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Research Article

Convergence Theorems with a Faster Iteration Process for Suzuki's Generalized Non-expansive Mapping with Numerical Examples

Osman Alagöz[1*] Birol Gündüz[2*] Sezgin Akbulut[3*]

[1]Bilecik Şeyh Edebali University, Department of Mathematics, Bilecik, Turkiye
[2]Erzincan University, Department of Mathematics, Erzurum, Turkiye
[2]Atatürk University, Department of Mathematics, Erzurum, Turkiye

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Abstract
In this paper firstly, we compared rates of convergences of some iteration processes which converge faster than Picard, Mann, Ishikawa and S-iteration processes. Then, we proved some strong and weak convergence theorems for the fastest iteration process for Suzuki's generalized non-expansive mapping in Banach spaces. We also supported our theoretical findings via numerical examples.

Keywords: Suzuki's non-expansive mapping, rate of convergence, fixed point

Öz

Bu makalede ilk olarak Picard, Mann, Ishikawa ve S-iterasyonlarından daha hızlı yakınsayan bazı iterasyonların yakınsaklık hızlarını karşılaştırdık. Daha sonra en hızlı iterasyon ile Banach uzaylarında Suzuki'nin genelleştirilmiş genişlemeyen dönüşümü için bazı güçlü ve zayıf yakınsaklık teoremleri ispatladık. Aynı zamanda teorik bulgularımızı nümerik örneklerle de destekledik.

Anahtar Kelimeler: Suzuki'nin genişlemeyen dönüşümü, yakınsaklık hızı, sabit nokta

1. Introduction

Fixed point theory for nonaxpansive mappings has been one of the most attractive area for many researchers for the last five decades. A nonexpansive mapping defined as:

Let C be a nonempty closed convex subset of Banach space X, a mapping $T: C \to C$ is said to be a nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

Corresponding Author: osman.alagoz@bilecik.edu.tr

If there exists a $x \in C$ such that $Tx = x$, then x is called a fixed point of T. In case of T has at least one fixed point, then it can be obtained by approximating fixed point by a certain iteration scheme. For instance, if T satisfies Banach contraction principle, the fixed points of T can be easily obtained by Picard iteration. One of the main conclusions which guarantees the existence of a fixed point was given by S. Banach in 1922 which is also called the Banach contraction principle and given as follows:

Theorem 1.1 Let (X, d) be a complete metric space and $T: X \to X$ be a contraction mapping, i.e., a mapping for which there exists a constant $k \in [0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point $x *\in X$ and the iteration of Picard converges to the fixed point $x *$. Moreover, the error estimation is given by;

$$d(T^n x, x^*) \leq \frac{k^n}{1-k} d(x, Tx) \tag{1.1}$$

for each $x \in X$.

Once T is a nonexpansive mapping, Picard iteration may fail to converge to a fixed point of T. In this sence, many authors worked on different iteration processes either for different classes of mappings of different spaces. During the years, many iteration processes have been searched or defined to approximate fixed point of contraction mappings, as the Picard (1890), the Mann (1953), the Ishikawa (1974) and the Noor (2000) iteration processes are defined respectively as;

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = Tx_n \end{cases} \tag{1.2}$$

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \end{cases} \tag{1.3}$$

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \tag{1.4}$$

and

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)x_n + \beta_n T z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

Some authors used the following definitions given by Berinde (2004) to compare the rates of convergence of the iteration process mentioned above.

2. Material and Methods

First we give two useful definitions that are used to determine the faster iteration which converge to the same point. The following definitions about the rate of convergence are given by (Berinde, 2004).

Definition 2.1. Let $\{a_n\}, \{b_n\}$ be two sequences of real numbers converging to a and b, resprectively. If $\lim_{n\to\infty} \frac{|a_n - b|}{|b_n - b|} = 0$, then $\{a_n\}$ converges faster than $\{b_n\}$.

Definition 2.2. Let p is a fixed point and let $\{x_n\}$ and $\{u_n\}$, both converging to p, the error estimates

$$\|x_n - p\| \leq a_n, \quad \forall n \geq 1,$$
$$\|u_n - p\| \leq b_n, \quad \forall n \geq 1,$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{x_n\}$ converges faster than $\{u_n\}$ to p.

Agarwal et al. (2007) desired to define a faster iteration and introduced the S-iteration process as follows;

$$(1.6)\qquad \begin{cases} x_1 = x \in C \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and they showed the S-iteration process converges at a same rate as Picard iteration and faster than Mann and Ishikawa iteration for contractions. From now on, researchers payed attention to produce new faster iterations.

