



Exponential Growth Of Solutions Of Higher-Order Viscoelastic Wave Equation With Logarithmic Term

Erhan Pişkin¹  and Nazlı Irkıl^{2*} 

¹Dicle University, Department of Mathematics, Diyarbakır, Turkey,

²Mardin Said Nursi Anatolian High School, Mardin, Turkey,

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Abstract

This paper deals with a higher-order viscoelastic wave equation with logarithmic source term. We prove, for suitable conditions, the exponential growth of solutions.

Keywords: Higher-order equation, Growth, Logarithmic nonlinearity.

Logaritmik Kaynak Terimli Yüksek Mertebeden Viskoelastik Dalga Denkleminin Çözümlerinin Üstel Büyümesi

Özet

Bu çalışma logaritmik kaynak terimli yüksek mertebeden viskoelastik dalga denklemi ile ilgilidir. Uygun koşullar altında çözümlerin üstel büyümesini ispatladık.

Anahtar Kelimeler: Yüksek mertebeden denklem, Büyüme, Logaritmik terim.

1. Introduction

In this paper, we deal with the exponential growth of solutions of the following higher-order viscoelastic wave equation with logarithmic term.

$$\begin{cases} u_{tt} + \Delta^2 u_{tt} + (-\Delta)^m u + u \\ - \int_0^t g(t-s) (-\Delta)^m u ds = k \ln|u|, & x \in \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, t) = \frac{\partial^i}{\partial v^i} u(x, t) = 0, \quad i = 1, 2, \dots, m-1, & x \in \partial\Omega \times (0, T), \end{cases} \quad (1)$$

where $\Omega \subset R^3$ is a bounded domain with smooth boundary $\partial\Omega$, v is the unit outer

normal, k is nonnegative constants to be chosen later and $P = (-\Delta)^m$, ($m \in N$). The kernel g has some conditions to be specified later.

The logarithmic nonlinear source term is related with many branches of physics. Cause of this is interest in it occurs naturally in inflation cosmology and supersymmetric field theories, quantum mechanics, nuclear physics (Bialynicki-Birula et al., 1975; Buljan et al., 2003; Martino et al., 2003; Gorke, 2009). A lot of authors studied hyperbolic type equation

*Corresponding Author: nazliirkil@gmail.com

with logarithmic term. (see; Bartkowski et al., 2008; Han, 2013; Messaoudi et al., 2017a and 2017b, Pişkin et al. 2019a and 2019b).

In 1970, the working of Dafermos with viscoelastic term provide a basis to the different papers (Dafermos, 1970). The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastic industry. Additionally, viscoelasticity influence part on working of biological phenomena. The other important property of viscoelastic material that return back to its original size after a impact force cut off (Cavalcanti, 2002).

When $m = 2$, problem (1) becomes the following

$$|u_t|^\rho u_{tt} + \Delta^2 u_{tt} + \Delta^2 u + u - \int_0^t g(t-s) \Delta^2 u ds = kuln|u| \quad (2)$$

Al-Gharabli et al. investigated the local existence, global existence and stability for problem (2) (Al-Gharabli et al.,). Later, they proved the existence and decay results of problem (2) for $\rho = 0$ and absence $\Delta^2 u_{tt}$ term (Al-Gharabli et al., 2019).

Peyravi consider

$$u_{tt} - \Delta u + u - \int_0^t g(t-s) \Delta u ds + h(u_t)u_t + |u|^2 u = uln|u|^k \quad (3)$$

in $\Omega \subset R^3$ with $h(s) = k_0 + k_1|s|^{m-1}$ (Peyravi 2018). He studied decay estimate and exponential growth of solutions for problem (3).

Motivated by the above studies, we asked that what results will be obtained if one revises the Laplace operator by other high order viscoelastic term. Additionally, to the

best of our knowledge, the growth of solution for the higher-order viscoelastic wave equation with logarithmic source term have not been well studied. Then, we established the exponential growth of the solution for problem (3).

The rest of our work is organized as follows. In section 2, we gave some notations and lemmas which will be used throughout this paper. In section 3, we get suitable conditions for growth of the solutions the problem.

