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# DUALITY IN THE PROBLEMS OF OPTIMAL CONTROL DESCRIBED BY CONVEX DIFFERENTIAL INCLUSIONS WITH DISTRIBUTED PARAMETERS

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***ABSTRACT***

*Sufficient condition for optimality is derived for the problem under consideration on the basis of the apparatus of locally conjugate mappings,and duality theorems are proved.A sufficient condition for an extremum is an extremal relation for the direct and dual problems.*

***Key words:****Multivalued mapping, subdifferential, conjugate function duality.*

## Introduction

It is known that optimization problems for differential inclusions constitute one of the intensively developing directions in optimal control theory.The reason is mainly the fact that a great number of problems in mathematical programming and economic dynamics,as well as classical problems on optimal control,differential games,and so on,can be reduced to such investigations [1]-[3].

The present paper is devoted to an investigation of problems of this kind,but with distributed parameters,where the treatment is in finite-dimensional Euclidean spaces.It can be divided conditionally into two parts.

In the first part (§2) a certain sufficient conditions is formulated for convex differential inclusions with first order partial derivatives.For such problem we use construction of contract analysis in terms of local conjugate mappings(LCM's) for convex problem to get sufficient conditions for optimality,that is based on some subtle computations with the help of the LCM apparatus.

At the end of §2 we consider on optimal control problem described by the linear equation.This example shows that in known problems the conjugate inclusion coincides with the conjugate equation which is traditionally obtained with the help of the Hamiltonian function.

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In the second part of the paper (§3) we construct the dual problem to convex problem for differential inclusion with distributed parameters.As is known,duality theory is by virtue of the importance of its applications one of the central directions in convex optimality problems ,and it is interpreted differently for different concrete cases.For example,in mathematical economics duality theory is interpreted in the form of prices ; in mechanics the potential energy and complementary energy are in a mutually dual relation the displacement field and the stress field are solutions of the direct and dual problems,respectively.Besides the indicated applications,duality often makes it possible to simplify the computational procedure and to construct a generalized solution of variational problems that do not have classical solutions.

The duality theorems proved allow one to conclude that a sufficient condition for an exremum is an exremal relation for the direct and dual problems.The latter means that if some pair of admissible solutions satisfies this relation,then each of them is a solutions of the corresponding (direct and dual) problem.We remark that a significant part of the investigations of Ekeland and Temam [5] for simple variational problems is connected with such problems,and there are similar results for differential inclusions with lumped parameters in [5]-[11].

As it was naturally expected,the suggested method of construction of dual problems permits to substitute the initial optimization problem with complex boundaries by a problem with the simplest boundaries,i.e. conjoint systems.

## §1 Necessary Information and Problem Statement

The basic concepts and definitions given below can be found in [1]. Let $R^n$ be the n-dimensional Euclidian space; $<x_1, x_2>$ is a pair of elements $x_1, x_2 \in R^n$; and $<x_1, x_2>$ is their inner product.We say that a multivalued mapping $a$: $R^{2n} \to 2^{R^n}$ is convex if its graph $\mathrm{gf}a = \{(x_1,x_2,v) : v \in a(x_1,x_2)\}$ is a convex subset of $R^{3n}$ .It is convex-valued if $a(x_1,x_2)$ is a convex set for each $(x_1, x_2) \in doma = \{(x_1,x_2) : a(x_1,x_2) \neq \phi\}$.

For such mappings we introduce the notation

$$W_a(x_1,x_2,v^*) = \inf\{< v,v^* >: v \in a(x_1,x_2)\}, v^* \in R^n,$$
$$a(x_1,x_2,v^*) = \{v \in a(x_1,x_2) :< v,v^* >= W_a(x_1,x_2,v^*)\}$$

For convex $a$ we let $W_a(x_1,x_2,v^*) = +\infty$ if $a(x_1,x_2) = \varnothing$ .

