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On Some Generalized Open Sets in Ideal Bitopological Spaces

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Abstract — In this article, we introduce and study the concepts of $\gamma_{ij}-I-$sets, $\gamma_{ij}-preI-$open sets, and $\gamma_{ij}-b-I-$open sets by generalizing $(i,j)-I-$open, $(i,j)-preI-$open, and $(i,j)-bI-$ open sets, respectively, in ideal bitopological spaces with an operation $\gamma:\tau\rightarrow P(X)$. Further, we describe and study $(\gamma,\delta)_{ij}-bI-$continuous functions in ideal bitopological spaces through this paper.

Keywords — *Ideal bitopological space, $\gamma_{ij}-I-$ open sets, $\gamma_{ij}-preI-$open sets, $\gamma_{ij}-bI-$open sets, $\gamma_{ij}-I-$continuous functions*

In 1963, Kelly [1] presented the concept of bitopological space (X,τ_1,τ_2) which is a nonempty set X endowed with two topologies τ_1 and τ_2. In 1966, Kuratowski [2] studied and applied the concept of ideals. An ideal on a topological space (X,τ) is a collection of subsets of X having the heredity property (i) if $A\in I$ and $B\subset A$, then $B\in I$ and (ii) if $A\in I$ and $B\in I$, then $A\cup B\in I$. If I is an ideal on X, then (X,τ_1,τ_2,I) is called an ideal bitopological space. In 1979, Kasahara [3] defined an operation $\gamma:\tau\rightarrow P(X)$ which is a mapping on τ such that $U\subseteq\gamma(U)$ for each $U\in\tau$. In 1984, Khedr [4,5], extended the operation γ to bitopological spaces. In 1996, Andrijevic [6] presented the field of topological space the concept of $b-$open sets. In 2007, Al-Hawary and AL-Omari [7] expanded the $b-$open sets to bitopological spaces in addition. Guler and Aslim [8] presented the idea of $bI-$open sets in ideal topological spaces. In 2012, Ekici [9] studied the concept of pre$-I-$open sets, $semi-I-$open sets and $bI-$open sets in ideal topological spaces. In 2011, Rajesh et al. [10] introduced the notion of $pre-I$-open sets in ideal bitopological spaces. In 2015, Ibrahim [11] introduced the concept of $\gamma-pre-I$-open sets in ideal topological spaces and Diganta [12] introduced the notion of $bI-$open sets in ideal bitopological spaces. In 2018, Hussain [13] presented the concept of $\gamma-pre-$open and $\gamma-b-$open sets in the field of topological space. In 2020, Bukhatwa and Demiralp in a similar study introduced and defend the notion of generalized β-open sets in ideal bitopological spaces [14]. For some more significant work in this direction we refer to [10,15–23].

Introduction

In this article, (X,τ_1,τ_2) always mean bitopological space with no separation axioms are supposed in this space, also (X,τ_1,τ_2,I) be an ideal bitopological space. Let A be a subset of X, by $int_i(A)$ and $cl_i(A)$ we denote respectively the interior and the closure of A with regard to τ_i for $i=1,2$. An operation γ on a bitopological space (X,τ_1,τ_2) is a mapping $\gamma:\tau_1\cup\tau_2\rightarrow P(X)$ such that $U\subseteq U^{\gamma}$ for all $U\in\tau_1\cup\tau_2$, where U^{γ} is denotes the value of γ at U. For example the operations $\gamma(U)=U$, $\gamma(U)=cl_i(U)$, $\gamma(U)=int_i(cl_j(U))$ for $U\in\tau_j$ are operations on $\tau_1\cup\tau_2$. If for each $x\in A$, there

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exists an $i-$open set U such that $x \in U$ and $U^{\gamma} \subseteq A$, then A is called γ_i-open set. τ_{γ_i} will denote the set of all γ_i-open set in X. Obviously we have $\tau_{\gamma_i} \subseteq \tau_i$. Complement of all γ_i-open sets are called γ_i-closed. Assumed (X, τ_1, τ_2, I) as an ideal bitopological space and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$ named the local function [24] of A with regard to τ_i and I. The definition of local function is giving as: for $A \subset X$, $A_i^*(\tau_i, I) = \{x \in X \mid U \cap A \notin I$, for all $U \in \tau_i(x)\}$ where $\tau_i(x) = \{U \in \tau_i \mid x \in U\}$. Observe additionally that closure operator for $\tau_i^*(I)$, accurate than τ_i, is defined by $Cl_i^*(A) = A \cup A_i^*$. $int_i^*(A)$ called the interior of A in $\tau_i(I)$ and $int_i^*(A_j^*)$ called the interior of A_j^* with regard to topology τ_i, where $A_j^* = \{x \in X \mid U \cap A \notin I$, for every $U \in \tau_j\}$. The $interior_{\gamma_i}$ of A is denoted by $int_{\gamma_i}(A)$ and defined to be the union of all γ_i-open sets of X contained in A and the $closure_{\gamma_i}$ of A is denote by $cl_{\gamma_i}(A)$ and defined to be the intersection of all γ_i-closed sets containing A.

