Techno-Science
Scientific Journal of Mehmet Akif Ersoy University
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Original Research Article

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# A Note on Hermite-Based Milne Thomson Type Polynomials Involving Chebyshev Polynomials and Other Polynomials

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**ARTICLE INFO**

**Article History**

| | | |
|---|---|---|
| Received | : | 24/02/2020 |
| Revised | : | 26/02/2020 |
| Accepted | : | 26/02/2020 |
| Available online | : | 26/02/2020 |

**Keywords**
Euler numbers and polynomials
Bernoulli numbers and polynomials
Chebyshev polynomials
Hermite-based Milne Thomson type polynomials
Generating function

**ABSTRACT**

*The aim of this paper is to investigate and survey some relations between new families of polynomials including r-parametric Hermite-based Milne Thomson type polynomials and other special numbers, the Bernoulli numbers, the Euler numbers, and the Chebyshev polynomials. By using generating functions and their functional equations of these polynomials are presented. Moreover, using Wolfram Mathematica 12.0 version, some plots and surface of these polynomials under the special conditions are shown. Finally, some remarks, comments and observations for these numbers and polynomials are given.*

## 1. INTRODUCTION

We use the following notations and definitions throughout of this paper.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, the set of integers, the set of real numbers, and the set of complex numbers, respectively. Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and also
$e^{xt}=exp(xt)$.

The Chebyshev polynomials of the first kind $T_n(x)$ and the Chebyshev polynomials of the second kind $U_n(x)$ are defined by following generating functions, respectively:

$$F_T(t;x)=\frac{1-xt}{1-2xt+t^2}=\sum_{n=0}^{\infty}T_n(x)\,t^n \tag{1}$$

and

$$F_U(t;x)=\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}U_n(x)\,t^n \tag{2}$$

see [1, 4, 5, 6, 7, 8, 9].

The cosine-Euler polynomials $E_n^{(C)}(x,y)$ and the sine-Euler polynomials $E_n^{(S)}(x,y)$ are defined by following generating functions, respectively:

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To cite this article: Kilar, N. and Simsek, Y. (2020). A Note on Hermite-Based Milne Thomson Type Polynomials Involving Chebyshev Polynomials and Other Polynomials. *Techno-Science* vol. 3, no. 1, p. 8-14.

$$F_{EC}(t;x,y)=\frac{2}{e^t+1}exp(xt)cos(yt)=\sum_{n=0}^{\infty}E_n^{(C)}(x,y)\frac{t^n}{n!} \tag{3}$$

and

$$F_{ES}(t;x,y)=\frac{2}{e^t+1}exp(xt)sin(yt)=\sum_{n=0}^{\infty}E_n^{(S)}(x,y)\frac{t^n}{n!} \tag{4}$$

see [10, 12, 15].

By using (3), we have

$E_n^{(C)}(x,0)=E_n(x)$,

which denotes the Euler polynomials [12]. When $x=0$, $E_n(0)=E_n$ are called the Euler numbers [1-20].

The cosine-Bernoulli polynomials $B_n^{(C)}(x,y)$ and the sine-Bernoulli polynomials $B_n^{(S)}(x,y)$ are defined by following generating functions, respectively:

$$F_{BC}(t;x,y)=\frac{t}{e^t-1}exp(xt)cos(yt)=\sum_{n=0}^{\infty}B_n^{(C)}(x,y)\frac{t^n}{n!} \tag{5}$$

and

$$F_{BS}(t;x,y)=\frac{t}{e^t-1}exp(xt)sin(yt)=\sum_{n=0}^{\infty}B_n^{(S)}(x,y)\frac{t^n}{n!} \tag{6}$$

see [10, 12, 15].

By using (5), we have

$B_n^{(C)}(x,0)=B_n(x)$,

which denotes the Bernoulli polynomials [12]. When $x=0$, $B_n(0)=B_n$ are called the Bernoulli numbers [1-20].

The polynomials $C_n(x,y)$ and the polynomials $S_n(x,y)$ are defined by following generating functions, respectively:

$$F_C(t;x,y)=exp(xt)cos(yt)=\sum_{n=0}^{\infty}C_n(x,y)\frac{t^n}{n!} \tag{7}$$

and

$$F_S(t;x,y)=exp(xt)sin(yt)=\sum_{n=0}^{\infty}S_n(x,y)\frac{t^n}{n!} \tag{8}$$

see [11, 12, 14, 15, 20, ].

