A Study over the Hyperbolic and Logarithmic Monomolecular, Logistic and Gompertz Growth Models

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ABSTRACT: In this study in addition to classical monomolecular, logistic and Gompertz models, their hyperbolic and logarithmic growth models were found. After that it is searched the effect of these hyperbolic and logarithmic growth models on the choice of appropriate growth model by using two separate data sets. For this purpose, classical monomolecular, logistic and Gompertz growth models and their hyperbolic and logarithmic growth models are compared with some model selection criteria such as coefficient of determination, error sum of squares. For two data sets it is found that the results of these hyperbolic and logarithmic growth models are better than the results of these growth models. Thus, it is considered that these hyperbolic and logarithmic growth models can be used in addition to these growth models. Even the results of these hyperbolic and logarithmic growth models were found the same for both data sets. In addition, some other hyperbolic and logarithmic growth models can be investigated for getting the best model choice.

Keywords: Hyperbolic, Logarithmic, Growth Models, Monomolecular, Logistic, Gompertz

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INTRODUCTION

Growth is one of the well-known features in biological creatures (Burkhart and Strub,1974). Growth models describe the changing size of something over time. The process of developing a mathematical model is called mathematical modeling. A model may help to explain a system and to study the effects of different components, and also to make predictions about behavior (Oyamakin and Chukwu, 2015).

A mathematical description of a real world system is often referred to as a mathematical model. A system can be formally defined as a set of elements also called components. A system can be formally defined as a set of elements also called components A set of trees in a forest stand, producers and consumers in an economic system are examples of components. The elements (components) have certain characteristics or attributes and these attributes have numerical or logical values. Among the elements, relationships exist and consequently the elements are interacting. The state of a system is determined by the numerical or logical values of the attributes of the system elements (Oyamakin and Chukwu, 2015). Experimenting on the state of a system with a model over time is termed simulation (Kansal et al., 2000).

In this study, an alternative nonlinear growth models called the hyperbolic and logarithmic monomolecular, Logistic and Gompertz were introduced and compared with the classical monomolecular (Brody,1945), Logistic (Ricker, 1979) and Gompertz models (Winsor, 1932) which are widely used.

Monomolecular Model

Since the origin of the monomolecular growth model suggested that the growth is directly proportional to the difference of the maximum sustainable constant A and the growth, this growth model starts the following differential equation:

$$\frac{\partial y}{\partial x} = r(A - y) \qquad (1)$$

where y is the value of growth, x is time, r is the proportion about growth, A is maximum sustainable growth

This differential equation produces a model of the following form:

$$y = A(1 - be^{-rx}) \qquad (2)$$

where b is a constant

The monomolecular model in equation 2 is found the following operations:

Separating the variables in equation 1 we get:

$$\frac{dy}{A-y} = rdx$$

Integrating both sides we get:

$$-\ln(A - y) = rx + C_1$$

$$\ln(A - y)^{-1} = rx + C_1$$

where C_1 is constant

Taking the exponential of both sides we get:

$$(A - y)^{-1} = Be^{rx}$$

$$\frac{1}{A-y} = Be^{rx}$$

where $B = e^{C_1}$

Multiply both sides by A-y we get:

$$1 = ABe^{rx} - yBe^{rx}$$

$$yBe^{rx} = ABe^{rx} - 1$$

Divide by Be^{rx} we get monomolecular model as

$$y = A(1 - be^{-rx})$$

where b=(AB)$^{-1}$

Logistic Model

The origin of the logistic growth model is described by the following equation:

$$\frac{\partial y}{\partial x} = ry\left(1 - \frac{y}{A}\right) \tag{3}$$

where y is the value of growth, x is time, r is the proportion about growth, A is maximum sustainable growth

This differential equation produces the model called logistic model in equation 4:

$$y = \frac{A}{1+be^{-rx}} \tag{4}$$

where b is a constant

The logistic model in equation 4 is found the following operations:

Separating the variables in equation 3 we get:

$$\frac{Ady}{y(A-y)} = rdx$$

Integrating both sides we get:

$$\int \frac{Ady}{y(A-y)} = \int rdx$$

$$\int \left(\frac{1}{y} + \frac{1}{A-y}\right)dy = \int rdx$$

$$\ln(y) - \ln(A-y) = rx + C_1$$

$$\ln\left(\frac{A-y}{y}\right) = -rx - C_1$$

where C_1 is constant

Taking the exponential of both sides we get:

$$\frac{A-y}{y} = be^{-rx}$$

$$\frac{A}{y} - 1 = be^{-rx}$$

where $b = e^{-C_1}$

After this we get:

$$\frac{A}{y} = be^{-rx} + 1$$

and then get the logistic model as

$$y = \frac{A}{1 + be^{-rx}}$$

Gompertz Model

The origin of the Gompertz growth model is described by the following equation:

$$\frac{\partial y}{\partial x} = ryln(\frac{A}{y}) \qquad\qquad (5)$$

where y is the value of growth, x is time, r is the proportion about growth, A is maximum sustainable growth