2. Preliminaries

In this part, we present some notations and lemmas which will be used throughout this paper. We denote $\|\cdot\|_2 = \|\cdot\|$ and $\|\cdot\|_p$ the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. We take by C and C_i ($i = 1, 2, \dots$) various positive constants.

Now, we give some important lemmas:

Lemma 1. (Gross,1975) (Logarithmic Sobolev Inequality). Let u be any function $u \in H_0^1(\Omega)$, $\Omega \subset R^3$ be a bounded smooth domain and $a > 0$ be any number. Then,

$$\int_\Omega \ln|u| u^2 dx \leq \left(\frac{3}{4} \ln \frac{4a}{e}\right) \|u\|^2 + \frac{a}{4} \|\nabla u\|^2 + \ln \|u\| \|u\|^2$$

Corollary 2. Let u be any function $u \in H_0^m(\Omega)$, $\Omega \subset R^3$ be a bounded smooth domain and $a > 0$ be any number and where c_p is the smallest positive number satisfying

$$\|\nabla u\|^2 \leq c_p \left\| |u|^{\frac{1}{2}} \right\|^2, \forall u \in H_0^m(\Omega).$$

Then, we obtain

$$\int_{\Omega} \ln|u| u^2 dx \leq \left(\frac{3}{4} \ln \frac{4a}{e}\right) \|u\|^2 + \frac{c_p a}{4} \|P^{\frac{1}{2}}u\|^2 + \ln \|u\| \|u\|^2 \quad (4)$$

Now, we present following assumptions:

(A1) $g: R^+ \rightarrow R^+$ is a C^1 nonincreasing function satisfying

$$g(s) \geq 0, \int_0^{\infty} g(s) ds < \infty, 1 - \int_0^{\infty} g(s) ds = l_0 > 0, \quad (5)$$

Lemma 3. The energy functional $E(t)$ is decreasing with respect to t . Where

$$\begin{aligned} E(t) &= \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u_t\|^2 \\ &+ \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \left\|P^{\frac{1}{2}}u\right\|^2 \\ &+ \frac{1}{2} g \circ P^{\frac{1}{2}}u + \frac{k+2}{4} \|u\|^2 - \frac{1}{2} \int_{\Omega} \ln|u|^k u^2 dx. \end{aligned} \quad (6)$$

Proof. We multiply both sides of (1) by u_t and then integrating over Ω , we have

$$\begin{aligned} &\int_{\Omega} u_{tt} u_t dx + \int_{\Omega} \Delta_{tt} \Delta u_t dx + \int_{\Omega} P^{\frac{1}{2}}u P^{\frac{1}{2}}u_t dx \\ &- \int_{\Omega} \int_0^t g(t-s) (-\Delta)^m u ds + \int_{\Omega} u u_t dx \\ &= \int_{\Omega} \ln|u|^k u u_t dx, \\ &\frac{d}{dt} \left(\frac{1}{2} \|u_t\|^2\right) + \frac{d}{dt} \left(\frac{1}{2} \|\Delta u_t\|^2\right) \\ &+ \frac{d}{dt} \left[\frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \left\|P^{\frac{1}{2}}u\right\|^2\right] \\ &+ \frac{d}{dt} \left(\frac{1}{2} g \circ P^{\frac{1}{2}}u\right) + \frac{d}{dt} \left(\frac{k+2}{4} \|u\|^2\right) \\ &- \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} \ln|u|^k u^2 dx\right) \\ &= \frac{1}{2} \left[g' \circ P^{\frac{1}{2}}u - g(t) \left\|P^{\frac{1}{2}}u\right\|^2\right], \end{aligned} \quad (7)$$

$$E'(t) = \frac{1}{2} \left[g' \circ P^{\frac{1}{2}}u - g(t) \left\|P^{\frac{1}{2}}u\right\|^2\right] \leq 0 \quad (8)$$

Next, we define the following functional on $H_0^m(\Omega)$,

$$\begin{aligned} I(t) &= \left(1 - \int_0^t g(s) ds\right) \left\|P^{\frac{1}{2}}u\right\|^2 + g \circ P^{\frac{1}{2}}u \\ &+ \|u\|^2 - \int_{\Omega} \ln|u|^k u^2 dx, \end{aligned}$$

$$J(t) = \frac{k}{4} \|u\|^2 + \frac{1}{2} I(t),$$

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u_t\|^2 + J(t) \quad (9)$$

where

$$g \circ P^{\frac{1}{2}}u = \int_0^t g(t-s) \left\|P^{\frac{1}{2}}u(t) - P^{\frac{1}{2}}u(s)\right\|^2 dt.$$