By using (3), (6), (7), and (8), we have the following indentities:

$$E_n^{(C)}(x,y)=\sum_{j=0}^{n}\binom{n}{j}C_j(x,y)E_{n-j} \tag{9}$$

and

$$B_n^{(S)}(x,y)=\sum_{j=0}^{n}\binom{n}{j}S_j(x,y)B_{n-j} \tag{10}$$

see [12].

By using (1), (2), (7), and (8), we have the following indentities:

$$\boldsymbol{T_n(x) = C_n\left(x, \sqrt{1-x^2}\right)} \tag{11}$$

and

$$\boldsymbol{U_{n-1}(x) = \frac{S_n\left(x, \sqrt{1-x^2}\right)}{\sqrt{1-x^2}}} \tag{12}$$

see [11].

Using Euler's formula, in [11], we defined two new families which are called *r*-parametric Hermite-based Milne Thomson type polynomials.

Firstly, we defined the following a new family of polynomials:

$$\begin{aligned}\boldsymbol{B(t,x,y,z,u,r,a,b)} &= \boldsymbol{2\left(b+f(t,a)\right)^z exp(xt) M_4(t,y,\vec{u},r)} \\ &= \sum_{n=0}^{\infty} \boldsymbol{\mathfrak{h}_1\,(n,x,y,z;\vec{u},r,a,b)\frac{t^n}{n!}}\end{aligned} \tag{13}$$

where

$$\boldsymbol{M_4(t,y,\vec{u},r) = exp\left(\sum_{j=1}^{r} u_j t^j\right) cos(yt) = \sum_{n=0}^{\infty} C_n(\vec{u},y;r)\frac{t^n}{n!}} \tag{14}$$

and $\boldsymbol{f(t,a)}$ denotes an analytic function or a meromorphic function, $\boldsymbol{a,b} \in \mathbb{N}$. Observe that, when $\boldsymbol{r=1}$, (14) reduces to the (7) see [11].

Secondly, we defined the following a new family of polynomials:

$$\begin{aligned}\boldsymbol{B_1(t,x,y,z,u,r,a,b)} &= \boldsymbol{2\left(b+f(t,a)\right)^z exp(xt) M_5(t,y,\vec{u},r)} \\ &= \sum_{n=0}^{\infty} \boldsymbol{\mathfrak{h}_2\,(n,x,y,z;\vec{u},r,a,b)\frac{t^n}{n!}}\end{aligned} \tag{15}$$

where

$$\boldsymbol{M_5(t,y,\vec{u},r) = exp\left(\sum_{j=1}^{r} u_j t^j\right) sin(yt) = \sum_{n=0}^{\infty} S_n(\vec{u},y;r)\frac{t^n}{n!}}. \tag{16}$$

Observe that, when $\boldsymbol{r=1}$, (16) reduces to the (8) see [11].

## 2. IDENTITIES AND RELATIONS

In this section, by using generating functions in Equation (1)-Equation (16), we obtain some identities, relations and integral representations including the *r*-parametric Hermite-based Milne Thomson type polynomials, the Chebyshev polynomials, the Euler numbers, the Bernoulli numbers, and other special polynomials. Moreover, using Wolfram Mathematica 12.0 version, we present some plots of the under the special conditions for aforementioned polynomials.

**Theorem 2.1.** Let $\boldsymbol{n} \in \mathbb{N}_{\boldsymbol{0}}$. Then we have

$$\boldsymbol{\mathfrak{h}_1\left(n+1, x, \sqrt{1-x^2}, 1; \vec{0}, r, -1, 0\right) = -(n+1)\sum_{j=0}^{n}\binom{n}{j} T_j(x) E_{n-j}}. \tag{17}$$

**Proof.** Substituting $\boldsymbol{f(t,-1) = \frac{t}{-exp(t)-1}}$, $\boldsymbol{b=0}$, $\boldsymbol{\vec{u} = \vec{0}}$ and $\boldsymbol{z=1}$ into (13), then using (3), we have

$$\mathfrak{h}_1\left(n,x,y,1;\vec{0},r,-1,0\right)=-nE_{n-1}^{(C)}(x,y). \tag{18}$$

Substituting $y=\sqrt{1-x^2}$ into (18), then using (9) and (11), yields the assertion of the theorem.

