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# HIERARCHIC GRAPHS BASED ON THE FIBONACCI NUMBERS

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## *ABSTRACT*

*In this paper, a new class of hierarchically definable graphs are proposed and they are proper subgraphs of Hierarchic Cubic graphs. These graphs are based on the Fibonacci series by changing initial conditions. When the initial conditions are changed, then the structure of obtained graph will be changed. Thus, we obtained a series of hierarchically definable graphs. The obtained graphs have logarithmic node degrees and diameters in terms of number of nodes. Thus they are comparable with incomplete hypercube graph. Sometimes, incomplete hypercube may include at least one node whose node degree is 1. This is an unwilling case, however, the obtained graphs do not have nodes of degree 1 except initial conditions graphs.*

*Hypercube graph and hierarchic cubic network are recursively definable graphs and the obtained graphs are proper subgraphs of hierarchic cubic network. Thus, it is important to verify that the constructed graphs are also recursively definable graphs. We prove that the obtained graphs are self-similar graphs or decomposable in terms of lower sized graphs in the same category.*

***Keywords:*** **:** *Hypercube Graphs, Hierarchical Cubic Network, Fibonacci Cube Graphs, Extended Fibonacci Cube Graphs.*

## 1. INTRODUCTION

Hypercube graph H(n) is a recursively definable graphs and it has been used for interconnection networks [1, 2, 3]. Its nice properties such as logarithmic diameter, regular graph, simple node labelling, good connectivity, recursive scalability, symmetry, sparesity, average distance and cost etc. makes it be a popular graph. Hierarchic cubic network graph uses H(n)s as building blocks and it is also recursively constructable graph [4]. Fibonacci and extended Fibonacci cube graphs are proper subgraphs of H(n) [5, 6, 7].

Efforts to improve some of these properties have lead to the evolution of hypercube variant graphs. Cube Connected Cycles, Folded Hypercube, Extended Hypercube are variants derived through the addition of extra nodes and/or links to the H(n) [3]. Another category of variants, which includes the Twisted n-cube graph [1] and the Multiply twisted cube graph [2], is derived by manipulating only the node-link incidences of the hypercube graph without the addition of extra nodes and links. Hierarchical Cubic Network graphs [4] is another derived network from hypercube graph by combining hypercube graphs in a hierarchical fashion.

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H(n) of dimension n connects up to $2^n$ nodes, each of which can be labelled by n-bit address uniquely, using a direct connection between two nodes if and only if their n-bit addresses differ in exactly one bit position. The reason for the popularity of the H(n) can be attributed to its topological properties, the ability to use simple routing algorithms and the ability to permit the embedding of commonly-rewired interconnection patterns.

Hierarchical Cubic Network graphs (HCN(n,n)) are constructed by using n-dimensional H(n)s as basic building blocks and these blocks are connected in a hierarchical manner. The HCN(n,n) uses almost half as many links as a comparable H(n) and yet emulates the desirable properties of a H(n) very efficiently. Moreover, the maximum internodes routing distance in a HCN(n,n) is about 3/4 of that in the comparable H(n).

A HCN(n,n) uses H(n)s as basic components and each such H(n) component is referred to as a cluster. The HCN(n,n) has $2^n$ clusters, where each cluster is a n-cube. Each node in the HCN(n,n) has (n+1) edges incident to it. Of these, n edges (links) are required for local connections within a cluster implementing the normal links in an n-cube. The additional link, called external link, is required to interconnect nodes in different clusters. Each node in the system can thus be uniquely associated with a pair of numbers (I,J), where I is a n-bit cluster number, and J is a n-bit address of the node within a cluster. A new link established between nodes (I,J), (K,L) where I=L and J=K or I=J and K=L= $\bar{I}$ .

It is also possible to have incomplete H(n) of dimension n and the derived HCN(m,m), where m<n, is an incomplete HCN(n,n). Thus, the derived interconnection network is a proper subgraph of HCN(n,n). Many such subgraphs can be obtained by changing the value of m in the interval [1,n-1]. In this paper, we will give a new class of hierarchical definable graphs based on Fibonacci cube and extended Fibonacci cube graphs, and these graphs are proper subgraphs of HCN(n,n) [6,7].

The H(n) is a powerful network that is able to perform various kinds of parallel computation and simulate many other networks. However, the number of nodes, which is a power of two limits its efficiency to perform a task of arbitrary size. Fibonacci Cube and $k^{th}$ order Extended Fibonacci Cubes ($EFC_k(n)$s) is a special subcube of a H(n) based on the Fibonacci number $f_n=f_{n-1}+f_{n-2}$, $ef_1(n)= ef_1(n-1)+ ef_1(n-2)$, respectively [5].

In this paper, we proposed a class of graphs based on FC(n) and $EFC_k(n)$ from the above reasons and these graphs are proper subgraphs of HCN(n,n). We called these graphs as Hierarchical Fibonacci Cube HFC(n) and Hierarchical Extended Fibonacci Cube graphs ($HEFC_k(n)$, $k\geq1$) [6, 7], and its properties and features are evaluated. Therefore, the objective of this paper is:

- to represent the construction of HFC(n)s, $HEFC_k(n)$s.
- to study the self-similarity properties of HFC(n)s, $HEFC_k(n)$s.

The rest of this paper is organized as follows. Section 2 describes the notations, the definitions, the outlines of H(n), HCN(n,n), FC(n), $EFC_k(n)$ and the way to make inter-block connections. Section 3 shows the construction of HFC(n)s, $HEFC_k(n)$s. Section 4 gives some structural properties of HFC(n)s, $HEFC_k(n)$s. Section 5 describes the decompositions of HFC(n)s, $HEFC_1(n)$s, …, $HEFC_k(n)$s in detail. Section 6 summarizes and concludes this paper.

## 2. DEFINITIONS AND NOTATIONS FOR H(n), HCN(n,n), FC(n), $EFC_k(n)$

First of all, we must briefly describe H(n), HCN(n,n), FC(n) and $EFC_k(n)$ for k<n-1.

### 2.1. H(n)

A H(3) can be represented as an ordinary cube in three dimensions where the vertices are the $8=2^3$ nodes of the 3-cube. In hypercube of dimension n, there are $2^n$ nodes, where each node is labelled with a unique label in sequence 0, 1, ..., $2^n$-1, and $n2^{n-1}$ edges. Two nodes i and j are directly connected if and only if the binary representations of i and j differ in exactly one bit. Thus in a H(n), each node is connected to n others. The distance between two nodes in H(n) is equal to the number of different bits in binary addresses of corresponding nodes.

Let u, v denote nodes u, v in H(n) or their addresses. Hamming distance is the exclusive-or operation on both addresses of nodes u, v and this distance is equal to Hamming distance. In other words, the Hamming distance between nodes u and v is the summation of different bit-position in addresses of nodes u and v, and it is denoted as H(u,v).

More generally, the definition of a H(n) of dimension n as a graph denoted by H(n)=(V,E) where $V=\{0,1\}^n$ is the set of vertices, represented by all the binary strings of length n, and the set of edges is

$E=\{(u,v)|u,v\in V$ such that u and v exactly differ in 1-bit position$\}$.

The node degree in H(n) is n and the diameter of H(n) is also n.

## 2.2. HCN(n,n)

HCN(n,n)s are constructed from H(n)s which are used as basic building blocks and addition of new edges between these building blocks. Each building block is referred to as a cluster. The HCN(n,n) has $2^n$ clusters, where each cluster is an n-cube. So there are $2^{2n}$ nodes and $(n+1)2^{2n-1}$ edges. The node degree in HCN(n,n) is n+1 and diameter of HCN(n,n) is $n+\left\lfloor \frac{n+1}{3} \right\rfloor+1$. n edges incident onto a node within a cluster are referred to as local links implementing the normal edges in an n-cube and the additional edges are needed to connect nodes within different cluster which are called external edges (links). The edges within a cluster are called non-diameter edges and the edges inter-clusters are called diameter edges. Each node in HCN(n,n*)* can be represented by a pair of numbers, (I,J) where I is the cluster number and J is the node number within a cluster.

Two nodes $(I_1,J_1)$ and $(I_2,J_2)$ $(I_1\neq I_2)$ are connected if and only if one of the following conditions is satisfied.

- $I_1=J_2$ and $I_2=J_1$
- $I_1=J_1$ and $I_2=J_2=\bar{I}_1$

## 2.3. FC(n)s and EFC$_k$(n)s

$f_n$ denotes a Fibonacci number and $f_n=f_{n-1}+f_{n-2}$ where initial condition $f_2=0$ and $f_3=2$. $ef_1(n)$ is also a Fibonacci number and it is called first order Fibonacci number where $ef_1(n)=ef_1(n-1)+ef_1(n-2)$ and initial condition is $ef_1(3)=2$ and $ef_1(4)=4$. $k^{th}$ order Fibonacci number is defined as $ef_k(n)$ and its initial condition is different. The initial condition for $ef_2(n)$ is $ef_2(4)=4$ and $ef_2(5)=8$. The initial condition for $ef_k(n)$ is $ef_k(k+2)=|\{\overbrace{dd\cdots d}^{k}\}|$ and $ef_k(k+3)=|\{\overbrace{dd\cdots d}^{k+1}\}|$ where $d\in\{0,1\}$.

FC(n), $EFC_1(n)$, …, $EFC_k(n)$ are defined by using $f_n$ and $ef_k(n)$, respectively.

***Definition 1.*** *Assume FC(n)=(V(n),E(n)), FC(n-1)=(V(n-1),E(n-1)) and FC(n-2)=(V(n-2),E(n-2)). The recursion for nodes set is* $V(n)=0||V(n-1)\cup 10||V(n-2)$, *where || denotes the concatenation of two bit-strings. Two nodes in FC(n) are connected by an edge in E(n) if and only if their labels differ exactly in 1-bit position. The initial condition for recursion is V(2)={} and V(3)={0,1}.*

***Definition 2.*** *Let* $EFC_1(n)=(V_1(n),E_1(n))$ *where* $V_1(n)$ *is the set of nodes and* $E_1(n)$ *is the set of edges in* $EFC_1(n)$, *and* $EFC_1(n-1)=(V_1(n-1),E_1(n-1))$, $EFC_1(n-2)=(V_1(n-2),E_1(n-2))$. $EFC_1(n)$ *can be defined recursively by using* $EFC_1(n-1)$ *and* $EFC_1(n-2)$. *Then* $V_1(n)=0||V_1(n-1)\cup 10||V_1(n-2)$ *where || denotes the concatenation of two strings. Two nodes in* $EFC_1(n)$ *are connected if and only if their address representations differ in exactly 1-bit position. An initial condition for recursion is* $V_1(3)=\{0,1\}$ *and* $V_1(4)=\{00,10,11,01\}$.

**Definition 3.** *Let* $EFC_k(n)=(V_k(n),E_k(n))$ *where* $V_k(n)$ *is the set of nodes and* $E_k(n)$ *is the set of edges in* $EFC_k(n)$, *and* $EFC_k(n-1)=(V_k(n-1),E_k(n-1))$, $EFC_k(n-2)=(V_k(n-2),E_k(n-2))$. $EFC_k(n)$ *can be defined recursively by using* $EFC_k(n-1)$ *and* $EFC_k(n-2)$. *Then* $V_k(n)=0||V_k(n-1)\cup 10||V_k(n-2)$ *where || denotes the concatenation of two strings. Two nodes in* $EFC_1(n)$ *are connected if and only if their address representations differ in exactly 1-bit position. An initial condition for recursion is* $V_k(k+2)=\{\overbrace{dd\cdots d}^{k}\}$ *and* $V_k(k+3)=\{\overbrace{dd\cdots d}^{k+1}\}$ *where* $d\in\{0,1\}$.

