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Common Fixed Point Results for a Class of (α,β) −Geraghty Contraction Type Mappings in Modular Metric Spaces

Merve Aktay*, Murat Özdemir

Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey.

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Abstract
In this paper, we introduce concepts of generalized $(\alpha,\beta)-$ Geraghty contraction type mappings in modular metric spaces via $(\alpha,\beta)-$ admissible pair in modular metric spaces are essentially weaker than the class of $\alpha-$ Geraghty contraction type mappings. We establish some fixed point and periodic point results for such contractions. Consequently, the obtained results encompass various generalizations of the Banach contraction principle.

Keywords: fixed point, modular metric, Geraghty contraction maps

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Öz

Bu makalede $\alpha-$ Geraghty daraltan tipi dönüşümlerin sınıfından daha zayıf olan $(\alpha,\beta)-$uygun çifti aracılığıyla genelleştirilmiş $(\alpha,\beta)-$ Geraghty daraltan tipi dönüşüm kavramı Modüler metrik uzaylarda tanıtıldı. Bu tarz dönüşümler için bazı sabit nokta ve periyodik nokta sonuçları verildi. Sonuç olarak elde edilen sonuçlar Banach daraltan ilkesinin çeşitli genelleştirmelerini kapsar.

Anahtar Kelimeler: Sabit nokta, modüler metrik, Geraghty daraltan dönüşümler

1. Introduction

It is well known that the Banach's contraction principle (Banach 1922), which is a useful tool in the study of many branches of mathematics and mathematical sciences, is one of the earlier and main results in fixed point theory. Geraghty (1973) proved an interesting generalization of Banach's contraction principle in the setting of complete metric spaces by considering an auxiliary function. Later, Amini-Harandi and Emami (2010) characterized the result of Geraghty in the context of a partially ordered complete metric space, and Caballero et al. (2012) gave some results. Gordji et al. (2012) defined the notion of ψ-Geraghty type contraction and Cho et al. (2013) gave some results for alpha-Geraghty contraction type maps. Also Karapınar and Samet (2014) proved that the results of Gordji et al. [14] and all results inspired by the paper of Gordji et al. (2012) are equivalent to existing results in the literature.

On the other hand, to deal with the problems of description of superposition operators, Chistyakov (2010) introduced the notion of

*Corresponding Author: merve.ozkan@atauni.edu.tr

modular metric spaces and gave some fundamental results on this topic, whereas some authors introduced the analog of the Banach contraction theorem in modular metric spaces and described the important aspects of applications of fixed point of mappings in modular metric spaces. Some recent results in this direction can be found in (Azadifar, 2013; Chaipunya et al., 2012; Hussain et al., 2015; Kuaket and Kumam 2011; Kumam, 2004; Padcharoen, 2016).

In this paper, we prove the existence and uniqueness of a fixed and common fixed point of generalized (α,β) −Geraghty contraction type maps via (α,β) −admissible and generalized (α,β) −admissible pair of mappings in the context of a complete modular metric space.

2. Material and Methods

Throughout this paper $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}$ will denote the set of natural numbers, positive real numbers, and real numbers, respectively. For this purpose, we remind the class of F all functions $\beta: [0,\infty) \to 0,1)$ which satisfies the condition: $\beta(t_n) \to 1$ implies $t_n \to 0$.

Let Ψ denote the class of functions $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

- ψ is strictly increasing,
- ψ is continuous,
- $\psi(t) = 0$ if and only if $t = 0$.

Now, we give some basic concepts and definitions about modular metric spaces.

Definition 2.1 (Chistyakov, 2010) *Let X be a nonempty set. A function $\omega: (0,\infty) \times X \times X \to 0,\infty]$ is said to be a metric modular on X if satisfying, for all $x,y,z \in X$ the following conditions hold:*

1. $\omega_\lambda(x,y) = 0$ *for all* $\lambda > 0$, $x = y$,
2. $\omega_\lambda(x,y) = \omega_\lambda(x,y)$ *for all* $\lambda > 0$,
3. $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,y) + \omega_\mu(x,y)$ *for all* $\lambda > 0$.

If instead of (i), *we have only the condition*

1. $\omega_\lambda(x,x) = 0$ *for all* $\lambda > 0$, *then* ω is *said to be a (metric) pseudomodular on* X.

For any x_0, the set

$$X_\omega(x_0) = \{x \in X: \lim_{\lambda\to\infty} \omega_\lambda(x,x_0) = 0\}$$

is called a modular metric space generated by x_0 and induced by ω. X_ω is independent of generators, we write X_ω instead of $X_\omega(x_0)$ and $\omega_\lambda(x,y) = \omega(\lambda,x,y)$, for all $\lambda > 0$.

The main property of a metric modular ω on a set X is the following: given $x,y \in X$, the function $0 < \lambda \mapsto \omega_\lambda(x,y) \in 0,\infty]$ is nonincreasing on $(0,\infty)$. In fact, if $0 < \mu < \lambda$, then

$$\begin{aligned}\omega_\lambda(x,y) &\le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y)\\ &= \omega_\mu(x,y).\end{aligned}$$

Obviously, a metric modular ω on a set X is nonincreasing with respect to $\lambda > 0$.

Chaipunya et al. (2012) changed the notion of convergent and Cauchy sequences in modular metric spaces under the direction of Mongkolkeha et al. (2011).