S. H. Khan (2013) introduced the normal S-iteration as follows;

$$(1.7)\qquad \begin{cases} x_1 = x \in C \\ x_{n+1} = T((1-\alpha_n)x_n + \alpha_n Tx_n) \end{cases}$$

where $\{\alpha_n\}$ is in $(0,1)$ and showed that the normal S-iteration converges at a rate than al of Picard, Mann and Ishikawa iterative processes for contractions.

Abbas and Nazir (2014) introduced the following iteration; for an Arbitrary $x_0 \in C$ constructed $\{x_n\}$ by

$$(1.8)\qquad \begin{cases} x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$. They also showed that this iteration process is faster than S-iteration process. In the same year, Thakur et al. (2014) gave a new iteration by getting inspired the iteration of (1.8) where the sequence $\{x_n\}$ is generated iteratively by $x_1 \in C$ and

$$(1.9)\qquad \begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$ and they proved that (1.9) is faster than Picard, Mann, Ishikawa, Noor, Agarwal et al. and Abbas et al. iteration processes for contractive mappings in the sense of (Berinde, 2004).

After this improvement, Kadioglu and Yildirim (2015) defined a new iteration as follows;

$$(1.10)\qquad \begin{cases} x_1 = x \in C \\ x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are in $(0,1)$ and they proved that this iteration process is faster than ever the S-iteration process or the normal S-iteration process.

On the other hand, Thakur et al. (2016) introduced a new modified iteration process for non-expansive mappings, where the sequence $\{x_n\}$ is generated iteratively by $x_1 \in C$ and

$$(1.11)\qquad \begin{cases} x_{n+1} = (1-\alpha_n)Tz_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$. They also proved the iteration (1.11) converges faster than Picard, Mann, Ishikawa, Noor, Agarwal et al., Abbas et al. iteration process, for contractive mappings.

On the other hand, the mapping T which is mentioned in the Theorem (1) is forced to be continuous on X by the contractive condition.

In this sense a question appears. Is there a condition which does not force T to be continuous? Kannan (1968) replied this open problem by determining a new condition for mappings that need not to be continuous; there exists $k \in [0, \frac{1}{2})$ such that

(1.12) $$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$.

Zamfirescu (1972) introduced a theorem as follows

Theorem 2.3. Let (X, d) be a metric space and $T: X \to X$ be a zamfirescu mapping, i.e., there exists the real numbers a, b amd c satisfying $a \in [0,1)$ and $b, c \in [0, \frac{1}{2})$ such that for each $x, y \in X$ at least one of the following is true;

Z1: $d(Tx, Ty) \leq ad(x, y)$
Z2: $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$
Z3: $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$

then T has a unique fixed point x^* and the Picard iteration $\{x_n\}$ converges to x^* for Arbitrary but fixed $x_0 \in X$.

Then again Sintunevarat and Pitea (2016) introduced a new iteration Scheme for nonlinear Zamrescu mapping and showed that this iteration process is faster than the S-iteration under a sufficient condition. The iteration is as follows;

(1.13) $$\begin{cases} x_1 = x \in C \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n \\ z_n = (1 - \gamma_n)x_n + \gamma_n T y_n \\ x_{n+1} = (1 - \alpha_n) T z_n + \alpha_n T y_n \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real conrol sequences in the interval $[0,1]$.

Noor (2000) presented the following process for any fixed $x_0 \in C$, construct $\{x_n\}$ by

(1.14) $$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T z_n \\ z_n = (1 - \gamma_n)x_n + \gamma_n T x_n \end{cases}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real control sequences in the interval $(0,1)$.

To sum up, there are four special iteration processes which are given by Kadioglu and Yildirim (1.10), Sintunavarat and Pitea (1.13), Thakur et al. (1.9) and Thakur et al. (1.11). These are said to be faster than other iteration processes like Picard, Mann, Ishikawa, Agarwal et al., Noor and Abbas et al. But till now, it is not studied that which iteration process is the fastest one? In this paper, we introduce the fastest iteration between these four challneging iterations for contractive mappings in the sense of Berinde(2004).

We choose the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be equal to a constant $\lambda \in (0,1)$ to compare the rates of convergence more easily and we Show that Kadioglu and yildirim iteration (1.10) is faster than other hree iteration under a condition with a numerical example. Finally we prove some convergence theorems by using the iteration of (1.10) Suzuki's generalized nonexpansive mappings.

3. Research Findings

In this section we compare the rates of convergence of the iterations and to support our claim we give a numerical example.

Theorem 3.1. Let C be a non empty closed convex subset of a normed space E. Let T be a contraction with a contraction factor $k \in (0,1)$ and fixed point p. Let $\{\alpha_n\}$ be defined by the iteration (1.10), $\{b_n\}$ by (1.9) $\{c_n\}$ by (1.11) and $\{d_n\}$ by (1.13), where $\{\alpha_n\}$, $\{b_n\}$ and $\{\gamma_n\}$ are in $[\lambda, 1 - \lambda]$ for all

$n \in \mathbb{N}$ and for some $\lambda \in (0,1)$, then the followings are satisfied.