It is immediately noticeable that FC(3)=H(1) and FC(2)=H(0). FC(n) is a proper subcube of H(n-2). FC(2) and FC(1) are null graphs. The node degree of FC(n) is between $\left\lfloor \frac{n-2}{3} \right\rfloor$ and n-2 and the diameter of FC(n) is n-2. The number of nodes in FC(n) is equal to $f_n$ and number of edges in FC(n) is

$$\frac{2(n-1)f_n - nf_{n-1}}{5}$$

The diameter of $EFC_1(n)$ is n-2 and the node degree of a node in $EFC_1(n)$ is between $\left\lceil \frac{n}{3} \right\rceil$ and n-2. The node degree of a node in $EFC_k(n)$ is between $\left\lceil \frac{n-(k-1)}{3} \right\rceil + (k-1)$ and n-2. The number of nodes in $EFC_1(n)$ is $ef_1(n)$ and the number of nodes in $EFC_k(n)$ is $ef_k(n)$. The numbers of edges for $EFC_1(n)$, $EFC_2(n)$, and so on, respectively, are

$$|E_1(n)| = 4f_{n-3} + f_{n-4} + \sum_{i=1}^{n-4} f_i ef_1(n-i-1) \quad \text{for}$$

n≥5

$$|E_2(n)| = 12f_{n-4} + 4f_{n-5} + \sum_{i=1}^{n-5} f_i ef_2(n-i-1)$$

for n≥6

…………..

$$|E_k(n)| = |E_k(k+3)|\, f_{n-k-2} + |E_k(k+2)|\, f_{n-k-3} + \sum_{i=1}^{n-k-3} f_i ef_k(n-i-1)$$

for n≥k+4.

## 3. STRUCTION OF HFC(n) and HEFC$_k$(n)s

A series of graphs, which proper subgraphs of HCN(n,n), can be built by using FC(n)s, $EFC_1(n)$s, $EFC_2(n)$s, …, $EFC_k(n)$s as building blocks. All of the obtained graphs (HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$, k<n-1) will not be explained in detail and we will only explain $HEFC_1(n)$ and the remaining graphs have similar properties and similar construction process.

The construction process can be explained on an example. The construction of new graph can be explained by using HCN(4,4). The clusters 0110, 0111, 1100, 1101, 1110, and 1111 are removed from HCN(4,4) and the edges in these clusters are also removed from HCN(4,4). The nodes 0110, 0111, 1100, 1101, 1110, and 1111 are removed from remaining clusters with incident edges. The last step for constructing Hierarchical Extended Fibonacci Cube- $HEFC_1(6)$ from HCN(4,4) is removing edges between the nodes (I,I) and nodes $(\bar{I},\bar{I})$ and derived $HEFC_1(6)$ is shown in Figure 2. $HEFC_1(3)$ is same as HCN(1,1) and $HEFC_1(4)$ is same as HCN(2,2) and the construction of $HEFC_1(5)$ is shown in Figure 2. Thus, constructing HEFC1(n) from HCN(n-2,n-2) can be summarized as follows.

- Removing the clusters whose node label is same as node label of node which is in HCN(n-2,n-2) and is not in $EFC_1(n)$.
- Removing the nodes which are in HCN(n-2,n-2) and are not in $EFC_1(n)$.
- Removing the edges of HCN(n-2,n-2) whose end points are (I,I) and $(\bar{I},\bar{I})$.

We called the obtained interconnection network as Hierarchical Extended Fibonacci Cube ($HEFC_1(n)$) or First Order Extended Fibonacci Cube [6,7]. The edges within a cluster are called horizontal edges and the edges between clusters are called diagonal edges. The graphs obtained by using $EFC_k(n)$ are called $k^{th}$ Order Hierarchic Extended Fibonacci Cubes – $HEFC_k(n)$s or simply Hierarchic Extended Fibonacci Cubes and the graph obtained by using FC(n) as building blocks is called Hierarchic Fibonacci Cube – HFC(n).

Two nodes (I,J) and (K,L) in $HEFC_1(n)$ are connected if and only if one of the following conditions holds.

- I=L and J=K.
- I=K and H(J,L)=1.

In Figure 2, dashed edges and nodes exist in HCN(3,3) and do not exist in $HEFC_1(5)$.

In the following sections, most of theorem's proofs are done for $HEFC_1(n)$ and proof for remaining graphs can be handled in the same way.

HFC(n), $HEFC_2(n)$, …, $HEFC_k(n)$ can be constructed in the same way and the only difference is that their initial conditions are different. The definitions of the remaining architectures can be expressed in the same way by changing $V_{H1}$ as $V_{Hk}$, 2≤k≤n+2 or $V_H$.

**Definition 4.** A HFC(n) is a graph and it contains FC(n) as basic building blocks and the node label is (I,J) where I is the label of building block and J is the node number in I[th] block. If HFC(n)=($V_H$(n),$E_H$(n)), then

$$V_H(n)=\bigcup_{\forall v\in V_1}\bigcup_{\forall u\in V_2}\{(v,u)\mid I=v \text{ and } J=u\}$$

and let (I,J), (K,L) be two nodes in HFC(n) and I, K are clusters' labels and J, L are nodes' labels, then

$$E_H(n)=\bigcup_{\forall(v_1,u_1),(v_2,u_2)\in V_H(n)}\left\{\begin{array}{l}((v_1,u_1),(v_2,u_2))\mid v_1=v_2\wedge u_1\neq u_2\\ \wedge H(u_1,u_2)=1\vee v_1=u_2\wedge v_2=u_1\end{array}\right\}$$

where I=$v_1$, J=$u_1$, K=$v_2$, and L=$u_2$.

**Definition 5.** A HEFC$_k$(n), 1≤k≤n-2, is a graph and it contains EFC$_k$(n) as basic building blocks and the node label is (I,J) where I is the label of building block and J is the node number in I[th] block. Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two EFC$_k$(n)s. If HEFC$_k$(n)=($V_{Hk}$(n),$E_{Hk}$(n)), then

$$V_{Hk}(n)=\bigcup_{\forall v\in V_1}\bigcup_{\forall u\in V_2}\{(v,u)\mid I=v \text{ and } J=u\}$$

and let (I,J), (K,L) be two nodes in HEFC$_k$(n) and I, K are clusters' labels and J, L are nodes' labels, then

$$E_{Hk}(n)=\bigcup_{\forall(v_1,u_1),(v_2,u_2)\in V_{Hk}(n)}\{((v_1,u_1),(v_2,u_2))\mid v_1=v_2\wedge u_1\neq u_2 \wedge H(u_1,u_2)=1\vee v_1=u_2\wedge v_2=u_1\}$$

where I=$v_1$, J=$u_1$, K=$v_2$, and L=$u_2$.and $v_1$, $u_1\in V_1$ and $v_2$,, $u_2\in V_2$.

## 4. TOPOLOGICAL PROPERTIES OF HFC(n)s, HEFC$_k$(n)s

This section describes the definitions, the structural properties of HFC(n), HEFC$_1$(n), …, HEFC$_k$(n) such as number of nodes, number of edges, node degree and connectivity, diameter.

The first properties of HFC(n), HEFC$_1$(n), …, HEFC$_k$(n) are the numbers of nodes and edges. These terms of any graph can be described as follows.

***Definition 6.*** *Let G be an undirected graph G=(V,E), where V is the vertex set and |V| is the number of nodes, E is the edge set and |E| is the number of edges. The degree of a node is the number of edges incident on to that node.*

Theorem 1. The number of edges in HFC(n) is

$$f_n\left[\frac{2(n-1)f_n-nf_{n-1}}{5}\right]+\frac{f_n^2-f_n}{2}$$

and the number of nodes is $(f_n)^2$.

**Proof.** The number of nodes in HFC(n) can be found by using Fibonacci series with given initial conditions. Once the number of nodes in any FC(n) is found, then the number of nodes in HFC(n) is the square of the number of nodes in FC(n). Similar case is also valid for obtaining a formula for number of edges in HFC(n).

A recursive formula for the number of edges in HFC(n) is as follows. While constructing FC(n)s recursively, two FC(n-1) and FC(n-2) are combined and a new FC(n)s is obtained so that all the edges in FC(n-1) and FC(n-2) are in constructed FC(n). The nodes in FC(n-2) are connected to nodes in FC(n-1) in one-to-one fashion. For this reason, the number of edges in FC(n) is

$$|E(n)|=|E(n-1)|+|E(n-2)|+|V(n-2)| =\frac{2(n-1)f_n-nf_{n-1}}{5}$$

where initial condition |E(3)|=1, |E(4)|=2.

For any HFC(n), |V(n)| FC(n)s are combined and a HFC(n) is yielded. All the edges in FC(n)s are in HFC(n) and any FC(n) in HFC(n) is connected to another FC(n) by the nodes (I,J) where I≠J. Thus, first cluster in HFC(n) is connected to all remaining clusters by an edge for each cluster. So there are |V(n)|-1 edges for first cluster. Second cluster is connected to all remaining clusters (except first cluster) by an edge and so there are |V(n)|-2 edges for second cluster, and so on. Last two clusters are connected by only one edge. Hence number of edges in HFC(n) is

$$|E_H(n))|=|V(n)|.|E(n)|+|V(n)|(|V(n)|-1)/2 =f_n\left[\frac{2(n-1)f_n-nf_{n-1}}{5}\right]+\frac{f_n^2-f_n}{2}$$

where initial condition $|E_H(3)|=3$, $|E_H(4)|=9$, and the number of nodes in HFC(n) is $(f_n)^2$ ♦

Similar case is also valid for obtaining a formula for numbers of nodes and edges in $HEFC_k(n)$.

**Theorem 2.** The number of edges in $HEFC_k(n)$ is

$$ef_k(n).\left[|E_k(k+3)|f_{n-k-2} + |E_k(k+2)|f_{n-k-3} + \sum_{i=1}^{n-k-3} f_i.ef_k(n-i-1)\right] + \frac{ef_k(n)(ef_k(n)-1)}{2}$$

and the number of nodes is $(ef_k(n))^2$.

**Proof.** A recursive formula for the number of edges in $HEFC_1(n)$ is as follows. While constructing $EFC_1(n)$s recursively, two $EFC_1(n-1)$ and $EFC_1(n-2)$ are combined and a new $EFC_1(n)$s is obtained so that all the edges in $EFC_1(n-1)$ and $EFC_1(n-2)$ are in constructed $EFC_1(n)$. The nodes in $EFC_1(n-2)$ are connected to nodes in $EFC_1(n-1)$ in one-to-one fashion. For this reason, the number of edges in $EFC_1(n)$ is

$$|E_1(n)|=|E_1(n-1)|+|E_1(n-2)|+|V_1(n-2)|$$
$$= 4.f_{n-3} + f_{n-4} + \sum_{i=1}^{n-2} f_i.ef_1(n-i-1)$$

with initial condition $|E(3)|=1$, $|E(4)|=4$ and $f_i$ stands for $i^{th}$ Fibonacci number.

For any $HEFC_1(n)$, the number of combined $EFC_1(n)$s is $|V_{H1}(n)|=ef_1(n)$ and a $HEFC_1(n)$ is yielded. All the edges in $EFC_1(n)$s are in $HEFC_1(n)$ and any $EFC_1(n)$ in $HEFC_1(n)$ is connected to another $EFC_1(n)$ by the nodes (I,J) where $I \neq J$. Thus, first cluster in $HEFC_1(n)$ is connected to all remaining clusters by an edge for each cluster. So there are $|V(n)|-1=ef_1(n)-1$ edges for first cluster. Second cluster is connected to all remaining clusters (except first cluster) by an edge and so there are $ef_1(n)-2$ edges for second cluster, and so on. Last two clusters are connected by only one edge. Hence number of edges in $HEFC_1(n)$ is

$$|E_{H1}(n))|=|V_1(n)|.|E_1(n)|+|V_1(n)|(|V_1(n)|-1)/2 =$$

$$ef_1(n).\left[4.f_{n-3} + f_{n-4} + \sum_{i=1}^{n-2} f_i.ef_1(n-i-1)\right] + \frac{ef_1(n)(ef_1(n)-1)}{2}$$

where initial condition $|E_{H1}(3)|=4$, $|E_{H1}(4)|=22$, and the number of nodes in $HEFC_1(n)$ is $(ef_1(n))^2$. The construction of $HEFC_2(n)$, …, $HEFC_k(n)$ are same as HFC(n) and HEFC1(n), so the numbers of nodes and edges can be expressed as follow.

$$|E_{H2}(n))|=|V_2(n)|.|E_2(n)|+|V_2(n)|(|V_2(n)|-1)/2 =$$

$$ef_2(n).\left[12.f_{n-4} + 4f_{n-5} + \sum_{i=1}^{n-5} f_i.ef_2(n-i-1)\right] + \frac{ef_2(n)(ef_2(n)-1)}{2}$$

and the number of nodes is $(ef_2(n))^2$.

