WEIGHTED MULTIRESOLUTION SPACES
AND WEIGHTED WAVELETS

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\textbf{ABSTRACT}

In this study it is shown that the orthogonal wavelet functions will be able to get for the space sequence which gives the multiresolution specifications in the weighted spaces for exponential kind weighted function.

\textbf{Keywords:} Wavelets, Multiresolution Analysis, Weighted Spaces.

\section{1. INTRODUCTION}

Wavelets are a powerful tool for signal processing application. Essentially their properties are; compact support, vanishing moments, smooth, fast transform, so that they are localized in space and frequency and wavelet coefficients of a function decay rapidly. In this study it is shown that there is orthonormal wavelet basis function in the weighted spaces $L_{2,\alpha}(\mathbb{R})$ with exponential weight function $e^{\alpha x}$. Our aim is to find equations dependence on weighted parameter $\alpha$. It is constructed multiresolution analysis for weighted spaces so that it can be obtained wavelet basis for weighted orthogonal complementary spaces. We used algebraic method for construction.

\section{2. MULTIRESOLUTION ANALYSIS IN WEIGHTED SPACES}

It was shown that there is multiresolution analysis for certain function spaces and weighted spaces [1],[2],[3],[4]. Let $L_{2,\alpha}(\mathbb{R})$ is space of measurable functions $f$ where

$$\|f\| = \left(\int_{-\infty}^{\infty} |f(x)|^2 e^{\alpha x} \, dx \right)^{\frac{1}{2}} < \infty.$$  \hspace{1cm}

We define a multiresolution analysis in the space $L_{2,\alpha}(\mathbb{R})$ as a sequence of closed subspaces \{\textit{V}_{i,\alpha}\}_{i=\mathbb{Z}}$

\[ \cdots V_{2,\alpha} \supset V_{1,\alpha} \supset V_{0,\alpha} \supset V_{-1,\alpha} \supset V_{-2,\alpha} \cdots \] (1)
\begin{align*}
\bigcup_{i \in \mathbb{Z}} V_{1,\alpha} &= L_{2,\alpha}(R) \\
\bigcap_{i \in \mathbb{Z}} V_{1,\alpha} &= \{0\} \\
f(x) \in V_{1,\alpha}, f(2x) \in V_{1-\alpha} \\
f(x) \in V_{1,\alpha}, f(x-n) \in V_{1,\alpha}, n \in \mathbb{Z}
\end{align*}

This implies there is a scaling function \( \{ \phi_{0,n,\alpha}(x) \}_{n \in \mathbb{Z}} \in V_{0,\alpha} \) and it is element of \( V_{0,\alpha} \) and it is Riesz basis set of \( V_{0,\alpha} \) and \( \phi_{i,n,\alpha}(x) = 2^{-\frac{i}{2}} \phi_{\alpha}(2^{-i} x-n), i,n \in \mathbb{Z} \) are basis of \( V_{1,\alpha} \). The scaling function \( \phi_{\alpha}(x) \) is element of both \( V_{0,\alpha} \) and \( V_{1,\alpha} \) so that it is written refinement equation

\[ \phi_{\alpha}(x) = \sum_{n=-\infty}^{\infty} h_{n,\alpha} \phi_{-1,\alpha}(x-n) \]

\[ \phi_{\alpha}(x) = \sqrt{2} \sum_{n} h_{n,\alpha} \phi_{\alpha}(2x-n) \]

where

\[ h_{n,\alpha} = \langle \phi_{\alpha}, \phi_{-1,\alpha} \rangle_{L_{2,\alpha}} \]

\[ h_{n,\alpha} = \sqrt{2} \int \phi_{\alpha}(x) \overline{\phi_{\alpha}(2x-n)} e^{\alpha x} dx \]

The Fourier transform of right and left sides are

\[ \hat{\phi}_{\alpha}(\omega) = \sqrt{2} \sum_{n} h_{n,\alpha} \hat{\phi}_{\alpha}(2x-n) e^{-j\omega x} dx \]

\[ \hat{\phi}_{\alpha}(\omega) = \frac{1}{\sqrt{2}} \sum_{n} h_{n,\alpha} e^{-j\frac{\omega}{2} n} \int \phi_{\alpha}(u) e^{-j\omega u} du \]

with

\[ m_{0,\alpha}(\omega) = \frac{1}{\sqrt{2}} \sum_{n} h_{n,\alpha} e^{-j\omega n} \]