3. Exponential Growth

In this part, we will prove exponential growth of solutions to problem (1). Now, we introduce the following

$$C_0 = 1 - \frac{3k}{4} \ln\left(\frac{4a}{e}\right), \gamma^* = e^{\frac{2C_0-k}{2k}}, E_1 = \frac{k}{4} (\gamma^*)^2 \quad (10)$$

Firstly we suppose that

$$(A2) \quad 0 < \alpha < \min\left\{\frac{4l_0}{k c_p}, \frac{1}{4} e^{\frac{3k+4}{3k}}\right\} \text{ and } k \geq 1.$$

Theorem 4. Suppose that (A1), (A2) hold. Assume further $\|u(x, 0)\|_2 > \gamma^*$ and $0 < E(0) < E_1$. Then the solution (1) grow up as an exponential function when $t \rightarrow \infty$.

Now, we give following two lemmas needed our proof for the theorem.

Lemma 5. Suppose that $I(t) < 0$ for all $t > 0$. Then $\|u(x, t)\|_2 > \gamma^*$ for all $t > 0$.

Proof. By using of (9) and Logarithmic Sobolev inequality, we obtain

$$\begin{aligned}
 J(t) &\geq I(t) \geq \left(1 - \int_0^t g(s) ds\right) \left\|P^{\frac{1}{2}}u\right\|^2 \\
 &\quad + \|u\|^2 - \int_{\Omega} \ln|u|^k u^2 dx \\
 &\geq \left(l_0 - \frac{kc_p \alpha}{4}\right) \left\|P^{\frac{1}{2}}u\right\|^2 \\
 &\quad - \left(1 - \frac{3k}{4} \ln \frac{4a}{e} - k \ln \|u\|\right) \|u\|^2. \quad (11)
 \end{aligned}$$

Using the condition in (A2) and definition of C_0 in (10), we get

$$I(t) > (C_0 - k \ln \|u\|) \|u\|^2. \quad (12)$$

By the negativity of $I(t)$ and using again (A2) we have

$$\|u(x, t)\|^2 > e^{\frac{C_0}{k}} > e^{\frac{2C_0 - k}{k}} = \gamma^*, \forall t \geq 0.$$

Lemma 6. Suppose that $\|u(x, 0)\|_2 > \gamma^*$ and $0 < E(0) < E_1$. Then $I(t) < 0$ for all $t > 0$.

Proof. We will use contradiction method in our proof. Firstly our aim to see that $I(0) < 0$. If not suppose that $I(0) \geq 0$. Then

$$E(0) \geq J(0) \geq \frac{k}{4} \|u\|^2 + \frac{1}{2} I(0) \geq \frac{k}{4} (\gamma^*)^2 = E_1, \quad (13)$$

which is impossible. Now suppose that $t_0 = \inf\{\tau > 0: I(\tau) = 0\}$. Hence $t_0 > 0$. So, $I(t) < 0$ over $(0, t_0)$ and by continuity $I(t_0) = 0$. Then, by Lemma 5 we have hence we have $\|u(x, t)\|_2 > \gamma^*$ for all $0 < t < t_0$ and from the continuity of $t \rightarrow \|u(x, t)\|_2^2$ we have $\|u(x, t_0)\|^2 \geq \gamma^*$. Using of (9) we have

$$\begin{aligned}
 E(t_0) &= \frac{1}{2} \|u_t(x, t_0)\|^2 + \frac{1}{2} \|\Delta u_t(x, t_0)\|^2 + J(t_0) \\
 &> \frac{k}{4} \|u(x, t_0)\|^2 + \frac{1}{2} I(t_0) \\
 &\geq \frac{k}{4} (\gamma^*)^2 = E_1, \quad (14)
 \end{aligned}$$

this is a contradiction.

Remark 7. By Lemma 5 and Lemma 6 we inferred that if $\|u(x, 0)\|_2 > \gamma^*$ and $0 < E(0) < E_1$ then $\|u(x, t)\|^2 > \gamma^*$ for all $t > 0$.