………..

$$|E_{Hk}(n))|=|V_k(n)|.|E_k(n)|+|V_k(n)|(|V_k(n)|-1)/2 =$$

$$ef_k(n).\left[|E_k(k+3)|f_{n-k-2} + |E_k(k+2)|f_{n-k-3} + \sum_{i=1}^{n-k-3} f_i.ef_k(n-i-1)\right] + \frac{ef_k(n)(ef_k(n)-1)}{2}$$

and the number of nodes is $(ef_k(n))^2$ ♦

**Theorem 3.** The node degree in HFC(n) is between $\left\lceil \frac{n+1}{3} \right\rceil - 1$ and n-1. The node degree in $HEFC_k(n)$ is between $\left\lceil \frac{n-(k-1)}{3} \right\rceil + (k-1)$ and n-1.

**Proof.** The second property of $HEFC_1(n)$ is the node degrees. The node degrees of $EFC_1(n)$s are between $\left\lceil \frac{n}{3} \right\rceil$ and n-2. While constructing $HEFC_1(n)$s from $EFC_1(n)$s, all nodes in each cluster have a diagonal link except (I,I) node for each cluster. This means that the node degrees of each node in each cluster except node (I,I) for each cluster increase by 1, and lower bound for degree does not change and upper bound increases by 1. So degrees of nodes for $HEFC_1(n)$ are between $\left\lceil \frac{n}{3} \right\rceil$ and n-1. Similar proof can be done for HFC(n), $HEFC_2(n)$, …, $HEFC_k(n)$. Similar proof can be handled for HFC(n) and $HEFC_2(n)$, …, $HEFC_k(n)$ ♦

Thus, the connectivity of $HEFC_1(n)$ is $\left\lceil \frac{n}{3} \right\rceil$, because at least removing $\left\lceil \frac{n}{3} \right\rceil$ edges from a node in $HEFC_1(n)$ may separate $HEFC_1(n)$ to two disjoint subgraphs. The connectivity of HFC(n) is $\left\lceil \frac{n+1}{3} \right\rceil - 1$ and connectivity of $HEFC_k(n)$ is $\left\lceil \frac{n-(k-1)}{3} \right\rceil + (k-1)$.

In order to determine the length of diameter, let us consider the properties of paths in $HEFC_1(n)$ (remaining graphs have similar properties and they will be depicted in short).

***Definition 7.*** *Given a graph G=(V,E), let a sequence of nodes* $P=v_1, v_2, ..., v_k$ ($v_i \in V$, $1 \le i \le k$) *be a path from node* $v_1$ *to* $v_k$ *where* $(v_i,v_{i+1}) \in E$ *for i=1, ..., k-1. For any pair of nodes u,* $v \in V$, *the distance between u and v is the length of the shortest path from u to v. The diameter of G is the maximum value among distances of all pairs of nodes u,* $v \in V$. *The average distance of G is the average of distances between any pair of nodes u,* $v \in V$.

The properties of paths in $HEFC_1(n)$ can be expressed in the following theorems. Same path properties are held for remaining graphs (HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$).

***Theorem 4.*** *Let labels of source and destination nodes be* $(S_c,S_n)$ *and* $(D_c,D_n)$, *respectively in* $HEFC_k(n)$. *If* $S_c=D_c$, *then the shortest path between* $(S_c,S_n)$ *and* $(D_c,D_n)$ *does not contain diagonal link.*

**Proof.** Let assume that the shortest path P between $(S_c,S_n)$ and $(D_c,D_n)$ contains at least one diagonal link. If the routing between $(S_c,S_n)$ and $(D_c,D_n)$ nodes contains one diagonal edge in P, it is impossible. In order to return back to $(S_c,S_n)$ node, P has to contain at least two diagonal edges. Let P contains r diagonal links. If r is r≥2, then P will be as follows.

$P \Rightarrow (S_c,S_n) \rightarrow \ldots \rightarrow (S_c,S_1) \rightarrow (S_1,S_c) \rightarrow \ldots \rightarrow (S_1,S_2) \rightarrow (S_2,S_1) \rightarrow \ldots \rightarrow (S_2,S_3) \rightarrow (S_3,S_2) \rightarrow \ldots \rightarrow (S_3,D_c) \rightarrow (D_c,S_3) \rightarrow \ldots \rightarrow (D_c,D_n)$.

This is not a shortest path and this is a contradiction. The longest path length in this case is the length of diameter of $EFC_1(n)$ which is n-2. This value can be evaluated by $H(S_n,D_n)$ ♦

Let us considered paths (path P) starting at node $(S_c,S_n)$ and ending at node $(D_c,D_n)$. All the following properties are considered for $S_c \ne D_c$ case. Proofs of these properties are not considered, because all of them will be proved in Theorem 3.

***Property 1.*** *If* $S_c=S_n$, $D_c \ne D_n$ *and* $S_c=D_n$ *then the shortest path between* $(S_c,S_n)$ *and* $(D_c,D_n)$ *contains one diagonal link and length of P is* $H(S_n,D_c)+1$.

***Property 2.*** *If* $S_c=S_n, S_c \ne D_n$, *and* $D_c=D_n$ *then the shortest path between* $(S_c,S_n)$ *and* $(D_c,D_n)$ *contains at most one diagonal link and length of P is* $H(S_n,D_c)+H(S_c,D_n)+1$.

***Property 3.*** *If* $S_c=S_n$, $S_c \ne D_n, S_n \ne D_c$, *and* $D_c \ne D_n$ *,then the shortest path between* $(S_c,S_n)$ *and* $(D_c,D_n)$ *contains at most one diagonal link and length of P is* $2H(S_c,D_c)+1$.

***Property 4.*** *If* $S_c \ne S_n$ *and* $S_n=D_c$, *and* $S_c=D_n$ *, then the shortest path between* $(S_c,S_n)$ *and* $(D_c,D_n)$ *consists of only one diagonal link and length of P is 1.*

***Property 5.*** *If* $S_c \ne S_n$,, $S_c=D_n, S_n \ne D_c$ *and* $D_c \ne D_n$, *then the shortest path between* $(S_c,S_n)$ *and* $(D_c,D_n)$ *contains at most one diagonal link and length of P is* $H(S_n,D_c)+1$.

***Property 6.*** *If* $S_c \ne S_n$ *and* $S_n=D_n$, *then the shortest path between* $(S_c,S_n)$ *and* $(D_c,D_n)$ *contains at most one diagonal link and length of P is* $H(S_c,D_n)+1$.

***Property 7.*** *If $S_c \neq S_n$, $S_n \neq D_c$, $S_n = D_n$, and $D_c \neq D_n$ then the shortest path between $(S_c,S_n)$ and $(D_c,D_n)$ contains at most two diagonal links and length of P is $H(S_c,D_c)+2$.*

***Property 8.*** *If $S_c \neq S_n$, $S_n \neq D_c$, $S_c \neq D_n$, and $D_n = D_c$ then the shortest path between $(S_c,S_n)$ and $(D_c,D_n)$ contains at most one diagonal link and length of P is $H(S_n,D_n)+H(S_c,D_n)+1$.*

***Theorem 5.*** *Let labels of source and destination nodes be $(S_c,S_n)$ and $(D_c,D_n)$, respectively in $HEFC_k(n)$. Let P be the shortest path between nodes $(S_c,S_n)$ and $(D_c,D_n)$, then P contains at most two diagonal links ($S_c \neq D_c$).*

**Proof.** There are eight cases (d(P) denotes the length of P), and when two nodes are in the same cluster, the shortest routing path is determined by conventional algorithms in $EFC_1(n)$ and denoted by $(S_c,S_n) \Rightarrow (D_c,D_n)$.

1-$S_c=S_n$, $D_c \neq D_n$ and $S_c=D_n$
P: $(S_c,S_n) \Rightarrow (S_c,D_c) \rightarrow (D_c,S_c)$ and $d(P)=H(S_n,D_c)+1$.

2-$S_c=S_n$, and $D_c=D_n$
P: $(S_c,S_n) \Rightarrow (S_c,D_c) \rightarrow (D_c,S_c) \Rightarrow (D_c,D_n)$ and $d(P)=H(S_n,D_c)+1+H(S_c,D_n)$.

3-$S_c=S_n$, $S_c \neq D_n$, $S_n \neq D_c$, and $D_c \neq D_n$,
P: $(S_c,S_n) \Rightarrow (S_c,D_c) \rightarrow (D_c,S_c) \Rightarrow (D_c,D_n)$ and $d(P)=H(S_n,D_c)+1+H(S_c,D_n)=2H(S_c,D_n)+1$.

4-$S_c \neq S_n$, $S_n=D_c$, and $S_c=D_n$,
P: $(S_c,S_n) \rightarrow (S_n,S_c)=(D_c,D_n)$ and $d(P)=1$.

5-$S_c \neq S_n$,, $S_c=D_n$, $S_n \neq D_c$ and $D_c \neq D_n$,
P: $(S_c,S_n) \Rightarrow (S_c,D_c) \rightarrow (D_c,S_c)$ and $d(P)=H(S_n,D_c)+1$.

6-$S_c \neq S_n$ and $S_n=D_n$
P: $(S_c,S_n) \rightarrow (S_n,S_c) \Rightarrow (S_n,D_n)$ and $d(P)=H(S_n,D_n)+1$.

7-$S_c \neq S_n$, $S_n \neq D_c$, and $S_n=D_n$
P: $(S_c,S_n) \rightarrow (S_n,S_c) \Rightarrow (S_n,D_c) \rightarrow (D_c,S_n)$ and $d(P)=H(S_c,D_c)+1$.

8-$S_c \neq S_n$, $S_n \neq D_c$, $S_c \neq D_n$, and $D_n=D_c$
P: $(S_c,S_n) \Rightarrow (S_c,D_c) \rightarrow (D_c,S_c) \Rightarrow (D_c,D_n)$ and $d(P)=1+H(S_n,D_c)+H(S_c,D_n)$.

When $S_c \neq S_n \neq D_c \neq D_n$, minimizing cluster is used to determine routing path and finding minimizing cluster will be discussed in the following section ♦

**Theorem 6.** The number of node disjoint paths between any pair of nodes in $HEFC_k(n)$ is equal to minimum node degree of corresponding nodes for $n \geq 5$.

**Proof.** The node degree of $HEFC_1(n)$ is between $\left\lceil \frac{n}{3} \right\rceil$ and n-1. So, the number of disjoint paths between two nodes is at least $\left\lceil \frac{n}{3} \right\rceil$. For example, the number of node disjoint paths between node (010,010) and (*,*) (* don't care) is equal to 2, since the node degree of node (010,010) is equal to 2. The node degree of node (101,000) is 4 and the node degree of node (000,001) is 4, so, there must be 4 disjoint paths between these nodes. The node disjoint paths between these nodes are as follow and they are also seen in Figure 4.

Path 1: (101,000)→(101,001)→(001,101)→(001,100)→(001,000)→(000,001).