Currently, several definitions from [11,12,15,16] are recalled to be used in this article.

Definition 1.1. A subset A of a bitopological space (X, τ_1, τ_2) with operation γ on $\tau_1 \cup \tau_2$ is named

i. $\gamma_{ij} - pre-$open set if $A \subseteq int_{\gamma_i}(cl_{\gamma_j}(A))$, where $i, j = 1, 2$ and $i \neq j$.

ii. $\gamma_{ij} - b-$open set if $A \subseteq int_{\gamma_i}(cl_{\gamma_j}(A)) \cup cl_{\gamma_j}(int_{\gamma_i}(A))$, where $i, j = 1, 2$ and $i \neq j$.

Definition 1.2. Let (X, τ_1, τ_2, I) be an ideal bitopological space with an operation γ on $\tau_1 \cup \tau_2$. The $\gamma-$local function of A with regard to γ and I is defined as giving: for $A \subset X$, $A_{\gamma_i}^*(\gamma, I) = \{x \in X \mid U \cap A \notin I$, for all $U \in \tau_{\gamma_i}(x)\}$ where $\tau_{\gamma_i}(x) = \{U \in \tau_{\gamma_i} \mid x \in U\}$.

In the case no ambiguity, we will replace $A_{\gamma_i}^*(\gamma, I)$ by $A_{\gamma_i}^*$.

Definition 1.3. *A* subset A of an ideal bitopological space (X, τ_1, τ_2, I) is called

i. $(i, j) - I-$open if $A \subseteq int_i(A_j^*)$, where $i, j = 1, 2$ and $i \neq j$.

ii. $(i, j) - pre - I-$open set if $A \subseteq int_i(cl_j^*(A))$, where $i, j = 1, 2$ and $i \neq j$.

iii. $(i, j) - bI-$open set if $A \subseteq int_i(cl_j^*(A)) \cup cl_j^*(int_i(A))$, where $i, j = 1, 2$ and $i \neq j$.

Definition 1.4. Let (X, τ_1, τ_2, I) be an ideal bitopological space with an operation γ and (Y, σ_1, σ_2) be a bitopological space with an operation β. Then a function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is called pairwise $(\gamma, \beta)_i$-continuous function if $f^{-1}(V)$ is γ_i-open in X for all β_i-open set V in Y.

Definition 1.5. *A* function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is called as an $(i, j) - I-$continuous function (resp. $(i, j) - preI-$continuous, $(i, j) - bI-$continuous) if $f^{-1}(V)$ is $(i, j) - I-$open (resp. $(i, j) - preI-$open, $(i, j) - bI-$open) in X for all σ_i-open set V in Y, where $i, j = 1, 2$ and $i \neq j$.

Throughout the article, we suppose that $i, j = 1, 2$ and $i \neq j$.

$\gamma_{ij} - preI-$Open Sets

In this section, the concept of $\gamma_{ij} - preI-$open sets in ideal bitopological spaces are presented and characterizations of their related notions are given.

Definition 2.1. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be $\gamma_{ij} - I-$open if $A \subseteq int_{\gamma_i}(A_{\gamma_j}^*)$. The set consisting of all $\gamma_{ij} - I-$open sets in X will be denoted by $\gamma_{ij} - IO(X)$.

Example 2.2. Let $X = \{a, b, c, d\}$ be a set and let $\tau_1 = \{\emptyset, X, \{b\}, \{c, d\}, \{b, c, d\}\}$, $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c, d\}\}$, and $I = \{\emptyset, \{a\}\}$ defined an operation $\gamma : \tau_1 \cup \tau_2 \to P(X)$ such that $\gamma(U) = Cl_i(A)$ for $U \in \tau_j$. Then we have, γ_{12}-I-open sets are $\{\emptyset, X, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$.

Definition 2.3. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be $\gamma_{ij} - preI-$open set if $A \subseteq int_{\gamma_i}(cl_{\gamma_j}^*(A))$. The set consisting of all $\gamma_{ij} - preI-$open sets in X will be denoted by $\gamma_{ij} - PIO(X)$.

Definition 2.4. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be γ_{ij}-$preI$-closed set if A^c is a γ_{ij}-$preI$-open set . Equivalently A is said to be γ_{ij}-$preI$-closed set if $A \supseteq cl_{\gamma_i}(int_{\gamma_j}^*(A))$. The set consisting of all γ_{ij}-$preI$-closed sets in X will be denoted by $\gamma_{ij} - PIC(X)$.

Theorem 2.5. Let (X,τ_1,τ_2,I) be an ideal bitopological space. Then,

i. Every γ_{ij}-I-open set is γ_{ij}-$preI$-open.

ii. Every γ_{ij}-$preI$-open set is γ_{ij}-pre-open.

