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Analytic solutions for the energy density and the temperature of the QGP in the early Universe

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Abstract — *After the Big Bang and at a time estimated to be a millionth of a second, the Universe was in a phase that filled with quark-gluon plasma. Due to the high temperature, the strong coupling constant that describes the strength of the strong force acting on the quarks and the gluons, was so small. As a result, quarks and gluons inside the plasma behaved as an ideal gas of gluons and massless quarks that weakly interact with each others. Hence, one can describe the characteristics of the plasma by the equations of states which relate energy density and pressure to its temperature as in the MIT bag model. Having these equations of states, one can solve Friedmann equations of the general relativity. In this work, we derive the analytic solutions for the Friedmann differential equations governing the time evolution of the energy density and the temperature of the quark-gluon plasma in that era of the early Universe adopting the MIT bag model.*

Keywords: quark-gluon plasma, Early Universe, Friedmann equations.
Mathematics Subject Classification: 80C50, 30A40, 90C26.

1 Introduction

Cosmological models are Mathematical models that are used to study the Universe as a whole. These studies include origin, nature and evolution of the Universe. Einstein formulated his cosmological model in the year 1917 [1, 2]. The Universe is isotropic and homogeneous implying that the space-time can be parametrized by the Friedmann-Lematre-Robertson-Walker (FLRW) metric [3, 4, 5, 6]. Upon inserting into the Einstein

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equations, one obtains the Friedmann equations [5]. Solving the Friedmann equations can shed light on the time evolution of the early Universe.

Quantum Chromodynamics (QCD) is widely accepted theory to describe the strong nuclear force acting on quarks and gluons [7]. The QCD coupling constant α_s, that determines the strength of the strong nuclear force, decreases at large momentum transfer or equivalently at short distances. As a consequence, quarks and gluons interact weakly which is referred as asymptotic freedom [8, 9]. In relation to this phenomena, Quark-Gluon plasma (QGP) was introduced as a new state of nuclear matter in the mid-seventies [10, 11]. At high temperatures and high densities, hadrons can break down into their constituents quarks and gluons that can lead to formation of QGP. Moreover, QGP was believed to fill the universe a millionth of a second after the Big Bang where the conditions of high temperatures and high densities were satisfied. At that era of early universe, the energy density, the pressure and the temperature of QGP were varying with time as Universe was cooling down. Studying this variation can be done through solving the Friedmann equations after specifying the equations of state of the QGP.

Mathematical expressions that relate two or more state functions of fluids, such as pressure, temperature, volume or internal energy, are known as equations of state. A brief discussion about the equations of state of the quark gluon plasma in the MIT bag model [12] will be presented in the next section. Then, in section 3, we will present our analytic solutions for the Friedmann differential equations to show the time evolution of the energy density and the temperature of the QGP in that era of the early Universe. It should be noted that, in Ref.[13] only the plots of the time evolution of the energy density, the temperature of the QGP in the MIT bag model obtained from numerical solutions of the differential equation were shown. Here, in this work, we show that analytic solutions can be obtained and derive these solutions and show that they lead to the same behavior shown in Ref.[13].

2 MIT bag model

The strong potential $\Phi(r)$ of the color field of two opposite color charges separated by a distance larger than a value r_0 can be expressed as

$$\phi(r) = ar \qquad r \gg r_0 \tag{1}$$

experimentally $a \simeq 0.6$ GeV/fm. Clearly, the gradient of $\Phi(r)$ is constant indicating that the strong force, which is the gradient of the potential, is constant. As a consequence, a quark will be influenced by a constant strong force if one tries to remove it from a hadron to distances larger than r_0 and in order to remove it completely an infinite amount of energy must be provided. However, long before that can happen, the color field will break up creating pair of quark-antiquark. This explains why colored particles are confined to each others in bound hadronic states. Bag models were proposed to solve for these bound states as it is virtually impossible to have solutions using exact QCD formalism. In the Bag models, quarks are assumed to be confined to a specific hadronic volume V. The part of the space that contains the hadron fields is called "bag" in which quarks do not have strong interactions and thus they can be treated as free point objects. The Dirac equation for the quarks reads