Definition 2.2 (Chaipunya et al., 2012; Cho et al., 2013) *Let X_ω be a modular metric space and $\{x_n\}$ be a sequence in X_ω.*

1. *A point $x \in X_\omega$ is called a limit of $\{x_n\}$ if, for each $\lambda, \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n,x) < \varepsilon$ for all $n \ge n_0$. A*

sequence that has a limit is said to be convergent (or converges to x), which is written as $\lim_{n\to\infty} x_n = x$.

2. *A sequence* $\{x_n\}$ *in* X_ω *is said to be a Cauchy sequence if, for each* $\lambda, \varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\omega_\lambda(x_n, x_m) < \varepsilon$ *for all* $n, m \geq n_0$.

3. *Let* X_ω *be a modular metric space. If every Cauchy sequences in* X_ω *converges,* X_ω *is said to be complete.*

Definition 2.3 (Chistyakov, 2011) *A modular* ω *on* X_ω *is said to satisfy the* Δ_2 −*condition if for a sequence* $\{x_n\} \subset X_\omega$ *and* x *in* X_ω, *there exists a number* $\lambda > 0$, *possibly depending on* $\{x_n\}$ *and* x, *such that* $lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, *then* $lim_{n\to\infty} \omega_{\frac{\lambda}{2}}(x_n, x) = 0$. *This implies that* $lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ *for all* $\lambda > 0$.

Henceforwards, in this paper we assume that ω is a modular on X_ω and satisfy in the Δ_2 −condition on X_ω.

3. Results

Samet et al. (2012) gave the definition of α −admissible and later Aydi (2015) gereneralized this definition. Also, Chandok (2015) introduced (α,β) −Geraghty type−I rational contractive mapping and (α,β) −admissible mapping. Regarding this, Chandok (2015) gave some results in complete metric spaces. Now we give the definition of (α,β) −admissible pair in modular metric spaces.

Definition 3.1 *Let* X_ω *be a modular metric space and* $T: X_\omega \to X_\omega$ *be a map and* $\alpha: X_\omega \times X_\omega \to \mathbb{R}^+$ *be a function. Then*

1. T *is said to be* α −*admissible if* $\alpha(x,y) \geq 1$ *implies* $\alpha(Tx,Ty) \geq 1$ *for all* $x, y \in X$.

2. *A self mapping* T *on* X_ω *is said to be triangular* α −*admissible if:*

(a) T *is* α −*admissible,*

(b) $\alpha(x,z) \geq 1$ *and* $\alpha(z,y) \geq 1$ *imply* $\alpha(x,y) \geq 1$.

3. *Let* $T: X_\omega \to X_\omega$ *be a map and* $\alpha: X_\omega \times X_\omega \to \mathbb{R}^+$ *be a function. Then* T *is said to be* α −*orbital admissible if* $\alpha(x,Tx) \geq 1$ *implies* $\alpha(Tx,T^2x) \geq 1$.

Definition 3.2 *Let* X_ω *be a modular metric space and* $T, S: X_\omega \to X_\omega$ *be maps and* $\alpha, \beta: X_\omega \times X_\omega \to \mathbb{R}^+$ *be two functions. Then*

1. (T,S) *is said to be* (α,β) −*admissible if* $\alpha(x,y) \geq 1$ *and* $\beta(x,y) \geq 1$ *implies* $\alpha(Tx,Sy) \geq 1$, $\alpha(Sx,Ty) \geq 1$ *and* $\beta(Tx,Sy) \geq 1$, $\beta(Sx,Ty) \geq 1$ *for all* $x, y \in X$.

2. T *is said to be* (α,β) −*admissible if* $\alpha(x,y) \geq 1$ *and* $\beta(x,y) \geq 1$ *implies* $\alpha(Tx,Ty) \geq 1$ *and* $\beta(Tx,Ty) \geq 1$ *for all* $x, y \in X$.

3. X_ω *is* (α,β)−*regular if* $\{x_n\}$ *is a sequence in* X_ω *such that* $x_n \to x \in X_\omega$, $\alpha(x_n, x_{n+1}) \geq 1$, $\beta(x_n, x_{n+1}) \geq 1$, *for all* n, *there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x_{n_k+1}) \geq 1$, $\beta(x_{n_k}, x_{n_k+1}) \geq 1$ *for all* $k \in \mathbb{N}$ *and* $\alpha(x,Tx) \geq 1$, $\alpha(x,Sx) \geq 1$, *and* $\beta(x,Tx) \geq 1$, $\beta(x,Sx) \geq 1$.

4. *Self mappings* T *and* S *on* X_ω *are said to be triangular* (α,β) −*admissible if:*

(a) (T,S) *is* (α,β) −*admissible,*

(b) $\alpha(x,z) \geq 1$ *and* $\alpha(z,y) \geq 1$ *imply* $\alpha(x,y) \geq 1$ *and* $\beta(x,z) \geq 1$ *and* $\beta(z,y) \geq 1$ *imply* $\beta(x,y) \geq 1$ *for all* $x, y, z \in X_\omega$.

Definition 3.3 *Let $T: X_\omega \to X_\omega$ be a map and $\alpha: X_\omega \times X_\omega \to \mathbb{R}^+$ be a function. Then T is said to be triangular α −orbital admissible if T is α −orbital admissible and $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ imply $\alpha(x,Ty) \geq 1$.*

Obviously, every (α,β) −admissible mapping is an α −admissible mapping. For examples, see (Samet et al., 2012 and Aydi, 2015) and for farther detail see (Arshad et al., 2016) and cited therein.

