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WEIGHTED MULTIRESOLUTION SPACES AND WEIGHTED WAVELETS

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ABSTRACT

In this study it is shown that the orthogonal wavelet functions will be able to got for the space sequence which gives the multiresolution spesifications in the weighted spaces for exponantial kind weighted function.

Keywords: *Wavelets, Multiresolution Analysis, Weighted Spaces.*

1. INTRODUCTION

Wavelets are a powerful tool for signal processing application. Essentially their properties are ; compact support,vanishing moments, smooth, fast transform, so that they are localized in space and frequency and wavelet coefficients of a function decay rapidly.

In this study it is shown that there is orthonormal wavelet basis function in the weighted spaces $L_{2,a}(R)$ with exponantial weight function e^{ax}.

Our aim is to find equations depence on weighted parameter a . It is constructed multiresolution analysis for weighted spaces so that it can be obtained wavelet basis for weighted orthogonal complementary spaces. We used algebric method for construction .

2. MULTIRESOLUTION ANALYSIS IN WEIGHTED SPACES

It was shown that there is multiresolution analysis for certain function spaces and weighted spaces [1],[2],[3],[4].

Let $L_{2,a}(R)$ is space of measurable functions f where

$$\|f\| = \left(\int_{-\infty}^{\infty} |f(x)|^2 e^{ax} dx \right)^{\frac{1}{2}} < \infty .$$

We define a multiresolution analysis in the space $L_{2,a}(R)$ as a sequence of closed subspaces $\{V_{i,a}\}_{i\in z}$

$$\cdots V_{2,a} \subset V_{1,a} \subset V_{0,a} \subset V_{-1,a} \subset V_{-2,a} \cdots \quad (1)$$

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$$\bigcup_{i\in Z} V_{i,a} = L_{2,a}(R) \quad (2)$$

$$\bigcap_{i\in Z} V_{i,a} = \{0\} \quad (3)$$

$$f(x)\in V_{i,a}\ ,\ f(2x)\in V_{i-1,a} \quad (4)$$

$$f(x)\in V_{i,a}\ ,\ f(x-n)\in V_{i,a}\ ,\ n\in Z \quad (5)$$

This implies there is a scaling function $\{f_{0,n,a}(x)\}_{n\in Z}\in V_{0,a}$ and it is element of

$V_{0,a}$ and it is Riesz basis set of $V_{0,a}$ and $f_{i,n,a}(x)=2^{-\frac{i}{2}}f_a(2^{-i}x-n),\quad i,n\in Z$ are basis of $V_{i,a}$. The scaling function $f_a(x)$ is element of both $V_{o,a}$ and $V_{-1,a}$ so that it is written refinement equation

$$f_a(x)=\sum_{n=-\infty}^{\infty} h_{n,a} f_{-1,a}(x-n)$$

$$f_a(x)=\sqrt{2}\sum_n h_{n,a} f_a(2x-n) \quad (6)$$

where

$$h_{n,a} =< f_a, f_{-1,n,a} >_{L_{2,a}}$$

$$h_{n,a} = \sqrt{2}\int_{-\infty}^{\infty} f_a(x)\overline{f_a(2x-n)}e^{ax}dx \quad (7)$$

The Fourier transform of right and left sides are

$$\hat{f}_a(w)=\sqrt{2}\sum_n h_{n,a}\int f_a(2x-n)e^{-jwx}dx$$

$$\hat{f}_a(w)=\frac{1}{\sqrt{2}}\sum_n h_{n,a}e^{-j\frac{w}{2}n}\int f_a(u)e^{-jw\frac{u}{2}}du \quad (8)$$

with

$$m_{0,a}(w)=\frac{1}{\sqrt{2}}\sum_n h_{n,a}e^{-jwn} \quad (9)$$

and $\hat{f}_a(w)=m_{0,a}\left(\frac{w}{2}\right)\hat{f}_a\left(\frac{w}{2}\right)$ (10)

is obtained. $m_{0,a}(w)$ is $2p$-periodic. The basis functions that is obtained by translating are orthonormal to each other in the weighted space $L_{2,a}(R)$ so that

$$d_{k,0} =< f_a(x), f_a(x-k) >_{L_{2,a}}$$

$$d_{k,0}=\int f_a(x)\overline{f_a(x-k)}e^{ax}dx$$

$$d_{k,0}=\int f_a(x)e^{\frac{a}{2}x}\overline{f_a(x-k)e^{\frac{a}{2}x}}dx$$

$$d_{k,0}=\int\left\{\frac{1}{2p}\int\hat{f}_a\left(w+j\frac{a}{2}\right)e^{jwx}dw\right\}\overline{f_a(x-k)e^{\frac{a}{2}x}}dx$$