Proof. Let A be a subset of X.

i. If A is $\gamma_{ij}-I-$open, then

$$A \subseteq int_{\gamma_i}(A^*_{\gamma_j}) \subseteq int_{\gamma_i}(A^*_{\gamma_j}\cup A) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A))$$

Therefore, A is a $\gamma_{ij}-preI-$open set.

ii. If A is γ_{ij}-$preI$-open, then

$$A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq int_{\gamma_i}(A^*_{\gamma_j}\cup A) \subseteq int_{\gamma_i}(cl_{\gamma_j}(A)\cup A) \subseteq int_{\gamma_i}(cl_{\gamma_j}(A))$$

Therefore, A is a γ_{ij}-pre-open set.

□

But generally the converse of this theorem is not true as giving in the next example.

Example 2.6. From example 2.2, take $A=\{a,b,d\}$. Calculations show that A is γ_{12}-$preI$-open however, it is not γ_{12}-I-open.

Theorem 2.7. Let (X,τ_1,τ_2,I) be an ideal bitopological space. Then,

i. The union of any $\gamma_{ij}-preI-$open sets is $\gamma_{ij}-preI-$open set.

ii. The intersection of any $\gamma_{ij}-preI-$closed sets is $\gamma_{ij}-preI-$closed sets.

Proof. i. Let $A_\alpha \in \gamma_{ij}-PIO(X)$ for each $\alpha\in\Lambda$, where Λ is an index set. $A_\alpha \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A_\alpha))$. Then,

$$\begin{aligned}\bigcup_{\alpha\in\Lambda} A_\alpha &\subseteq \bigcup_{\alpha\in\Lambda}\{int_{\gamma_i}(cl^*_{\gamma_j}(A_\alpha))\}\\ \bigcup_{\alpha\in\Lambda} A_\alpha &\subseteq \{int_{\gamma_i}(\bigcup_{\alpha\in\Lambda} cl^*_{\gamma_j}(A_\alpha))\} \subseteq \{int_{\gamma_i}((\bigcup_{\alpha\in\Lambda}(A_\alpha)^*_{\gamma_j}\cup A_\alpha)))\}\\ \bigcup_{\alpha\in\Lambda} A_\alpha &\subseteq \{int_{\gamma_i}(cl^*_{\gamma_j}(\bigcup_{\alpha\in\Lambda} A_\alpha))\}\end{aligned}$$

Then, $\bigcup_{\alpha\in\Lambda} A_\alpha$ is $\gamma_{ij}-preI-$open.

ii. The proof follows by using (i) and taking complement.

□

Definition 2.8. Let (X,τ_1,τ_2,I) be an ideal bitopological space and $A\subset X$. Then,

i. $\gamma_{ij}-preI-$closure of A is represented by $pI-cl_{\gamma_{ij}}(A)$, is defined as the intersection of each $\gamma_{ij}-preI-$closed subset of X containing A. That is

$$pI-cl_{\gamma_{ij}}(A)=\bigcap\{U \mid U\supseteq A \text{ with } U^c\in\gamma_{ij}-PIO(X)\}.$$

ii. $\gamma_{ij}-preI-$interior of A is represented by $pI-int_{\gamma_{ij}}(A)$, is defined as the union of all $\gamma_{ij}-preI-$open subset of X contained in A. That is

$$pI-int_{\gamma_{ij}}(A)=\bigcup\{U \mid U\subseteq A \text{ with } U\in\gamma_{ij}-PIO(X)\}.$$

Theorem 2.9. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $pI - cl_{\gamma_{ij}}(A^c) = (pI - int_{\gamma_{ij}}(A))^c$

ii. $pI - int_{\gamma_{ij}}(A^c) = (pI - cl_{\gamma_{ij}}(A))^c$

iii. A is $\gamma_{ij} - preI-$open$\Leftrightarrow A = pI - int_{\gamma_{ij}}(A)$

iv. A is $\gamma_{ij} - preI-$closed$\Leftrightarrow A = pI - cl_{\gamma_{ij}}(A)$

The proof can be obtained directly by the definition 2.8 and omitted.

Lemma 2.10. Let X be a space and $A \subset X$. Then,

i. $cl_{\gamma_{ij}}(A) \cap U \subseteq cl_{\gamma_{ij}}(A \cap U)$, for any γ_i-open set U in X.

ii. $int_{\gamma_{ij}}(A \cap V) \subseteq int_{ij}(A) \cap V$, for any γ_i-closed set V in X.