Lemma 3.4 *Let $T: X_\omega \to X_\omega$ be a triangular α −admissible mapping. Assume that there exists $x_1 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then, we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.*

Proof Since T is triangular α −admissible and $\alpha(x_0,Tx_0) \geq 1$, we deduce that $\alpha(x_1,x_2) = \alpha(Tx_0,Sx_1) \geq 1$. By continuing this process, we get $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \geq 0$. Assume that $\alpha(x_n,x_m) \geq 1$. We shall prove that $\alpha(x_n,x_{m+1}) \geq 1$, where $m > n$. Since T is triangular α −admissible and $\alpha(x_m,x_{m+1}) \geq 1$, we get that $\alpha(x_n,x_{m+1}) \geq 1$. Therefore, we obtained that $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.

Samet et al. (2012) gave generalized α −Geraghty contraction type map. Later Arshad et al. (2016) give some generalizations in metric spaces.

Now, we give the definition of generalized (α,β) −Geraghty contraction type mappings in modular metric spaces.

Definition 3.5 *Let X_ω be a complete modular metric space, $\alpha,\beta: X_\omega \times X_\omega \to \mathbb{R}^+$ be functions, and two maps $T,S: X_\omega \to X_\omega$ is called a pair of generalized (α,β) −Geraghty contraction type mappings if there exists $\gamma \in \mathrm{F}$ such that for all $x,y \in X_\omega$,*

$$\alpha(x,y)\beta(x,y)\psi(\omega_1(Tx,Sy)) \leq \gamma(\psi(M_{T,S}(x,y)))\psi(M_{T,S}(x,y)),$$

where

$$M_{T,S}(x,y) = \max\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Sy), \frac{\omega_2(x,Sy)+\omega_2(y,Tx)}{2}\}. \tag{3.1}$$

Theorem 3.6 *Let X_w be a complete modular metric space, $\alpha,\beta: X_w \times X_w \to \mathbb{R}^+$ be functions, and let T, S be self-mappings on X_w satisfying the following conditions:*

1. *(T,S) is a pair of generalized (α,β) −geraghty contraction type mappings,*
2. *(T,S) is triangular (α,β) −admissible pair,*
3. *There exists $x_0 \in X_w$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\beta(x_0,Tx_0) \geq 1$,*
4. *T and S are continuous.*

Then (T,S) has a common fixed point.

Proof From the hypothesis (3) of Theorem 3.6, there exists a point $x_0 \in X_w$ such that $\alpha(x_0,Tx_0) \geq 1$. Let $x_1 = Tx_0$ and $x_2 = Sx_1$. Define a sequence $\{x_n\}$ by $x_{2n+2} = Sx_{2n+1}$ and $x_{2n+1} = Tx_{2n}$ for all $n \geq 0$. First of all, we show that $\alpha(x_n,x_{n+1}) \geq 1$ and $\alpha(x_{n+1},x_n) \geq 1$ for all $n \geq 0$. Since $\alpha(x_0,Tx_0) \geq 1$, we have $\alpha(x_0,x_1) \geq 1$. Since (T,S) is a generalized (α,β) −admissible pair of mappings, we have

$$\alpha(x_1,x_2) = \alpha(Tx_0,Sx_1) \geq 1,$$

and

$$\alpha(x_3,x_2) = \alpha(Tx_2,Sx_1) \geq 1.$$

By induction, we get

$$\alpha(x_n,x_{n+1}) \geq 1, \quad (3.2)$$

for all $n \geq 0$. Similarly,

$$\beta(x_n,x_{n+1}) \geq 1, \quad (3.3)$$

for all $n \geq 0$. Since $\gamma \in 0,1)$ and from (3.2) and (3.3), we have

$$\begin{aligned}
&\psi(\omega_1(x_{2n+1},x_{2n+2})) \\
&\leq \alpha(x_{2n},x_{2n+1})\beta(x_{2n},x_{2n+1}) \\
&\quad \psi(\omega_1(Tx_{2n},Sx_{2n+1})) \\
&\leq \gamma(\psi(M_{T,S}(x_{2n},x_{2n+1})))\psi(M_{T,S}(x_{2n},x_{2n+1}))
\end{aligned} \quad (3.4)$$

where

$$\begin{aligned}
M_{T,S}(x_{2n},x_{2n+1}) &= \max\{\omega_1(x_{2n},x_{2n+1}), \\
&\quad \omega_1(x_{2n},Tx_{2n}), \omega_1(x_{2n+1},Sx_{2n+1}) \\
&\quad \frac{\omega_2(x_{2n},Sx_{2n+1}) + \omega_2(x_{2n+1},Tx_{2n})}{2}\} \\
&= \max\{\omega_1(x_{2n},x_{2n+1}), \omega_1(x_{2n+1},x_{2n+2}), \\
&\quad \frac{\omega_2(x_{2n},x_{2n+2})}{2}\} \\
&\leq \max\{\omega_1(x_{2n},x_{2n+1}), \omega_1(x_{2n+1},x_{2n+2}), \\
&\quad \frac{\omega_1(x_{2n},x_{2n+1}) + \omega_1(x_{2n+1},x_{2n+2})}{2}\} \\
&= \max\{\omega_1(x_{2n},x_{2n+1}), \omega_1(x_{2n+1},x_{2n+2})\}.
\end{aligned} \quad (3.5)$$