This differential equation produces the model called the Gompertz model in equation 6:

$$y = Ae^{-be^{-rx}} \qquad\qquad (6)$$

where b is a constant

The Gompertz model in equation 6 is found the following operations:

Separating the variables in equation 5 we get:

$$\frac{dy}{yln(\frac{A}{y})} = rdx$$

Integrating both sides we get:

$$\int \frac{dy}{yln(\frac{A}{y})} = \int rdx$$

If we say $u = \ln\left(\frac{A}{y}\right)$ $\ and\ du = \frac{-\frac{A}{y^2}}{\frac{A}{y}}dy = \frac{-dy}{y}$, then we can get

$$-\int \frac{du}{u} = \int rdx$$

and

$$-lnu = rx + C_2$$

Taking the exponential of both sides we get:

$$u = be^{-rx}$$

$$\ln\left(\frac{A}{y}\right) = be^{-rx}$$

where $b = e^{-C_2}$

Again taking the exponential of both sides we get:

$$\frac{A}{y} = e^{be^{-rx}}$$

and then get the Gompertz model as

$$y = Ae^{-be^{-rx}}$$

MATERIALS AND METHODS

Materials

Two different data sets were used in this study. The first one is the mean fork length measurements of female and male flathead trout (*Salmo platycephalus*). The second one is the length of the eucalyptus plant.

In this study, the mean fork length measurements of the flathead trout (*Salmo platycephalus*), which is listed in the IUCN red list of threatened species, which were sampled from the Zamanti Stream of Seyhan River were used (Kara et al., 2011). For the presentation of the models, the measurements of the fork lengths (cm) in the age-structured female and male of *S. platycephalus* from Zamanti Stream of the River Seyhan in Table 1 in their articles were used in this study in Table 1.

Table 1. Observed mean fork lengths of *S. platycephalus* According to the Gender

Sex/Age (year)	1	2	3	4	5	6	7	8	9	10
Female	13.68	18.03	21.63	25.55	28.29	30.85	33.37	36.03	38.30	40.0
Male	15.15	18.72	20.91	25.11	27.97	30.88	33.13	35.94	37.04	39.20

In this study, the data taken from the tree, *E. Camaldulensis Dehn*. were used for the growth model in Table 2. The data set was taken from the study of Yıldızbakan (2015).

Table 2. The height growth value of the trees (*E. Camaldulensis* Dehn) according to year

The height growth value of the trees (*E. Camaldulensis* Dehn)									
Planting Age (year)	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Height Growth (m)	0.41	3.23	7.45	11.41	14.83	18.11	18.95	19.69	21.50	23.40

Methods

The hyperbolic functions have similar names to the trigonometric functions, but they are defined in terms of the exponential function (Oyamakin and Chukwu, 2015).

Thus the hyperbolic sine function and its inverse provide an alternative method for evaluating:

$$\int \frac{1}{\sqrt{1 + x^2}} dx \qquad (7)$$

say x=sinh(u) and then dx=cosh(u) du

$$\int \frac{1}{\sqrt{1 + x^2}} dx = \text{arcsinh}(x) + C_3 \qquad (8)$$

where C_3 is constant

Also say x=tanu , $(\frac{-\pi}{2} < u < \frac{\pi}{2})$ dx=(1+tan²u)du

$$\int \frac{1}{\sqrt{1 + x^2}} dx = ln\left|x + \sqrt{1 + x^2}\right| + C_4 \qquad (9)$$

where C_4 is constant

From equations 8 and 9, we get:

$$\text{arcsinh}(x) = ln\left|x + \sqrt{1 + x^2}\right| + C_4 - C_3 \qquad (10)$$

for all x. Evaluating both sides of this equality at x=0, we get

$$0 = \text{arcsinh}(0) = ln|1| + C_3 - C_2$$

So $C_4 - C_3 = 0$ and

$$\text{arcsinh}(x) = ln\left|x + \sqrt{1 + x^2}\right| \qquad (11)$$

for all x. Since the hyperbolic sine function is defined in terms of the exponential function, we should not find it surprising that the inverse hyperbolic sine function may be expressed in terms of the natural logarithm function (Oyamakin and Chukwu, 2015).

