_3^5 + 2\theta_3^4 + 3\theta_3^3 + 4\theta_3^2 + 5\theta_3 + 6} \\
\frac{1}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)(\theta_4-\theta_6)} &= \frac{\theta_4}{2\theta_4^5 + 2\theta_4^4 + 3\theta_4^3 + 4\theta_4^2 + 5\theta_4 + 6} \\
\frac{1}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)(\theta_5-\theta_6)} &= \frac{\theta_5}{2\theta_5^5 + 2\theta_5^4 + 3\theta_5^3 + 4\theta_5^2 + 5\theta_5 + 6} \\
\frac{1}{(\theta_6-\theta_1)(\theta_6-\theta_2)(\theta_6-\theta_3)(\theta_6-\theta_4)(\theta_6-\theta_5)} &= \frac{\theta_6}{2\theta_6^5 + 2\theta_6^4 + 3\theta_6^3 + 4\theta_6^2 + 5\theta_6 + 6}
\end{aligned}$$

Using the above identities, we can give the Binet formulas of sixth-order Pell, Pell-Lucas and modified Pell sequences in the following form: Binet formulas of sixth-order Pell, Pell-Lucas and modified Pell sequences are

$$P_n = \sum_{k=1}^{6} \frac{\theta_k^{n+5}}{(2\theta_k^5 + 2\theta_k^4 + 3\theta_k^3 + 4\theta_k^2 + 5\theta_k + 6)}$$

and

$$Q_n = \theta_1^n + \theta_2^n + \theta_3^n + \theta_4^n + \theta_5^n + \theta_6^n,$$

and

$$E_n = \sum_{k=1}^{6} \frac{(\theta_k - 1)\theta_k^{n+4}}{(2\theta_k^5 + 2\theta_k^4 + 3\theta_k^3 + 4\theta_k^2 + 5\theta_k + 6)}$$

respectively.

4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized Hexanacci sequence $\{W_n\}$.

Theorem 5 (Simson Formula of Generalized Hexanacci Numbers). *For all integers n we have*

(4.1)
$$\begin{vmatrix} W_{n+5} & W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} & W_{n-5} \end{vmatrix} = (-1)^n r_6^n \begin{vmatrix} W_5 & W_4 & W_3 & W_2 & W_1 & W_0 \\ W_4 & W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_3 & W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} & W_{-5} \end{vmatrix}.$$

Proof. (4.1) is given in Soykan [14], see also [15].

A special case of the above theorem is the following Theorem which gives Simson formula of the generalized sixth-order Pell sequence $\{V_n\}$.

Theorem 6 (Simson Formula of Generalized Sixth-Order Pell Numbers). *For all integers n we have*

$$\begin{vmatrix} V_{n+5} & V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} & V_{n-5} \end{vmatrix} = (-1)^n \begin{vmatrix} V_5 & V_4 & V_3 & V_2 & V_1 & V_0 \\ V_4 & V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_3 & V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} & V_{-5} \end{vmatrix}.$$

The previous Theorem gives the following results as particular examples.

Corollary 7. *Simson formula of sixth-order Pell, Pell-Lucas and modified Pell numbers are given as*

$$\begin{vmatrix} P_{n+5} & P_{n+4} & P_{n+3} & P_{n+2} & P_{n+1} & P_n \\ P_{n+4} & P_{n+3} & P_{n+2} & P_{n+1} & P_n & P_{n-1} \\ P_{n+3} & P_{n+2} & P_{n+1} & P_n & P_{n-1} & P_{n-2} \\ P_{n+2} & P_{n+1} & P_n & P_{n-1} & P_{n-2} & P_{n-3} \\ P_{n+1} & P_n & P_{n-1} & P_{n-2} & P_{n-3} & P_{n-4} \\ P_n & P_{n-1} & P_{n-2} & P_{n-3} & P_{n-4} & P_{n-5} \end{vmatrix} = (-1)^n,$$

$$\begin{vmatrix} Q_{n+5} & Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} & Q_{n-5} \end{vmatrix} = -884552(-1)^n = 884552(-1)^{n+1},$$

$$\begin{vmatrix} E_{n+5} & E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} & E_{n-5} \end{vmatrix} = 6(-1)^n.$$

5. Some Identities

In this section, we obtain some identities of sixth order Pell, sixth order Pell-Lucas and modified sixth order Pell numbers. First, we can give a few basic relations between $\{P_n\}$ and $\{Q_n\}$.