Now, we can state proof of the Theorem 4.

Proof. Let us to define

$$L(t) = E_1 + \varepsilon H(t), \quad (15)$$

and

$$H(t) = 2 \int_{\Omega} u_t(t) u(t) dx + 2 \int_{\Omega} \Delta^2 u_t(t) \Delta u(t) dx, \quad (16)$$

where $\varepsilon > 0$ is a sufficiently small number to be determined later.

By differentiating with respect to t , we obtain

$$\begin{aligned}
 H'(t) &= 2 \int_{\Omega} (u_{tt} u + u_t^2) dx + 2 \int_{\Omega} (\Delta^2 u_{tt} u + \Delta^2 u_t u_t) dx, \\
 &= 2 \int_{\Omega} (-\Delta^2 u_{tt} - P u - \int_0^t g(t-s) P u ds - u + k \ln|u|) u dx \\
 &\quad + 2 \int_{\Omega} \Delta^2 u_{tt} u + 2 \|u_t\|^2 + 2 \|\Delta u_t\|^2, \\
 &= 2 \|u_t\|^2 + 2 \|\Delta u_t\|^2 - 2 \|u\|^2 - 2 \left\|P^{\frac{1}{2}}u\right\|^2 \\
 &\quad - 2 \int_{\Omega} P^{\frac{1}{2}}u \int_0^t g(t-s) P^{\frac{1}{2}}u ds + 2k \int_{\Omega} u^2 \ln|u| dx. \quad (17)
 \end{aligned}$$

By the using of Young's inequality for all $\varepsilon > 0$,

$$\begin{aligned}
 \int_{\Omega} P^{\frac{1}{2}}u \int_0^t g(t-s) P^{\frac{1}{2}}u ds &\leq \frac{\delta}{2} \left\|P^{\frac{1}{2}}u(t)\right\|^2 \\
 &\quad + \frac{1}{2\delta} \left(\int_0^t g(s) ds\right) \int_{\Omega} \int_0^t g(t-s) \left|P^{\frac{1}{2}}u(t) - P^{\frac{1}{2}}u(s)\right|^2 ds dx, \\
 &\leq \frac{\delta}{2} \left\|P^{\frac{1}{2}}u(t)\right\|^2 + \frac{1}{2\delta} \left(\int_0^t g(s) ds\right) g \circ P^{\frac{1}{2}}u(t). \quad (18)
 \end{aligned}$$

Insert (18) into (17), we have

$$\begin{aligned}
 H'(t) &\geq 2 \|u_t\|^2 - 2 \left\|P^{\frac{1}{2}}u\right\|^2 - 2 \|u\|^2 + 2 \|\Delta u_t\|^2 \\
 &\quad - \delta \left\|P^{\frac{1}{2}}u(t)\right\|^2 - \frac{1}{\delta} \left(\int_0^t g(s) ds\right) g \circ P^{\frac{1}{2}}u(t) \\
 &\quad + 2k \int_{\Omega} u^2 \ln|u| dx.
 \end{aligned}$$

$$\begin{aligned} &\geq 2\|u_t\|^2 + 2\|\Delta u_t\|^2 - (2 + \delta)\|P^{\frac{1}{2}}u\|^2 - 2\|u\|^2 \\ &\quad - \frac{1}{\delta} \left(\int_0^t g(s) ds \right) g \circ P^{\frac{1}{2}}u(t) + 2k \int_{\Omega} u^2 \ln|u| dx. \end{aligned} \quad (19)$$

Adding and subtracting $\eta I(t)$ into (19), we have

$$\begin{aligned} H'(t) &\geq 2\|u_t\|^2 + 2\|\Delta u_t\|^2 + \left(\eta - \frac{1}{\delta} \left(\int_0^t g(s) ds \right) \right) g \circ P^{\frac{1}{2}}u(t) \\ &\quad + [\eta - 2]\|u\|^2 + k(2 - \eta) \int_{\Omega} u^2 \ln|u| dx. \\ &\quad + [\eta(1 - \int_0^t g(s) ds) - (2 + \delta)] \|P^{\frac{1}{2}}u\|^2 - \eta I(t), \end{aligned} \quad (20)$$

where $\eta > 2$. For estimating the fifth term (20), we use Corollary 2, we obtain

$$\begin{aligned} \int_{\Omega} \ln|u| u^2 dx &\leq \left(\frac{3}{4} \ln \frac{4a}{e} \right) \|u\|^2 \\ &\quad + \frac{c_p a}{4} \|P^{\frac{1}{2}}u\|^2 + \ln \|u\| \|u\|^2 \end{aligned} \quad (21)$$