$$d_{k,0}=\int\hat{f}_a\left(w+j\frac{a}{2}\right)\left\{\frac{1}{2p}\int\overline{f_a(x-k)e^{\frac{a}{2}x}}e^{jwx)}dx\right\}dw$$

$$d_{k,0}=\int\hat{f}_a\left(w+j\frac{a}{2}\right)\left\{\frac{1}{2p}\int\overline{f_a(u)e^{\frac{a}{2}(u+k)}}e^{jw(u+k)}dx\right\}dw$$

$$d_{k,0}=e^{\frac{a}{2}k}\int e^{jwk}\hat{f}_a\left(w+j\frac{a}{2}\right)\left\{\frac{1}{2p}\int\overline{f_a(u)e^{-ju\left(w+j\frac{a}{2}\right)}}dx\right\}dw$$

$$d_{k,0}=e^{\frac{a}{2}k}\int e^{jwk}\hat{f}_a\left(w+j\frac{a}{2}\right)\overline{\hat{f}_a\left(w+j\frac{a}{2}\right)}dw$$

$$d_{k,0}=e^{\frac{a}{2}k}\int e^{jwk}\left|\hat{f}_a\left(w+j\frac{a}{2}\right)\right|^2dw \quad (11)$$

we have for $k\neq 0$

$$\int e^{jwk}\left|\hat{f}_a\left(w+jw\frac{a}{2}\right)\right|^2dw=0 \quad (12)$$

and for $k=0$

$$\int\left|\hat{f}_a\left(w+j\frac{a}{2}\right)\right|^2dw=1 \quad (13)$$

For k=0 ; integral is splitted into $2p$-length sequences as

$$\int_0^{2p}\sum_l\left|\hat{f}_a\left(w+j\frac{a}{2}+2pl\right)\right|^2dw=1 \quad (14)$$

so that

$$\sum_l\left|\hat{f}_a\left(w+j\frac{a}{2}+2pl\right)\right|^2=\frac{1}{2p}$$

(15)

Substituting (10) into (15)

$$\sum_l \left| m_{0,a}\left(\frac{w}{2}+j\frac{a}{4}+pl\right)\right|^2 \left|\hat{f}_a\left(\frac{w}{2}+j\frac{a}{4}+pl\right)\right|^2 = \frac{1}{2p} \quad (16)$$

is obtained. In this summation, we split into even and odd index, using periodicity of $m_{f,a}(w)$ and $m_{0,a}(w)$, using equation (15)

$$\left| m_{0,a}\left(\frac{w}{2}+j\frac{a}{4}\right)\right|^2 + \left| m_{0,a}\left(\frac{w}{2}+j\frac{a}{4}+p\right)\right|^2 = 1 \quad (17)$$

is obtained. Equation 17 is simplyfied as

$$\left| m_{0,a}\left(t+j\frac{a}{4}\right)\right|^2 + \left| m_{0,a}\left(t+j\frac{a}{4}+p\right)\right|^2 = 1 \quad (18)$$

where $t = \frac{w}{2}$.

3. WEIGHTED ORTHOGONAL DIFFERENCE SPACE AND WEIGHTED WAVELET BASIS FUNCTIONS

Let's consider orthogonal complemantary space sequence of space sequence $\{V_{i,a}\}_{i\in z}$

For a space in this space sequence

$$V_{-1,a} = V_{0,a} \oplus W_{0,a} \quad 19)$$

is obtained by direct sum. For a function f which elements of difference space

$$f \in W_{0,a} \;,\; f \in V_{-1,a} \;,\; f \perp V_{0,a}$$

so that f is represented with basis function of $V_{-1,a}$

$$f = \sum_n f_{n,a} f_{-1,n,a} \quad (20)$$

The coefficient $f_{n,a}$ is obtained with weighted inner product as

$$f_{n,a} = < f, f_{-1,n,a} >_{L_{2,a}}$$

$$f_{n,a} = \sqrt{2}\int f\, \overline{f(2x-n)} e^{ax} dx \quad (21)$$

The Fourier transform of the double side of equation is

$$\hat{f}(w) = \sqrt{2}\sum_n f_{n,a} \int f(2x-n) e^{-jwx} dx$$

$$\hat{f}(w) = \frac{1}{\sqrt{2}}\sum_n f_{n,a}\, e^{-j\frac{w}{2}n} \hat{f}_a\left(\frac{w}{2}\right) \quad (22)$$

We define

$$m_{f,a} = \frac{1}{\sqrt{2}}\sum_n f_{n,a} e^{-jwn}$$

and

$$\hat{f}(w) = m_{f,a}\left(\frac{w}{2}\right)\hat{f}_a\left(\frac{w}{2}\right) \quad (23)$$

is obtained. $m_{f,a}(w)$ is $2p$-periodic.