For a convex mapping $a$ the cone of tangent directions at a point $(x_1{}^0, x_2{}^0, v^0) \in \mathrm{gf}a$ will be denoted by $K_a(x_1{}^0,x_2{}^0,v^0)$ .

$$K_a(x_1{}^0,x_2{}^0,v^0) = \mathrm{con}\left(gfa - (x_1^0,x_2^0,v^0)\right)$$
$$= \{(\bar{x}_1,\bar{x}_2,\bar{v}) : \bar{x}_1 = \lambda(\bar{x}_1 - x_1{}^0), \bar{x}_2 = \lambda(x_2 - x_2{}^0), \bar{v} = \lambda(v - v^0), \lambda > 0, (x_1,x_2,v) \in gfa\}$$

Moreover for a convex mapping $a$:

$$\Omega_a\left(x_1{}^*,x_2{}^*,v^*\right) = \inf_{x_1,x_2,v}\left\{- < x_1,x_1{}^* > - < x_2,x_2{}^* > + < v,v^* >: (x_1,x_2,v) \in gfa\right\}$$

A mapping

$$a^*(v^*;(x_1,x_2,v)) = \{(x_1^*,x_2^*) : (-x_1^*,-x_2^*,v^*) \in K_a^*(x_1,x_2,v)\}$$

is called the locally conjugate mapping(LCM) to $a$ at the point $(x_1, x_2, v)$, where $K_a^*(x_1, x_2, v)$ is the cone dual to the cone $K_a(x_1, x_2, v)$.

**According to the definition in [1] , [12] for a function g:** $R^n \to R^1 \cup \{\pm\infty\}$

$$g^*(x^*) = \sup_x \{< x, x^* > - g(x)\},$$

$$dom g = \{x : g(x) < +\infty\}$$

Here $g^*$ is called the conjugate function of a function g.

A function is said to be proper if it does not take the value -∞ and is not identically equal to +∞.Subdifferential of a some convex proper function $F(., y^*), y^* \in R^n$ defined on $R^n$ at $x_0 \in dom F(., y^*)$ is denoted

$$\partial_x F(x_0, y^*) = \{x^* : F(x, y^*) - F(x_0, y^*) \geq < x^*, x - x_0 >, \forall x \in R^n\}$$

In the next section we study the convex problem for differential inclusions with first order partial derivatives:

$$(1.1) \qquad I(x(.,.)) = \iint_Q g(x(t,\tau), t, \tau) dt d\tau + \int_0^1 g_0(x(1,\tau), \tau) dt \to \inf$$

$$(1.2) \qquad \frac{\partial x(t,\tau)}{\partial t} \in a\left(\frac{\partial x(t,\tau)}{\partial \tau}, x(t,\tau)\right), \ 0 < t \leq 1, \ \ 0 \leq \tau < 1$$

$$(1.3) \qquad x(t,1) = 0, \ x(0,\tau) = 0, \ Q = [0,1] \times [0,1]$$

Here $a$: $R^{2n} \to 2^{R^n}$ is a convex multivalued mapping, $g$ is continuous function that is convex with respect to $x$, $g : R^n \times Q \to R^1$, $g_0 : R^n \times [0,1] \to R^1$.

The problem is to find a solution $\tilde{x}(t,\tau)$ of the first boundary value problem (1.2), (1.3) that minimizes (1.1). Here an admissible solution is understood to be an absolutely continuous functions with summable first partial derivatives.However,as will be seen from the context ,the definition of a solution in this or that sense (classical,generalized etc.) is introduced only for simlicity and does not in any way restrict the class of problems under consideration.