Theorem 2.11. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $pI\text{-}int_{\gamma_{ij}}(A) = A \cap int_{\gamma_i}(cl^*_{\gamma_j}(A))$

ii. $pI\text{-}cl_{\gamma_{ij}}(A) = A \cup cl_{\gamma_i}(int^*_{\gamma_j}(A))$

Proof. i. We have $pI\text{-}int_{\gamma_{ij}}(A) \subseteq A$ and since $pI\text{-}int_{\gamma_{ij}}(A)$ is γ_{ij}-$preI$-open, then

$$pI - int_{\gamma_{ij}}(A) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(pI - int_{\gamma_{ij}}(A)))$$

Therefore, $pI\text{-}int_{\gamma_{ij}}(A) \subseteq A \cap int_{\gamma_i}(cl^*_{\gamma_j}(A))$. Also,

$$int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq cl^*_{\gamma_j}(int_{\gamma_i}(cl^*_{\gamma_j}(A)))$$

Then we have,

$$int_{\gamma_i}(int_{\gamma_i}(cl^*_{\gamma_j}(A))) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(int_{\gamma_i}(cl^*_{\gamma_j}(A))))$$
$$int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(int_{\gamma_i}(cl^*_{\gamma_j}(A))))$$

Therefore, $int_{\gamma_i}(cl^*_{\gamma_j}(A))$ is γ_{ij}-$preI$-open. Thus,

$$A \cap (int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq pI - int_{\gamma_{ij}}(A)$$

ii. The proof follows from (i) and by taking complement.

□

Theorem 2.12. Let (X,τ_1,τ_2,I) be an ideal bitopological space with an operation γ on τ and $A \subset X$. Then,

i. If $I = \{\varnothing\}$, then A is $\gamma_{ij} - preI-$open if and only if A is $\gamma_{ij} - pre-$open.

ii. If $I = P(X)$ then A is γ_{ij}-$preI$-open if and only if A is γ_i-open.

Proof. i. We have just to show that if $I = \{\varnothing\}$ and A is $\gamma_{ij} - pre-$open, then A is $\gamma_{ij} - preI-$open. If $I = \{\varnothing\}$, then $A^*_{\gamma_i} = cl_{\gamma_i}(A)$ for every subset A of X. Assumed A to be $\gamma_{ij} - pre-$open set, then

$$A \subseteq int_{\gamma_i}(cl_{\gamma_j}(A)) \subseteq int_{\gamma_i}(A^*_{\gamma_j}) \subseteq int_{\gamma_i}(A^*_{\gamma_j} \cup A) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A))$$

Therefore, A is $\gamma_{ij} - preI-$open.

ii. Let $I = P(X)$, then $A^*_{\gamma_j} = \varnothing$ for all subset A of X. Let A be any $\gamma_{ij} - preI-$open set, then $A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) = int_{\gamma_i}(A \cup A^*_{\gamma_j}) = int_{\gamma_i}(A)$. Therefore, A is γ_i-open. Opposite is obvious.

□

$\gamma_{ij} - bI-$Open Sets

In this section, the concept of $\gamma_{ij} - bI-$open sets in ideal bitopological spaces are presented and characterizations of their related notions are given.

Definition 3.1. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be $\gamma_{ij} - bI-$open set if $A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A))$. The set consisting of all $\gamma_{ij} - bI-$open sets in X will be symbolized by $\gamma_{ij} - BIO(X)$.

Definition 3.2. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is called $\gamma_{ij} - bI-$closed set if A^c is an $\gamma_{ij} - bI-$open set. Equivalently A is called $\gamma_{ij} - bI-$closed set if $A \supseteq cl_{\gamma_i}(int^*_{\gamma_j}(A)) \cap int^*_{\gamma_j}(cl_{\gamma_i}(A))$. The set consisting of all $\gamma_{ij} - bI-$closed sets in X will be symbolized by $\gamma_{ij}-$BIC(X).

Theorem 3.3. Let (X, τ_1, τ_2, I) be an ideal bitopological space and A be a subset of X. Then,

i. A is $\gamma_{ij} - bI-$open if and only if A^c is $\gamma_{ij} - bI-$closed.

ii. If A is $\gamma_{ij} - preI-$open then A is $\gamma_{ij} - bI-$open.

iii. If A is γ_{ij}-I-open then A is γ_{ij}-bI-open.

iv. If A is γ_{ij}-bI-open then A is γ_{ij}-b-open.

Proof. i. This can be obtained directly by the ii and taking complement.

ii. Let A be $\gamma_{ij} - preI-$open set. Then,

$$A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A))$$

iii. By theorem 2.5.

iv. The prove will be obtained directly by using the fact that $\tau^*(I)$ is accurate than τ. But generally, the converse of this theorem is not true as giving in the next examples.

□

Example 3.4. Let $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{a\}\}$, and $I = \{\emptyset, \{a\}\}$. For $U \in \tau_i$, let

$$\gamma(U) = \begin{cases} int_i(Cl_j(U)), & a \notin U \\ U \quad , & a \in U \end{cases}$$

Take $A = \{b, c\}$ then A is γ_{12}-bI-open in X but neighter γ_{12}-I-open nor γ_{12}-$preI$-open in X.