If $M_{T,S}(x_{2n},x_{2n+1}) = \omega_1(x_{2n+1},x_{2n+2})$, then we have

$$\begin{aligned}
&\psi(\omega_1(x_{2n+1},x_{2n+2})) \\
&\leq \gamma(\psi(M_{T,S}(x_{2n},x_{2n+1})))\psi(M_{T,S}(x_{2n},x_{2n+1})) \\
&\leq \gamma(\psi(M_{T,S}(x_{2n},x_{2n+1})))\psi(\omega_1(x_{2n+1},x_{2n+2})) \\
&< \psi(\omega_1(x_{2n+1},x_{2n+2})).
\end{aligned}$$

Since ψ is strictly increasing, $\omega_1(x_{2n+1},x_{2n+2}) < \omega_1(x_{2n+1},x_{2n+2})$, which is a contradiction, and this implies that $M_{T,S}(x_{2n},x_{2n+1}) = \omega_1(x_{2n},x_{2n+1})$. Therefore, we get $\omega_1(x_{n+1},x_{n+2}) < \omega_1(x_n,x_{n+1})$. Further, the sequence $\omega_1(x_n,x_{n+1})$ is positive and nonincreasing. Hence, there exists $r \geq 0$ such that $\lim_{n\to\infty}\omega_1(x_n,x_{n+1}) = r$. Let us show that $r = 0$. To the contrary, assume that $r > 0$. Then, we have

$$\begin{aligned}
\frac{\psi(\omega_1(x_{n+1},x_{n+2}))}{\psi\left(M_{T,S}(x_n,x_{n+1})\right)} &\leq \gamma(\psi(M_T(x_n,x_{n+1})) \\
&< 1.
\end{aligned} \quad (3.6)$$

On taking the limit in (3.6), we get $\lim_{n\to\infty}\gamma(\psi(M_T(x_n,x_{n+1}))) = 1$. Since $\gamma \in$ F, we get

$$\lim_{n\to\infty}(\psi(M_T(x_n,x_{n+1})) = 0, \quad (3.7)$$

which implies that

$$r = \lim_{n\to\infty}\omega_1(x_n,x_{n+1}) = 0.$$

This is a contradiction. Next, we shall show that $\{x_n\}$ is a Cauchy sequence. Assume that, $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that, for all $k \geq 1$, there exists $m(k) > n(k) > k$ with $\omega_2(x_{n(k)},x_{m(k)}) \geq \varepsilon$. Since $\omega_1(x_{n(k)},x_{m(k)}) \geq \omega_2(x_{n(k)},x_{m(k)})$, then we have $\omega_1(x_{n(k)},x_{m(k)}) \geq \varepsilon$. Let $m(k)$ be the smallest number satisfying the conditions above. Hence, we have $\omega_1(x_{n(k)},x_{m(k)-1}) < \varepsilon$. Using the property (3) of modular metric, we have

$$\begin{aligned}
\varepsilon &\leq \omega_2(x_{n(k)},x_{m(k)}) \\
&\leq \omega_1(x_{n(k)},x_{m(k)-1}) + \omega_1(x_{m(k)-1},x_{m(k)}) \\
&\leq \varepsilon + \omega_1(x_{m(k)-1},x_{m(k)}).
\end{aligned} \quad (3.8)$$

On taking the limit in (3.8) as $k \to \infty$, we have

$$\lim_{k\to\infty}\omega_2(x_{n(k)},x_{m(k)}) = \varepsilon. \quad (3.9)$$

Since

$$\begin{aligned}
&|\omega_1(x_{n(k)},x_{m(k)-1}) - \omega_2(x_{n(k)},x_{m(k)})| \\
&\qquad \leq \omega_1(x_{m(k)},x_{m(k)-1}),
\end{aligned}$$

we have

$$\lim_{k\to\infty}\omega_1\left(x_{n(k)},x_{m(k)-1}\right)=\varepsilon. \quad (3.10)$$

Similarly, we obtain

$$\lim_{k\to\infty}\omega_1\left(x_{m(k)},x_{n(k)-1}\right) = \lim_{k\to\infty}\omega_1\left(x_{n(k)-1},x_{m(k)-1}\right) = \varepsilon. \quad (3.11)$$

From Lemma 3.4, we obtain

$$\begin{aligned}\psi(\omega_1\left(x_{n(k)+1},x_{m(k)+2}\right)) &= \psi(\omega_1\left(Tx_{n(k)},Sx_{m(k)+1}\right)) \\ &\le \alpha\left(x_{n(k)},x_{m(k)+1}\right)\beta\left(x_{n(k)},x_{m(k)+1}\right)\psi\left(\omega_1\left(Tx_{n(k)},Sx_{m(k)+1}\right)\right) \\ &\le \gamma(\psi\left(M_{T,S}\left(x_{n(k)},x_{m(k)+1}\right)\right)\psi(M_{T,S}\left(x_{n(k)},x_{m(k)+1}\right)),\end{aligned}$$

where

$$\begin{aligned}M_{T,S}\left(x_{n(k)},x_{m(k)+1}\right) &= \max\{\omega_1\left(x_{n(k)},x_{m(k)+1}\right),\omega_1\left(x_{n(k)},Tx_{n(k)}\right), \omega_1\left(x_{m(k)+1},Sx_{m(k)+1}\right), \\ &\qquad \frac{\omega_2\left(x_{n(k)},Sx_{m(k)+1}\right)+\omega_2\left(x_{m(k)+1},Tx_{n(k)}\right)}{2}\} \\ &= \max\{\omega_1\left(x_{n(k)},x_{m(k)+1}\right),\omega_1\left(x_{n(k)},x_{n(k)+1}\right), \omega_1\left(x_{m(k)+1},x_{m(k)+2}\right), \\ &\qquad \frac{\omega_2\left(x_{n(k)},x_{m(k)+2}\right)+\omega_2\left(x_{m(k)+1},x_{n(k)+1}\right)}{2}\}.\end{aligned}$$