(1) $\{a_n\}$ converges faster than $\{c_n\}$
(2) $\{c_n\}$ converges faster than $\{d_n\}$
(3) $\{b_n\}$ converges faster than $\{d_n\}$
(4) $\{a_n\}$ converges faster than $\{b_n\}$ and $\{c_n\}$ converges faster than $\{b_n\}$ if

$$(1+k)\lambda^2 < 1.$$

Proof. As proved in Theorem (5) of Kadioglu and Yildirim (2014),

$$\|a_{n+1} - p\| \le [k.(1-(1-k)\lambda)^2]^n \|a_1 - p\|$$

for all $n \in \mathbb{N}$. Let

$$a'_n = [k.(1-(1-k)\lambda)^2]^n \|a_1 - p\|$$

for all $n = 1,2,\ldots$

As proved in Theorem (2.3) of Thakur et al. (2016),

$$\|b_{n+1} - p\| \le k^n(1-(1-k^2)\alpha\beta\gamma)^n \|b_1 - p\|$$

Let $\alpha = \beta = \gamma = \lambda$, thus

$$b'_n = k^n(1-(1-k^2)\lambda^3)^n \|b_1 - p\|$$

for all $n = 1,2,\ldots$

As proved in Theorem (3.1) of Thakur et al. (2016),

$$\|c_{n+1} - p\| \le k^n(1-(1-k)\gamma)\text{^}n \|c_1 - p\|$$

Let $c'_n = k^n(1-(1-k)\gamma)\text{^}n \|c_1 - p\|$

for all $n = 1,2,\ldots$

Similarly as proved in Theorem (2.1) of Sintunavarat and Pitea ( ),

$$\|d_{n+1} - p\| = \{1-(1-k)\beta[\gamma - \alpha + \alpha.\gamma]\}^n \|d_1 - p\|$$

Let

$$d'_n = \{1-(1-k)\beta[\gamma - \alpha + \alpha.\gamma]\}^n \|d_1 - p\|$$

for all $n = 1,2,\ldots$

(1) $\frac{a'_n}{c'_n} = \frac{[k.(1-(1-k)\lambda)^2]^n \|a_1-p\|}{k^n(1-(1-k)\gamma)\text{^}n\|c_1-p\|} \to 0$

as $n \to \infty$. Thus $\{a_n\}$ converges faster than $\{c_n\}$ to p.

(2) $\frac{c'_n}{d'_n} = \frac{k^n(1-(1-k)\gamma)\text{^}n\|c_1-p\|}{\{1-(1-k)\beta[\gamma-\alpha+\alpha.\gamma]\}^n\|d_1-p\|} \to 0$

as $n \to \infty$. Thus $\{c_n\}$ converges faster than $\{d_n\}$ to p.

(3) $\frac{b'_n}{d'_n} = \frac{k^n(1-(1-k^2)\lambda^3)^n\|b_1-p\|}{(1-(1-k)\lambda^3)^n\|d_1-p\|} \to 0$

as $n \to \infty$. Thus $\{b_n\}$ converges faster than $\{d_n\}$ to p.

(4) Since $(1+k)\lambda^2 < 1$ we get

$$(1.14) \qquad (1-(1-k^2)\lambda^3 > 1-(1-k)\lambda)$$

So,

$$\frac{a'_n}{b'_n} = \frac{[k(1-(1-k)\lambda)^2]^n\|a_1-p\|}{k^n(1-(1-k^2)\lambda^3)^n\|b_1-p\|}$$
$$= (1-(1-k)\lambda)^n.\left[\frac{1-(1-k)\lambda}{1-(1-k^2)\lambda^3}\right]^n.\frac{\|a_1-p\|}{\|b_1-p\|}$$
$$\to 0$$

as $n \to \infty$. Thus $\{a_n\}$ converges faster than $\{b_n\}$ to p. Similarly by (1.14) we get,

$$\frac{c'_n}{b'_n} = \frac{[k(1-(1-k)\lambda)]^n\|c_1-p\|}{k^n(1-(1-k^2)\lambda^3)^n\|b_1-p\|}$$
$$= \left[\frac{(1-(1-k)\lambda)}{(1-(1-k^2)\lambda^3)}\right]^n.\frac{\|c_1-p\|}{\|b_1-p\|}$$
$$\to 0$$

this yields that $\{c_n\}$ converges faster than $\{b_n\}$ to p.