and

\[ \hat{\phi}_{\alpha}(\omega) = m_{0,\alpha}(\frac{\omega}{2}) \hat{\phi}_{\alpha}(\frac{\omega}{2}) \]

is obtained. \( m_{0,\alpha}(\omega) \) is \( 2\pi \)-periodic. The basis functions that is obtained by translating are orthonormal to each other in the weighted space \( L_{2,\alpha}(R) \) so that

\[ \delta_{k,0} = \langle \phi_{\alpha}(x), \phi_{\alpha}(x-k) \rangle_{L_{2,\alpha}} \]

\[ \delta_{k,0} = \int \phi_{\alpha}(x) \overline{\phi_{\alpha}(x-k)} e^{\alpha x} dx \]

\[ \delta_{k,0} = \int \phi_{\alpha}(x) e^{\frac{\alpha x}{2}} \overline{\phi_{\alpha}(x-k)} e^{\frac{\alpha x}{2}} dx \]

\[ \delta_{k,0} = \int \left\{ \frac{1}{2\pi} \int \phi_{\alpha}(\omega + j\frac{\alpha}{2}) e^{\alpha \omega} d\omega \right\} e^{\frac{\alpha x}{2}} \overline{\phi_{\alpha}(x-k)} e^{\frac{\alpha x}{2}} dx \]

\[ \delta_{k,0} = \int \phi_{\alpha}(\omega + j\frac{\alpha}{2}) \left\{ \frac{1}{2\pi} \int \phi_{\alpha}(x-k) e^{\alpha \omega} d\omega \right\} e^{\frac{\alpha x}{2}} dx \]

\[ \delta_{k,0} = e^{\frac{\alpha x}{2}} \int e^{\frac{\alpha \omega}{2}} \phi_{\alpha}(\omega + j\frac{\alpha}{2}) \overline{\phi_{\alpha}(x-k)} e^{\frac{\alpha \omega}{2}} dx \]

\[ \delta_{k,0} = e^{\frac{\alpha x}{2}} \int e^{\frac{\alpha \omega}{2}} \left| \frac{\phi_{\alpha}(\omega + j\frac{\alpha}{2})}{\phi_{\alpha}(\omega)} \right|^2 d\omega \]

we have for \( k \neq 0 \)

\[ \int e^{j\omega k} \left| \phi_{\alpha}(\omega + j\frac{\alpha}{2}) \right|^2 d\omega = 0 \]

and for \( k = 0 \)

\[ \int \left| \phi_{\alpha}(\omega + j\frac{\alpha}{2}) \right|^2 d\omega = 1 \]

For \( k=0 \) ; integral is splitted into \( 2\pi \)-length sequences as

\[ \int_{0}^{2\pi} \sum_{l} \left| \frac{\phi_{\alpha}(\omega + j\frac{\alpha}{2} + 2\pi l)}{\phi_{\alpha}(\omega + j\frac{\alpha}{2})} \right|^2 d\omega = 1 \]

so that

\[ \sum_{l} \left| \frac{\phi_{\alpha}(\omega + j\frac{\alpha}{2} + 2\pi l)}{\phi_{\alpha}(\omega + j\frac{\alpha}{2})} \right|^2 = \frac{1}{2\pi} \]

Substituting (10) into (15)
The Fourier transform of the double side of equation is

\[ \sum_{l} m_{\alpha l} \left( \frac{\omega + j \alpha}{4} + \pi \right)^{2} \hat{\Phi} \left( \frac{\omega + j \alpha}{4} + \pi \right)^{2} = \frac{1}{2\pi} \] (16)

is obtained. In this summation, we split into even and odd index, using periodicity of \( m_{\alpha l} (\omega) \) and \( m_{0,\alpha} (\omega) \), using equation (15)

\[ |m_{\alpha l} \left( \frac{\omega + j \alpha}{4} \right)^{2} + m_{\alpha l} \left( \frac{\omega + j \alpha + \pi}{4} \right)^{2}| = 1 \] (17)

is obtained. Equation 17 is simplyfied as

\[ |m_{\alpha l} (\tau + j \frac{\alpha}{4})^{2} + m_{\alpha l} (\tau + j \frac{\alpha}{4} + \pi)| = 1 \] (18)

where \( \tau = \frac{\omega}{2} \).

3. WEIGHTED ORTHOGONAL DIFFERENCE SPACE AND WEIGHTED WAVELET BASIS FUNCTIONS

Let's consider orthogonal complementry space sequence of space sequence \( \{ V_{\alpha l} \} \).