Lemma 8. *The following equalities are true:*

(5.1)
$$\begin{aligned} Q_n &= -6P_{n+6} + 11P_{n+5} + 7P_{n+4} + 8P_{n+3} + 9P_{n+2} + 17P_{n+1}, \\ Q_n &= -P_{n+5} + P_{n+4} + 2P_{n+3} + 3P_{n+2} + 11P_{n+1} - 6P_n, \\ Q_n &= -P_{n+4} + P_{n+3} + 2P_{n+2} + 10P_{n+1} - 7P_n - P_{n-1}, \\ Q_n &= -P_{n+3} + P_{n+2} + 9P_{n+1} - 8P_n - 2P_{n-1} - P_{n-2}, \\ Q_n &= -P_{n+2} + 8P_{n+1} - 9P_n - 3P_{n-1} - 2P_{n-2} - P_{n-3}, \end{aligned}$$

and

$$
\begin{aligned}
442276P_n &= -39853Q_{n+6}+96245Q_{n+5}+14588Q_{n+4}+15965Q_{n+3}+14650Q_{n+2}+10285Q_{n+1}, \\
442276P_n &= 16539Q_{n+5}-25265Q_{n+4}-23888Q_{n+3}-25203Q_{n+2}-29568Q_{n+1}-39853Q_n, \\
442276P_n &= 7813Q_{n+4}-7349Q_{n+3}-8664Q_{n+2}-13029Q_{n+1}-23314Q_n+16539Q_{n-1}, \\
442276P_n &= 8277Q_{n+3}-851Q_{n+2}-5216Q_{n+1}-15501Q_n+24352Q_{n-1}+7813Q_{n-2}, \\
442276P_n &= 15703Q_{n+2}+3061Q_{n+1}-7224Q_n+32629Q_{n-1}+16090Q_{n-2}+8277Q_{n-3}.
\end{aligned}
$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$Q_n = a\times P_{n+5}+b\times P_{n+4}+c\times P_{n+3}+d\times P_{n+2}$$

and solving the system of equations

$$
\begin{aligned}
Q_0 &= a\times P_6+b\times P_5+c\times P_4+d\times P_3+e\times P_2+f\times P_1 \\
Q_1 &= a\times P_7+b\times P_6+c\times P_5+d\times P_4+e\times P_3+f\times P_2 \\
Q_2 &= a\times P_8+b\times P_7+c\times P_6+d\times P_5+e\times P_4+f\times P_2 \\
Q_3 &= a\times P_9+b\times P_8+c\times P_7+d\times P_6+e\times P_5+f\times P_4 \\
Q_4 &= a\times P_{10}+b\times P_9+c\times P_8+d\times P_7+e\times P_6+f\times P_5 \\
Q_5 &= a\times P_{11}+b\times P_{10}+c\times P_9+d\times P_8+e\times P_7+f\times P_6
\end{aligned}
$$

we find that $a=-6, b=11, c=7, d=8, e=9, f=17$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{P_n\}$ and $\{E_n\}$.

Lemma 9. *The following equalities are true:*

$$
\begin{aligned}
E_n &= 2P_{n+6}-5P_{n+5}-P_{n+3}-P_{n+2}-P_{n+1}, \\
E_n &= -P_{n+5}+2P_{n+4}+P_{n+3}+P_{n+2}+P_{n+1}+2P_n, \\
E_n &= P_n-P_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
6P_n &= E_{n+6}-E_{n+5}-2E_{n+4}-3E_{n+3}-4E_{n+2}-5E_{n+1}, \\
6P_n &= E_{n+5}-E_{n+4}-2E_{n+3}-3E_{n+2}-4E_{n+1}+E_n, \\
6P_n &= E_{n+4}-E_{n+3}-2E_{n+2}-3E_{n+1}+2E_n+E_{n-1}, \\
6P_n &= E_{n+3}-E_{n+2}-2E_{n+1}+3E_n+2E_{n-1}+E_{n-2}, \\
6P_n &= E_{n+2}-E_{n+1}+4E_n+3E_{n-1}+2E_{n-2}+E_{n-3}.
\end{aligned}
$$

Thirdly, we give a few basic relations between $\{Q_n\}$ and $\{E_n\}$.