Example 3.5. Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X, \{a\}\}$, $\tau_2 = \{\varnothing, X, \{b\}\}$, and $I = \{\varnothing, \{b\}\}$. Let define an operation $\gamma : \tau_1 \cup \tau_2 \to P(X)$ such that $\gamma(U) = U$ for all $U \in \tau_i$. Then we have, $\gamma_{12} - bI-$open sets are $\{\varnothing, X, \{a\}, \{a, c\}, \{a, b\}, \{b, c\}\}$. Take $A = \{b\}$. Calculations show that A is $\gamma_{12} - b-$open but not $\gamma_{12} - bI-$open.

Theorem 3.6. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Let A be a subset of X. If A is $\gamma_{ij} - bI-$open with $int_{\gamma_i}(A) = \varnothing$, then A is $\gamma_{ij} - preI-$open.

Proof. Let A be $\gamma_{ij} - bI-$open set, then

$$A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A)) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(\varnothing) \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A))$$

□

Theorem 3.7. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Then,

i. The union of any $\gamma_{ij} - bI-$open sets is a $\gamma_{ij} - bI-$open set.

ii. The intersection of any $\gamma_{ij} - bI-$closed sets is a $\gamma_{ij} - bI-$closed set.

PROOF. i. Let $A_\alpha \in \gamma_{ij} - BIO(X)$ for each $\alpha \in \Lambda$, where Λ is an index set.

$$A_\alpha \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A_\alpha)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A_\alpha))$$

Thus,

$$\begin{aligned}
\bigcup_{\alpha\in\Lambda} A_\alpha &\subseteq \bigcup_{\alpha\in\Lambda}\{int_{\gamma_i}(cl^*_{\gamma_j}(A_\alpha)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A_\alpha))\} \\
&\subseteq \bigcup_{\alpha\in\Lambda}\{int_{\gamma_i}(A_\alpha \cup (A_\alpha)^*_{\gamma_j}) \cup (int_{\gamma_i}(A_\alpha)) \cup (int_{\gamma_i}(A_\alpha))^*_{\gamma_j}\} \\
&\subseteq \{int_{\gamma_i}(\bigcup_{\alpha\in\Lambda} A_\alpha) \cup (\bigcup_{\alpha\in\Lambda}(A_\alpha)^*_{\gamma_j})\} \\
&\quad \cup\, int_{\gamma_i}(\bigcup_{\alpha\in\Lambda} A_\alpha) \cup int_{\gamma_i}(\bigcup_{\alpha\in\Lambda}(A_\alpha)^*_{\gamma_j}) \\
&\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(\bigcup_{\alpha\in\Lambda} A_\alpha)) \cup\; cl^*_{\gamma_j}(int_{\gamma_i}(\bigcup_{\alpha\in\Lambda} A_\alpha))
\end{aligned}$$

Then, $\bigcup_{\alpha\in\Lambda} A_\alpha$ is $\gamma_{ij} - bI-$open.

ii. The proof follows by using (i) and taking complement.

□

Theorem 3.8. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Let A and U be subsets of X. If A is $\gamma_{ij} - bI-$open and $U \in \tau_1 \cap \tau_2$,then $A \cap U$ is $\gamma_{ij} - bI-$open.

PROOF. Let A be $\gamma_{ij} - bI-$open set, then $A \subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A))$.

$$\begin{aligned}
A \cap U &\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A)) \cap U \\
&\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A) \cap U) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A) \cap U) \\
&\subseteq int_{\gamma_i}((A \cup A^*_{\gamma_j}) \cap U) \cup ((int_{\gamma_i}(A) \cup int(A)^*_{\gamma_j}) \cap U) \\
&\subseteq int_{\gamma_i}((A \cap U) \cup (A \cap U)^*_{\gamma_j}) \cup ((int_{\gamma_i} A \cap U) \cup int(A \cap U)^*_{\gamma_j}) \\
A \cap U &\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A \cap U)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A \cap U))
\end{aligned}$$

□

Definition 3.9. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $\gamma_{ij} - bI-$closure of A, symbolized by $bI - cl_{\gamma_{ij}}(A)$, is defined as the intersection of all $\gamma_{ij} - bI-$closed subset of X containing A. That is

$$bI - cl_{\gamma_{ij}}(A) = \bigcap\{U \mid U \supseteq A \text{ with } U^c \in \gamma_{ij} - BIO(X)\}.$$

ii. $\gamma_{ij} - bI-$interior of A, symbolized by $bI - int_{\gamma_{ij}}(A)$, is defined as the union of each $\gamma_{ij} - bI-$open subset of X contained in A. That is

$$bI - int_{(\gamma_{ij})}(A) = \bigcup\{U \mid U \subseteq A \text{ with } U \in \gamma_{ij} - BIO(X)\}.$$

Theorem 3.10. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then,

i. $bI - cl_{\gamma_{ij}}(A) = A \cup \{cl_{\gamma_i}(int^*_{\gamma_j}(A)) \cap int^*_{\gamma_j}(cl_{\gamma_i}(A))\}$

ii. $bI - int_{\gamma_{ij}}(A) = A \cap \{int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A))\}$

The proof follows from Theorem 2.11.