Conclusion 3.2. Under the condition (1.14), we compare the rates of convergence of the iterations of Kadioglu et al. (1.10), Thakur et al. (1.11), Thakur et al. (1.9) and Sintunavarat et al. (1.13), respectively. If the condition (1.14) fails, since $\frac{c'_n}{a'_n} \nrightarrow 0$ when $n \to \infty$. It can not be compared the rates of convergence of Kadioglu et al. (1.10) and Thakur et al. (1.11) by the definition of Berinde (2004). However, it can be obtain a partly sorting Thakur et al.

(1.9), Thakur et al. (1.11) and Sintunavarat et al. (1.13), respectively.

Now we support our theorem by giving a numerical example.

Example 3.3. Let $X = \mathbb{R}$ and $K = [0,100]$, Let $t: K \to K$ be a mapping defined by $Tx = \sqrt{x^2 - 5x + 20}$ with a contraction factor $k = \frac{39}{352}\sqrt{595}$ for all $x \in K$. Choose $x_1 = 40.00$ as initial value and $\alpha_n = \beta_n = \gamma_n = \lambda = 0.45$, $n = 1,2,\ldots$ our corresponding iteration process, the Kadioglu and Yildirim iteration process (1.10), the Thakur et al. iteration process (1.9), the Thakur et al. teration process (1.11) and the Sintunavarat and Pitea iteration process (1.13) are, respectively, given in Table (1).

All sequences converge to $p = 4$. Even the contraction factor k or λ values satisfy $(1+k)\lambda^2 < 1$, so by the Theorem (3.1), the Kadioglu and Yildirim iteration process is faster than the iterations Thakur et al. (1.11), Thakur et al. (1.9) and Sintunavarat and Pitea (1.13).

4. Results

In this section we introduce weak and strong convergence theorems of the iteration of Kadioglu and Yildirim for suzuki-generalized nonexpansive mappings in uniformly convex Banach spaces. First we give some definitions and useful lemmas.

Definition 4.1. (Suzuki,2008) Let C be a nonempty subset of a Banach space X, a mapping $T: C \to C$ is said to satisfy condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$.

Puengrattana (2011) proved some convergence theorems for such mappings using the Ihikawa iteration in uniformly convex Banach spaces and CAT(0) spaces.

Definiton 4.2. (Goebel and Kirk, 1990) A Banach space X is called uniformly convex if for each $\epsilon \in (0,2]$ there is a $\delta > 0$ such that for $x, y \in X$,

$$\left.\begin{array}{r}\|x\| \leq 1\\ \|y\| \leq 1\\ \|x-y\| > \epsilon\end{array}\right\} \Rightarrow \left\|\frac{x+y}{2}\right\| \leq \delta$$

Proposition 4.3. (Suzuki, 2008) Let C be a nonempty subset of a Banach space X and $T: C \to C$.

(1) If T is nonexpansive then T satisfies condition (C)
(2) If T satisfies condition (C) and has a fixed point, then T is a quasi nonexpansive mapping.
(3) If T satisfies condition (C), then $\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$

for all $x, y \in C$

Lemma 4.4. (Suzuki, 2008) Let T be a mapping on a subset of a Banach space X with the Opial property. Assume that T satisfies condition (C). If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tx = z$. That is, $I - T$ is demiclosed at zero.

Lemma 4.5. (Suzuki, 2008) Let C be a weakly compact convex subset of a uniformly convex Banach space X. Let T be a mapping on C. Assume that T satisfies condition (C). Then T has a fixed point.

Lemma 4.6. (Schu, 1991) Suppose that X is a uniformly convex Banach space and $0 \leq t_n \leq 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequence of X such that $limsup_{n\to\infty}\|x_n\| \leq r$, $limsup_{n\to\infty}\|y_n| \leq r$ and $limsup_{n\to\infty}\|t_n x_n + (1-t_n)y_n| = r$ hold for some $r \geq 0$. Then $limsup_{n\to\infty}\|x_n - y_n\| = 0$.

Let $\{x_n\}$ be a bounded sequence in a Banach space X and a nonempty closed convex subset of X. Let r be a continuous functional $r(.,\{x_n\}): X \to [0,\infty)$ given by

$$r(x,\{x_n\}) = limsup_{n\to\infty}\|x - x_n\|$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ according to C is given by

$$r(C,\{x_n\}) = \inf\{r(x,\{x_n\}): x \in C\}$$

The asymptotic center $A_C(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to C is the set

$$A_C(\{x_n\}) = \{x \in X: \ r(x,\{x_n\}) \le r(y,\{x_n\})$$

for all $y \in K$.

If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that in a uniformly convex Banach space, $A_C(\{x_n\})$ consists of exactly one point.

Lemma 4.7. (Mann, 1953) Let C be a nonempty closed subset of a uniformly convex Banach space X and $\{x_n\}$ be a bounded sequence in C such that $A(\{x_n\}) = \{y_n\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that

$$\lim_{n\to\infty} r(y_m,\{x_n\}) = \rho,$$

then $\lim_{m\to\infty} y_m = y$.