**Theorem 2.2.** Let $n\in\mathbb{N}$. Then we have

$$\mathfrak{h}_2\left(n,x,\sqrt{1-x^2},1;\vec{0},r,1,0\right)=2\sqrt{1-x^2}\sum_{j=1}^{n}\binom{n}{j}U_{j-1}(x)B_{n-j}. \tag{19}$$

**Proof.** Substituting $f(t,1)=\frac{t}{exp(t)-1}$, $b=0,\vec{u}=\vec{0}$ and $z=1$ into (15), then using (6), we have

$$\mathfrak{h}_2\left(n,x,y,1;\vec{0},r,1,0\right)=2B_n^{(S)}(x,y). \tag{20}$$

Substituting $y=\sqrt{1-x^2}$ into (20), then using (10) and (12), yields the assertion of the theorem.

Let's give different plots of the polynomials $\mathfrak{h}_1(n,x,y,z;\vec{u},r,a,b)$ and the polynomials $\mathfrak{h}_2(n,x,y,z;\vec{u},r,a,b)$ under some special conditions.

Fig. 1. is obtained by $n\in\{0,1,2,3,4,5\}$ using (17) for $x\in[-1,1]$.

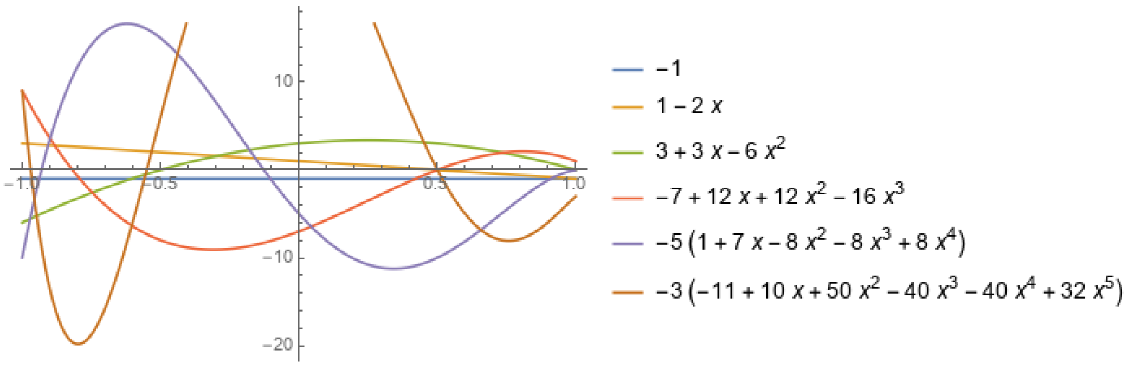



**Fig. 1.** Plots of the polynomials $\mathfrak{h}_1\left(n+1,x,\sqrt{1-x^2},1;\vec{0},r,-1,0\right)$ for varing $x$ values and $n\in\{0,1,2,3,4,5\}$.

Fig. 2. is obtained by $n\in\{0,1,2,3,4,5\}$ using (19) for $x\in[-1,1]$.