## §2. Sufficient Conditions For Optimality For Differential Inclusions

**Theorem 2.1.** Suppose that $g(x,t,\tau)$ and $g_0(x,\tau)$ are jointly continuous functions convex with respect to $x$, and $a$ is a convex closed mapping i.e. $gfa$ is a convex closed subset of $R^{3n}$ .Then for the optimality of the solution $\tilde{x}(t,\tau)$ among all admissible solutions it is sufficient that there exist an absolutely continuous functions $\{u^*(t,\tau), x^*(t,\tau)\}$ with summable first partial derivatives such that the conditions a) – c) hold:

$$a)\ \left(u^*(t,\tau),\ \frac{\partial u^*(t,\tau)}{\partial\tau}\right) \in a^*\left(x^*(t,\tau); \frac{\partial\tilde{x}(t,\tau)}{\partial\tau}, \tilde{x}(t,\tau), \frac{\partial\tilde{x}(t,\tau)}{\partial t}\right)$$

$$+\{0\}\times\left(\partial g(\tilde{x}(t,\tau),t,\tau)+\frac{\partial x^*(t,\tau)}{\partial t}\right)$$

b) $u^*(t,0)=0, x^*(1,\tau)\in\partial g_0(\tilde{x}(1,\tau),\tau)$

c) $\dfrac{\partial\tilde{x}(t,\tau)}{\partial t}\in a\left(\dfrac{\partial\tilde{x}(t,\tau)}{\partial\tau}, \tilde{x}(t,\tau), x^*(t,\tau)\right)$

Proof. By Theorem 2.1. III in [1]

$$a^*(v^*; x_1, x_2, v)=\partial_{(X_1,X_2)}W_a(x_1,x_2,v^*),\ v^*\in a(x_1,x_2,v^*)$$

Then by using the Moreau-Rockafeller theorem $([1],[9])$, from condition a) we obtain the Inclusion

$$\left(u^*(t,\tau), \frac{\partial u^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}\right)\in\partial_{(X_1,X_2)}\Big[W_a\Big(\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}, \tilde{x}(t,\tau), x^*(t,\tau)\Big)$$

$$+g_1\Big(\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}, \tilde{x}(t,\tau), t,\tau\Big)\Big],\ g_1(x_1,x_2,t,\tau)\equiv g(x,t,\tau),(t\tau)\in Q$$

Using the definitions of subdifferential and $W_a$, we rewrite the last relation in the form

$$\left\langle\frac{\partial x(t,\tau)}{\partial t}, x^*(t,\tau)\right\rangle-\left\langle\frac{\partial\tilde{x}(t,\tau)}{\partial t}, x^*(t,\tau)\right\rangle+g(x(t,\tau),\tau)-g(\tilde{x}(t,\tau),t,\tau)$$

$$\geq\left\langle u^*(t,\tau), \frac{\partial x(t,\tau)}{\partial\tau}-\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}\right\rangle+\left\langle\frac{\partial u^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}, x(t,\tau)-\tilde{x}(t,\tau)\right\rangle,$$

$$g(x(t,\tau),\tau)-g(\tilde{x}(t,\tau),t,\tau)\geq\left\langle\frac{\partial}{\partial t}\big(\tilde{x}(t,\tau)-x(t,\tau)\big), x^*(t,\tau)\right\rangle$$

$$+\frac{\partial}{\partial\tau}\left\langle u^*(t,\tau), x(t,\tau)-\tilde{x}(t,\tau)\right\rangle-\left\langle\frac{\partial x^*(t,\tau)}{\partial t}, x(t,\tau)-\tilde{x}(t,\tau)\right\rangle$$

On the other hand,by the second condition in b)

$$g_0(x(1,\tau),\tau)-g_0(\tilde{x}(1,\tau),\tau)\geq\left\langle x^*(1,\tau), x(1,\tau)-\tilde{x}(1,\tau)\right\rangle$$

Integrating the preceding relation over the domain *Q,* and the latter over the interval [0,1] and then adding them,we get

$$\iint_Q [g(x(t,\tau),t,\tau) - g(\widetilde{x}(t,\tau),t,\tau)]dtd\tau + \int_0^1 [g_0(x(1,\tau)),\tau - g_0(\widetilde{x}(1,\tau)\tau)]d\tau$$