Path 2: (101,000)→(000,101)→(000,001).

Path 3: (101,000)→(101,010)→(101,011)→(011,101)→(011,100)→(011,000)→(000,011)→(000,001).

Path4:(101,000)→(101,100)→(100,101)→(100,100)→(100,000)→(000,100)→(000,000)→(000,001) ♦

The fourth analysed structural property in this paper is the diameter of $HEFC_1(n)$. The diameter of an $EFC_1(n)$ is n-2 and the diameter of $HEFC_1(n)$ can be derived from diameter of $EFC_1(n)$.

The maximum length of a shortest path in $HEFC_1(n)$ occurs in the case of source and destination nodes in different clusters. From Theorem 3, it can easily seen that cases 1, 4, 5, 6, and 7 may not determine the length of diameter of $HEFC_1(n)$, because the shortest paths in these cases generally have shorter length than diameter. The cases 2, 3, or 8 may determine the diameter of $HEFC_1(n)$. The diameter of $HEFC_k(n)$ is independent on k.

***Theorem 7.*** *The upper bound of diameters of HFC(n), HEFC$_1$(n), …, HEFC$_k$(n) is 2n-3.*

**Proof.** The diameter of HEFC$_1$(n) traverses between two clusters. The length of diameter can be derived from the following routing steps. Let ($S_c$, $S_n$) be source node and ($D_c$, $D_n$) be destination node. Routing steps are taken from proof of Theorem 3.

1-d(P)=H($S_n$,$D_c$)+1≤n-2+1=n-1.
2-d(P)=H($S_n$,$D_c$)+1+H($S_c$,$D_n$)≤n-2+n-2+1=2n-3.
3-d(P)=H($S_n$,$D_c$)+1+H($S_c$,$D_n$)=2H($S_n$,$D_c$)+1=2H($S_c$,$D_n$)+1≤2(n-2)+1=2n-3.
4-d(P)=1.
5-d(P)= H($S_c$,$D_n$)+1≤n-2+1=n-1.
6-d(P)=H($S_n$,$D_n$)+1≤n-2+1=n-1.
7-d(P)=H($S_c$,$D_c$)+1≤n-2+1=n-1.
8-d(P)=H($S_n$,$D_c$)+1+H($S_c$,$D_n$)≤n-2+1+n-2=2n-3.

The length of shortest path in case of $S_c \neq S_n \neq D_c \neq D_n$ discussed in next section ♦

## 5. SELF-SIMILARITY OF HFC(n), HEFC$_1$(n), …, HEFC$_k$(n)

In this section, most of the explanations of decompositions will be done on HEFC$_1$(n) and HFC(n), since HEFC$_2$(n), …, HEFC$_k$(n) have similar decompositions. Because of the different coefficients used in decompositions of HFC(n) and HEFC$_k$(n)s, some theorem for HFC(n) will be given. However, all explanations will be done on HEFC$_1$(n). Each HEFC$_1$(n) can be decomposed to lower sized HEFC$_1$(r)s, r<n. Before giving decomposition of HEFC$_1$(n)s, some definitions must be given.

- If graphs $G_1$ and $G_2$ are **isomorphic**, then it is denoted as $G_1 \approx G_2$. If $G_1$ is a subgraph of $G_2$, it is denoted as $G_1 \subseteq G_2$.
- A subgraph of a graph G=(V,E) **induced** by a subset of its vertices, V'⊆V, is the graph (V',E'), where E'={(i,j)∈E| i,j∈V'}.
- We write $G_1 \cup G_2$ to denote the graph ($V_1 \cup V_2$, $E_1 \cup E_2$) and $G_1 \cap G_2$ to denote the graph ($V_1 \cap V_2$, $E_1 \cap E_2$), and $\bigcup_{i=1}^{n} G_i = G_1 \cup G_2 \cup \cdots \cup G_n$ .
- If $G_1 \cap G_2 = (\emptyset,\emptyset)$, then we write $G_3 = G_1 \nabla G_2$, instead of $G_1 \cup G_2$ to emphasize the $G_3$ consists of two disjoint subgraphs. If all graphs are isomorphic, then $G_1 \nabla G_2 \nabla ... \nabla G_m = m.G$.
- A graph $G_1$ is said to be **directly embedded** in $G_2$, denoted $G_1 \prec G_2$ if and only if there is a subgraph G=(V,E) induced by a subset of its vertices, V'⊆V, is the graph (V',E'), where E'={(i,j)∈E:i,j∈E'}.
- $\nabla_{i=1}^{n} FC(i) = FC(1) \nabla FC(2) \nabla \cdots \nabla FC(n)$.
- e(i) means that if i is an even number, then e(i) returns true, otherwise returns false, and similarly, o(i) returns true, if I is odd, otherwise returns false.

Hsu [5] developed FC(n)s and denoted that all FC(n)s can be decomposed into two smaller different FC(n)s, and he denoted these subcubes as LOW(n) and HIGH(n) which denote the subgraph induced by the set of nodes in {0,1,…,$f_{n-1}$-1}, {$f_{n-1}$,…,$f_n$-1}, respectively. Then

- LOW(n)≈FC(n-1)
- HIGH(n) ≈FC(n-2)
- LOW(n)∩HIGH(n)=(∅,∅)

He defined another important point such as LINK(n)={(i,j): |i-j|=$f_{n-1}$, (i,j)∈E(n)}. FC(n) can be decomposed into FC(n-1) and FC(n-2) and are connected exactly by the set of edges in LINK(n). It is clear that each edge (i,j) in LINK(n) connects a node j in HIGH(n) to a node i=j-$f_{n-1}$ in LOW(n), and no other edges exist between LOW(n) and HIGH(n).

Thus, FC(n) can be decomposed into a subgraph FC(n-1) and a subgraph FC(n-2); moreover, there are exactly $f_{n-2}$ links between the two subgraphs, and this decomposition can be handled recursively. This property is useful when deriving substructures or embeddings of other types of graphs.

FC(6) consists of FC(5) and FC(4), and FC(5) consists of FC(4) and FC(3). So, FC(6)≻(2.FC(4)∇FC(3)). This decomposition can be generalized as follows (k≤n), for all FC(n), n≥5.

- $FC(n) \succ (f_j.FC(n-j+1) \nabla f_{j-1}.FC(n-j))$
- $FC(n) \succ (f_{n-j+1}.FC(j) \nabla f_{n-j}.FC(j-1))$
- $FC(2n) \succ \nabla_{i=1}^{n} FC(2i-1)$
- $FC(2n+1) \succ \nabla_{i=1}^{n} FC(2i)$
- $FC(n+2) \succ \nabla_{i=1}^{n} FC(i)$

In the following, we show that $HEFC_1(n)$ contains disjoint subgraphs $HEFC_1(n-1)$, $HEFC_1(n-2)$, $ef_1(n-1).EFC_1(n-2)$, and $ef_1(n-2).EFC_1(n-1)$. For example, $HEFC_1(5)$ consists of $HEFC_1(4)$, $HEFC_1(3)$, $ef_1(4).EFC_1(3)$, and $ef_1(3).EFC_1(4)$ as shown in Figure 5.

Let us redefine LOW(n), and HIGH(n) with respect to $HEFC_1(n)$. Let $HEFC_1(n)=(V_{H1}(n),E_{H1}(n))$ be a hierarchical Fibonacci cube of dimension n. $LOW_{H1}(n)$ denotes the subgraph induced by the set of nodes in the Cartesian product of the set $\{0,1,...,ef_1(n-1)-1\}$ by itself, $HIGH_{H1}(n)$ denotes the subgraph induced by the set of nodes in the Cartesian product of the set $\{ef_1(n-1),..., ef_1(n)-1\}$ by itself, $LOW_{F1}(n)$ denotes the subgraph induced by the set of nodes in the set $\{ef_1(n-1),..., ef_1(n)-1\}X\{0,1,...,ef_1(n-1)-1\}$, and $HIGH_{F1}(n)$ denotes the subgraph induced by the set of nodes in the set $\{0,1,...,ef_1(n-1)-1\}X\{ef_1(n-1),..., ef_1(n)-1\}$.

***Theorem 8.*** *HFC(n) consists of $LOW_H(n)$, $HIGH_H(n)$, $f_{n-2}LOW_F(n)$, and $f_{n-1}HIGH_F(n)$.*

**Proof.** HFC(n) consists of $f_n.FC(n)$ and $f_n=f_{n-1}+f_{n-2}$.

$$
\begin{aligned}
HFC(n) &\approx f_n.FC(n)\\
&=(f_{n-1}+f_{n-2})FC(n)\\
&=(f_{n-1}+f_{n-2})(FC(n-1)\nabla FC(n-2))\\
&=f_{n-1}.FC(n-1)\nabla f_{n-1}.FC(n-2)\nabla f_{n-2}.FC(n-1)\\
&\quad \nabla f_{n-2}.FC(n-2)\\
&=\underbrace{f_{n-1}.FC(n-1)}_{LOW_H(n)}\nabla\underbrace{f_{n-1}.FC(n-2)}_{HIGH_F(n)}\\
&\quad \nabla\underbrace{f_{n-2}.FC(n-2)}_{HIGH_H(n)}\nabla\underbrace{f_{n-2}.FC(n-1)}_{LOW_F(n)} \quad \blacklozenge
\end{aligned}
$$

***Corollary 1.*** *If HFC(n) consists of $LOW_H(n)$, $HIGH_H(n)$, $f_{n-2}LOW_F(n)$, and $f_{n-1}HIGH_F(n)$, then*

- *$LOW_H(n)\approx HFC(n-1)$*
- *$LOW_F(n)\approx f_{n-2}FC(n-1)$*
- *$HIGH_H(n)\approx HFC(n-2)$*
- *$HIGH_F(n)\approx f_{n-1}FC(n-2)$*

***Theorem 9.*** *$HEFC_k(n)$ consists of $LOW_{Hk}(n)$, $HIGH_{Hk}(n)$, $ef_k(n-2).LOW_{Fk}(n)$, and $ef_k(n-1).HIGH_{Fk}(n)$.*

**Proof.** $HEFC_1(n)$ consists of $ef_1(n).EFC_1(n)$ and $ef_1(n)=ef_1(n-1)+ef_1(n-2)$.

$$
\begin{aligned}
HEFC_1(n) &\approx ef_1(n).EFC_1(n)\\
&=(ef_1(n-1)+ef_1(n-2))EFC_1(n)\\
&=(ef_1(n-1)+ef_1(n-2))(EFC_1(n-1)\nabla EFC_1(n-2))\\
&=ef_1(n-1).EFC_1(n-1)\nabla ef_1(n-1).EFC_1(n-2)\\
&\quad \nabla ef_1(n-2).EFC_1(n-1)\nabla ef_1(n-2).EFC_1(n-2)\\
&=\underbrace{ef_1(n-1).EFC_1(n-1)}_{LOW_{H1}(n)}\nabla\underbrace{ef_1(n-1).EFC_1(n-2)}_{HIGH_{F1}(n)}\\
&\quad \nabla\underbrace{ef_1(n-2).EFC_1(n-2}_{HIGH_{H1}(n)}\nabla\underbrace{ef_1(n-2).EFC_1(n-1)}_{LOW_{F1}(n)}
\end{aligned}
$$

$HEFC_2(n)$, …, $HEFC_k(n)$ have similar structures ♦

***Corollary 2.*** *If $HEFC_k(n)$ consists of $LOW_{Hk}(n)$, $HIGH_{Hk}(n)$, $ef_k(n-2).LOW_{Fk}(n)$, and $ef_k(n-1).HIGH_{Fk}(n)$, then*

- *$LOW_{Hk}(n)\approx HEFC_k(n-1)$*
- *$LOW_{Fk}(n)\approx ef_k(n-2).EFC_k(n-1)$*
- *$HIGH_{Hk}(n)\approx HEFC_k(n-2)$*
- *$HIGH_{Fk}(n)\approx ef_k(n-1).EFC_k(n-2)$*

This property of $HEFC_1(n)$ is useful when deriving substructures or embedding of other types of graphs. It is also a basis for divide-and-conquer algorithms on the $HEFC_1(n)$. So, $HEFC_1(n)$ can be decomposed with respect to Theorem 9 and Corollary 2. LINK(n) defined by Hsu [5] can be redefined for $HEFC_1(n)$, and

$LINK_{H1}(n)=\{((I,J),(K,L)):|I-L|=0,\ |J-K|=0,\ (I,J)\in \{ef_1(n-1),..., ef_1(n)-1\}X\{0,1,...,ef_1(n-1)-1\},\ (K,L)\in \{0,1,...,ef_1(n-1)-1\}\ X\{ef_1(n-1),..., ef_1(n)-1\},\ (I,J)\in E_{H1}(n), (K,L)\in E_{H1}(n)\}$.