LEMMA 10. *The following equalities are true:*

$$\begin{aligned}
3Q_n &= 5E_{n+6} - 8E_{n+5} - 10E_{n+4} - 9E_{n+3} - 5E_{n+2} + 23E_{n+1}, \\
3Q_n &= 2E_{n+5} - 5E_{n+4} - 4E_{n+3} + 28 \times E_{n+1} + 5E_n, \\
3Q_n &= -E_{n+4} - 2E_{n+3} + 2E_{n+2} + 30E_{n+1} + 7E_n + 2E_{n-1}, \\
3Q_n &= -4E_{n+3} + E_{n+2} + 29E_{n+1} + 6E_n + E_{n-1} - E_{n-2}, \\
3Q_n &= -7E_{n+2} + 25E_{n+1} + 2E_n - 3E_{n-1} - 5E_{n-2} - 4E_{n-3},
\end{aligned}$$

and

$$\begin{aligned}
221138E_n &= -25069Q_{n+6} + 78334Q_{n+5} - 35686Q_{n+4} + 5831Q_{n+3} + 4485Q_{n+2} + 2960Q_{n+1}, \\
221138E_n &= 28196Q_{n+5} - 60755Q_{n+4} - 19238Q_{n+3} - 20584Q_{n+2} - 22109Q_{n+1} - 25069Q_n, \\
221138E_n &= -4363Q_{n+4} + 8958Q_{n+3} + 7612Q_{n+2} + 6087Q_{n+1} + 3127Q_n + 28196Q_{n-1}, \\
221138E_n &= 232Q_{n+3} + 3249Q_{n+2} + 1724Q_{n+1} - 1236Q_n + 23833Q_{n-1} - 4363Q_{n-2}, \\
221138E_n &= 3713Q_{n+2} + 1956Q_{n+1} - 1004Q_n + 24065Q_{n-1} - 4131Q_{n-2} + 232Q_{n-3}.
\end{aligned}$$

We now present a few special identities for the modified sixth order Pell sequence $\{E_n\}$.

THEOREM 11. *(Catalan's identity) For all natural numbers n and m, the following identity holds*

$$E_{n+m}E_{n-m} - E_n^2 = (P_{n+m} - P_{n+m-1})(P_{n-m} - P_{n-m-1}) - (P_n - P_{n-1})^2$$

Proof. We use the identity

$$E_n = P_n - P_{n-1}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified sixth order Pell sequence.

COROLLARY 12. *(Cassini's identity) For all natural numbers n and m, the following identity holds*

$$E_{n+1}E_{n-1} - E_n^2 = (P_{n+1} - P_n)(P_{n-1} - P_{n-2}) - (P_n - P_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n = P_n - P_{n-1}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified sixth order Pell sequence $\{E_n\}$.

THEOREM 13. *Let n and m be any integers. Then the following identities are true:*

(a): *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = (P_{m+1} - P_m)(P_n - P_{n-1}) - (P_m - P_{m-1})(P_{n+1} - P_n).$$

(b): *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2}-E_n^4=(P_{n+2}-P_{n+1})(P_{n+1}-P_n)(P_{n-1}-P_{n-2})(P_{n-2}-P_{n-3})-(P_n-P_{n-1})^4$$

(c): *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6}-E_{n+3}^3=(P_{n+1}-P_n)(P_{n+2}-P_{n+1})(P_{n+6}-P_{n+5})-(P_{n+3}-P_{n+2})^3$$

Proof. Use the identity $E_n=P_n-P_{n-1}$.

6. Linear Sums

The following Theorem presents summing formulas of generalized sixth order Pell numbers.