$f \perp V_{0,a}$ so that f is orthogonal to basis function of $f \perp V_{0,a}$.

$$f \perp f_{a,k}(x)$$

$$< f, f_{k,a} >_{L_{2,a}} = 0 \quad (24)$$

$$\int f(x)\overline{f_a(x-k)} e^{ax} dx = 0 \quad (25)$$

$$\int f(x) e^{\frac{a}{2}x} \overline{f_a(x-k) e^{\frac{a}{2}x}} dx = 0 \quad (26)$$

Using equation (26)

$$\int\left\{\frac{1}{2p}\int \hat{f}\left(w+j\frac{a}{2}\right)e^{jwx} dw\right\}\overline{f_a(x-k)e^{\frac{a}{2}x}} dx = 0$$

$$\int \hat{f}\left(w+j\frac{a}{2}\right)\left\{\frac{1}{2p}\int \overline{f_a(x-k)e^{\frac{a}{2}x}} e^{jwx)} dx\right\} dw = 0$$

is obtained and conclusion

$$\int e^{jwk}\hat{f}\left(w+j\frac{a}{2}\right)\overline{\hat{f}_a\left(w+j\frac{a}{2}\right)} dw = 0 \quad (27)$$

is obtained. This equation is splitted $2p$-length

$$\int_0^{2p} e^{jwk}\sum_l \hat{f}\left(w+j\frac{a}{2}+2pl\right)\overline{\hat{f}_a\left(w+j\frac{a}{2}+2pl\right)} dw = 0$$

$$\sum_l \hat{f}\left(w+j\frac{a}{2}+2pl\right)\overline{\hat{f}_a\left(w+j\frac{a}{2}+2pl\right)} = 0 \quad (28)$$

substituting (10) and (23) into (28)

$$\sum_l m_{f,a}\left(\frac{w}{2}+j\frac{a}{4}+pl\right)\hat{f}_a\left(w+j\frac{a}{4}+pl\right) + \overline{m_{0,a}\left(\frac{w}{2}+j\frac{a}{4}+pl\right)}\;\overline{\hat{f}_a\left(w+j\frac{a}{4}+pl\right)} = 0 \quad (29)$$

is obtained. We split the summation into odd and even index and using periodicity of $m_{f,a}(w)$ and $m_{0,a}(w)$ and using equation (15)

$$\overline{m_{f,a}\left(\frac{w}{2}+j\frac{a}{4}\right)}m_{0,a}\left(\frac{w}{2}+j\frac{a}{2}\right)+ \overline{m_{f,a}\left(\frac{w}{2}+j\frac{a}{4}+p\right)}\,\overline{m_{0,a}\left(\frac{w}{2}+j\frac{a}{2}+p\right)}=0 \quad (30)$$

is obtained and with $t=\frac{w}{2}$

$$\overline{m_{f,a}\left(t+j\frac{a}{4}\right)}m_{0,a}\left(t+j\frac{a}{2}\right)+ \overline{m_{f,a}\left(t+j\frac{a}{4}+p\right)}\,\overline{m_{0,a}\left(t+j\frac{a}{2}+p\right)}=0 \quad (31)$$

is obtained . For this equation, we can find a function $l(t)$ with $2p$ -periodic as

$$m_{f,a}(t+j\frac{a}{4})=l(t)\overline{m_{0,a}(t+j\frac{a}{4}+p)} \quad (32)$$

and we substituting $l(t)$

$$l(t)\overline{m_{0,a}(t+j\frac{a}{4}+p)}m_{0,a}(t+j\frac{a}{4})+ l(t+p)\overline{m_{0,a}(t+j\frac{a}{4})}m_{0,a}(t+j\frac{a}{4}+p)=0 \quad (33)$$

$$\overline{m_{0,a}(t+j\frac{a}{4})}\;\overline{m_{0,a}(t+j\frac{a}{4}+p)}\left[l(t)+l(t+p)\right]=0 \quad (34)$$

and $l(t)$ satisfies

$$l(t)+l(t+p)=0 \quad (35)$$

$l(t)$ is choosen as

$$l(t)=e^{jt}g(2t) \quad (36)$$

substituting (32) and (36) into (23)

$$\hat{f}(w)=e^{j\frac{w}{2}}e^{\frac{a}{4}}g\left(w-j\frac{a}{2}\right)\overline{m_{0,a}\left(\frac{w}{2}+p\right)}\hat{f}_a\left(\frac{w}{2}\right) \quad (37)$$