All $HEFC_k(n)$ contains $LOW_{Hk}(n)$, $HIGH_{Hk}(n)$, $LOW_{Fk}(n)$, and $HIGH_{Fk}(n)$. A node in $LOW_{Fk}(n)$ is connected to a node in $HIGH_{Fk}(n)$ by a link in $LINK_{Hk}(n)$, and this case is correct for all nodes in $LOW_{Fk}(n)$, and $HIGH_{Fk}(n)$.

**Theorem 10.** HFC(n) has the embedding for $n\geq 5$ such as below.

- $HFC(n)\succ(f_j.HFC(n-j+1)\nabla f_{j-1}.HFC(n-j))$

$$\nabla\left[\bigcup_{i=n-j}^{n-2} f_{n-i-1}\left(f_i.FC(i+1)\nabla f_{i+1}.FC(i)\right)\right]$$

for $2\leq j\leq n-3$.

- $HFC(n) \succ (f_{n-j}.HFC(j+1) \nabla f_{n-j-1}.HFC(j))$

$$\nabla \left[ \bigcup_{i=j}^{n-2} f_{n-i-1}\left(f_i.FC(i+1) \nabla f_{i+1}.FC(i)\right) \right]$$

for $3 \le j \le n-2$.

**Proof.** Proof the first case can be done by using induction on the pair (n,j).

(Base step) When j=2,

$HFC(n) \succ (f_2.HFC(n-1) \nabla f_1.HFC(n-2) \nabla f_3(f_{n-2}.FC(n-1)$
$\nabla f_{n-1}.FC(n-1))$.

(hypothesis) Assume that given relation satisfied for $j \le n-3$, and then

$HFC(n) \succ (2HFC(n-2) \nabla HFC(n-3) \nabla f_{n-3}.FC(n-2)$
$\nabla f_{n-2}.FC(n-3) \nabla f_{n-2}.FC(n-1) \nabla f_{n-1}.FC(n-2))$.

(Concluding step) Hypothesis step can be used to obtain result of concluding step.

$HFC(n) \succ (2HFC(n-2) \nabla HFC(n-3) \nabla f_{n-3}.FC(n-2)$
$\nabla f_{n-2}.FC(n-3) \nabla f_{n-2}.FC(n-1) \nabla f_{n-1}.FC(n-2)$
$\succ ([HFC(n-2) \nabla HFC(n-3) \nabla f_{n-3}.FC(n-2)$
$\nabla f_{n-2}.FC(n-3)] \nabla HFC(n-2) \nabla f_{n-2}.FC(n-1)$
$\nabla f_{n-1}.FC(n-2))$
$\succ (HFC(n-1) \nabla HFC(n-2) \nabla f_{n-2}.FC(n-1)$
$\nabla f_{n-1}.FC(n-2))$.

Similar proof can be done for second case ♦

**Theorem 11.** $HEFC_k(n)$ has the embeddings seen in Fig. 6 for $n \ge 5$ $(1 \le k \le n-2)$.

**Proof.** Proof of the first case can be done by using induction on the pair (n,j).

(Base step) When j=4,

$HEFC_1(n) \succ (f_4.HEFC_1(n-3) \nabla f_3.HEFC_1(n-4)$
$\nabla (f_1.ef_1(n-1)+f_2.ef_1(n-3))EFC_1(n-2)$
$\nabla (f_2.ef_1(n-2)+f_3.ef_1(n-4))EFC_1(n-3)$
$\nabla f_1.ef_1(n-2).EFC_1(n-1) \nabla f_3.ef_1(n-3).$
$EFC_1(n-4))$.

(Hypothesis) Assume that given relation satisfied for $j \le n-4$, and then

$HEFC_1(n) \succ (f_{n-4}.HEFC_1(5) \nabla f_{n-5}.HEFC_1(4)$
$\nabla (f_1.ef_1(n-1)+f_2.ef_1(n-3))EFC_1(n-2)$
$\nabla (f_2.ef_1(n-2)+f_3.ef_1(n-4))EFC_1(n-3)$
$\nabla f_1.ef_1(n-2).EFC_1(n-1) \nabla f_3.ef_1(n-3).$
$EFC_1(n-4))$.

(Concluding step) Hypothesis step can be used to obtain result of concluding step.

$HEFC_1(n) \succ (f_{n-4}.HEFC_1(5) \nabla f_{n-5}.HEFC_1(4)$
$\nabla (f_1.ef_1(n-1)+f_2.ef_1(n-3))EFC_1(n-2)$
$\nabla (f_2.ef_1(n-2)+f_3.ef_1(n-4))EFC_1(n-3)$
$\nabla \ldots \nabla (f_{n-6}.ef_1(6)+f_{n-5}.ef_1(4))EFC_1(5)$
$\nabla f_1.ef_1(n-2).EFC_1(n-1) \nabla f_{n-5}.ef_1(5).$
$EFC_1(4))$.
$\succ (f_{n-3}.HEFC_1(4) \nabla f_{n-4}.HEFC_1(3)$
$\nabla (f_1.ef_1(n-1)+f_2.ef_1(n-3))EFC_1(n-2)$
$\nabla (f_2.ef_1(n-2)+f_3.ef_1(n-4))EFC_1(n-3)$
$\nabla \ldots \nabla (f_{n-6}.ef_1(6)+f_{n-5}.ef_1(4))EFC_1(5)$
$\nabla (f_{n-5}.ef_1(5)+f_{n-4}.ef_1(3))EFC_1(4)$
$\nabla f_1.ef_1(n-2).EFC_1(n-1)$
$\nabla f_{n-4}.ef_1(4).EFC_1(3))$.

Similar proof can be done for second case ♦

**Theorem 12.** Assume that $n \ge 7$ and HFC(n) is a hierarchical Fibonacci cube, then the the embeddings of HFC(n) depend on the value of j either odd or even.. When j is even, the embeddings in Fig. 7 can be obtrained.

**Proof.** If we prove first case, then all other cases can be proved in similar way. When k is even, 2n-j+1 is odd and 2n-j is even. Due to values of j, first case is based on the following recurrences.

$$f_j \sum_{m=1}^{\frac{2n-j}{2}} f_{2m} + 1 + f_{j-1} \sum_{m=1}^{\frac{2n-j}{2}} f_{2m-1} +$$
$$\sum_{i=2n-j}^{2n-2} f_{2n-i-1}\left(\left(\sum_{r=0}^{i-1} f_r + 1\right) + \left(\sum_{m=0}^{i-2} f_m + 1\right)\right)$$

All other cases can be proved in similar ways ♦

**Theorem 13.** Assume that n≥7 and $HEFC_k(n)$ is a hierarchical extended Fibonacci cube, then the embeddings of $HEFC_k(n)$ depend on the value of j either odd or even. When j is even, the embeddings in the Fig. 8 can be obtained.

**Proof.** If we prove first case, then all other cases can be proved in similar way. When j is even, 2n-j+1 is odd and 2n-j is even. Due to values of j, first case is based on the following recurrences.

$$f_j \sum_{m=1}^{\frac{2n-j}{2}} f_{2m} + 1 + f_{j-1} \sum_{m=1}^{\frac{2n-j}{2}} f_{2m-1} +$$
$$\sum_{i=i}^{j-2}\left( (f_i.ef_k(2n-i) + f_{i+1}.ef_k(2n-i-2)) \sum_{r=0}^{n-i-3}(f_r+1)\right)$$
$$+ f_1.ef_k(2n-2)\sum_{r=1}^{n-1}(f_{2r}+1)$$
$$+ f_{j-1}.ef_k(2n-j+1)\sum_{r=0}^{2n-j-1} f_{2r-1}$$

All other cases can be proved in similar ways ♦

**Theorem 14.** Assume that n≥5 and HFC(n) is a hierarchical Fibonacci cube, then the embeddings in Fig.9 can be obtained (j is odd).

**Proof.** HFC(n)s are based on Fibonacci series $f_n^2 = (f_{n-1} + f_{n-2})^2$. This series has the following recursive properties (for 2≤j≤n-3).

$$f_{2n} = \sum_{m=1}^{n-1} f_{2m-1} + \sum_{i=2n-j}^{2n-2} f_{2n-i-1}\left(\left(\sum_{s=0}^{i-1} f_s + 1\right) + \left(\sum_{r=0}^{i-2} f_r + 1\right)\right)$$

$$f_{2n+1} = \sum_{m=1}^{n} f_{2m} + 1 + \sum_{i=2n-j+1}^{2n-1} f_{2n-i}\left(\left(\sum_{s=0}^{i-1} f_s + 1\right) + \left(\sum_{r=0}^{i-2} f_r + 1\right)\right)$$

$$f_{n+2} = \sum_{m=0}^{n} f_m + 1 + \sum_{i=2n-j}^{2n-2} f_{2n-i-1}\left(\left(\sum_{s=0}^{i-1} f_s + 1\right) + \left(\sum_{r=0}^{i-2} f_r + 1\right)\right)$$

Thus, HFC(n)s can be also constructed in same manner ($f_m$s are used for HFC(n)s; $f_s$ and $f_r$ are used for FC(n)s ) ♦

**Theorem 15.** Assume that n≥7 and $HEFC_k(n)$ is a hierarchical extended Fibonacci cube. The embeddings in the Fig.10 can be obtained, when j is odd.

**Proof.** If we prove first case, then all other cases can be proved in similar way. When j is odd, 2n-j+1 is even and 2n-j is odd. Due to the values of j, first case is based on the following recurrences.

$$f_j \sum_{m=1}^{\frac{2n-j}{2}} f_{2m} + 1 + f_{j-1} \sum_{m=1}^{\frac{2n-j}{2}} f_{2m-1} +$$
$$\sum_{i=i}^{j-2}\left( (f_i.ef_k(2n-i) + f_{i+1}.ef_k(2n-i-2)) \sum_{r=0}^{n-i-3}(f_r+1)\right)$$
$$+ f_1.ef_k(2n-2)\sum_{r=1}^{n-1}(f_{2r}+1)$$
$$+ f_{j-1}.ef_k(2n-j+1)\sum_{r=0}^{2n-j-1} f_{2r-1}$$

All other cases can be proved in similar ways ♦

**Theorem 16.** Assume that n≥7 and HFC(n) is a hierarchical Fibonacci cube. Then embeddings in the Fig. 11 are valid, while j is odd.

**Proof.** If we prove first case, then all other cases can be proved in similar way. When j is odd, 2n-j+1 is even and 2n-j is odd. Due to values of j, first case is based on the following recurrences.

j is odd: First case is based on the recurrence

$$f_j \sum_{m=1}^{\frac{2n-j+1}{2}} f_{2m-1} + f_{j-1} \sum_{m=1}^{\frac{2n-j-1}{2}} f_{2m} + 1 +$$
$$\sum_{i=2n-j}^{2n-2} f_{2n-i-1}\left(\left(\sum_{r=0}^{i-1} f_r + 1\right) + \left(\sum_{m=0}^{i-2} f_m + 1\right)\right)$$

All other cases can be proved in similar ways ♦

**Theorem 17.** Assume that n≥5 and $HEFC_k(n)$ is a hierarchical extended Fibonacci cube. The embeddings in Fig.12 are valid for $HEFC_k(n)$, while j is odd.