From (3.9), (3.10) and (3.11), we make an inference that

$$\lim_{k\to\infty}M_T\left(x_{n(k)-1},x_{m(k)-1}\right)=\varepsilon. \quad (3.12)$$

Hence, we have

$$\frac{\psi(\omega_1\left(x_{n(k)+1},x_{m(k)+2}\right))}{\psi(M_{T,S}\left(x_{n(k)},x_{m(k)+1}\right))} \le \gamma(\psi\left(M_{T,S}\left(x_{n(k)},x_{m(k)+1}\right)\right) < 1. \quad (3.13)$$

On taking the limit in (3.13) as $k\to\infty$, we have

$$\lim_{k\to\infty}\gamma(\psi\left(M_{T,S}\left(x_{n(k)},x_{m(k)+1}\right)\right)=1, \quad (3.14)$$

which implies that

$$\lim_{k\to\infty}M_{T,S}\left(x_{n(k)},x_{m(k)+1}\right)=0. \quad (3.15)$$

From (3.12) and (3.15), we have $\varepsilon=0$. This is a contradiction. Thus, we obtain that $\{x_n\}$ is a Cauchy sequence. In a similar way, obviously for other cases $\{x_n\}$ is a Cauchy sequence. Since X_ω is a complete modular metric space, it follows that there exists $x_\iota\in X_\omega$ such that $\lim_{n\to\infty}\omega_1(x_n,x_\iota)=0$. Finally, we shall show that x_ι is a common fixed point of T and S. Since $\lim_{n\to\infty}\omega_1(x_n,x_\iota)=0$, then we have $\lim_{n\to\infty}\omega_1(x_{2n},x_\iota)=\lim_{n\to\infty}\omega_1(x_{2n+1},x_\iota)=0$. By the continuity of T and S, we get

$$\lim_{n\to\infty}\omega_1(x_{2n+1},Tx_\iota)=\lim_{n\to\infty}\omega_1(Tx_{2n},Tx_\iota) = 0$$

and

$$\lim_{n\to\infty}\omega_1(x_{2n+2},Sx_\iota)=\lim_{n\to\infty}\omega_1(Sx_{2n+1},Sx_\iota) = 0$$

Hence, $x_\iota=Tx_\iota=Sx_\iota$, it follows that (T,S) has common fixed point.

Theorem 3.7 *Let X_w be a complete modular metric space, $\alpha,\beta: X_w\times X_w\to\mathbb{R}^+$ be two functions, and let T, S be self-mappings on X_w satisfying the following conditions:*

1. *(T,S) is a pair of generalized (α,β) −Geraghty contraction type mappings,*
2. *(T,S) is triangular (α,β) −admissible pair,*
3. *There exists $x_0\in X_w$ such that $\alpha(x_0,Tx_0)\ge 1$ and $\beta(x_0,Tx_0)\ge 1$,*
4. *If $\{x_n\}$ is a sequence in X_w such that $\alpha(x_n,x_{n+1})\ge 1$ and $\beta(x_n,x_{n+1})\ge 1$ for all n and $x_n\to x_\iota\in X_w$ as $n\to\infty$, then there*

exists a subsequence $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k},x_\iota)\geq 1$ *and* $\beta(x_{n_k},x_\iota)\geq 1$ *for all* k.

Then (T,S) *have a common fixed point.*

Proof Similar to the proof of Theorem 3.6, we define a sequence $x_{2n+2}=Sx_{2n+1}$ and $x_{2n+1}=Tx_{2n}$ for all $n\geq 0$. This sequence converges to $x_\iota\in X_w$. By the hypothesis (4) of Theorem 3.7, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x_\iota)\geq 1$ and $\beta(x_{n_k},x_\iota)\geq 1$ for all k. From the hypothesis (4) of Theorem 3.7, we have

$$\psi(\omega_1(x_{2n_k+1},Tx_\iota))=\psi(\omega_1(Sx_{2n_k},Tx_\iota))$$
$$\leq \alpha(x_{2n_k},x_\iota)\beta(x_{2n_k},x_\iota)\psi(\omega_1(Sx_{2n_k},Tx_\iota))$$
$$\leq \gamma(\psi\left(M_{T,S}(x_{2n_k},x_\iota)\right)\psi(M_{T,S}(x_{2n_k},x_\iota)),$$

where

$$M_{T,S}(x_{2n_k},x_\iota)=\max\{\omega_1(x_{2n_k},x_\iota),\ \omega_1(x_{2n_k},Tx_{2n_k}),\omega_1(x_\iota,Sx_\iota),\ \frac{\omega_2(x_{2n_k},Sx_\iota)+\omega_2(x_\iota,Tx_{2n_k})}{2}\}. \quad (3.16)$$