Theorem 3.11. For an ideal bitopological space (X, τ_1, τ_2, I) with an operation γ on τ and $A \subset X$, we have, *i.* If $I = \{\varnothing\}$, then A is $\gamma_{ij} - bI-$open if and only if A is $\gamma_{ij} - b-$open.

ii. If $I = P(X)$, then A is γ_i-open.

PROOF. i. We have just to show that if $I = \{\varnothing\}$ and A is $\gamma_{ij} - b-$open, then A is $\gamma_{ij} - bI-$open. If $I = \{\varnothing\}$, then $A^*_{\gamma_j} = cl_{\gamma_i}(A)$ for every subset A of X . Let A be $\gamma_{ij} - b-$open set. Then,

$$\begin{aligned} A &\subseteq int_{\gamma_i}(cl_{\gamma_j}(A)) \cup cl_{\gamma_j}(int_{\gamma_i}(A)) \\ &\subseteq int_{\gamma_i}(A^*_{\gamma_j}) \cup (int_{\gamma_i}(A))^*_{\gamma_j} \\ &\subseteq int_{\gamma_i}(A \cup A^*_{\gamma_j}) \cup int_{\gamma_i}(A) \cup (int_{\gamma_i}(A))^*_{\gamma_j} \\ &\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A)) \end{aligned}$$

So A is $\gamma_{ij} - bI-$open.

ii. Let $I = P(X)$, then$A^*_{\gamma_j} = \varnothing$ for every subset A of X. Let A be any $\gamma_{ij} - bI-$open set. Then,

$$\begin{aligned} A &\subseteq int_{\gamma_i}(cl^*_{\gamma_j}(A)) \cup cl^*_{\gamma_j}(int_{\gamma_i}(A)) \\ &= int_{\gamma_i}(A \cup A^*_{\gamma_j}) \cup int_{\gamma_i}(A) \cup (int_{\gamma_i}(A))^*_{\gamma_j} \\ &= int_{\gamma_i}(A) \cup int_{\gamma_i}(A) \end{aligned}$$

So A is γ_i-open. Opposite is obvious.

□

$(\gamma,\beta)_{ij} - I-$Continuous Functions

In this section, the concept of $(\gamma,\beta)_{ij} - I-$continuous functions in ideal bitopological spaces are introduced and characterizations of their related notions are given.

Throughout this section, (X, τ_1, τ_2, I) be an ideal bitopological space with an operation γ and (Y, σ_1,σ_2) be a bitopological space with an operation β.

Definition 4.1. A function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is said to be $(\gamma, \beta)_{ij} - I-$continuous function (resp. $(\gamma,\beta)_{ij} - preI-$continuous, $(\gamma, \beta)_{ij} - bI-$continuous) if $f^{-1}(V)$ is $\gamma_{ij} - I-$open (resp. $\gamma_{ij} - preI-$open, $\gamma_{ij} - bI-$open) in X for every β_i-open set V in Y, for $i, j = 1, 2$ and $i \neq j$.

It is clear that every $(\gamma,\beta)_{ij} - I-$continuous functions is $(\gamma,\beta)_{ij} - preI-$continuous and $(\gamma,\beta)_{ij} - bI-$continuous but the converse is not true as shown in the example.

Example 4.2. Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_2 = \{\varnothing, X, \{a\}\}$, $I = \{\varnothing, \{a\}\}$, and let for $U \in \tau_i$,

$$\gamma(U) = \begin{cases} int_i(Cl_j(U)), & a \notin U \\ U \quad , & a \in U \end{cases}$$

Let $\sigma_1 = \{\varnothing, X, \{b\}, \{a, b\}\}, \sigma_2 = \{\varnothing, X, \{c\}\}, \beta(V) = V$ for $V \in \sigma_i$. Let define a function $f : (X, \tau_1, \tau_2, I) \to (X, \sigma_1, \sigma_2)$ such that $f(a) = c, f(b) = b, f(c) = a$. Then f is $(\gamma, \beta)_{12}$-b-I-continuous but neither $(\gamma, \beta)_{12}$-I-continuous nor $(\gamma, \beta)_{12}$-$preI$-continuous because $\{a, b\}$ is σ_1-open set and $f^{-1}(\{a, b\}) = \{b, c\}$ which is γ_{12}-b-I-open in X but neither γ_{12}-I-open nor γ_{12}-$preI$-open in X.

Definition 4.3. A function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1,\sigma_2)$ is said to be pairwise $(\gamma,\beta)_i - I-$continuous function if $f^{-1}(V)$ is $\gamma_i - I-$open in X for every β_i-open set V in Y.

Note that the concept of pairwise $(\gamma,\beta)_i - I-$continuous and $(\gamma, \beta)_{ij} - I-$continuous are independent.