(2.1)

$$\geq \iint_Q \frac{\partial}{\partial t}\langle \widetilde{x}(t,\tau) - x(t,\tau), x^*(t,\tau)\rangle dtd\tau + \iint_Q \frac{\partial}{\partial \tau}\langle u^*(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau)\rangle dtd\tau + \int_0^1 \langle x^*(1,\tau), x(1,\tau) - \widetilde{x}(1,\tau)\rangle d\tau$$

It is clear that

$$(2.2)\quad \iint_Q \frac{\partial}{\partial t}\langle \widetilde{x}(t,\tau) - x(t,\tau), x^*(t,\tau)\rangle .dtd\tau = \int_0^1 \langle x^*(1,\tau), \widetilde{x}(1,\tau) - x(1,\tau)\rangle d\tau$$

$$- \int_0^1 \langle x^*(0,\tau), \widetilde{x}(0,\tau) - x(0,\tau)\rangle d\tau$$

where, since $\widetilde{x}(0,\tau) = x(0,\tau) = 0(see(1.3))$

$$\int_0^1 \langle x^*(0,\tau), \widetilde{x}(0,\tau) - x(0,\tau)\rangle d\tau = 0$$

Analogously

(2.3)

$$\iint_Q \frac{\partial}{\partial \tau}\langle u^*(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau)\rangle dtd\tau = \int_0^1 \langle u^*(t,1), x(t,1) - \widetilde{x}(t,1)\rangle dt - \int_0^1 \langle u^*(t,0), x(t,0) - \widetilde{x}(t,0)\rangle dt$$

and since $x(t,1) = \widetilde{x}(t,1) = 0$ and $u^*(t,0) = 0$

by condition b)

$$\int_0^1 < u^*(t,1), x(t,1) - \widetilde{x}(t,1) > dt = 0$$

$$\int_0^1 < u^*(t,0), x(t,0) - \widetilde{x}(t,0) > dt = 0 \text{ and } u^*(t,0) = 0$$

*Then forom (2.2) and (2.3) we obtain that the right-hand side of the inequality (2.1) is equal to zero.Thus,we have finally*

$$\iint_Q g(x(t,\tau),t,\tau)dtd\tau \geq \iint_Q g_0(\widetilde{x}(1,\tau),\tau)d\tau$$

For all admissible solutions $x(t,\tau),(t,\tau) \in Q.$ The theorem is proved.

In the conclusion of this section we consider an example:

(2.4)

$$\mathrm{I}(x(t,\tau)) \to \inf ,$$

$$\frac{\partial x(t,\tau)}{\partial t} = A_1 \frac{\partial x(t,\tau)}{\partial \tau} + A_2 x(t,\tau) + Bu(t,\tau), \quad u(t,\tau) \in U$$

$x(t,1)=0, x(0,\tau)=0$

where $A_1$ and $A_2$ are $n\times n$ matrices, B is a rectangular $n\times r$ matrix, $U\subset R^r$ is a convex closed set, and g and $g_0$ is continuously differentiable function of $x$ .It is required to find a controlling parameter $\tilde{u}(t,\tau) \in U$ such that the solution $\tilde{x}(t,\tau)$ corresponding to it minimizes $I(x(.,.))$ .
In this case

$a(x_1, x_2)=A_1x_1+A_2x_2+BU$

By elementary computations we find that

$$a^*(v^*;(x_1,x_2,v)) = \begin{cases} (A_1^* v^*, A_2^* v^*), & B^* v^* \in [con(U-u)]^* \\ \varnothing & , B^* v^* \notin [con(U-u)]^* \end{cases}$$

where $v=A_1x_1+A_2x_2+Bu$, and $[\mathrm{con}M]^*$ is the cone dual to the cone con$M$.
Then, using Theorem 2.1, we get the relations