## 6. CONCLUSION AND FUTURE RESEARCH

The constructed graphs HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ are special proper subgraphs of HCN(n-2,n-2) for k<n-1. $HEFC_k(n)$s can be constructed recursively from $HEFC_k(n-1)$,

$HEFC_k(n-2)$, $EFC_k(n-1)$, and $EFC_k(n-2)$ and HFC(n)s can be constructed recursively from HFC(n-1), HFC(n-2), FC(n-1), and FC(n-2). The obtained graphss are more sparse than HCN(n-2,n-2) and they have self-similarity property. The properties of HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ can be summarized as follows.

- The number of nodes in HFC(n) is $(f_n)^2$ and the number of nodes in $HEFC_k(n)$ is $(ef_k(n))^2$
- The number of edges in HFC(n) is $f_n\left[\frac{2(n-1)f_n - nf_{n-1}}{5}\right] + \frac{f_n^2 - f_n}{2}$ and the number of edges in $HEFC_k(n)$ is

  $$ef_k(n).\left[|E_k(k+3)|f_{n-k-2} + |E_k(k+2)|f_{n-k-3} + \sum_{i=1}^{n-k-3} f_i.ef_k(n-i-1)\right] + \frac{ef_k(n)(ef_k(n)-1)}{2}.$$

- HFC(n)s have node degrees between $\left\lceil\frac{n+1}{3}\right\rceil - 1$ and n-1 and the node degree for $HEFC_k(n)$ is between $\left\lceil\frac{n-(k-1)}{3}\right\rceil + (k-1)$ and n-1.
- All HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ can be decomposed to lower sized HFC(r), $HEFC_1(r)$, …, $HEFC_k(r)$, r<n. Thus, HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ have self-similarity property. These graphs have recurrent structures which are essential in developing fault-tolerant schemes.
- Due to HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ having recurrent structures, recursive-descent and recursive-doubling algorithms can be developed on HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ easily, if these graphs are used as interconnection networks.
- If path P in one of HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ contains three or more diagonal links, then P is not a shortest path.
- Any shortest path in one of HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ contains at most two diagonal links.
- The upper bound for the shortest paths in one of HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ and diameter of HFC(n), $HEFC_1(n)$, …, $HEFC_k(n)$ are 2n-3.

## REFERENCES

[1] A. Esfahanian, L.M. Ni and B. E. Sagan, "The Twisted n-cube with application to multiprocessing", IEEE Transactions on Computers, vol.40, pp.88-93, 1991.

[2] K. Efe, "A variation on the hypercube with lower diameter", IEEE Transactions on Computers, vol. 40, pp. 1312-1316, 1991.

[3] A. S. Vaidya, P. S. N. Rao, and S. R. Shankar, "A class of hypercube-like networks", Tech. Rep. EE/01/93, Dept. of Electrical Eng., Indian Institute of Science, Bangalore, 1993.

[4] K. Chose, and K. R. Desai, "Hierarchical Cubic Networks", IEEE Transactions on Parallel and Distributed Systems, vol. 6, pp.427-435, 1995.

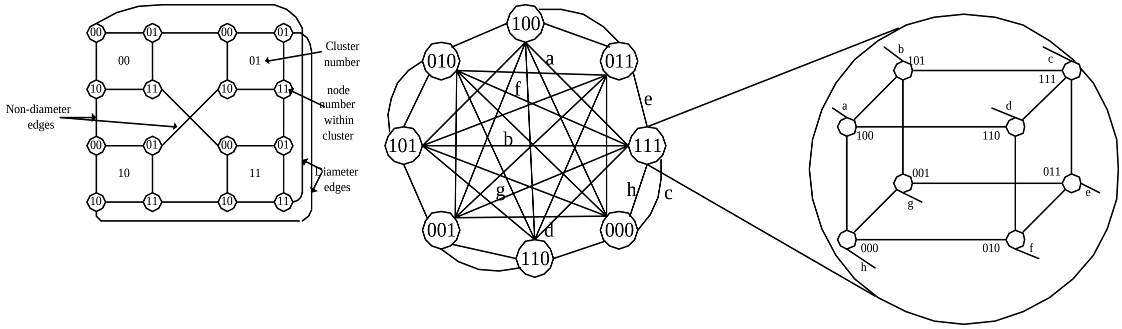
[5] W.-J. Hsu, "Fibonacci Cubes – A New Interconnection Topology", IEEE Transactions on Parallel and Distributed Systems, vol. 4, pp. 3-12, 1993.

[6] A. Karci, "New Interconnection Netwroks: Fibonacci Cube and Extended Fibonacci Cubes Based Hierarchic Networks", IEEE: The 15th International Conference on Information Networking (ICOIN-15), Jan. 31 - Feb. 2, 2001, Beppu Convention(B-Con) Plaza, Beppu City, Japan.

[7] A. Karci, "Recursive Construction of Hierarchical Fibonacci Cubes and Hierarchical Extended Fibonacci Cubes", IEEE: 2001 International Conference on Parallel and Distributed Systems (ICPADS-2001), June 26-29, 2001, KyongJu city, Korea.

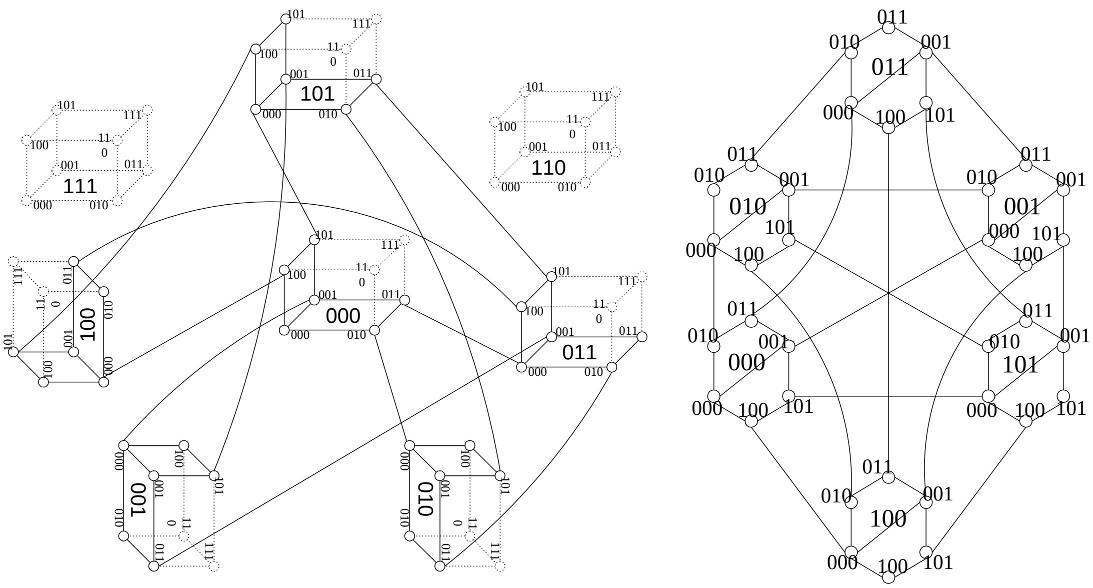
[8] J. Duato, S. Yalamanchili, and L. Ni, "Interconnection networks – An engineering approach", IEEE Computer Society Press, 1997.

## Appendix




(a) (b)

***Figure 1*** *(a) HCN(2,2), (b) HCN(3,3).*



(a) (b)

**Figure 2.** HEFC$_1$(5)s: (a) Deriving HEFC$_1$(5) from HCN(3,3); (b) HEFC$_1$(5).

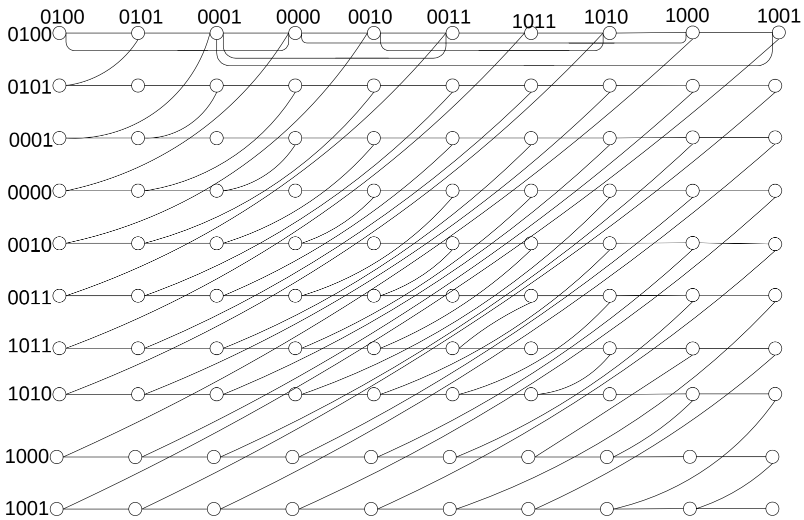



***Figure 3.*** *HEFC$_1$(6).*

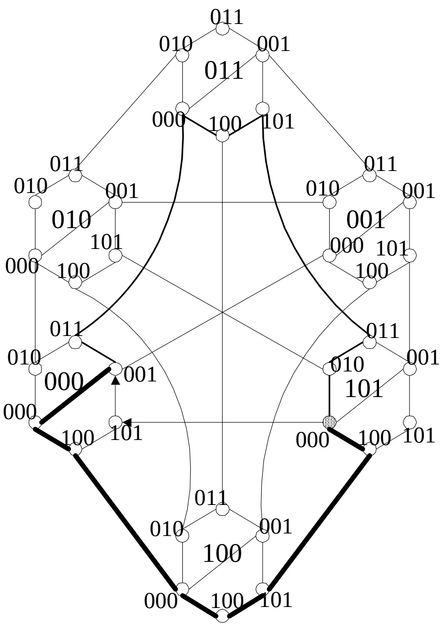



***Figure 4.*** *Disjoint paths between nodes (101,000) and (000,001)*

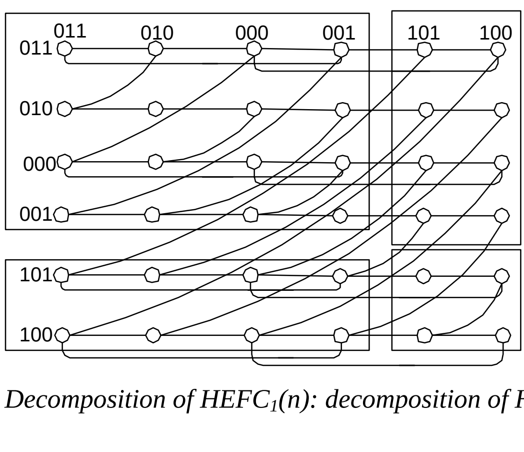



***Figure 5.*** *Decomposition of HEFC$_1$(n): decomposition of HEFC$_1$(5).*

- $HEFC_k(n) \succ (f_j.HEFC_k(n-j+1) \nabla f_{j-1}.HEFC_k(n-j)) \nabla \left[ \bigcup_{i=1}^{j-2} \left( \left( f_i.ef_k(n-i) + f_{i+1}.ef_k(n-i-2) \right) EFC_k(n-i-1) \right) \right]$
  $\nabla f_1.ef_j(n-2).EFC_j(n-1) \nabla f_{j-1}.ef_j(n-j+1).EFC_j(n-j)$

for 4≤j≤n-3.