Lemma 4.8. Let X be a Banach space and $C \ne \emptyset$ be a closed convex subset of X and let $T: C \to C$ be a mapping satisfying condition (C) such that $F(T) \ne \emptyset$. Let the sequence $\{x_n\}$ be given in (1.10), then $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F(T)$.

Proof. Let $p \in F(T)$ and $z \in C$, since T satisfies condition (C),

$$\frac{1}{2}\|p - Tp\| = 0 \le \|p - z\|$$

İmplies that $\|Tp - Tz\| \le \|p - z\|$. From the Proposition (4.3) , we get

$$\begin{aligned}\|z_n - p\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &\le \|x_n - p\| \qquad (4.1)\end{aligned}$$

By (4.1), we obtain

$$\begin{aligned}\|y_n - p\| &= \|(1-\alpha_n)z_n + \alpha_n Tz_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\| \\ &\le \|z_n - p\| \\ &\le \|x_n - p\| \qquad (4.2)\end{aligned}$$

Next, by (4.29), we get

$$\begin{aligned}\|x_{n+1} - p\| &= \|Ty_n - p| \\ &\le \|y_n - p\| \\ &\le \|x_n - p\| \qquad (4.3)\end{aligned}$$

This implies that $\lim_{n\to\infty}\|x_n - p\|$ exists.

Theorem 4.9. Let X be a uniformly convex Banach space and $C \ne \emptyset$ be a closed convex subset of X and let $T: C \to C$ be a mapping satisfying condition (C). For any $x_1 \in C$, let the sequence $\{x_n\}$ be given in (1.10). Then $F(T) \ne \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Proof. Assume that $F(T) \ne \emptyset$ and let p is a fixed point of T. By Lemma (4.5), we say that $\lim_{n\to\infty}\|x_n - p\|$ exists and also the sequence $\{x_n\}$ is bounded. Set

$$\lim_{n\to\infty}\|x_n - p\| = r \qquad (4.4)$$

By (4.1), we get

$$limsup_{n\to\infty}\|z_n - p\| \le limsup_{n\to\infty}\|x_n - p\| \qquad (4.5)$$

from the Proposition (4.3), we obtain

$$limsup_{n\to\infty}\|Tx_n - p\| \le limsup_{n\to\infty}\|x_n - p\| \qquad (4.6)$$

Furthermore,

$$\begin{aligned}\|x_{n+1} - p\| &\le \|y_n - p\| \\ &\le (1-\alpha_n)z_n + \alpha_n\|Tz_n - p\| \\ &\le (1-\alpha_n)\|z_n - p\| + \alpha_n\|Tz_n - p\| \\ &\le \|z_n - p\| - \alpha_n\|z_n - p\| + \alpha_n\|Tz_n - p\| \\ &\le \|x_n - p\| - \alpha_n\|x_n - p\| + \alpha_n\|z_n - p\|\end{aligned}$$

so this inequality requires that

$$\frac{\|x_{n+1} - x_n\| - \|x_n - p\|}{\alpha_n} \le \|z_n - p\| - \|x_n - p\| \qquad (4.7)$$

since $\alpha_n \in (0,1)$ and by (4.7), we conclude,

$$\begin{aligned}\|x_{n+1} - p\| - \|x_n - p\| &\le \frac{\|x_{n+1} - x_n\| - \|x_n - p\|}{\alpha_n} \\ &\le \|z_n - p\| - \|x_n - p|\end{aligned}$$

thus, we obtain

$$\|x_{n+1}-p\| \le \|z_n - p\| \quad (4.8)$$

By (4.8), we have

$$r \le \liminf_{n\to\infty}\|z_n - p\| \quad (4.9)$$

now using (4.5) and (4.9), we obtain

$$\lim_{n\to\infty}\|(1-\beta_n)(x_n-p)+\beta_n(Tx_n-p)\| = \lim_{n\to\infty}\|z_n-p\| = r. \quad (4.10)$$

thus, by (4.4), (4.6), (4.10) and Lemma (4.6), we get $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Now we prove the sufficient part. To do this, let suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. By the Proposition (4.3) and the concept of asymptotic center we have,

$$\begin{aligned} r(Tp,\{x_n\}) &= \limsup_{n\to\infty}\|x_n - Tp\| \\ &\le \limsup_{n\to\infty}(3\|Tx_n - x_n\| + \|x_n - p|) \\ &= \limsup_{n\to\infty}\|x_n - p\| \\ &= r(p,\{x_n\}) \end{aligned}$$

this implies

$$\|r(Tp,\{x_n\}) - r(p,\{x_n\})\| \to 0$$

for $m\to\infty$. By Lemma (4.7), w get $Tp = p$. This means T has at least one fixed point. i.e. $F(T)\neq\emptyset$.