Example 4.4. Let $X = \{a,b,c\}$ be a set and $\tau_1 = \{\varnothing, X, \{b\}, \{b, c\}\}$, $\tau_2 = \{\varnothing, X, \{b, c\}\}$, $I = \{\varnothing, \{a\}\}$, with operation $\gamma(U) = U$ for $U \in \tau_i$ and $\sigma_1 = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}, \sigma_2 = \{\varnothing, X, \{b, c\}\}$, with operation

$$\beta(V) = \begin{cases} Cl_j(V), & c \notin V \\ V \quad , & c \in V \end{cases}$$

for $V \in \sigma_i$ and define $f : (X, \tau_1, \tau_2, I) \to (X, \sigma_1, \sigma_2)$ such that $f(a) = b, f(b) = c$ and $f(c) = a$. Then f is $(\gamma, \beta)_{12}$-I-continuous function but it is not $(\gamma, \beta)_1$-I-continuous because $\{a\}$ is σ_1-open set and $f^{-1}(\{a\}) = \{c\}$ which is γ_{12}-I-open but not γ_1-I-open in X.

Theorem 4.5. For the function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent.

i. f is $(\gamma,\beta)_{ij} - I-$continuous,

ii. For all x in X and each β_i-open set V in Y containing $f(x)$, there exists a $\gamma_{ij} - I-$open set U of X containing x such that $f(U) \subset V$.

PROOF. (i⇒ii) Let V be a β_i-open set in Y such that $f(x) \in V$. Since f is $(\gamma,\beta)_{ij} - I-$continuous, $f^{-1}(V)$ is $\gamma_{ij} - I-$open set in X. Let $U = f^{-1}(V)$, then $f(x) \in f(U) \subset V$.

(ii⇒i) Let V be a β_i-open set in Y and let $x \in f^{-1}(V)$. Then we have $f(x) \in V$. By (ii), there exists an $\gamma_{ij} - I-$open set U in X containing x such that $f(U) \subset V$. Therefore, $x \in U \in f^{-1}(V)$. Hence $f^{-1}(V)$ is $\gamma_{ij} - I-$open set in X, so f is $(\gamma, \beta)_{ij} - I-$continuous.

□

Theorem 4.6. Let $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ be a $(\gamma,\beta)_{ij} - I-$continuous function. Then, the following statements are equivalent:

i. The inverse image of every β_i-closed set in Y is $\gamma_{ij} - I-$closed set in X.

ii. For each subset U of X, $f(I - cl_{\gamma_{ij}}(U)) \subset cl_{\beta_i}(f(U))$.

iii. For each subset V of Y, $I - cl_{\gamma_{ij}}(f^{-1}(V)) \subset f^{-1}(cl_{\beta_i}(V))$.

PROOF. (i⇒ii) Let $U \subset X$. Since $cl_{\beta_i}(f(U))$ is β_i-closed set in Y, therefore by (i), we have $f^{-1}(cl_{\sigma_i}(f(U)))$ is $\gamma_{ij} - I-$closed set in X. Also $U \subset f^{-1}$ $(cl_{\beta_i}(f(U)))$ and $I - cl_{\beta_i}(U)$ is the smallest set $\gamma_{ij} - I-$closed set containing U. Therefore,

$$I - cl_{\beta_i}(U) \subset f^{-1}\ (cl_{\beta_i}(f(U)))$$

This implies that $f(I - cl_{\gamma_{ij}}(U)) \subset cl_{\beta_i}(f(U))$.

(ii⇒iii) Let $V \subset Y$. Then $f^{-1}(V) \subset X$ by (ii), that

$$f(I - cl_{\gamma_{ij}}(f^{-1}(V))) \subset cl_{\beta_i}(f(f^{-1}(V))) \subset cl_{\beta_i}(V)$$

Hence $I - cl_{\gamma_{ij}}\ (f^{-1}(V)) \subset f^{-1}(cl_{\beta_i}(V))$.

(iii⇒i) Let V be a β_i-closed set in Y. By (iii),

$$I - cl_{\gamma_{ij}}\ (f^{-1}(V)) \subset f^{-1}(cl_{\beta_i}(V)) = f^{-1}(V)$$

Therefore, $f^{-1}(V) = I - cl_{\gamma_{ij}}\ (f^{-1}(V))$ and so $f^{-1}(V)$ is $\gamma_{ij} - I-$closed set in X. □

Theorem 4.7. The function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is $(\gamma, \beta)_{ij} - I-$ continuous function if and only if $f^{-1}(int_{\beta_i}(V)) \subset I-\ int_{\gamma_{ij}}(f^{-1}(V))$ for every subset V of Y.