Taking the limit in (3.16) as $k\to\infty$, we get

$$M_{T,S}(x_{2n_k},x_\iota)=\omega_1(x_\iota,Sx_\iota).$$

Since $\omega_1(x_\iota,Sx_\iota)>0$ for enough large k, we have

$$M_{T,S}(x_{2n_k},x_\iota)>0.$$

Since $\gamma\in(0,1)$, we get

$$\gamma(\psi(M_{T,S}(x_{2n_k},x_\iota)))<\psi(M_{T,S}(x_{2n_k},x_\iota)).$$

So we get

$$\omega_1(x_{2n_k},Sx_\iota)<M_{T,S}(x_{2n_k},x_\iota). \quad (3.17)$$

Letting $k\to\infty$ in (3.17), we obtain

$$\omega_1(x_\iota,Sx_\iota)<\omega_1(x_\iota,Sx_\iota),$$

which is a contradiction. Thus we obtain that $\omega_1(x_\iota,Sx_\iota)=0$. Similarly, we have $\omega_1(x_\iota,Tx_\iota)=0$. Thus, $x_\iota=Sx_\iota=Tx_\iota$. That is, (T,S) have common fixed point. Finally, we shall show that, (T,S) have a unique common fixed point. To the contrary, assume that x_ι and y_ι are two common fixed point of (T,S) and $x_\iota\neq y_\iota$. Using the similar process of (3.4) and (3.5), we get

$$\psi(\omega_1(x_\iota,y_\iota))$$
$$\leq\gamma(\psi\left(M_{T,S}(x_\iota,y_\iota)\right)\psi(M_{T,S}(x_\iota,y_\iota)),$$

where

$$M_{T,S}(x_\iota,y_\iota)$$
$$=\max\{\omega_1(x_\iota,y_\iota),\omega_1(x_\iota,Tx_\iota),\omega_1(y_\iota,Sy_\iota),\ \frac{\omega_2(x_\iota,Sy_\iota)+\omega_2(y_\iota,Tx_\iota)}{2}\}.$$

Thus,

$$\psi(\omega_1(x_\iota,y_\iota))$$
$$\leq\gamma(\psi\left(M_{T,S}(x_\iota,y_\iota)\right)\psi(\omega_1(x_\iota,y_\iota)).$$

$\omega_1(x_\iota,y_\iota)=0$ if not, this is a contradiction. Therefore (T,S) have a unique common fixed point.

If $S=T$ in Theorem 3.6, then we have

$$M_T(x,y)=\max\{\omega_1(x,y),\omega_1(x,Tx),\ \omega_1(y,Ty),\frac{\omega_2(x,Ty)+\omega_2(y,Tx)}{2}\}.$$

We have the following corollary.

Corollary 3.8 *Let* X_w *be a complete modular metric space,* $\alpha,\beta: X_w\times X_w\to\mathbb{R}^+$ *be functions, and let* T *be a self-mapping on* X_w *satisfying the following conditions:*

1. T *is a generalized* (α,β) *−Geraghty contraction type mapping,*
2. T *is triangular* (α,β) *−admissible mapping,*

3. *There exists* $x_0 \in X_w$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\beta(x_0, Tx_0) \geq 1$,

4. *If* $\{x_n\}$ *is a sequence in* X_w *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $\beta(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x_\iota \in X_w$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x_\iota) \geq 1$ *and* $\beta(x_{n_k}, x_\iota) \geq 1$ *for all* k.

Then T *has a fixed point* $x_\iota \in X_w$ *and* $\{T^n x_0\}$ *converges to* x_ι.

Corollary 3.8 generalize results of [12] and [2]. T is continuous instead of the hypothesis (4) of Corollary 3.8, then, T has a fixed point.

If $T = S$ and

$$\alpha(x,y)\omega_1(Tx,Ty) \leq \gamma(M_T(x,y)))M_T(x,y),$$

where

$$M_T(x,y) = \max\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \frac{\omega_2(x,Ty)+\omega_2(y,Tx)}{2}\},$$

Then we get the following corollary.

Corollary 3.9 *Let* X_w *be a complete modular metric space,* $\alpha: X_w \times X_w \to \mathbb{R}^+$ *be a function, and* T *be a self-mapping on* X_w *satisfying the following conditions:*

1. T *is a generalized* α −*Geraghty contraction type mapping,*

2. T *is a triangular* $\alpha-$ *admissible mapping,*

3. *there exists* $x_0 \in X_w$ *such that* $\alpha(x_0, Tx_0) \geq 1$,

4. *if* $\{x_n\}$ *is a sequence in* X_w *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x_\iota \in X_w$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x_\iota) \geq 1$ *for all* k. *Then* T *has a fixed point* $x_\iota \in X_w$ *and* $\{T^n x_0\}$ *converges to* x_ι

If $S = T$, $M(x,y) = \max\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty)\}$

and $\beta(x,y) = 1$ for all $x, y \in X_w$ in Theorem 3.6, then we get the following corollary.

Corollary 3.10 *Let* X_w *be a complete modular metric space,* $\alpha: X_w \times X_w \to \mathbb{R}^+$ *be a function, and* T *be a self-mapping on* X_w *satisfying the following conditions:*

1. T *is a* α −*Geraghty contraction type mapping,*

2. T *is a triangular* α −*admissible mapping,*

3. *There exists* $x_0 \in X_w$ *such that* $\alpha(x_0, Tx_0) \geq 1$,

4. *If* $\{x_n\}$ *is a sequence in* X_w *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x_\iota \in X_w$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x_\iota) \geq 1$ *for all* k.

Then T *has a fixed point* $x_\iota \in X_w$ *and* $\{T^n x_0\}$ *converges to* x_ι.

The corollary is similar when T is assumed to be continuous in Corollary 3.9 and Corollary 3.10, and so T has a fixed point.