THEOREM 14. *For $n\geq 0$ we have the following formulas:*

(a): *(Sum of the generalized sixth order Pell numbers)*

$$\sum_{k=0}^{n}V_k=\frac{1}{6}(V_{n+6}-V_{n+5}-2V_{n+4}-3V_{n+3}-4V_{n+2}-5V_{n+1}-V_5+V_4+2V_3+3V_2+4V_1+5V_0).$$

(b):

$$\sum_{k=0}^{n}V_{2k}=\frac{1}{6}(-V_{2n+2}+4V_{2n+1}+2V_{2n}+3V_{2n-1}+V_{2n-2}+2V_{2n-3}-2V_5+5V_4-2V_3+6V_2-V_1+7V_0).$$

(c):

$$\sum_{k=0}^{n}V_{2k+1}=\frac{1}{6}(2V_{2n+2}+V_{2n+1}+2V_{2n}+V_{2n-2}-V_{2n-3}+V_5-4V_4+4V_3-3V_2+5V_1-2V_0).$$

Proof.

(a): Using the recurrence relation

$$V_n=2V_{n-1}+V_{n-2}+V_{n-3}+V_{n-4}+V_{n-5}+V_{n-6}$$

i.e.

$$V_{n-6}=V_n-2V_{n-1}-V_{n-2}-V_{n-3}-V_{n-4}-V_{n-5}$$

we obtain

$$\begin{aligned}
V_0 &= V_6 - 2V_5 - V_4 - V_3 - V_2 - V_1 \\
V_1 &= V_7 - 2V_6 - V_5 - V_4 - V_3 - V_2 \\
V_2 &= V_8 - 2V_7 - V_6 - V_5 - V_4 - V_3 \\
&\vdots \\
V_{n-6} &= V_n - 2V_{n-1} - V_{n-2} - V_{n-3} - V_{n-4} - V_{n-5} \\
V_{n-5} &= V_{n+1} - 2V_n - V_{n-1} - V_{n-2} - V_{n-3} - V_{n-4} \\
V_{n-4} &= V_{n+2} - 2V_{n+1} - V_n - V_{n-1} - V_{n-2} - V_{n-3} \\
V_{n-3} &= V_{n+3} - 2V_{n+2} - V_{n+1} - V_n - V_{n-1} - V_{n-2} \\
V_{n-2} &= V_{n+4} - 2V_{n+3} - V_{n+2} - V_{n+1} - V_n - V_{n-1} \\
V_{n-1} &= V_{n+5} - 2V_{n+4} - V_{n+3} - V_{n+2} - V_{n+1} - V_n \\
V_n &= V_{n+6} - 2V_{n+5} - V_{n+4} - V_{n+3} - V_{n+2} - V_{n+1}.
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
\sum_{k=0}^{n} V_k &= (V_{n+6} + V_{n+5} + V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} - V_5 - V_4 - V_3 - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
&\quad -2(V_{n+5} + V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} - V_4 - V_3 - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
&\quad -(V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} - V_3 - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
&\quad -(V_{n+3} + V_{n+2} + V_{n+1} - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
&\quad -(V_{n+2} + V_{n+1} - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
&\quad -(V_{n+1} - V_0 + \sum_{k=0}^{n} V_k)
\end{aligned}$$

Then, solving the above equality we obtain

$$\sum_{k=0}^{n} V_k = \frac{1}{6}(V_{n+6} - V_{n+5} - 2V_{n+4} - 3V_{n+3} - 4V_{n+2} - 5V_{n+1} - V_5 + V_4 + 2V_3 + 3V_2 + 4V_1 + 5V_0).$$

(b) and (c): Using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6}$$

i.e.