Theorem 4.10. Let X be a uniformly convex Banach space and $C\neq\emptyset$ be a closed convex subset of X with the Opial property and let T and $\{x_n\}$ be defined in Theorem (4.9) and $F(T)\neq\emptyset$, then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Uniformly convexity of X implies that X is reflexive. Now, by Eberlin's theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ Converging weakly to some $w_1\in X$. Since C is closed and convex, then $w_1\in C$. From Mazur's Theorem and by Lemma (4.4),

$w_1\in F(T)$. Our goal is to show that $\{x_n\}$ converges weakly to w_1 Let suppose that there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $w_2\in C$ and $w_2\neq w_1$. From Lemma (4.4), $w_2\in F(T)$. Since $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p\in F(T)$. By Theorem (4.9) and Opial's property we have,

$$\begin{aligned} \lim_{n\to\infty}\|x_n - w_1\| &= \lim_{k\to\infty}\|x_{n_k} - w_1\| \\ &< \lim_{k\to\infty}\|x_{n_k} - w_2\| \\ &= \lim_{n\to\infty}\|x_n - w_2\| \\ &= \lim_{i\to\infty}\|x_{n_i} - w_2\| \\ &< \lim_{i\to\infty}\|x_{n_i} - z_1\| \\ &= \lim_{n\to\infty}\|x_n - z_1\| \end{aligned}$$

This is a contradiction. Thus, we get $w_1 = w_2$. In this sense, $\{x_n\}$ converges weakly to a fixed point of T.

Theorem 4.11. Let X be a uniformly convex Banach space and $C\neq\emptyset$ be a compact convex subset of X and let $T: C\to C$ be a mapping satisfying condition (C). and let T and $\{x_n\}$ be as defined in Theorem (4.9), then $\{x_n\}\to p\in F(T)$.

Proof. We showed that $F(T)\neq\emptyset$ and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ in Theorem (4.9). From compactness of C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges strongly to a fixed point $p\in C$. By using the Proposition (4.3)

$$\|x_{n_k} - Tp\| \le 3\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p|$$

for all $n\ge 1$. If we take the limit of each side of this inequality we get $\lim_{n\to\infty}\|x_{n_k} - Tp\| = 0$. This implies that $Tp = p$ and p is the strong limit of the sequence $\{x_n\}$.

Recall that a mapping $T: C\to X$ is said to be demicompact provided whenever $\{x_n\}\subset C$ is bounded and $\{x_n - Tx_n\}$ converges, then there is a subsequence $\{x_{n_i}\}$ which converges.

A mapping $T: C \to C$ is said to satisfy condition (I), if there exists a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq F(d(x, F(T))$ for all $x \in C$, where $d(x, F(T)) = \inf_{p \in F(T)} d(x, p)$. (Senter and Dotson 1974).
They also proved that For a nonexpansive mapping T has fixed points, condition (I) is weaker than the requirement that T is demicompact.

Theorem 4.12. Let X be a uniformly convex Banach space and let $C \neq \emptyset$ closed and convex subset of X and let $\{x_n\}$ be as Theorem (4.9) and $F(T)\} \neq \emptyset$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$ by Lemma (4.8). We say $\lim_{n\to\infty} d(x_n, F(T))$ exists. Put $\lim_{n\to\infty} d(x_n, F(T)) = r$ for some $r \geq 0$. In case of $r = 0$, the proof is trivial. We consider the case of $r \neq 0$. Since T satisfies the condition (I), we have

$$f(d(x_n, F(T)) \leq \|Tx_n - x_n\|$$

By theorem (4.9), $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and this yields that

$$\lim_{n\to\infty} f(d(x_n, F(T)) = 0$$

Since f is a nondecresing function and $f(0) = 0$, we obtain $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that $\|x_{n_k} - y_k\| < \frac{1}{2^k}$ for $k \in \mathbb{N}$. By (4.3), we get

$$\|x_{n+1} - y_k\| \leq \|x_{n_k} - y_k\| < \frac{1}{2^k}$$

By using the known triangel inequalty, we obtain

$$\begin{aligned}\|y_{k+1} - y_k\| &\leq \|y_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}-y_k}\| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &\leq \frac{1}{2^{k-1}} \to \infty\end{aligned}$$

Thus $\{y_k\}$ is a Cauchy sequence in $F(T)$ and it converges to a point p. Since $F(T)$ is closed, then $p \in F(T)$ and s $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} \|x_n - p\|$ exists and we have $x_n \to p \in F(T)$.

We give some numerical results in Table (2) for the iterations of (1.10),(1.9),(1.11) and (1.13) with different initial points and different control sequences as in case1, case2, case3, case4.

From the Table (2), we show that the iteration of (1.10) is faster than the iterations of (1.9), (1.11) and (1.13).