is obtained. In this formula we define

$$\hat{y}_a(w)=e^{j\frac{w}{2}}e^{\frac{a}{4}}\overline{m_{0,a}\left(\frac{w}{2}+p\right)}\hat{f}_a\left(\frac{w}{2}\right) \quad (38)$$

and we rewrite $\hat{f}(w)$ as

$$\hat{f}(w)=g\left(w-j\frac{a}{2}\right)\hat{y}_a(w) \quad (39)$$

where

$$g(w-j\frac{a}{2})=\sum_k g(k)e^{-jk\left(w-j\frac{a}{2}\right)} \quad (40)$$

Hence

$$\hat{f}(w)=\left(\sum_k g(k)e^{-jk\left(w-j\frac{a}{2}\right)}\right)\hat{y}_a(w) \quad (41)$$

$$\hat{f}(w)=\sum_k g(k)\hat{y}_a(w)e^{-jk\left(w-j\frac{a}{2}\right)} \quad (42)$$

and using inverse Fourier transform ,we obtain $f(x)$ with weighted wavelet basis functions

$$f(x)=\frac{1}{2p}\int\sum_k g(k)\hat{y}_a(w)e^{-jky}e^{-k\frac{a}{2}}e^{jwx}dx$$

$$f(x)=\sum_k g(k)e^{-k\frac{a}{2}}\frac{1}{2p}\int\hat{y}_a(w)e^{jw(x-k)}dx$$

$$f(x)=\sum_k g(k)y_a(x-k)e^{-k\frac{a}{2}} \quad (43)$$

this implies that a function $f(x)$ which is element of $W_{0,a}$ is represented with $y_a(x)$.

4. ORTHOGONAL WAVELET BASIS FUNCTIONS

Analysing orthonormalty of wavelet, we have

$$<y_a(x),y_a(x-k)>_{L_{2,a}}=e^{\frac{a}{2}k}\int e^{jwk}\left|\hat{y}_a\left(w+j\frac{a}{2}\right)\right|^2dw$$

$$<y_a(x),y_a(x-k)>_{L_{2,a}}= e^{\frac{a}{2}k}\int_0^{2p}e^{jwk}\sum_l\left|\hat{y}_a\left(w+j\frac{a}{2}+2pl\right)\right|^2dw \quad (44)$$

substituting (38)

$$<\psi_\alpha(x),\psi_\alpha(x-k)>_{L_{2,\alpha}}=$$

$$e^{\frac{\alpha}{2}k}\int_0^{2\pi} e^{j\omega k}\sum_l \left| e^{j\left(\frac{\omega}{2}+\pi l\right)} m_{0,\alpha}\left(\frac{\omega}{2}+j\frac{\alpha}{4}+\pi l+\pi\right)\hat{\varphi}_\alpha\left(\frac{\omega}{2}+j\frac{\alpha}{4}+\pi l\right)\right|^2 d\omega$$

$$=e^{\frac{\alpha}{2}k}\int_0^{2\pi} e^{j\omega k}\sum_l \left| m_{0,\alpha}\left(\frac{\omega}{2}+j\frac{\alpha}{4}+\pi l+\pi\right)\right|^2 \left|\hat{\varphi}_\alpha\left(\frac{\omega}{2}+j\frac{\alpha}{4}+\pi l\right)\right|^2 d\omega$$

$$=e^{\frac{\alpha}{2}k}\int_0^{2\pi} e^{j\omega k}\sum_l \left| m_{0,\alpha}\left(\frac{\omega}{2}+j\frac{\alpha}{4}+\pi l+\pi\right)\right|^2 \left|\hat{\varphi}_\alpha\left(\frac{\omega}{2}+j\frac{\alpha}{4}+\pi l\right)\right|^2 d\omega \quad (45)$$

splitting sum into odd and even index and using periodicity of $m_{\varphi,\alpha}(\omega)$ and $m_{0,\alpha}(\omega)$ fand using equation (17)

$$<\psi_\alpha(x),\psi_\alpha(x-k)>_{L_{2,\alpha}}=\delta_{k,0}$$

is obtained. Hence we show that we obtain orthonormal weighted wavelet basis function for weighted spaces which satify condition of weighted multiresolution with weight $e^{\alpha x}$.

5. CONCLUSION

In this study it is shown that the orthogonal wavelet functions will be able to got for the space sequence which gives the multiresolution spesifications in the weighted spaces for exponantial kind weighted function. The weighted wavelet functions can be used for the rapid numerical solution of certain integral equation and degenerate differential equations with boundary conditions.

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