$$2V_{n-1} = V_n - V_{n-2} - V_{n-3} - V_{n-4} - V_{n-5} - V_{n-6}$$

we obtain

$$
\begin{aligned}
2V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} - V_{-2} \\
2V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 - V_0 \\
2V_7 &= V_8 - V_6 - V_5 - V_4 - V_3 - V_2 \\
2V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 - V_4 \\
&\vdots \\
2V_{2n-1} &= V_{2n} - V_{2n-2} - V_{2n-3} - V_{2n-4} - V_{2n-5} - V_{2n-6} \\
2V_{2n+1} &= V_{2n+2} - V_{2n} - V_{2n-1} - V_{2n-2} - V_{2n-3} - V_{2n-4} \\
2V_{2n+3} &= V_{2n+4} - V_{2n+2} - V_{2n+1} - V_{2n} - V_{2n-1} - V_{2n-2} \\
2V_{2n+5} &= V_{2n+6} - V_{2n+4} - V_{2n+3} - V_{2n+2} - V_{2n+1} - V_{2n}.
\end{aligned}
$$

Now, if we add the above equations by side by, we get

$$
\begin{aligned}
(6.1)\quad 2(-V_1 + \sum_{k=0}^{n} V_{2k+1}) &= (V_{2n+2} - V_2 - V_0 + \sum_{k=0}^{n} V_{2k}) - (-V_0 + \sum_{k=0}^{n} V_{2k}) \\
&\quad -(-V_{2n+1} + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} + \sum_{k=0}^{n} V_{2k}) - (-V_{2n+1} - V_{2n-1} + V_{-1} \\
&\quad + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} - V_{2n-2} + V_{-2} + \sum_{k=0}^{n} V_{2k}).
\end{aligned}
$$

Similarly, using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6}$$

i.e.

$$2V_{n-1} = V_n - V_{n-2} - V_{n-3} - V_{n-4} - V_{n-5} - V_{n-6}$$

we write the following obvious equations;

$$\begin{aligned}
2V_2 &= V_3 - V_1 - V_0 - V_{-1} - V_{-2} - V_{-3} \\
2V_4 &= V_5 - V_3 - V_2 - V_1 - V_0 - V_{-1} \\
2V_6 &= V_7 - V_5 - V_4 - V_3 - V_2 - V_1 \\
2V_8 &= V_9 - V_7 - V_6 - V_5 - V_4 - V_3 \\
&\vdots \\
2V_{2n-2} &= V_{2n-1} - V_{2n-3} - V_{2n-4} - V_{2n-5} - V_{2n-6} - V_{2n-7} \\
2V_{2n} &= V_{2n+1} - V_{2n-1} - V_{2n-2} - V_{2n-3} - V_{2n-4} - V_{2n-5} \\
2V_{2n+2} &= V_{2n+3} - V_{2n+1} - V_{2n} - V_{2n-1} - V_{2n-2} - V_{2n-3} \\
2V_{2n+4} &= V_{2n+5} - V_{2n+3} - V_{2n+2} - V_{2n+1} - V_{2n} - V_{2n-1} \\
2V_{2n+6} &= V_{2n+7} - V_{2n+5} - V_{2n+4} - V_{2n+3} - V_{2n+2} - V_{2n+1}
\end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$\begin{aligned}
(6.2)\ 2(-V_0 + \sum_{k=0}^{n} V_{2k}) &= (-V_1 + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n+1} + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} + \sum_{k=0}^{n} V_{2k}) \\
&\quad -(-V_{2n+1} - V_{2n-1} + V_{-1} + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} - V_{2n-2} + V_{-2} + \sum_{k=0}^{n} V_{2k}) \\
&\quad -(-V_{2n+1} - V_{2n-1} - V_{2n-3} + V_{-3} + V_{-1} + \sum_{k=0}^{n} V_{2k+1}).
\end{aligned}$$

Then, solving the system (6.1)-(6.2) using

$$\begin{aligned}
V_{-1} &= (-V_0 - V_1 - V_2 - V_3 - 2V_4 + V_5) \\
V_{-2} &= (-V_5 + 3V_4 - V_3) \\
V_{-3} &= (-V_4 + 3V_3 - V_2)
\end{aligned}$$

the required result of (b) and (c) follow.

As special cases of above Theorem, we have the following three Corollaries. First one presents some summing formulas of sixth order Pell numbers.