By considering the four cases with particular initial points. We examine the algorithms. We now test fastness of these iterations are given in Table (2). We take avarege of number iterations for different initial points. The results are given in Figure (2). Thus we observe that the iteration of Kadioglu et al. (1.10) converges faster than other iterations mentioned in Table (2).

Table 1: the Kadioglu and Yildirim iteration process (1.10), the Thakur et al. iteration process (1.9), the Thakur et al. iteration process (1.11) and the Sintunavarat and Pitea iteration process (1.13) are, respectively, given in Example (3.2)

Step	(1.10)	(1.11)	(1.9)	(1.13)
1	40.00000000000	40.00000000000	40.00000000000	40.00000000000
2	35.61052759902	36.17970240708	36.75032546912	36.95921377178
3	31.27043067254	32.39551576817	33.52602968711	33.94015238463
4	26.99439863247	28.65637599982	30.33224134972	30.94683951314
5	22.80435177375	24.97479944534	27.17573436544	27.98447386191
6	18.73470250416	21.36895663878	24.06567504407	25.05991190832
7	14.84284005376	17.86636155975	21.01481759137	22.18241156610
8	11.23105294092	14.51073929102	18.04148199344	19.36481252519
9	8.090682464284	11.37518257991	15.17292923713	16.62546679883
10	5.752420636601	8.586216537653	12.45122302364	13.99147806010
11	4.514274785800	6.352772990786	9.943122070908	11.50418357296
12	4.113719398071	4.916080608998	7.753793199129	9.228015726308
13	4.022697430285	4.270641010318	6.029944065781	7.261703127837
14	4.004423433044	4.068420766968	4.899144247386	5.737195934374
15	4.000857931682	4.016405660000	4.327379670259	4.758684226446
16	4.000166240904	4.003879295910	4.104730092960	4.275480126743
17	4.000032206528	4.000914214146	4.031626628148	4.089287607739
18	4.000006239281	4.000215276156	4.009362061639	4.027572923019
19	4.000001208710	4.000050682974	4.002754266687	4.008374304477
20	4.000000234158	4.000011931883	4.000808797306	4.002530111420
21	4.000000045362	4.000002808997	4.000237376251	4.000763194838
22	4.000000008787	4.000000661291	4.000069657109	4.000230102217
23	4.000000001702	4.000000155680	4.000020439640	4.000069365365
24	4.000000000329	4.000000036649	4.000005997567	4.000020909584
25	4.000000000063	4.000000008628	4.000001759848	4.000006302928
26	4.000000000012	4.000000002031	4.000000516386	4.000001899929
27	4.000000000002	4.000000000478	4.000000151521	4.000000572706
28	4.000000000000	4.000000000112	4.000000044460	4.000000172634
29	4.000000000000	4.000000000020	4.000000013045	4.000000052038
33	4.000000000000	4.000000000000	4.000000000096	4.000000000429
38	4.000000000000	4.000000000000	4.000000000000	4.000000000001
40	4.000000000000	4.000000000000	4.000000000000	4.000000000000

Table 2: Different initial points and different control sequences as in case1, case2, case3, case4

Influence of initial points				
Case1: $\alpha_n = \frac{1}{n+1}$ $\beta_n = \frac{1}{3n+1}$ $\gamma_n = \frac{1}{5n+1}$				
Initial value	(1.10)	(1.9)	(1.11)	(1.13)
0.12	21	23	23	24
0.17	21	23	23	24
0.23	21	23	23	23
0.48	21	23	23	23
0.75	21	22	22	22
0.92	21	22	21	22
Case2: $\alpha_n = \frac{1}{\sqrt{n+1}}$ $\beta_n = \frac{1}{\sqrt[3]{n+1}}$ $\gamma_n = \frac{1}{\sqrt[4]{n+1}}$				
Initial value	(1.10)	(1.9)	(1.11)	(1.13)
0.12	15	20	16	19
1.17	15	20	16	19
0.23	15	20	16	18
0.48	15	20	15	19
0.75	15	19	15	19
0.92	14	18	15	18
Case3: $\alpha_n = \frac{2n}{7n+9}$ $\beta_n = \frac{1}{(3n+5)^{\frac{5}{2}}}$ $\gamma_n = \frac{1}{(n+1)^{\frac{7}{2}}}$				
Initial value	(1.10)	(1.9)	(1.11)	(1.13)
0.12	21	23	23	23
0.17	21	23	23	24
0.23	21	23	23	23
0.48	20	23	23	23
0.75	20	23	22	23
0.92	19	22	22	22
Case4: $\alpha_n = \frac{1}{n^2}$ $\beta_n = \frac{1}{3n^2+1}$ $\gamma_n = \frac{1}{5n^2+1}$				
0.12	20	21	21	22
0.17	20	21	21	22
0.23	20	21	21	22
0.48	20	21	21	22
0.75	19	20	20	21
0.92	18	20	20	21