- $HEFC_k(n) \succ (f_{n-(j-1)}.HEFC_k(j) \nabla f_{n-j}.HEFC_k(j-1)) \nabla \left[ \bigcup_{i=n-(j+1)}^{n-3} \left( \left( f_{i-2}.ef_k(n-i+2) + f_{i-1}.ef_k(n-i) \right) EFC_k(n-i+1) \right) \right]$
  $\nabla f_1.ef_k(n-2).EFC_k(n-1) \nabla f_j.ef_k(n-j).EFC_k(n-j-1)$

for 6≤k≤n-3.

***Figure 6**. The embeddings of HEFC$_k$(n) for n≥5 (Theorem 11).*

- $HFC(2n) \succ \left( f_j \nabla_{m=1}^{\frac{2n-j}{2}} HFC(2m) \right) \nabla \left( f_{j-1} \nabla_{m=1}^{\frac{2n-j}{2}} HFC(2m-1) \right) \nabla \left[ \bigcup_{i=2n-j}^{2n-2} f_{2n-i-1} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$

for 2≤j≤n-5.

- $HFC(2n+1) \succ \left( f_j \nabla_{m=0}^{\frac{2n-j}{2}} HFC(2m+1) \right) \nabla$
  $\left( f_{j-1} \nabla_{m=1}^{\frac{2n-j}{2}} HFC(2m) \right) \nabla \bigcup_{i=2n-j+1}^{2n-1} f_{2n-i} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\}$

for 2≤j≤n-5.

- $HFC(n+2) \succ \left( f_j \nabla_{m=1}^{n-j+1} HFC(m) \right) \nabla \left( f_{j-1} \nabla_{m=1}^{n-j} HFC(m) \right) \nabla \left[ \bigcup_{i=n-j+2}^{n} f_{n-i+1} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$

for 2≤j≤n-5.

- $HFC(2n) \succ \left( f_{2n-j} \nabla_{m=1}^{\frac{j}{2}} HFC(2m) \right) \nabla \left( f_{2n-j-1} \nabla_{m=1}^{\frac{j}{2}} HFC(2m-1) \right) \nabla \left[ \bigcup_{i=j}^{2n-2} f_{2n-i-1} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$

for 3≤j≤n-2.

- $HFC(2n+1) \succ \left( f_{2n+1-j} \nabla_{m=1}^{\frac{j}{2}} HFC(2m) \right) \nabla \left( f_{2n-j} \nabla_{m=1}^{\frac{j}{2}} HFC(2m-1) \right) \nabla \left[ \bigcup_{i=j}^{2n-1} f_{2n-i} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$

for 3≤j≤n-2.

- $HFC(n+2) \succ \left( f_{n+2-j} \nabla_{m=1}^{\frac{j}{2}} HFC(2m) \right) \nabla \left( f_{n-j+1} \nabla_{m=1}^{\frac{j}{2}} HFC(2m-1) \right) \nabla \left[ \bigcup_{i=j}^{n} f_{n-i+1} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$

for 3≤j≤n-2.

***Figure 7**. Embeddings of HFC(n) for even values of j ( Theorem 12).*

- $HEFC_k(2n) \succ \left(f_j \nabla_{m=2}^{\frac{2n-j}{2}} HEFC_k(2m)\right) \nabla \left(f_{j-1} \nabla_{m=2}^{\frac{2n-j}{2}} HEFC_k(2m-1)\right)$

$$\nabla\left[\bigcup_{i=1}^{j-2}\left\{\left(f_i.ef_k(2n-i)+f_{i+1}.ef_k(2n-i-2)\right)\left(\nabla_{m=3}^{2n-i-3}EFC_k(m)\right)\right\}\right]$$

$$\nabla\left(f_1.ef_k(2n-2)\nabla_{m=2}^{n-1}EFC_k(2m)\right)\nabla\left(f_{j-1}ef_k(2n-j+1)\nabla_{m=2}^{\frac{2n-j}{2}}EFC_k(2m-1)\right) \text{ for } 4\le j\le n\text{-}3.$$

- $HEFC_k(2n+1) \succ \left(f_j \nabla_{m=1}^{\frac{2n-j}{2}} HEFC_k(2m+1)\right) \nabla \left(f_{j-1} \nabla_{m=2}^{\frac{2n-j}{2}} HEFC_k(2m)\right)$

$$\nabla\left[\bigcup_{i=1}^{j-2}\left\{\left(f_i.ef_k(2n-i+1)+f_{i+1}.ef_k(2n-i-1)\right)\nabla_{m=3}^{2n-i-2}EFC_k(m)\right\}\right]$$

$$\nabla\left(f_1ef_k(2n-1)\nabla_{m=1}^{n-1}EFC_k(2m+1)\right)\nabla\left(f_{j-1}ef_k(2n-j+2)\nabla_{m=2}^{\frac{2n-j}{2}}EFC_k(2m)\right) \text{ for } 4\le j\le n\text{-}3.$$

- $HEFC_k(n+2) \succ \left(f_j \nabla_{m=3}^{n-j+1} HEFC_k(m)\right) \nabla \left(f_{j-1} \nabla_{m=3}^{n-j} HEFC_k(m)\right)$

$$\nabla\left[\bigcup_{i=1}^{j-2}\left\{\left(f_i.ef_k(n-i+2)+f_{i+1}.ef_k(n-i)\right)\nabla_{m=3}^{n-i-1}EFC_k(m)\right\}\right]$$

$$\nabla\left(f_1ef_k(n)\nabla_{m=3}^{n-1}EFC_k(m)\right)\nabla\left(f_{j-1}ef_k(n-j+3)\nabla_{m=3}^{n-j}EFC_k(m)\right) \text{for } 4\le j\le n\text{-}3.$$

- $HEFC_k(2n) \succ \left(f_{2n-j+1} \nabla_{m=2}^{\frac{j}{2}} HEFC_k(2m-1)\right) \nabla \left(f_{2n-j} \nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m)\right)$

$$\nabla\left[\bigcup_{i=2n-(j+1)}^{2n-3}\left\{\left(f_i.ef_k(2n-i+2)+f_{i+1}.ef_k(2n-i)\right)\nabla_{m=3}^{2n-i-1}EFC_k(m)\right\}\right]$$

$$\nabla\left(f_1ef_k(2n-2)\nabla_{m=2}^{n-1}EFC_k(2m)\right)\nabla\left(f_{j-1}ef_k(2n-j)\nabla_{m=2}^{\frac{2n-j-2}{2}}EFC_k(2m)\right) \text{ for } 7\le j\le n\text{-}3.$$

- $HEFC_k(2n+1) \succ \left(f_{2n-j+2} \nabla_{m=2}^{\frac{j}{2}} HEFC_k(2m-1)\right) \nabla \left(f_{2n-j+1} \nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m)\right)$

$$\nabla\left[\bigcup_{i=2n-j}^{2n-2}\left\{\left(f_i.ef_k(2n-i+3)+f_{i+1}.ef_k(2n-i+1)\right)\nabla_{m=3}^{2n-i}EFC_k(m)\right\}\right]$$

$$\nabla\left(f_1ef_k(2n-1)\nabla_{m=2}^{n}EFC_k(2m-1)\right)\nabla\left(f_{j-1}ef_k(2n-j+1)\nabla_{m=2}^{\frac{2n-j}{2}}EFC_k(2m-1)\right) \text{ for } 7\le j\le n\text{-}3.$$

- $HEFC_k(n+2) \succ \left(f_{n-j+1} \nabla_{m=2}^{\frac{j}{2}} HEFC_k(2m-1)\right) \nabla \left(f_{n-j+2} \nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m)\right)$

$$\nabla\left[\bigcup_{i=n-j+1}^{n-1}\left\{\left(f_i.ef_k(n-i+4)+f_{i+1}.ef_k(n-i+2)\right)\nabla_{m=3}^{n-i+1}EFC_k(m)\right\}\right]$$

$$\nabla\left(f_1ef_k(n)\nabla_{m=3}^{n-1}EFC_k(m)\right)\nabla\left(f_{j-1}ef_k(n-j+2)\nabla_{m=3}^{n-j-1}EFC_k(m)\right) \text{for } 7\le j\le n\text{-}3.$$

***Figure 8.*** *Embeddings of $HEFC_k(n)$ for even values of j (Theorem 13).*

- $HFC(2n) \succ \left(\nabla_{m=1}^{n} HFC(2m-1)\right) \nabla \left[ \bigcup_{i=2n-j}^{2n-2} f_{2n-i-1} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$ for $2 \le j \le n-3$.
- $HFC(2n+1) \succ \left(\nabla_{m=1}^{n} HFC(2m)\right) \nabla \left[ \bigcup_{i=2n-j+1}^{2n-1} f_{2n-i} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$ for $2 \le j \le n-3$.
- $HFC(n+2) \succ \left(\nabla_{m=1}^{n} HFC(m)\right) \nabla \left[ \bigcup_{i=n-j+2}^{n} f_{n-i+1} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$ for $2 \le j \le n-3$.
- $HFC(2n) \succ \left(f_{n-k+1} \nabla_{m=1}^{n} HFC(2m-1)\right) \nabla \left[ \bigcup_{i=j}^{2n-2} f_{2n-i-1} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$ for $3 \le j \le n-2$.
- $HFC(2n+1) \succ \left(\nabla_{m=1}^{n} HFC(2m)\right) \nabla \left[ \bigcup_{i=j}^{2n-1} f_{2n-i} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$ for $3 \le j \le n-2$.
- $HFC(n+2) \succ \left( \nabla_{m=1}^{\frac{n}{2}} HFC(2m) \right) \nabla \left[ \bigcup_{i=j}^{n} f_{n-i+1} \left\{ f_i \nabla_{m=1}^{i-1} FC(m) \right\} \nabla \left\{ f_{i+1} \nabla_{m=1}^{i-2} FC(m) \right\} \right]$ for $3 \le j \le n-2$.

***Figure 9.*** *Embeddings of HFC(n) for odd values of j (Theorem 14).*

- $$HEFC_k(2n) \succ \left( f_j \nabla_{m=1}^{\frac{2n-j-1}{2}} HEFC_k(2m+1) \right) \nabla \left( f_{j-1} \nabla_{m=2}^{\frac{2n-j-1}{2}} HEFC_k(2m) \right)$$
$$\nabla \left[ \bigcup_{i=1}^{j-2} \left\{ \left( f_i.ef_k(2n-i) + f_{i+1}.ef_k(2n-i-2) \right) \left( \nabla_{m=3}^{2n-i-3} EFC_k(m) \right) \right\} \right]$$
$$\nabla \left( f_1 ef_k(2n-2) \nabla_{m=2}^{n-1} EFC_k(2m) \right) \nabla \left( f_{j-1} ef_k(2n-j+1) \nabla_{m=2}^{\frac{2n-j-1}{2}} EFC_k(2m) \right) \text{ for } 4 \le j \le n\text{-}3.$$

- $$HEFC_k(2n+1) \succ \left( f_j \nabla_{m=2}^{\frac{2n-j+1}{2}} HEFC_k(2m) \right) \nabla \left( f_{j-1} \nabla_{m=1}^{\frac{2n-j-1}{2}} HEFC_k(2m+1) \right)$$
$$\nabla \left[ \bigcup_{i=1}^{j-2} \left\{ \left( f_i.ef_k(2n-i+1) + f_{i+1}.ef_k(2n-i-1) \right) \left( \nabla_{m=3}^{2n-i-2} EFC_k(m) \right) \right\} \right]$$
$$\nabla \left( f_1 ef_k(2n-1) \nabla_{m=1}^{n-1} EFC_k(2m+1) \right) \nabla \left( f_{j-1} ef_k(2n-j+2) \nabla_{m=1}^{\frac{2n-j-1}{2}} EFC_k(2m+1) \right) \text{ for } 4 \le j \le n\text{-}3$$