(2.5) $$u^*(t,\tau) = A_1^* x^*(t,\tau)$$

(2.6) $$\frac{\partial u^*(t,\tau)}{\partial \tau} - \frac{\partial x^*(t,\tau)}{\partial t} = A_2^* x^*(t,\tau) + g'(\tilde{x}(t,\tau),t,\tau)$$

(2.7) $$\left\langle u - \tilde{u}(t,\tau), B^* x^*(t,\tau) \right\rangle \geq 0, u \in U$$

(2.8) $$x^*(1,\tau) = g_0'(\tilde{x}(1,\tau),\tau) \,,\; u^*(t,0) = 0$$

substituting (2.5) in (2.6),we have

(2.9) $$-\frac{\partial x^*(t,\tau)}{\partial t} = -A_1^* \frac{\partial x^*(t,\tau)}{\partial \tau} + A_2^* x^*(t,\tau) + g'(\tilde{x}(t,\tau),t,\tau)$$

Obviously, (2.7) and second condition of the (2.8) can be written in the form

$$\left\langle B\tilde{u}(t,\tau), x^*(t,\tau) \right\rangle = \inf_{u\in U} \left\langle Bu, x^*(t,\tau) \right\rangle$$

$$x^*(t,0) = 0$$

Thus, we have obtained the following result.
Theorem 2.2. The solution $\tilde{x}(t,\tau)$ corresponding to the control $\tilde{u}(t,\tau)$ minimizes I($x$(.,.)) in the problem (2.4) if there exists a function $x^*(t,\tau)$ satisfying the conditions (2.8),(2.9),(2.10).

## §3. On Duality In Differential Inclusions with Distributed Parameters

The problem of determining the supremum

(3.1) $$\sup_{\substack{u^*(t,\tau),x^*(t,\tau),z^*(t,\tau)\\ u^*(t,0)=0}} I_*(u^*(t,\tau),x^*(t,\tau),z^*(t,\tau))$$

is called the dual problem to the direct problem (1.1)-(1.3) where

$$I_*(u^*(t,\tau),x^*(t,\tau),z^*(t,\tau))=\iint_Q\left[\Omega_a\left(u^*(t,\tau),\frac{\partial u^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}-z^*(t,\tau),x^*(t,\tau)\right)-g^*\left(z^*(t,\tau),t,\tau\right)\right]dtd\tau$$
$$-\int_0^1 g_0{}^*\left(x^*(1,\tau),\tau\right)d\tau$$

It is assumed that the functions $u^*(t,\tau),x^*(t,\tau),z^*(t,\tau)$ are an absolutely continuous functions with summable first partial derivatives on Q and $z^*(t,\tau)$ is an absolutely continuous function on Q.

**Theorem 3.1** The inequality

$$I(\widetilde{x}(t,\tau))\geq I_*((u^*(t,\tau),x^*(t,\tau),z^*(t,\tau))$$

is valid for all admissible solutions $x(t,\tau)$ and $\{u^*(t,\tau),x^*(t,\tau),z^*(t,\tau)\}$ of the direct problem (1.1)-(1.3) and the dual problem (3.1),respectively.

Proof.It is clear from the definitions of the functions $\Omega_a$, $g^*$ and $g_0^*$ *that*

(3.2)
$$\Omega_a\left(u^*(t,\tau),\frac{\partial u^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}-z^*(t,\tau),x^*(t,\tau)\right)$$
$$\leq-\left\langle u^*(t,\tau),\ \frac{\partial x^*(t,\tau)}{\partial\tau}\right\rangle-\left\langle\frac{\partial u^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}-z^*(t,\tau),x^*(t,\tau)\right\rangle$$
$$+\left\langle\frac{\partial x(t,\tau)}{\partial t},x^*(t,\tau)\right\rangle=-\frac{\partial}{\partial\tau}\left\langle u^*(t,\tau),x^*(t,\tau)\right\rangle+\frac{\partial}{\partial t}\left\langle x^*(t,\tau),x(t,\tau)\right\rangle+\left\langle z^*(t,\tau),x(t,\tau)\right\rangle$$