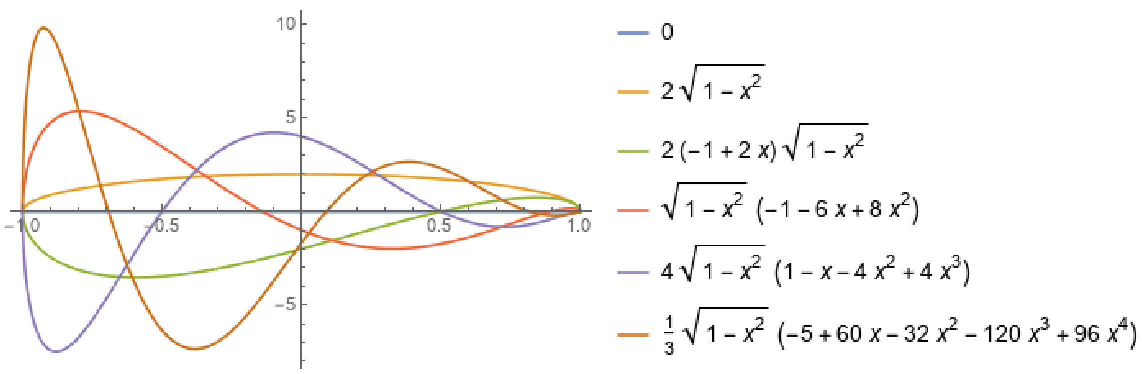
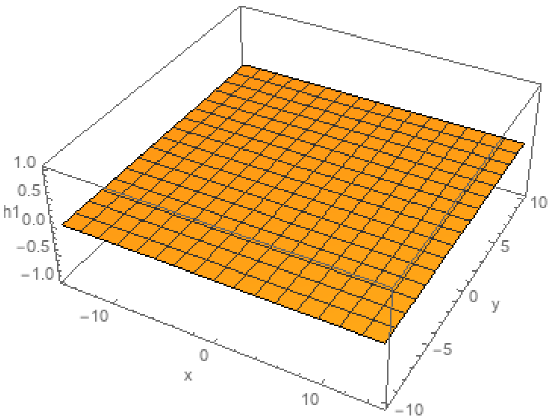
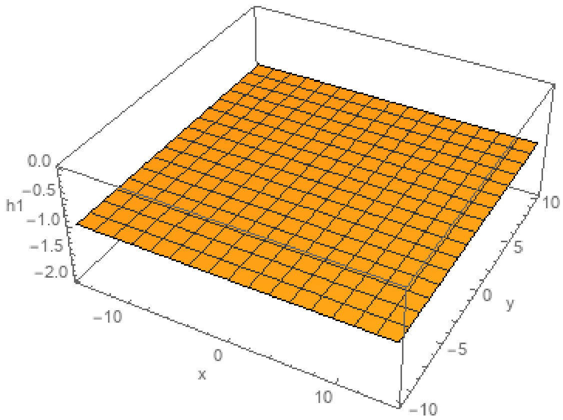



**Fig. 2.** Plots of the polynomials $\mathfrak{h}_2\left(n,x,\sqrt{1-x^2},1;\vec{0},r,1,0\right)$ for varing $x$ values and $n\in\{0,1,2,3,4,5\}$.

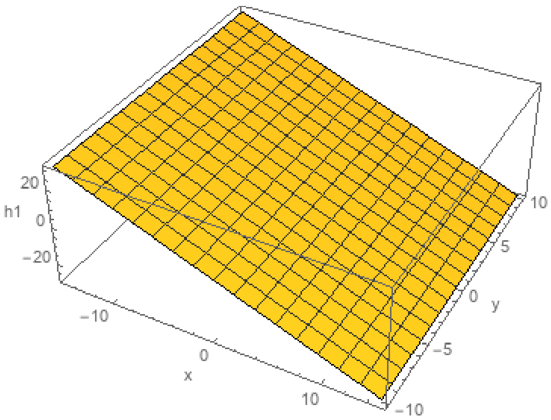
Fig. 3. is obtained by $n\in\{0,1,2,3,4,5\}$ using (18) for $x\in[-15,15]$ and $y\in[-10,10]$.



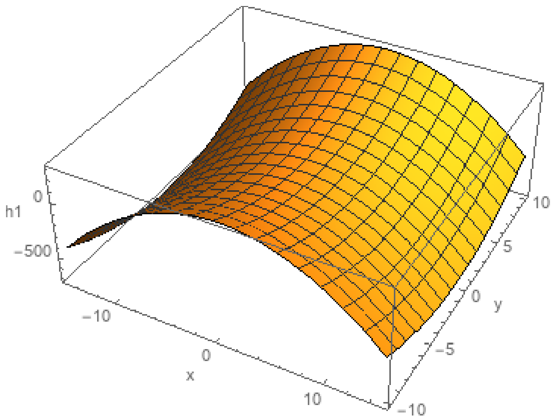

(a) Surface obtained by varying $x$ and $y$ values for $n$ = 0.