Corollary 15. *For $n \geq 0$ we have the following formulas:*

(a): *(Sum of the sixth order Pell numbers)*

$$\sum_{k=0}^{n} P_k = \frac{1}{6}(P_{n+6} - P_{n+5} - 2P_{n+4} - 3P_{n+3} - 4P_{n+2} - 5P_{n+1} - 1).$$

(b): $\sum_{k=0}^{n} P_{2k} = \frac{1}{6}(-P_{2n+2} + 4P_{2n+1} + 2P_{2n} + 3P_{2n-1} + P_{2n-2} + 2P_{2n-3} - 2).$

(c): $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{6}(2P_{2n+2} + P_{2n+1} + 2P_{2n} + P_{2n-2} - P_{2n-3} + 1).$

Second one presents some summing formulas of sixth order Pell-Lucas numbers.

Corollary 16. *For $n \geq 0$ we have the following formulas:*

(a): *(Sum of the sixth order Pell-Lucas numbers)*

$$\sum_{k=0}^{n} Q_k = \frac{1}{6}(Q_{n+6} - Q_{n+5} - 2Q_{n+4} - 3Q_{n+3} - 4Q_{n+2} - 5Q_{n+1} + 14).$$

(b): $\sum_{k=0}^{n} Q_{2k} = \frac{1}{6}(-Q_{2n+2} + 4Q_{2n+1} + 2Q_{2n} + 3Q_{2n-1} + Q_{2n-2} + 2Q_{2n-3} + 28).$

(c): $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{6}(2Q_{2n+2} + Q_{2n+1} + 2Q_{2n} + Q_{2n-2} - Q_{2n-3} - 14).$

Last one presents some summing formulas of modified fourth order Pell numbers.

Corollary 17. *For $n \geq 0$ we have the following formulas:*

(a): *(Sum of the modified sixth order Pell numbers)*

$$\sum_{k=0}^{n} E_k = \frac{1}{6}(E_{n+6} - E_{n+5} - 2E_{n+4} - 3E_{n+3} - 4E_{n+2} - 5E_{n+1}).$$

(b): $\sum_{k=0}^{n} E_{2k} = \frac{1}{6}(-E_{2n+2} + 4E_{2n+1} + 2E_{2n} + 3E_{2n-1} + E_{2n-2} + 2E_{2n-3} - 3).$

(c): $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{6}(2E_{2n+2} + E_{2n+1} + 2E_{2n} + E_{2n-2} - E_{2n-3} + 3).$

7. Matrices Related with Generalized Sixth-Order Pell numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+5} \\ W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_5 \\ W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix} \tag{7.1}$$

For matrix formulation (7.1), see [7]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \\ W_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{pmatrix}.$$

We define the square matrix A of order 6 as:

$$A=\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det M=-1$. From (1.2) we have

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix}=\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix}=\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} V_5 \\ V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V=P$ in (7.2) we have