$$\Big(i\gamma^{\mu}\partial_{\mu} - m\Big)\psi = 0 \tag{2}$$

On the other hand, outside the bag we have $\psi = 0$. Consequently, the quark current, $j_{\mu} = (\bar{\psi}\gamma_{\mu}\psi)$, through the surface of the bag is zero and the boundary condition reads

$$n^{\mu} j_{\mu} = n^{\mu}(\bar{\psi}\gamma_{\mu}\psi) = 0 \tag{3}$$

where the vector n^{μ} represents the normal to the surface at a given point. First bag model was formulated by Bogolioubov and had a problem that energy momentum is not conserved at the bag surface. To avoid this problem, the MIT bag model had been introduced as an alternative to Bogolioubov bag model [12]. In order to derive the equations of state of the quark gluon plasma, in the MIT bag model, several assumptions were taking into account. Firstly, the hot quark-gluon plasma with energy scale $\sim$ 200 MeV just contains massless u and d quarks with neglected strong interactions inside the plasma. Secondly, no quarks are allowed to leave the bag and no quarks can be found outside the bag. Thirdly, inside the bag, quarks are treated as massless particles. Lastly, the applied internal pressure, that arises from the kinetic energy of the quarks inside the bag, on the surface of the bag is balanced by an external pressure $\mathcal{B}$ exerted by the vacuum on the same surface leading to the conservation of the energy momentum. This pressure balance is the reason for confinement in the MIT bag model.

The derivation of the equations of the state of the QGP that relates energy density and pressure to temperature in the MIT bag model can be found in Refs. [12, 14]. They are given as

$$\begin{aligned} \varepsilon &= \frac{37\pi^2}{30} T^4 + \mathcal{B} \\ p &= \frac{37\pi^2}{90} T^4 - \mathcal{B}. \end{aligned} \tag{4}$$

where $\mathcal{B} = 150\,\mathrm{MeV/fm^3}$ is a bag constant parameter. In the following section we will use natural units in our analysis. In natural units $\hbar = c = 1$ where $\hbar = h/2\pi$ is the reduced Planck constant and c is the speed of light in vacuum.

3 Results and discussion

In this section, we present our results for the time evolution of the total energy density (ε) and temperature (T) in the early universe in the MIT model. We will solve the corresponding differential equation to obtain analytic solutions for the time evolution of the energy density and the time evolution of the temperature T. In our analysis, we use the following initial conditions [13, 15]:

$$\varepsilon_i(t_i) = 10^7\,\mathrm{MeV/fm^3} \qquad \text{at} \qquad t_i = 10^{-9}\ s \tag{5}$$

and run the evolution from the time of the electroweak phase transition, $t_i = 10^{-9}\ s$, to the time of the QCD phase transition, $t_f = 10^{-4}\ s$.

3.1 Time evolution of energy density

The time evolution of the energy density in the early Universe can be expressed as [16]

$$-\frac{d\varepsilon}{3\sqrt{\varepsilon}\,(\varepsilon+p)} = \sqrt{\frac{8\pi G}{3}}dt \tag{6}$$

which allows us to find the time evolution of the energy density ε. After substitution, in the above equation, of the pressure p from Eq.(4)

$$p = \frac{1}{3}(\varepsilon - 4\mathcal{B}) \tag{7}$$

we find that

$$-\frac{d\varepsilon}{\sqrt{\varepsilon}\,(\varepsilon-\mathcal{B})} = 4\sqrt{\frac{8\pi G}{3}}dt \tag{8}$$