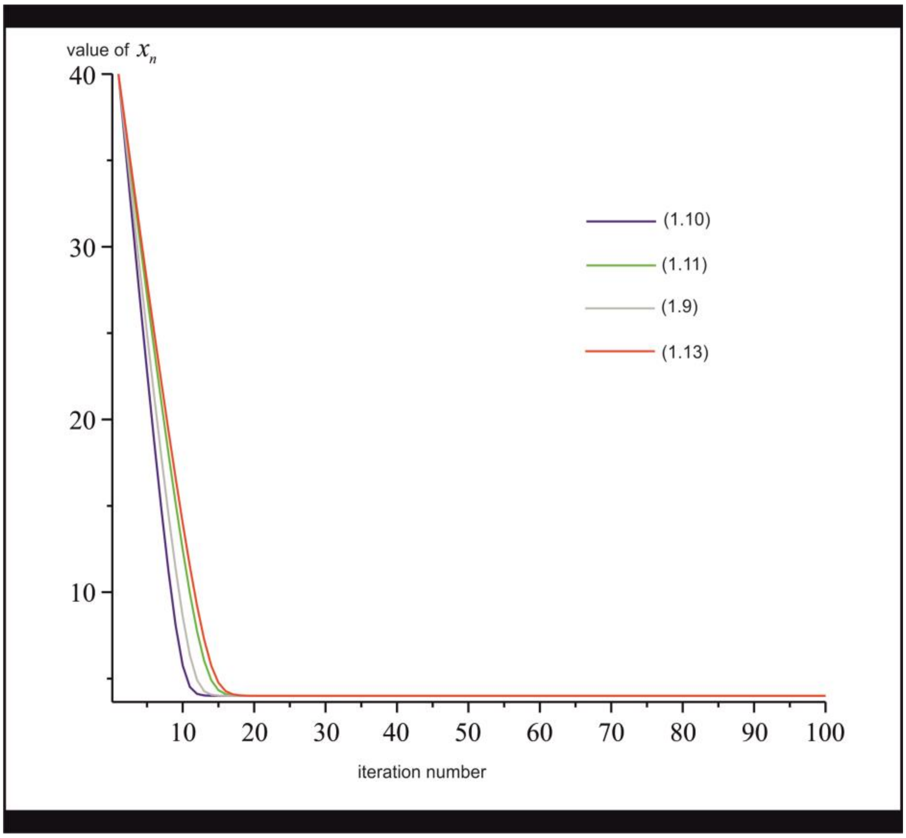

Figure 1: Convergence rates of iterations given in Exampmle (3.2)

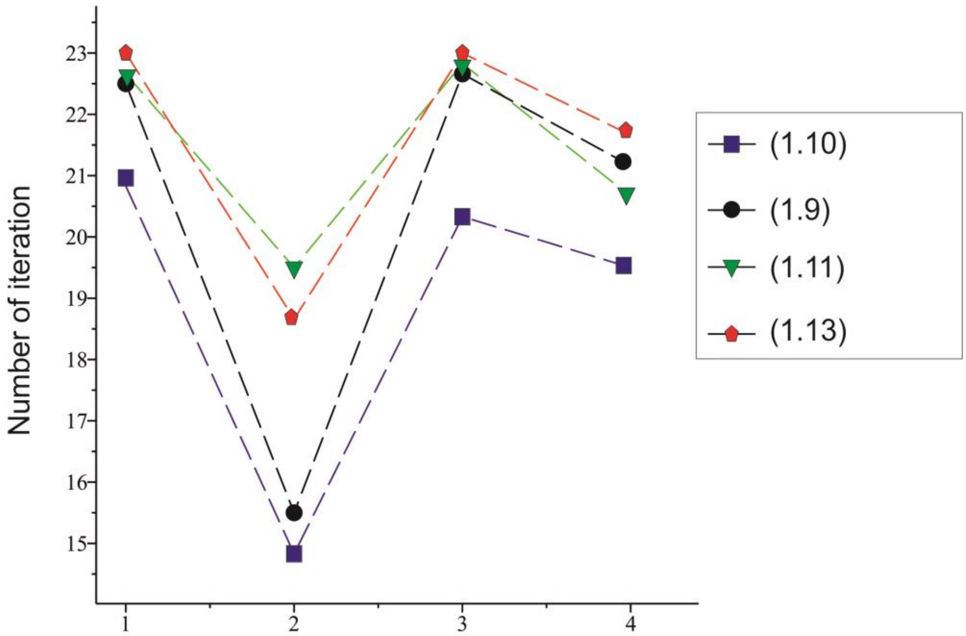

Figure 2: By considering the four cases with particular initial points.

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