For a space in this space sequence \( V_{-1,\alpha} = V_{0,\alpha} \oplus W_{0,\alpha} \) is obtained by direct sum. For a function \( f \) which elements of difference space \( f \in W_{0,\alpha} \), \( f \in V_{-1,\alpha} \), \( f \perp V_{0,\alpha} \) so that \( f \) is represented with basis function of \( V_{-1,\alpha} \)

\[ f = \sum_{n} f_{n,\alpha} \Phi_{-1,n,\alpha} \] (20)

The coefficient \( f_{n,\alpha} \) is obtained with weighted inner product as

\[ f_{n,\alpha} = \langle f, \Phi_{-1,n,\alpha} \rangle \] (21)

The Fourier transform of the double side of equation is

\[ \hat{f}(\omega) = \sqrt{2} \sum_{n} f_{n,\alpha} \hat{\Phi}(2x-n)e^{-j\omega x} \] (22)

We define

\[ m_{f,\alpha} = \frac{1}{\sqrt{2}} \sum_{n} f_{n,\alpha} e^{-j\omega n} \] and

\[ \hat{f}(\omega) = m_{f,\alpha} \hat{\Phi}_{\alpha} \frac{\omega}{2} \] (23)

is obtained. \( m_{f,\alpha} (\omega) \) is \( 2\pi \)-periodic.

\[ f \perp V_{0,\alpha} \) so that \( f \) is orthogonal to basis function of \( f \perp V_{0,\alpha} \).

\[ \int f(x)\Phi_{\alpha}(x-k)e^{j\omega x} dx = 0 \] (25)

Using equation (26)

\[ \int \left( \frac{1}{2\pi} \int f(\omega + j\frac{\alpha}{2})e^{j\omega x} d\omega \right) \Phi_{\alpha}(x-k)e^{-j\omega x} dx = 0 \] (26)

is obtained and conclusion

\[ \int e^{-j\omega x} \Phi_{\alpha}(x-k)e^{j\omega x} d\omega = 0 \] (27)

is obtained. This equation is splitted \( 2\pi \)-length

\[ \sum_{l} \int f(\omega + j\frac{\alpha}{2} + 2\pi l)\hat{\Phi}_{\alpha}(x) \Phi_{\alpha}(x-k)e^{j\omega x} dx = 0 \] (28)

substituting (10) and (23) into (28)

\[ \sum_{l} \int m_{f,\alpha}(\omega + j\frac{\alpha}{4} + 2\pi l)\hat{\Phi}_{\alpha}(x) \Phi_{\alpha}(x-k)e^{j\omega x} dx = 0 \] (29)
Weighted Multiresolution Spaces And Weighted Wavelets

Veli SHAKHMUROV and Hasan DEM IR

is obtained. We split the summation into odd and even index and using periodicity of \( m_{f,\alpha}(\omega) \) and using equation (15)

\[
m_{f,\alpha}\left(\frac{\omega}{2} + j\frac{\alpha}{4}\right) m_{0,\alpha}\left(\frac{\omega}{2} + j\frac{\alpha}{4}\right) +
\]

\[
m_{f,\alpha}\left(\frac{\omega}{2} + j\frac{\alpha}{4} + \pi\right) m_{0,\alpha}\left(\frac{\omega}{2} + j\frac{\alpha}{4} + \pi\right) = 0
\]

is obtained. For this equation, we can find a function \( \lambda(\tau) \) with \( 2\pi \)-periodic as

\[
m_{f,\alpha}(\tau + j\frac{\alpha}{4}) = \lambda(\tau)m_{0,\alpha}(\tau + j\frac{\alpha}{4} + \pi)
\]

and we substituting \( \lambda(\tau) \)

\[
\lambda(\tau)m_{0,\alpha}(\tau + j\frac{\alpha}{4} + \pi) m_{0,\alpha}(\tau + j\frac{\alpha}{4}) +
\]

\[
\lambda(\tau + \pi)m_{0,\alpha}(\tau + j\frac{\alpha}{4} + \pi) = 0
\]

\[
\lambda(\tau + \pi) = 0
\]

and \( \lambda(\tau) \) satisfies

\[
\lambda(\tau) + \lambda(\tau + \pi) = 0
\]

\[
\lambda(\tau) \text{ is chosen as}
\]

\[
\lambda(\tau) = e^{j\pi/2}\gamma(2\tau)
\]

substituting (32) and (36) into (23)

\[
\hat{f}(\omega) = \sum_{k} \gamma(k) e^{-j\pi k/\alpha} \hat{\psi}_{\alpha}(\omega)
\]