PROOF. (⇒) Let V be a β_i-open set in Y and f is a $(\gamma,\beta)_{ij} - I-$ continuous function. Then, $f^{-1}(int_{\beta_i}(V)$ is $\gamma_{ij} - I$-open set in X and we have

$$f^{-1}(int_{\beta_i}(V)) \subset I - int_{\gamma_{ij}}(f^{-1}(int_{\beta_i}(V))) \subset I-\ int_{\gamma_{ij}}(f^{-1}(V))$$

(⇐) Let V be a β_i-open set in Y, then $int_{\beta_i}(V) = V$. Therefore,

$$f^{-1}(V) \subset f^{-1}(int_{\beta_i}(V)) \subset I-\ int_{\gamma_{ij}}(f^{-1}(V))$$

Then, $f^{-1}(V) = I - int_{\gamma_{ij}}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is $\gamma_{ij} - I-$open set in X and so f is a $(\gamma, \beta)_{ij} - I-$ continuous function. □

Note that,in generally the composition of two $(\gamma,\beta)_{ij} - I-$ continuous functions need not to be $(\gamma,\beta)_{ij} - I-$ continuous function, as giving in the next examples for that defined (Z, ℓ_1, ℓ_2) to be a bitopological space with an operation δ.

Example 4.8. Let (X, τ_1, τ_2, I) and $(X, \sigma_1, \sigma_2, J)$ be two ideal bitopological spaces such that $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X, \{a, b\}\}$, $\tau_2 = \{\varnothing, X, \{a\}, \{a, b\}\}$, and $I = \{\varnothing, \{b\}\}$

$$\gamma(U) = \begin{cases} cl_j(U) & \text{, if } b \notin U \\ U & \text{, if } b \in U \end{cases}$$

for $U \in \tau_i$. Let $\sigma_2 = \{\varnothing, X, \{b\}, \{b, c\}\}$, $\sigma_2 = \{\varnothing, X, \{b, c\}\}$, $J = \{\varnothing, \{a\}\}$, $\beta(V) = V$ for $V \in \sigma_i$. Let define a function $f : (X, \tau_1, \tau_2, I) \to (X,\sigma_1,\sigma_2)$ such that $f(a) = b, f(b) = a$ and $f(c) = c$ and let (X, ℓ_1, ℓ_2) be a bitoplogical space such that $\ell_1 = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}, \ell_2 = \{\varnothing, X, \{b, c\}\}$,

$$\delta(W) = \begin{cases} cl_j(W) & \text{, if } c \notin W \\ W & \text{, if } c \in W \end{cases}$$

for $W \in \ell_i$ and define $g : (X, \sigma_1, \sigma_2, J) \to (X, \ell_1, \ell_2)$ such that $g(a) = b$, $g(b) = a$ and $g(c) = a$. Then f is $(\gamma, \beta)_{12}-I-$continuous function and g is $(\beta, \delta)_{12}-I-$continuous function but the composition $g\circ f$ is not $(\gamma, \delta)_{12}-I-$continuous function because $\{a\}$ is δ_i-open set and $(g\circ f)^{-1}(\{a\}) = \{c\} \notin \gamma_{12}-IO(X)$.

Theorem 4.9. Let f be a function from (X, τ_1, τ_2, I) to (Y, σ_1, σ_2) and g from $(Y, \sigma_1, \sigma_2, J)$ to (Z, ℓ_1, ℓ_2). Then $g \circ f$ is $(\gamma, \delta)_{ij} - I-$continuous if f is $(\gamma,\beta)_{ij} - I-$continuous and g is pairwise $(\beta, \delta)_i - I-$continuous.

Proof. Let w be any δ_i-open set in Z. Since g is pairwise $(\beta, \delta)_i$-continuous, then $g^{-1}(w)$ is a β_i-open set in Y. On the other hand, since f is $(\gamma,\delta)_{ij}$-I-continuous, we have $f^{-1}(g^{-1}(w)) \in \gamma_{ij} - IO(X)$. Therefore $g \circ f$ is $(\gamma, \delta)_{ij} - I-$continuous. □

Conclusion

In this study, we defend the notion of $\gamma_{ij} - I-$open sets, $\gamma_{ij} - perI-$open sets and $\gamma_{ij} - bI$open by generalizing by $(i, j) - I-$open, $(i, j) - preI-$open, $(i, j) - bI-$open sets respectively, in ideal bitopological spaces with an operation $\gamma : \tau \to P(X)$. We show that every $\gamma_{ij} - I-$open set is a $\gamma_{ij} - preI-$open sets and $\gamma_{ij} - bI-$open sets but the converse is nt always true.

Consequently the following diagrams are true:

$$\gamma_{ij} - I - open \to \gamma_{ij} - preI - open \to \gamma_{ij} - bI - open$$

$$\gamma_{ij} - preI - open(\gamma_{ij} - bI - open) \leftrightarrow \gamma_{ij} pre - open(\gamma_{ij} - b - open)(I = \{\emptyset\})$$

$$(\gamma, \delta)_{ij} - I - continuous \to (\gamma, \delta)_{ij} - preI - Continuous \to (\gamma, \delta)_{ij} - bI - Continuous$$

These notations, defined in this study, can be extended to other practicable researched of topology such as fuzzy topology, soft topology, intuitionistic topology and so on. Also generalized separation axioms can be introduced by the concept of generalized $I-$open set, $pre-$open sets and $b-$open sets.

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