$$\begin{pmatrix} P_{n+5} \\ P_{n+4} \\ P_{n+3} \\ P_{n+2} \\ P_{n+1} \\ P_n \end{pmatrix}=\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} P_{n+4} \\ P_{n+3} \\ P_{n+2} \\ P_{n+1} \\ P_n \\ P_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n = \begin{pmatrix} P_{n+1} & \sum\limits_{k=0}^{4} P_{n-k} & \sum\limits_{k=0}^{3} P_{n-k} & \sum\limits_{k=0}^{2} P_{n-k} & \sum\limits_{k=0}^{1} P_{n-k} & P_n \\ P_n & \sum\limits_{k=1}^{5} P_{n-k} & \sum\limits_{k=1}^{4} P_{n-k} & \sum\limits_{k=1}^{3} P_{n-k} & \sum\limits_{k=1}^{2} P_{n-k} & P_{n-1} \\ P_{n-1} & \sum\limits_{k=2}^{6} P_{n-k} & \sum\limits_{k=2}^{5} P_{n-k} & \sum\limits_{k=2}^{4} P_{n-k} & \sum\limits_{k=2}^{3} P_{n-k} & P_{n-2} \\ P_{n-2} & \sum\limits_{k=3}^{7} P_{n-k} & \sum\limits_{k=3}^{6} P_{n-k} & \sum\limits_{k=3}^{5} P_{n-k} & \sum\limits_{k=3}^{4} P_{n-k} & P_{n-3} \\ P_{n-3} & \sum\limits_{k=4}^{8} P_{n-k} & \sum\limits_{k=4}^{7} P_{n-k} & \sum\limits_{k=4}^{6} P_{n-k} & \sum\limits_{k=4}^{5} P_{n-k} & P_{n-4} \\ P_{n-4} & \sum\limits_{k=5}^{9} P_{n-k} & \sum\limits_{k=5}^{8} P_{n-k} & \sum\limits_{k=5}^{7} P_{n-k} & \sum\limits_{k=5}^{6} P_{n-k} & P_{n-5} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & \sum\limits_{k=0}^{4} V_{n-k} & \sum\limits_{k=0}^{3} V_{n-k} & \sum\limits_{k=0}^{2} V_{n-k} & \sum\limits_{k=0}^{1} V_{n-k} & V_n \\ V_n & \sum\limits_{k=1}^{5} V_{n-k} & \sum\limits_{k=1}^{4} V_{n-k} & \sum\limits_{k=1}^{3} V_{n-k} & \sum\limits_{k=1}^{2} V_{n-k} & V_{n-1} \\ V_{n-1} & \sum\limits_{k=2}^{6} V_{n-k} & \sum\limits_{k=2}^{5} V_{n-k} & \sum\limits_{k=2}^{4} V_{n-k} & \sum\limits_{k=2}^{3} V_{n-k} & V_{n-2} \\ V_{n-2} & \sum\limits_{k=3}^{7} V_{n-k} & \sum\limits_{k=3}^{6} V_{n-k} & \sum\limits_{k=3}^{5} V_{n-k} & \sum\limits_{k=3}^{4} V_{n-k} & V_{n-3} \\ V_{n-3} & \sum\limits_{k=4}^{8} V_{n-k} & \sum\limits_{k=4}^{7} V_{n-k} & \sum\limits_{k=4}^{6} V_{n-k} & \sum\limits_{k=4}^{5} V_{n-k} & V_{n-4} \\ V_{n-4} & \sum\limits_{k=5}^{9} V_{n-k} & \sum\limits_{k=5}^{8} V_{n-k} & \sum\limits_{k=5}^{7} V_{n-k} & \sum\limits_{k=5}^{6} V_{n-k} & V_{n-5} \end{pmatrix}$$

THEOREM 18. *For all integer $m, n \geq 0$, we have*

(a): $B_n = A^n$

(b): $C_1 A^n = A^n C_1$

(c): $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a): By expanding the vectors on the both sides of (7.3) to 6-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b): Using (a) and definition of C_1, (b) follows.

(c): We have $AC_{n-1} = C_n$, i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of A^n matris can be given as

$$A^n = 2A^{n-1} + A^{n-2} + A^{n-3} + A^{n-4} + A^{n-5} + A^{n-6}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

for all integer m and n.

THEOREM 19. *For $m, n \geq 0$ we have*

$$\begin{aligned} V_{n+m} &= V_n P_{m+1} + V_{n-1}(P_m + P_{m-1} + P_{m-2} + P_{m-3} + P_{m-4}) \\ &\quad + V_{n-2}(P_m + P_{m-1} + P_{m-2} + P_{m-3}) + V_{n-3}(P_m + P_{m-1} \\ &\quad + P_{m-2}) + V_{n-4}(P_m + P_{m-1}) + V_{n-5} P_m \\ &= V_n P_{m+1} + P_{m-4} V_{n-1} + P_{m-1}(V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4}) \\ &\quad + P_{m-2}(V_{n-1} + V_{n-2} + V_{n-3}) + P_{m-3}(V_{n-1} + V_{n-2}) \\ &\quad + P_m (V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}) \end{aligned} \tag{7.4}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m.We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof.