Example 3.11 *Let* $X_\omega = [0,\infty)$, $\omega_1(x,y) = |x-y|$, $\omega_\lambda(x,y) = \frac{1}{\lambda}|x-y|$. *Define the mappings* $\alpha, \beta: X_w \times X_w \to \mathbb{R}^+$, *and consider the mappings* $T, S: X_\omega \to X_\omega$, $\gamma \in$ F, *such that* $\gamma(0) = 0$, $\gamma(t) = \frac{7}{8}$ *for all* $t \geq 0$ *and let* $\psi(t) = \frac{t}{2}$. *For all* $x, y \in X_\omega$,

$$Tx = \begin{cases} \frac{x}{5}, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases},$$

$$Sx = \begin{cases} \frac{x}{2}, & x \in [0,1] \\ 2x-1, & \text{otherwise} \end{cases}$$

and

$$\alpha(x,y)=\beta(x,y)=\begin{cases}1, & (x,y)\in[0,1]\\ 0, & \text{otherwise}\end{cases}$$

Now, we show that (T,S) is an (α,β) −admissible mapping. $X_\omega=[0,1]$ is a complete modular metric space. Let $x,y\in X_\omega$, $\alpha(x,y)\geq 1$ and $\beta(x,y)\geq 1$, $x,y\in$ 0,1]. For all $x\in$ 0,1], we have $Tx\leq 1$ and $Sx\leq 1$. Then $\alpha(Tx,Sy)\geq 1$, $\alpha(Sx,Ty)\geq 1$ and $\beta(Tx,Sy)\geq 1$, $\beta(Sx,Ty)\geq 1$. For $x_0=0$, we have $\alpha(x_0,Tx_0)\geq 1$ and $\beta(x_0,Tx_0)\geq 1$. Therefore, the assertions hold. T and S are not continuous mappings. Let $\{x_n\}$ is a sequence in X_w such that $\alpha(x_n,x_{n+1})\geq 1$ and $\beta(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x_\iota\in X_w$ as $n\to\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x_\iota)\geq 1$ and $\beta(x_{n_k},x_\iota)\geq 1$ for all k. Later, we show that (T,S) is a pair of generalized (α,β) −Geraghty contraction type mapping. For all $x,y\in$ 0,1],

$$\begin{aligned}M_{T,S}(x,y)&=\max\{\omega_1(x,y),\omega_1(x,Tx),\\&\qquad\omega_1(y,Sy),\frac{\omega_2(x,Sy)+\omega_2(y,Tx)}{2}\}\\&=\begin{cases}|x-y|, & 0\leq y\leq\frac{x}{5}\\ \frac{4x}{5}, & \frac{x}{5}<y\leq 1\end{cases}\end{aligned}$$

and

$$\alpha(x,y)\beta(x,y)\psi(\omega_1(Tx,Sy))=\psi\left(\left|\frac{x}{5}-\frac{y}{2}\right|\right)=\frac{1}{2}\left|\frac{x}{5}-\frac{y}{2}\right|$$

Therefore, (T,S) holds (3.1), and so, (T,S) is a pair of generalized (α,β) −Geraghty contraction type mapping. Obviously, a common fixed point of (T,S) is $x_\iota=0$.

Periodic Point Results

In this section we prove some periodic point results for self-mappings on a complete modular metric space. It is an obvious fact that, if x is a fixed point of T (i.e. $Fix(T):=\{x\in X: Tx=x\}$), then x is also a fixed point of T^n for every $n\in N$. Now, we give the following definition.

Definition 4.1 (Jeong and Rhoades, 2005*) A mapping $T: X\to X$ is said to have property (P) if $Fix(T^n)=Fix(T)$ for every $n\in\mathbb{N}$.*

For further details on these property, we refer to (Jeong and Rhoades, 2005). Now, we use the property (P) in modular metric spaces.

Theorem 4.2 *Let X_w be a complete modular metric space, and T be self-mappings on X_w satisfying the following conditions:*

1. *there exists $\tau>0$ and a function $\psi\in$ F and $\alpha,\beta: X_w\times X_w\to\mathbb{R}^+$ be two functions such that*

$$\alpha(x,Tx)\beta(x,Tx)\psi(\omega_1(Tx,T^2x))\leq\gamma\left(\psi(M_T(x,Tx))\right)\psi(M_T(x,Tx)),$$

where

$$M_T(x,Tx)=\max\{\omega_1(x,Tx),\omega_1(x,Tx),\omega_1(Tx,T^2x),\frac{\omega_2(x,T^2x)+\omega_2(Tx,Tx)}{2}\},$$

holds for all $x\in X_\omega$ with, $\omega_1(Tx,T^2x)>0$,

2. *There exists $x_0\in X_w$ such that $\alpha(x_0,Tx_0)\geq 1$,*
3. T is a triangular α −admissible mapping,
4. *If $\{x_n\}$ is a sequence in X_w such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$ and $\omega_1(x_n,x)\to 0$, as $n\to\infty$, then $\omega_1(Tx_n,Tx)\to 0$ as $n\to\infty$.*
5. *If $t\in Fix(T^n)$ and $t\notin Fix(T)$, then $\alpha(T^{n-1}t,T^nt)\geq 1$.*

Then T has property (P).