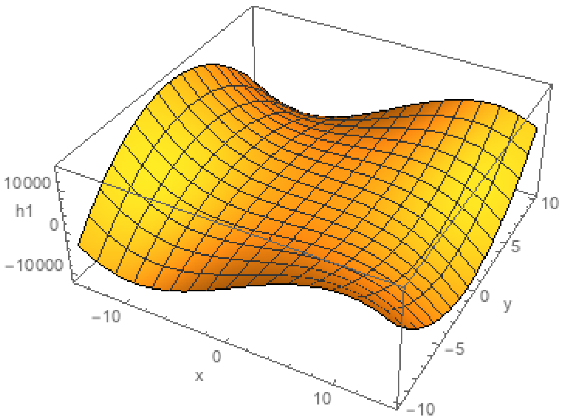

(b) Surface obtained by varying $x$ and $y$ values for $n$ = 1.



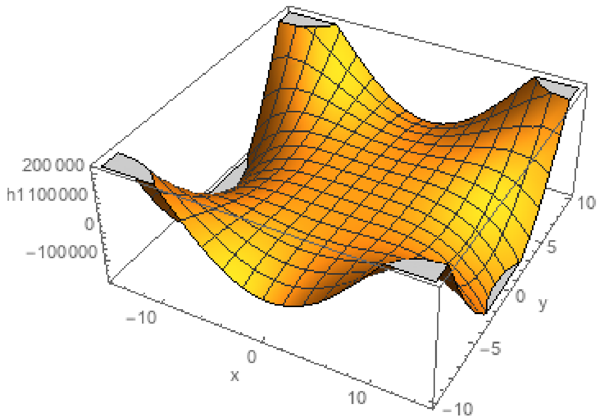

(c) Surface obtained by varying $x$ and $y$ values for $n$ = 2.




(d) Surface obtained by varying $x$ and $y$ values for $n$ = 3.



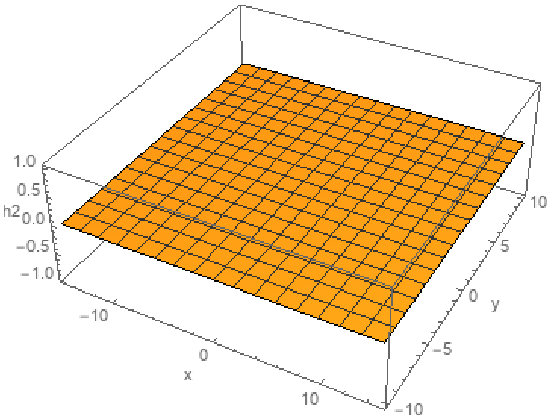

(e) Surface obtained by varying $x$ and $y$ values for $n$ = 4.



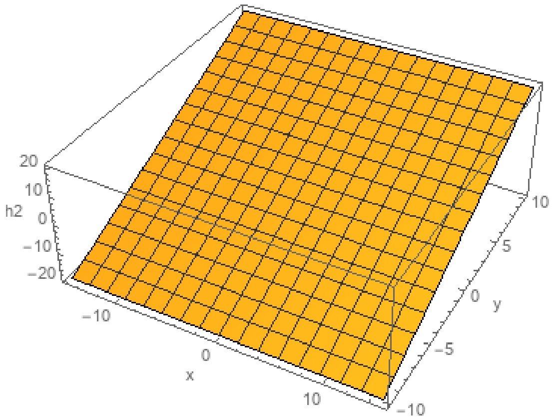

(f) Surface obtained by varying $x$ and $y$ values for $n$ = 5.

**Fig. 3.** Surface figures of the polynomials $\mathfrak{h}_1\left(n, x, y, 1; \vec{0}, r, -1,0\right)$ for $n \in \{0,1,2,3,4,5,6\}$.

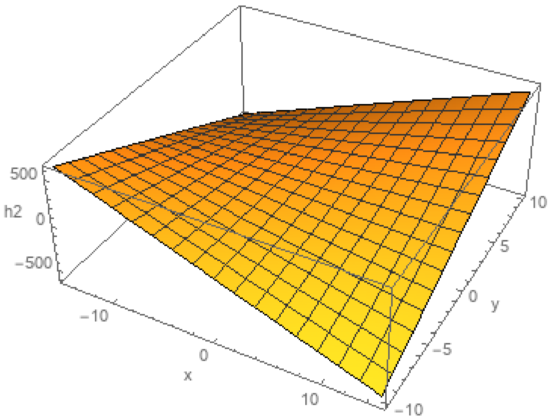
Fig. 4. is obtained by $n \in \{0,1,2,3,4,5\}$ using (20) for $x \in [-15,15]$ and $y \in [-10,10]$.