By the combining (20) and (21) we have

$$\begin{aligned} H'(t) &\geq 2\|u_t\|^2 + 2\|\Delta u_t\|^2 \\ &\quad + \left(\eta - \frac{1}{\delta} \left(\int_0^t g(s) ds \right) \right) g \circ P^{\frac{1}{2}}u(t) - \eta I(t) \\ &\quad + k(2 - \eta) \left[\left(\frac{3}{4} \ln \frac{4a}{e} \right) \|u\|^2 + \frac{c_p a}{4} \|P^{\frac{1}{2}}u\|^2 + \ln \|u\| \|u\|^2 \right] \\ &\quad + [\eta - 2]\|u\|^2 + [\eta(1 - \int_0^t g(s) ds) - (2 + \delta)] \|P^{\frac{1}{2}}u\|^2 \\ &\geq 2\|u_t\|^2 + 2\|\Delta u_t\|^2 + \left(\eta + \frac{1}{\delta} - \frac{l_0}{\delta} \right) g \circ P^{\frac{1}{2}}u(t) \\ &\quad + (2 - \eta) \left[\left(\frac{3k}{4} \ln \frac{4a}{e} + k \ln \|u\| \right) - 1 \right] \|u\|_2^2 \\ &\quad + [\eta l_0 - (2 + \delta) + k(2 - \eta) \frac{c_p a}{4}] \|P^{\frac{1}{2}}u\|^2 - \eta I(t). \end{aligned} \quad (22)$$

Using the condition in (A2) and definition of C_0 in (10), we can select η large so that

$$\eta > \max \left\{ 2, \frac{l_0 - 1}{\delta}, \frac{8 - 8k + 4\delta}{4l_0 - kc_p a} \right\}$$

Consequently because of Lemma 6, for some $c_i > 0$. ($i = 0, 1, 2 \dots$), we can write (22)

$$\begin{aligned} H'(t) &\geq 2\|u_t\|^2 + 2\|\Delta u_t\|^2 + c_0 \|u\|^2 \\ &\quad + c_1 (g \circ P^{\frac{1}{2}}u)(t) + c_2 \|P^{\frac{1}{2}}u\|^2. \end{aligned} \quad (23)$$

Thus, we have

$$L'(t) = \varepsilon H'(t) > 0.$$

Hence, L is an increasing function on $(0, \infty)$ and taking ε small enough such that $L(0) > 0$ so that $L(t) > 0$ for all $t \geq 0$.

On the other hand, by using the Hölder, Young inequalities and Poincare inequalities in (15) and then by Remark 7 and (23) we obtain

$$\begin{aligned} L(t) &\leq \frac{k}{4} (\gamma^*)^2 + 2\varepsilon \|u\|^2 \\ &\quad + \varepsilon \|u_t\|^2 + 2\varepsilon \|\Delta u\|^2 + \varepsilon \|\Delta u_t\|^2 \\ &\leq \frac{k}{4} \|u\|^2 + 2\varepsilon \|u\|^2 + \varepsilon \|u_t\|^2 \\ &\quad + 2\varepsilon \|\Delta u\|^2 + \varepsilon \|\Delta u_t\|^2 \\ &\leq \left(\frac{k}{4} + 2\varepsilon \right) \|u\|^2 + \varepsilon \|u_t\|^2 \\ &\quad + 2\varepsilon \|P^{\frac{1}{2}}u\|^2 + \varepsilon \|\Delta u_t\|^2 \\ &\leq \vartheta(\varepsilon, k) L'(t). \end{aligned} \quad (24)$$

Integrate (24), yields

$$L(t) > e^{\frac{t}{\vartheta(\varepsilon, k)}} L(0).$$

This completes the our proof.

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