(3.3) $$-g^*\left(z^*(t,\tau),t,\tau\right)\leq g\left(x(t,\tau),t,\tau\right)-\left\langle x(t,\tau),z^*(t,\tau)\right\rangle,$$
$$-g_0^*\left(x^*(1,\tau),\tau\right)\leq g_0\left(x(1,\tau),\tau\right)-\left\langle x(1,\tau),x^*(1,\tau)\right\rangle$$

Using (3.2) and (3.3) , we have

(3.4)
$$I_*\left(u^*(t,\tau),x^*(t,\tau),z^*(t,\tau)\right)\leq-\iint_Q\frac{\partial}{\partial\tau}\left\langle u^*(t,\tau),x(t,\tau)\right\rangle dtd\tau+\iint_Q\frac{\partial}{\partial t}\left\langle x^*(t,\tau),x(t,\tau)\right\rangle dtd\tau$$

$$+\iint_Q g(x(t,\tau),t,\tau)dtd\tau+\int_0^1\left[g_0(x(1,\tau),\tau)-\langle x(1,\tau),x^*(1,\tau)\rangle\right]d\tau$$

But since

$$\iint_Q \frac{\partial}{\partial\tau}\langle u^*(t,\tau),x(t,\tau)\rangle dtd\tau=\int_0^1\left[\langle u^*(t,1),x(t,1)\rangle-\langle u^*(t,0),x(t,0)\rangle\right]dt,$$

$$\iint_Q \frac{\partial}{\partial t}\langle x^*(t,\tau),x(t,\tau)\rangle dtd\tau=\int_0^1\left[\langle x^*(1,\tau),x(1,\tau)\rangle dt-\langle x^*(0,\tau),x(0,\tau)\rangle\right]d\tau$$

and since the solution $x(t,\tau)(x(0,\tau)=0,x(t,1)=0)$ is admissible,and $u^*(t,0)=0,$ we get from the preceding inequality (3.4)

$$I_*(u^*(t,\tau),x^*(t,\tau),z^*(t,\tau))\le\int_0^1\langle x^*(1,\tau),x(1,\tau)\rangle d\tau+I(x(t,\tau))-\int_0^1\langle x^*(1,\tau),x(1,\tau)\rangle d\tau=I(x(t,\tau)$$

Thus

$$I_*(u^*(t,\tau),x^*(t,\tau),z^*(t,\tau))\le\ I(x(t,\tau)),$$

which is what was required.

Theorem 3.2 If the functions $\tilde{x}(t,\tau)$ and $\{u^*(t,\tau),x^*(t,\tau),z^*(t,\tau)\}$,where $\tilde{z}(t,\tau)\in\partial g(\tilde{x}(t,\tau),t,\tau)$ and $\tilde{x}^*(1,\tau)\in\partial g_0(\tilde{x}(1,\tau),\tau)$ satisfy the conditions a) – c) of Theorem 3.1,then they are solutions of the direct and dual problems,respectively,and their values coincide.

Proof. The fact that $\tilde{x}(t,\tau)$ is a solution of the direct problem was proved in Theorem 2.1.We study the remaining assertions.By the definition of an LCM,the condition a)is equiavalent to the inequality

$$-\left\langle u^*(t,\tau),\mathrm{x}_1-\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}\right\rangle-\left\langle\frac{\partial u^*(t,\tau)}{\partial\tau}-\frac{\partial\tilde{x}^*(t,\tau)}{\partial t}-\tilde{z}^*(t,\tau),x_2-\tilde{x}(t,\tau)\right\rangle$$
$$+\left\langle\tilde{x}^*(t,\tau),v-\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}\right\rangle\ge 0\ ,\ (x_1,x_2,v)\in gfa$$