Proof Let $x_0 \in X_w$ such that $\alpha(x_0,Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_n = Tx_{n-1}$. By the hypothesis (2) of the Theorem 4.2 and using (3.2), we obtain

$$\alpha(x_n,x_{n+1}) \geq 1,$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is a fixed point of T and the proof is finished. Therefore, we suppose $x_n \neq x_{n+1}$ or $\omega_1(Tx_{n-1},T^2x_{n-1}) > 0$ for all $n \in \mathbb{N}$. In view of (3.2) and (3.3), we have

$$\psi(\omega_1(x_n,x_{n+1})) \leq \gamma(\psi(M_T(x_{n-1},Tx_{n-1})))\psi(M_T(x_{n-1},Tx_{n-1}))$$

By using a similar reasoning as in the proof of Theorem 3.6, we get that the sequence $\{x_n\}$ is a ω −Cauchy sequence. Hence, the ω −completeness of X_w ensures that there exists $x_\iota \in X_\omega$ such that $x_n \to x_\iota$ as $n \to \infty$. The hypothesis (4) of Theorem 4.2, we get $\omega_1(x_{n+1},Tx_\iota) = \omega_1(Tx_n,Tx_\iota) \to 0$ as $n \to \infty$, that is $x_\iota = Tx_\iota$. Hence, T has a fixed point and $Fix(T^n) = Fix(T)$ is true for $n = 1$. Let $n > 1$ and assume that $t \in Fix(T^n)$ and $t \notin Fix(T)$ such that $w_1(t,Tt) > 0$. From the hypothesis (1) of Theorem 4.2, we get

$$\begin{aligned}
&\psi(\omega_1(t,Tt)) \\
&= \psi(\omega_1(T(T^{n-1}t),T^2(T^{n-1}t)) \\
&\leq \alpha(T^{n-1}t,T(T^{n-1}t))\beta(T^{n-1}t,T(T^{n-1}t)) \\
&\quad \psi(\omega_1(T(T^{n-1}t),T^2(T^{n-1}t))) \\
&\leq \gamma\left(\psi\left(M_T(T^{n-1}t,T(T^{n-1}t))\right)\right) \\
&\quad \psi(M_T(T^{n-1}t,T(T^{n-1}t))),
\end{aligned}$$

where

$$\begin{aligned}
&M_T(T^{n-1}t,T(T^{n-1}t)) \\
&= \max\{\omega_1(T^{n-1}t,T(T^{n-1}t)), \\
&\quad \omega_1(T^{n-1}t,T(T^{n-1}t)), \\
&\quad \omega_1(T(T^{n-1}t),T^2(T^{n-1}t)), \\
&\quad \tfrac{\omega_2(T^{n-1}t,T^2(T^{n-1}t))+\omega_2(T(T^{n-1}t),T(T^{n-1}t))}{2}\}. \\
&= \max\{\omega_1(T^{n-1}t,T(T^{n-1}t)), \\
&\quad \omega_1((T(T^{n-1}t),T^2(T^{n-1}t))\}.
\end{aligned}$$

For $\gamma \in$ F, and if

$$\max\{\omega_1(T^{n-1}t,T(T^{n-1}t)), \omega_1(T(T^{n-1}t),T^2(T^{n-1}t))\} = \omega_1(t,Tt),$$

then, we obtain

$$\psi(\omega_1(t,Tt)) \leq \gamma(\psi(M_T(T^{n-1}t,T(T^{n-1}t))))\psi(\omega_1(t,Tt)),$$

which is a contradiction. Thus, we deduce that $\omega_1(t,Tt) = 0$. If

$$\max\{\omega_1(T^{n-1}t,T(T^{n-1}t)), \omega_1(T(T^{n-1}t),T^2(T^{n-1}t))\} = \omega_1(T^{n-1}t,T(T^{n-1}t)),$$

then, we write

$$\begin{aligned}
&\psi(\omega_1(t,Tt)) \\
&\leq \gamma\left(\psi\left(M_T(T^{n-1}t,T(T^{n-1}t))\right)\right) \\
&\quad \psi(\omega_1(T^{n-1}t,T(T^{n-1}t))) \\
&\leq \psi(\omega_1(T^{n-1}t,T(T^{n-1}t))).
\end{aligned}$$

Since ψ is strictly increasing, we have

$$\omega_1(T(T^{n-1}t),T^2(T^{n-1}t)) \leq \omega_1(T^{n-1}t,T(T^{n-1}t)).$$

Thus, $\omega_1(T(T^{n-1}t),T^2(T^{n-1}t))$ is positive and nonincreasing. So, there exists $r \geq 0$ such that $\lim_{n\to\infty}\omega_1(T^{n-1}t,T(T^{n-1}t)) = r$. Let us show that $r = 0$. To the contrary, assume that $r > 0$. Using the similar process of (3.6) and (3.7), we have that $\lim_{n\to\infty}\gamma(\psi(M_T(T^{n-1}t,T(T^{n-1}t)))) = 1$. Since $\gamma \in$ F, we obtain

$$\lim_{n\to\infty}\psi(M_T(T^{n-1}t,T(T^{n-1}t))) = 0.$$

In the sequel, we obtain $r = \lim_{n\to\infty}\omega_1(T^{n-1}t, T(T^{n-1}t)) = 0$, and so,

$$\lim_{n\to\infty}\omega_1(T(T^{n-1}t), T^2(T^{n-1}t)) = 0,$$

which is a contradiction. Thus, we make an inference that $\omega_1(t, Tt) = 0$. Therefore,

$Fix(T^n) = Fix(T)$ for all $n \in \mathbb{N}$ and T has property (P).

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