REMARK 20. *By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So, for all integers m, n (7.4) is true.*

Corollary 21. *For all integers m, n, we have*

$$\begin{aligned} P_{n+m} &= P_nP_{m+1}+P_{n-1}(P_m+P_{m-1}+P_{m-2}+P_{m-3}+P_{m-4}) \\ &\quad +P_{n-2}(P_m+P_{m-1}+P_{m-2}+P_{m-3})+P_{n-3}(P_m+P_{m-1}+P_{m-2}) \\ &\quad +P_{n-4}(P_m+P_{m-1})+P_{n-5}P_m, \\ Q_{n+m} &= Q_nP_{m+1}+Q_{n-1}(P_m+P_{m-1}+P_{m-2}+P_{m-3}+P_{m-4}) \\ &\quad +Q_{n-2}(P_m+P_{m-1}+P_{m-2}+P_{m-3})+Q_{n-3}(P_m+P_{m-1}+P_{m-2}) \\ &\quad +Q_{n-4}(P_m+P_{m-1})+Q_{n-5}P_m, \\ E_{n+m} &= E_nP_{m+1}+E_{n-1}(P_m+P_{m-1}+P_{m-2}+P_{m-3}+P_{m-4}) \\ &\quad +E_{n-2}(P_m+P_{m-1}+P_{m-2}+P_{m-3})+E_{n-3}(P_m+P_{m-1}+P_{m-2}) \\ &\quad +E_{n-4}(P_m+P_{m-1})+E_{n-5}P_m. \end{aligned}$$

REFERENCES

[1] Bicknell, N., A primer on the Pell sequence and related sequence, Fibonacci Quarterly, 13(4), 345-349, 1975.

[2] Dasdemir, A., On the Pell, Pell-Lucas and Modi.ed Pell Numbers By Matrix Method, Applied Mathematical Sciences, 5(64), 3173-3181, 2011.

[3] Ercolano, J., Matrix generator of Pell sequence, Fibonacci Quarterly, 17(1), 71-77, 1979.

[4] Gökbas, H., Köse, H., Some sum formulas for products of Pell and Pell-Lucas numbers, Int. J. Adv. Appl. Math. and Mech. 4(4), 1-4, 2017.

[5] Hanusa, C., A Generalized Binet.s Formula for kth Order Linear Recurrences: A Markov Chain Approach, Harvey Mudd College, Undergraduate Thesis (Math Senior Thesis), 2001.

[6] Horadam, A. F., Pell identities, Fibonacci Quarterly, 9(3), 245-263, 1971.

[7] Kalman, D., Generalized Fibonacci Numbers By Matrix Methods, Fibonacci Quarterly, 20(1), 73-76, 1982.

[8] Kiliç, E., Ta¸sçi, D., The Linear Algebra of The Pell Matrix, Boletín de la Sociedad Matemática Mexicana, 3(11), 2005.

[9] Kiliç, E., Ta¸sçi, D., The Generalized Binet Formula, Representation and Sums of the Generalized Order-k Pell Numbers, Taiwanese Journal of Mathematics, 10(6), 1661-1670, 2006.

[10] Kiliç, E., Stanica, P., A Matrix Approach for General Higher Order Linear Recurrences, Bulletin of the Malaysian Mathe-matical Sciences Society, (2) 34(1), 51.67, 2011.

[11] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[12] Melham, R., Sums Involving Fibonacci and Pell Numbers, Portugaliae Mathematica, 56(3), 309-317, 1999.

[13] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/

[14] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, arXiv:1903.01313v1 [math.NT], 2019.

[15] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56, 2019.

[16] Soykan, Y., On Generalized Third-Order Pell Numbers, Asian Journal of Advanced Research and Reports, 6(1): 1-18, 2019.

[17] Soykan, Y., A Study of Generalized Fourth-Order Pell Sequences, Journal of Scienti.c Research and Reports, 25(1-2), 1-18, 2019.

[18] Soykan, Y., Properties of Generalized Fifth-Order Pell Numbers, Asian Research Journal of Mathematics, 15(3), 1-18, 2019.

[19] Ya¼gmur, T., New Approach to Pell and Pell-Lucas Sequences, Kyungpook Math. J. 59, 23-34, 2019.