This means that

$$(3.5)\qquad\left(\tilde{u}^*(t,\tau),\frac{\partial\tilde{u}^*(t,\tau)}{\partial\tau}-\frac{\partial\tilde{x}^*(t,\tau)}{\partial t}-\tilde{z}^*(t,\tau),\tilde{x}^*(t,\tau)\right)\in dom\Omega_a$$

where

$$dom\Omega_a=\{(-x_1^*,-x_2^*,v^*):\Omega_a(x_1^*,x_2^*,v^*)>-\infty\}$$

Further,since $\partial g(x,t,\tau) \subset dom g^*(.,t,\tau)$ and $\partial g_0(x,\tau) \subset dom g_0^*(.,\tau)$, it is clear that

(3.6) $\tilde{z}^*(t,\tau) \in dom g^*(.,t,\tau)$, $\tilde{x}^*(1,\tau) \in dom g^*(.,\tau)$

Then it can be concluded from (3.5) and (3.6) that the indicated functions $\{\tilde{u}^*(t,\tau), \tilde{x}^*(t,\tau), \tilde{z}^*(t,\tau)\}$ form an admissible solution.It remains to show that it is optimal.Using Lemma 2.2.III in [1] (p.105) , we get

(3.7)
$$\Omega_a\left(\tilde{u}^*(t,\tau), \frac{\partial \tilde{u}^*(t,\tau)}{\partial \tau} - \frac{\partial \tilde{x}^*(t,\tau)}{\partial t} - \tilde{z}^*(t,\tau), \tilde{x}^*(t,\tau)\right)$$
$$= W_a\left(\frac{\partial \tilde{x}(t,\tau)}{\partial \tau}, \tilde{x}(t,\tau), \tilde{x}^*(t,\tau)\right) - \left\langle \tilde{u}^*(t,\tau), \frac{\partial \tilde{x}(t,\tau)}{\partial t}\right\rangle$$
$$- \left\langle \frac{\partial \tilde{u}^*(t,\tau)}{\partial \tau} - \frac{\partial \tilde{x}^*(t,\tau)}{\partial t} - \tilde{z}^*(t,\tau), \tilde{x}^*(t,\tau)\right\rangle$$

*By condition c)*

(3.8)
$$W_a\left(\frac{\partial \tilde{x}(t,\tau)}{\partial \tau}, \tilde{x}(t,\tau), \tilde{x}^*(t,\tau)\right) = \left\langle \frac{\partial \tilde{x}(t,\tau)}{\partial t}, \tilde{x}^*(t,\tau)\right\rangle$$

Moreover we can write

$$\left\langle \tilde{x}(t,\tau), \tilde{z}^*(t,\tau)\right\rangle - g(\tilde{x}(t,\tau),t,\tau) = g^*(\tilde{z}^*(t,\tau),t,\tau),$$
$$\left\langle \tilde{x}(1,\tau), \tilde{x}^*(1,\tau)\right\rangle - g_0(\tilde{x}(1,\tau),\tau) = g_0^*(\tilde{x}^*(1,\tau),\tau),$$

where it is taken into account that

$$\tilde{z}^*(t,\tau) \in \partial g(\tilde{x}(t,\tau),t,\tau), \ \tilde{x}^*(1,\tau) \in \partial g_0(\tilde{x}(1,\tau),\tau)$$

Then in view of (3.7)-(3.9) it is easy to establish as in the proof of Theorem 3.1 that

$$I(\tilde{x}(t,\tau)) \geq I_*((u^*(t,\tau), x^*(t,\tau), z^*(t,\tau)).$$

The proof is complete.

## Conclusions

In the first part a certain sufficient conditions is formulated for convex differential inclusions with first order partial derivatives. For such problem is used construction of convex analysis in terms of local conjugate mappings for convex problem to get sufficient conditions for optimality. In the second part of the paper is constructed the dual problem to convex problem for first order differential inclusions. The duality theorems proved allow one to conclude that a sufficient condition for an extremum is an extremal relation for the direct and dual problems.

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