- $$HEFC_k(n+2) \succ \left( f_j \nabla_{m=3}^{n-j+1} HEFC_k(m) \right) \nabla \left( f_{j-1} \nabla_{m=3}^{n-j} HEFC_k(m) \right)$$
$$\nabla \left[ \bigcup_{i=1}^{j-2} \left\{ \left( f_i.ef_k(n-i+2) + f_{i+1}.ef_k(n-i) \right) \left( \nabla_{m=3}^{n-i-1} EFC_k(m) \right) \right\} \right]$$
$$\nabla \left( f_1 ef_k(n) \nabla_{m=3}^{n-1} EFC_k(m) \right) \nabla \left( f_{j-1} ef_k(n-j+3) \nabla_{m=3}^{n-j} EFC_k(2m) \right) \text{for } 4 \le j \le n\text{-}3$$

- $$HEFC_k(2n) \succ \left( f_{2n-j+1} \nabla_{m=1}^{\frac{j}{2}} HEFC_k(2m+1) \right) \nabla \left( f_{2n-j} \nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m) \right)$$
$$\nabla \left[ \bigcup_{i=2n-(j+1)}^{2n-3} \left\{ \left( f_i.ef_k(2n-i+2) + f_{i+1}.ef_k(2n-i) \right) \left( \nabla_{m=3}^{2n-i-1} EFC_k(m) \right) \right\} \right]$$

- $$\nabla \left( f_1 ef_k(2n-2) \nabla_{m=2}^{n-1} EFC_k(2m) \right) \nabla \left( f_{j-1} ef_k(2n-j) \nabla_{m=1}^{\frac{2n-j-3}{2}} EFC_k(2m+1) \right) \text{ for } 7 \le j \le n\text{-}3.$$

- $$HEFC_k(2n+1) \succ \left( f_{2n-j+2} \nabla_{m=1}^{\frac{j-2}{2}} HEFC_k(2m+1) \right) \nabla \left( f_{2n-j+1} \nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m) \right)$$
$$\nabla \left[ \bigcup_{i=2n-j}^{2n-2} \left\{ \left( f_i.ef_k(2n-i+3) + f_{i+1}.ef_k(2n-i+1) \right) \left( \nabla_{m=3}^{2n-i} EFC_k(m) \right) \right\} \right]$$
$$\nabla \left( f_1 ef_k(2n-1) \nabla_{m=1}^{n-1} EFC_k(2m+1) \right) \nabla \left( f_{j-1} ef_k(2n-j+1) \nabla_{m=2}^{\frac{2n-j-1}{2}} EFC_k(2m) \right) \text{ for } 7 \le j \le n\text{-}3$$

- $$HEFC_k(n+2) \succ \left( f_{n-j+1} \nabla_{m=1}^{\frac{j-2}{2}} HEFC_k(2m+1) \right) \nabla \left( f_{n-j+2} \nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m) \right)$$
$$\nabla \left[ \bigcup_{i=n-j+1}^{n-1} \left\{ \left( f_i.ef_k(n-i+4) + f_{i+1}.ef_k(n-i+2) \right) \left( \nabla_{m=3}^{n-i+1} EFC_k(m) \right) \right\} \right]$$
$$\nabla \left( f_1 ef_k(n) \nabla_{m=3}^{n-1} EFC_k(m) \right) \nabla \left( f_{j-1} ef_k(n-j+2) \nabla_{m=3}^{n-j-1} EFC_k(2m) \right) \text{for } 7 \le j \le n\text{-}3$$

***Figure 10.*** *Theorem 15.*

- $$\mathrm{HFC}(2n)\succ\left(f_j\nabla_{m=1}^{\frac{2n-j+1}{2}}\mathrm{HFC}(2m-1)\right)\nabla\left(f_{j-1}\nabla_{m=1}^{\frac{2n-j-1}{2}}\mathrm{HFC}(2m)\right)\nabla\left[\bigcup_{i=2n-j}^{2n-2}f_{2n-i-1}\left\{f_i\nabla_{m=1}^{i-1}\mathrm{FC}(m)\right\}\nabla\left\{f_{i+1}\nabla_{m=1}^{i-2}\mathrm{FC}(m)\right\}\right]$$ for $2\le j\le n-5$.
- $$\mathrm{HFC}(2n+1)\succ\left(f_j\nabla_{m=1}^{\frac{2n-j+1}{2}}\mathrm{HFC}(2m)\right)\nabla\left(f_{j-1}\nabla_{m=1}^{\frac{2n-j+1}{2}}\mathrm{HFC}(2m-1)\right)\nabla\left[\bigcup_{i=2n-j+1}^{2n-1}f_{2n-i}\left\{f_i\nabla_{m=1}^{i-1}\mathrm{FC}(m)\right\}\nabla\left\{f_{i+1}\nabla_{m=1}^{i-2}\mathrm{FC}(m)\right\}\right]$$ for $2\le j\le n-5$.
- $$\mathrm{HFC}(n+2)\succ\left(f_j\nabla_{m=1}^{n-j+1}\mathrm{HFC}(m)\right)\nabla\left(f_{j-1}\nabla_{m=1}^{n-j}\mathrm{HFC}(m)\right)\nabla\left[\bigcup_{i=n-j+2}^{n}f_{n-i+1}\left\{f_i\nabla_{m=1}^{i-1}\mathrm{FC}(m)\right\}\nabla\left\{f_{i+1}\nabla_{m=1}^{i-2}\mathrm{FC}(m)\right\}\right]$$ for $2\le j\le n-5$.
- $$\mathrm{HFC}(2n)\succ\left(f_{2n-j}\nabla_{m=1}^{\frac{j}{2}}\mathrm{HFC}(2m)\right)\nabla\left(f_{2n-j-1}\nabla_{m=1}^{\frac{j}{2}}\mathrm{HFC}(2m-1)\right)\nabla\left[\bigcup_{i=j}^{2n-2}f_{2n-i-1}\left\{f_i\nabla_{m=1}^{i-1}\mathrm{FC}(m)\right\}\nabla\left\{f_{i+1}\nabla_{m=1}^{i-2}\mathrm{FC}(m)\right\}\right]$$ for $3\le j\le n-2$.
- $$\mathrm{HFC}(2n+1)\succ\left(f_{2n-j+1}\nabla_{m=1}^{\frac{j}{2}}\mathrm{HFC}(2m)\right)\nabla\left(f_{2n-j}\nabla_{m=1}^{\frac{j}{2}}\mathrm{HFC}(2m-1)\right)\nabla\left[\bigcup_{i=j}^{2n-1}f_{2n-i}\left\{f_i\nabla_{m=1}^{i-1}\mathrm{FC}(m)\right\}\nabla\left\{f_{i+1}\nabla_{m=1}^{i-2}\mathrm{FC}(m)\right\}\right]$$ for $3\le j\le n-2$.
- $$\mathrm{HFC}(n+2)\succ\left(f_{n-j+2}\nabla_{m=1}^{\frac{j}{2}}\mathrm{HFC}(2m)\right)\nabla\left(f_{n-j+1}\nabla_{m=1}^{\frac{j}{2}}\mathrm{HFC}(2m-1)\right)\nabla\left[\bigcup_{i=j}^{n}f_{n-i+1}\left\{f_i\nabla_{m=1}^{i-1}\mathrm{FC}(m)\right\}\nabla\left\{f_{i+1}\nabla_{m=1}^{i-2}\mathrm{FC}(m)\right\}\right]$$ for $3\le j\le n-2$.

***Figure 11.*** *Theorem 16.*

- $HEFC_k(2n) \succ \left(\nabla_{m=1}^{n-1} HEFC_k(2m+1)\right) \nabla \left[\bigcup_{i=1}^{j-2} \left\{\left(f_i.ef_k(2n-i) + f_{i+1}.ef_k(2n-i-2)\right)\nabla_{m=3}^{2n-i-3} EFC_k(m)\right\}\right]$

  $\nabla\left(f_1 ef_k(2n-2)\nabla_{m=2}^{n-1} EFC_k(2m)\right)\nabla\left(f_{j-1} ef_k(2n-j+1)\nabla_{m=2}^{\frac{2n-j-1}{2}} EFC_k(2m)\right)$ for 4≤j≤n-3.

- $HEFC_k(2n+1) \succ \left(\nabla_{m=2}^{n} HEFC_k(2m)\right) \nabla \left[\bigcup_{i=1}^{j-2} \left\{\left(f_i.ef_k(2n-i+1) + f_{i+1}.ef_k(2n-i-1)\right)\nabla_{m=3}^{2n-i-2} EFC_k(m)\right\}\right]$

  $\nabla\left(f_1 ef_k(2n-1)\nabla_{m=1}^{n-1} EFC_k(2m+1)\right)\nabla\left(f_{j-1} ef_k(2n-j+2)\nabla_{m=1}^{\frac{2n-j-1}{2}} EFC_k(2m+1)\right)$ for 4≤j≤n-3.

- $HEFC_k(n+2) \succ \left(\nabla_{m=3}^{n} HEFC_k(m)\right) \nabla \left[\bigcup_{i=1}^{j-2} \left\{\left(f_i.ef_k(n-i+2) + f_{i+1}.ef_k(n-i)\right)\nabla_{m=3}^{n-i-1} EFC_k(2m+1)\right\}\right]$

  $\nabla\left(f_1 ef_k(n)\nabla_{m=3}^{n-1} EFC_k(m)\right)\nabla\left(f_{j-1} ef_k(n-j+3)\nabla_{m=3}^{n-j} EFC_k(m)\right)$ for 4≤j≤n-3.

- $HEFC_k(2n) \succ \left(f_{2n-j+1}\nabla_{m=1}^{\frac{j-2}{2}} HEFC_k(2m+1)\right)\nabla\left(f_{2n-j}\nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m)\right)$

  $\nabla\left[\bigcup_{i=2n-(j+1)}^{2n-3} \left\{\left(f_i.ef_k(2n-i+2) + f_{i+1}.ef_k(2n-i)\right)\nabla_{m=3}^{2n-i-1} EFC_k(m)\right\}\right]$

  $\nabla\left(f_1 ef_k(2n-2)\nabla_{m=2}^{n-1} EFC_k(2m)\right)\nabla\left(f_{j-1} ef_k(2n-j)\nabla_{m=1}^{\frac{2n-j-3}{2}} EFC_k(2m+1)\right)$ for 7≤j≤n-3.

- $HEFC_k(2n+1) \succ \left(f_{2n-j+2}\nabla_{m=1}^{\frac{j-2}{2}} HEFC_k(2m+1)\right)\nabla\left(f_{2n-j+1}\nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m)\right)$

  $\nabla\left[\bigcup_{i=2n-j}^{2n-2} \left\{\left(f_i.ef_k(2n-i+3) + f_{i+1}.ef_k(2n-i+1)\right)\nabla_{m=3}^{2n-i} EFC_k(m)\right\}\right]$

  $\nabla\left(f_1 ef_k(2n-1)\nabla_{m=1}^{n-1} EFC_k(2m+1)\right)\nabla\left(f_{j-1} ef_k(2n-j+1)\nabla_{m=2}^{\frac{2n-j-1}{2}} EFC_k(2m)\right)$ for 7≤j≤n-3.

- $HEFC_k(n+2) \succ \left(f_{n-j+1}\nabla_{m=1}^{\frac{j-2}{2}} HEFC_k(2m+1)\right)\nabla\left(f_{n-j+2}\nabla_{m=2}^{\frac{j-2}{2}} HEFC_k(2m+1)\right)$

  $\nabla\left[\bigcup_{i=n-j+1}^{n-1} \left\{\left(f_i.ef_k(n-i+4) + f_{i+1}.ef_k(n-i+2)\right)\nabla_{m=3}^{n-i+1} EFC_k(m)\right\}\right]$

  $\nabla\left(f_1 ef_k(n)\nabla_{m=3}^{n-1} EFC_k(m)\right)\nabla\left(f_{j-1} ef_k(n-j+2)\nabla_{m=3}^{n-j-1} EFC_k(m)\right)$ for 7≤j≤n-3.

***Figure 12.*** *Theorem 17.*

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