This can be rewritten as

$$\frac{d\varepsilon}{dt} = -4\sqrt{\frac{8\pi G}{3}}\sqrt{\varepsilon}\,(\varepsilon-\mathcal{B}) \tag{9}$$

After doing the integration and making some simplifications, the exact analytic solution of Eq.(6) can be written as

$$\varepsilon(t) = -(B + 2\sqrt{B\,\varepsilon(t)}) + (\varepsilon(t) - B)\exp\left(4\sqrt{\frac{8\pi B\,G}{3}}\;t + a\right) \tag{10}$$

where a is a constant. The value of a can be determined using initial conditions [13]:

$$\varepsilon(t_i) = 10^7\,\text{MeV/fm}^3 \quad \text{at} \quad t_i = 10^{-9}\;s \tag{11}$$

and thus we find that

$$a = \log\left(\frac{\sqrt{\varepsilon(t_i)} + \sqrt{B}}{\sqrt{\varepsilon(t_i)} - \sqrt{B}}\right) - 4\sqrt{\frac{8\pi B\,G}{3}}\;t_i \tag{12}$$

We need to solve the previous equation to obtain $\varepsilon(t)$. We start by rewrite the above equation as

$$\varepsilon(t)-\varepsilon(t)\exp\left(4\sqrt{\frac{8\pi B\,G}{3}}\;t+a\right) = -2\sqrt{B\,\varepsilon(t)}-B-B\exp\left(4\sqrt{\frac{8\pi B\,G}{3}}\;t+a\right) \tag{13}$$

which can be further rewritten as

$$(1-\exp\left(4\sqrt{\frac{8\pi B\,G}{3}}\;t+a\right))\varepsilon(t) = -2\sqrt{B\,\varepsilon(t)}-(1+\exp\left(4\sqrt{\frac{8\pi B\,G}{3}}\;t+a\right))B \tag{14}$$

and thus we get

$$\varepsilon(t) = \frac{-2\sqrt{B\,\varepsilon(t)}}{(1-\exp\left(4\sqrt{\frac{8\pi B G}{3}}\;t+a\right))} - \frac{(1+\exp\left(4\sqrt{\frac{8\pi B G}{3}}\;t+a\right))}{(1-\exp\left(4\sqrt{\frac{8\pi B G}{3}}\;t+a\right))}B \tag{15}$$

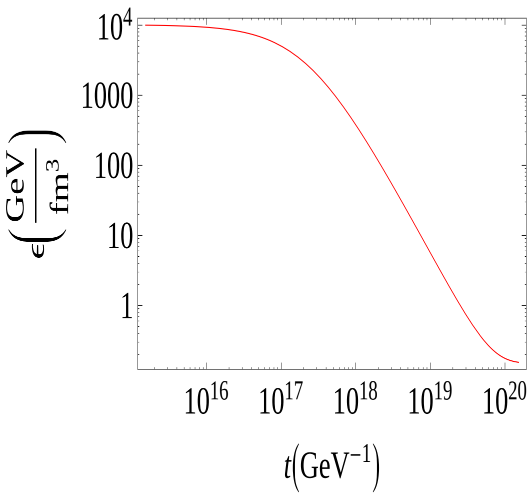

Figure 1: Time evolution of the energy density in MIT bag model.

let us define

$$k(t) = \frac{2\sqrt{B}}{\left(1-\exp\left(4\sqrt{\frac{8\pi B G}{3}}\, t + a\right)\right)}$$

$$b(t) = -\frac{\left(1+\exp\left(4\sqrt{\frac{8\pi B G}{3}}\, t + a\right)\right)}{\left(1-\exp\left(4\sqrt{\frac{8\pi B G}{3}}\, t + a\right)\right)} B \tag{16}$$

Thus we get

$$\varepsilon(t) + k(t)\sqrt{\varepsilon(t)} = b(t) \tag{17}$$

This equation has two solutions given as

$$\varepsilon_{\pm}(t) = \frac{1}{2}\left(k^2(t) + 2b(t) \pm \sqrt{k^4(t) + 4k^2(t)\, b(t)}\right) \tag{18}$$

At $t_i = 10^{-9}\, s$ we find that $\varepsilon_+(t_i) = 10^7\,\mathrm{MeV}/\mathrm{fm}^3$ while the other solution gives $\varepsilon_-(t_i) = 150\,\mathrm{MeV}/\mathrm{fm}^3$. Clearly, $\varepsilon_+(t)$ is the desired solution that we search for. In Fig.1, we present our results for the time evolution of the energy density.