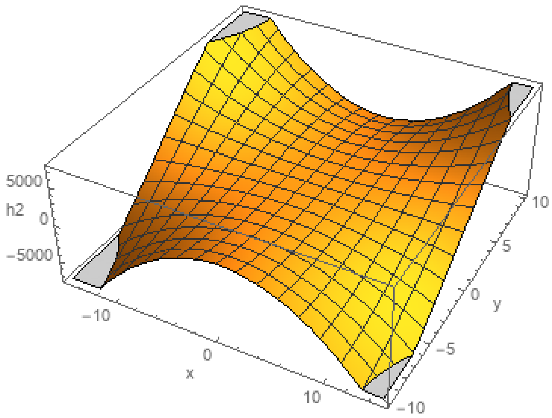

(a) Surface obtained by varying $x$ and $y$ values for $n = 0$.



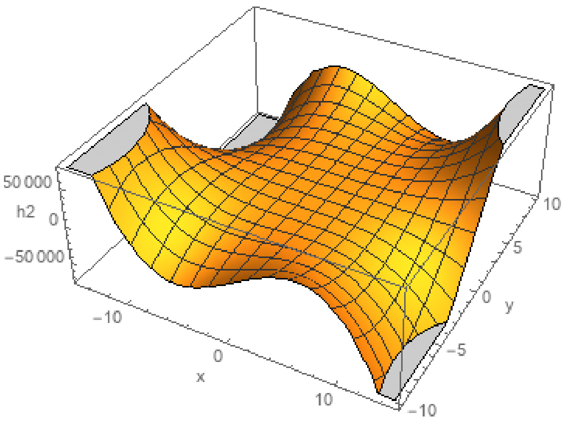

(b) Surface obtained by varying $x$ and $y$ values for $n = 1$.



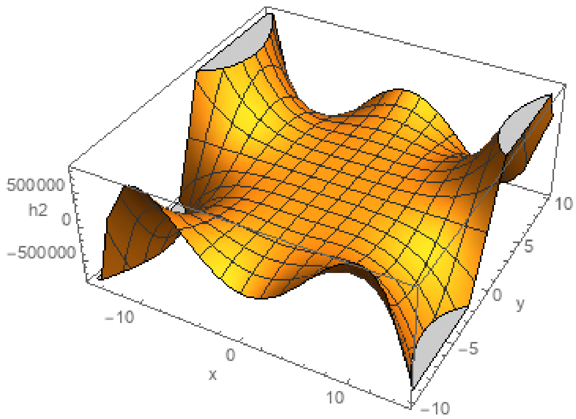

(c) Surface obtained by varying $x$ and $y$ values for $n = 2$.




(d) Surface obtained by varying $x$ and $y$ values for $n = 3$.




(e) Surface obtained by varying $x$ and $y$ values for $n = 4$.




(f) Surface obtained by varying $x$ and $y$ values for $n = 5$.

**Fig. 4.** Surface figures of the polynomials $\mathfrak{h}_2(n, x, y, 1; \vec{0}, r, 1, 0)$ for $n \in \{0,1,2,3,4,5,6\}$.

## 3. CONCLUSIONS

Special numbers, special polynomials and their generating functions, and their functional equations have very important role in mathematics, mathematical physics, probability and statistics, engineering and other related science and social sciences. For instance, the Hermite polynomials and the Chebyshev polynomials and their applications are useful in many physics and engineering problems, which are also used to solve different real world problems such as heat equation.

Thus, the results of this article have potential to motivate many researchers for future research on these aforementioned numbers and polynomials. Consequently, the results of this article may be potentially used in mathematics, in mathematical physics, in engineering, in social sciences.

## 4. ACKNOWLEDGEMENTS

The paper was supported by *Scientific Research Project Administration of Akdeniz University Project ID FDK 5276*.

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**Techno-Science** *Paper ID:* 693541