is obtained. In this formula we define

\[
\hat{\psi}_{\alpha}(\omega) = e^{j\frac{\alpha}{2} \frac{\omega}{2}} m_{0,\alpha}\left(\frac{\omega}{2} + \pi\right) \hat{\psi}_{0}\left(\frac{\omega}{2}\right)
\]

and we rewrite \( \hat{f}(\omega) \) as

\[
\hat{f}(\omega) = \sum_{k} \gamma(k) e^{-j\pi k/\alpha} \hat{\psi}_{\alpha}(\omega)
\]

where

\[
\gamma(\omega - j\frac{\alpha}{2}) = \sum_{k} \gamma(k)e^{-j\pi k/\alpha}
\]

Hence

\[
\hat{f}(\omega) = \sum_{k} \gamma(k) \hat{\psi}_{\alpha}(\omega) e^{-j\pi k/\alpha}
\]

and using inverse Fourier transform, we obtain \( f(x) \) with weighted wavelet basis functions

\[
f(x) = \sum_{k} \gamma(k) e^{-j\pi k/\alpha} \frac{1}{2\pi} \int \hat{\psi}_{\alpha}(\omega) e^{j\omega x} e^{j\pi k/\alpha} d\omega
\]

this implies that a function \( f(x) \) which is element of \( W_{0,\alpha} \) is represented with \( \psi_{\alpha}(x) \).

4. ORTHOGONAL WAVELET BASIS FUNCTIONS

Analysing orthogonality of wavelet, we have

\[
<\psi_{\alpha}(x), \psi_{\alpha}(x-k)>_{L_{2,\alpha}} = \frac{\alpha}{2\pi} e^{j\frac{\alpha}{2} \frac{k}{2}} \int \hat{\psi}_{\alpha}(\omega) e^{j\omega x} e^{j\omega (x-k)} d\omega
\]

\[
<\psi_{\alpha}(x), \psi_{\alpha}(x-k)>_{L_{2,\alpha}} = e^{j\frac{\alpha}{2} \frac{k}{2}} \int e^{j\omega x} e^{j\omega (x-k)} d\omega
\]

substituting (38)

\[
\frac{\alpha}{2\pi} e^{j\frac{\alpha}{2} \frac{k}{2}} \int \hat{\psi}_{\alpha}(\omega) e^{j\omega x} e^{j\omega (x-k)} d\omega
\]

\[
e^{j\frac{\alpha}{2} \frac{k}{2}} \int e^{j\omega x} \int e^{j\omega (x-k)} d\omega
\]
\[ \langle \psi(x), \psi(x-k) \rangle_{L^2} = \]
\[ e^{2\pi i k x} \sum_{n \in \mathbb{Z}} \left( m_{0,\alpha} \left( \frac{\omega}{2} + j \frac{\alpha}{4} + \pi \right) \right) \left( m_{0,\alpha} \left( \frac{\omega}{2} + j \frac{\alpha}{4} + \pi \right) \right)^* \]
\[ = e^{2\pi i k x} \sum_{n \in \mathbb{Z}} \left( m_{0,\alpha} \left( \frac{\omega}{2} + j \frac{\alpha}{4} + \pi \right) \right) \left( m_{0,\alpha} \left( \frac{\omega}{2} + j \frac{\alpha}{4} + \pi \right) \right)^* \]
\[ \int_{-\infty}^{\infty} \left[ \left( m_{0,\alpha} \left( \frac{\omega}{2} + j \frac{\alpha}{4} + \pi \right) \right) \left( m_{0,\alpha} \left( \frac{\omega}{2} + j \frac{\alpha}{4} + \pi \right) \right)^* \right] \, d\alpha \quad (45) \]

splitting sum into odd and even index and using

periodicity of \( m_{f,\alpha} (\omega) \) and \( m_{0,\alpha} (\omega) \) and using equation (17)

\[ \langle \psi(x), \psi(x-k) \rangle_{L^2} = \delta_{k,0} \]
is obtained. Hence we show that we obtain orthonormal weighted wavelet basis function for weighted spaces which satisfy condition of weighted multiresolution with weight \( e^{\alpha x} \).

5. CONCLUSION

In this study it is shown that the orthogonal wavelet functions will be able to get for the

space sequence which gives the multiresolution specifications in the weighted spaces for exponential kind weighted function. The weighted wavelet functions can be used for the rapid numerical solution of certain integral equation and degenerate differential equations with boundary conditions.

REFERENCES


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