3.2 Time evolution of Temperature

The time evolution of the temperature can be evaluated from Eqs. (4,6)

$$\frac{3}{\sqrt{\frac{37\pi^2}{30}}} \int_{T_0}^{T} \frac{dT}{T\sqrt{T^4 + \frac{30B}{37\pi^2}}} = -2\sqrt{6\pi G}\,(t - t_0)\,, \tag{19}$$

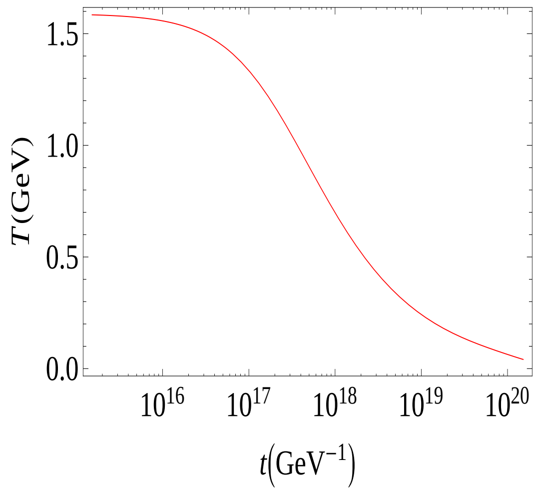

Figure 2: Time evolution of the temperature in MIT bag model.

where t_0 is the initial time and T_0 is the initial temperature. Defining $\alpha^2 = \frac{30B}{37\pi^2}$ we can write

$$\frac{3}{\sqrt{\frac{37\pi^2}{30}}}\int_{T_0}^{T}\frac{dT}{T\sqrt{T^4+\alpha^2}} = -2\sqrt{6\pi G}\,(t-t_0)\,, \tag{20}$$

$$\int_{T_0}^{T}\frac{dT}{T\sqrt{T^4+\alpha^2}} = -\frac{2}{3\,\alpha}\sqrt{6\pi BG}\,(t-t_0)\,, \tag{21}$$

This differential equation has a solution in the form

$$\ln\left(\frac{T^2}{\alpha(\alpha+\sqrt{T^4+\alpha^2})}\right) - \ln\left(\frac{T_0^2}{\alpha(\alpha+\sqrt{T_0^4+\alpha^2})}\right) = -\frac{4}{3}\sqrt{6\pi BG}\,(t-t_0)\,, \tag{22}$$

which can be expressed as

$$\frac{T^2}{\alpha+\sqrt{T^4+\alpha^2}} = \lambda_0\exp[-\frac{4}{3}\sqrt{6\pi BG}(t-t_0)], \tag{23}$$

where $\lambda_0 = \frac{T_0^2}{\alpha+\sqrt{T_0^4+\alpha^2}}$ is a positive parameter. Defining $\kappa = \lambda_0\exp[-\frac{4}{3}\sqrt{6\pi BG}\,(t-t_0)]$ we find that Eq.(23) has two solutions $T_\pm = \pm\sqrt{\frac{2\,\alpha\,\kappa}{1-\kappa^2}}$. The desired and accepted solution is T_+ as the other solution T_- lead to negative values of temperature. In Fig.2 we present our results for the time evolution of the temperature where the initial temperature $T_0 \simeq 1.58$ as can be deuced from Eq(4).

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