Hyperbolic and Logarithmic Monomolecular Growth Model

The origin of the hyperbolic and logarithmic monomolecular growth models suggested that in addition to constant r in the original monomolecular model and the term

$$\frac{d}{\sqrt{1 + x^2}}$$

which is multiplied by the difference of the maximum sustainable constant A and the growth is added where d is proportion about the hyperbolic and logarithmic monomolecular growth, respectively.
And so the hyperbolic and logarithmic monomolecular growth models start the following differential equation:

$$\frac{\partial y}{\partial x} = (r + \frac{d}{\sqrt{1+x^2}})(A - y) \qquad (12)$$

This differential equation produces two types of models in equation 13 and equation 14, respectively:

$$y = A(1 - be^{-rx - darcsinh(x)}) \qquad (13)$$

$$y = A(1 - be^{-rx}\left(x + \sqrt{1 + x^2}\right)^{-d}) \qquad (14)$$

The hyperbolic and logarithmic monomolecular models in equation 13 and equation 14 are found the following operations, respectively:
Separating the variables in equation 12 we get:

$$\frac{dy}{A - y} = (r + \frac{d}{\sqrt{1+x^2}})dx$$

Integrating both sides we get:

$$-\ln(A - y) = rx + darcsinh(x) + C_5$$

$$\ln(A - y)^{-1} = rx + darcsinh(x) + C_5 \qquad (15)$$

and

$$-\ln(A - y) = rx + dln(x + \sqrt{1 + x^2}) + C_5$$

$$\ln(A - y)^{-1} = rx + dln(x + \sqrt{1 + x^2}) + C_5 \qquad (16)$$

where C_5 is constant
Taking the exponential of both sides of equation 15 and equation 16, respectively we get:

$$(A - y)^{-1} = Be^{rx + darcsinh(x)}$$

$$\frac{1}{A - y} = Be^{rx + darcsinh(x)} \qquad (17)$$

and

$$(A - y)^{-1} = Be^{rx + dln(x + \sqrt{1 + x^2})}$$

$$\frac{1}{A - y} = Be^{rx}(x + \sqrt{1 + x^2})^d \qquad (18)$$

respectively where $B = e^{C_5}$

Multiply both sides of equation 17 and equation 18 by (A-y), respectively we get:

$1 = ABe^{rx+darcsinh(x)} - yBe^{rx+darcsinh(x)}$

$yBe^{rx+darcsinh(x)} = ABe^{rx+darcsinh(x)} - 1$ (19)

and

$1 = ABe^{rx}(x + \sqrt{1 + x^2})^d - yBe^{rx}(x + \sqrt{1 + x^2})^d$

$yBe^{rx}(x + \sqrt{1 + x^2})^d = ABe^{rx}(x + \sqrt{1 + x^2})^d - 1$ (20)

respectively.

Divide equation 19 and equation 20 by $Be^{rx+darcsinh(x)}$ and $e^{rx}(x + \sqrt{1 + x^2})^d$, respectively we get the hyperbolic and logarithmic monomolecular models as

$y = A(1 - be^{-rx-darcsinh(x)})$

and

$y = A(1 - be^{-rx}(x + \sqrt{1 + x^2})^{-d}$

respectively where b=(AB)$^{-1}$

Hyperbolic and Logarithmic Logistic Growth Model

The origin of the hyperbolic and logarithmic logistic growth models suggested that in addition to constant r in the original logistic model and the term

$$\frac{d}{\sqrt{1 + x^2}}$$

which is multiplied by the multiplication of the difference of the maximum sustainable constant A and the growth and the division of the growth and the maximum sustainable constant A   is added where d is proportion about hyperbolic and logarithmic logistic growth.

And so the hyperbolic and logarithmic logistic growth models start the following differential equation:

$$\frac{\partial y}{\partial x} = \left(r + \frac{d}{\sqrt{1+x^2}}\right)\frac{y}{A}(A - y) \qquad (21)$$

This differential equation produces two types of models in equation 22 and equation 23, respectively:

$$y = \frac{A}{1 + be^{-rx-darcsinh(x)}} \qquad (22)$$

$$y = \frac{A}{1 + be^{-rx}(x + \sqrt{1 + x^2})^{-d}} \qquad (23)$$

The hyperbolic and logarithmic logistic growth models in equation 22 and equation 23 are found the following operations, respectively:

Separating variables in equation 21 we get:

$$\frac{Ady}{y(A-y)} = (r + \frac{d}{\sqrt{1+x^2}})dx$$

Integrating both sides we get:

$$\int \frac{Ady}{y(A-y)} = \int (r + \frac{d}{\sqrt{1+x^2}})dx$$

$$\ln(y) - \ln(A-y) = rx + darcsinh(x) + C_6$$

$$\ln\left(\frac{A-y}{y}\right) = -rx - darcsinh(x) - C_6 \qquad (24)$$

and

$$\ln(y) - \ln(A-y) = rx + dln(x + \sqrt{1+x^2}) + C_6$$

$$\ln\left(\frac{A-y}{y}\right) = -rx - dln(x + \sqrt{1+x^2}) - C_6 \qquad (25)$$

where C_6 is constant

Taking the exponential of both sides of equation 24 and equation 25, respectively we get:

$$\frac{A-y}{y} = be^{-rx-darcsinh(x)}$$

$$\frac{A}{y} - 1 = be^{-rx-darcsinh(x)} \qquad (26)$$

and

$$\frac{A-y}{y} = be^{-rx-dln(x+\sqrt{1+x^2})}$$

$$\frac{A}{y} - 1 = be^{-rx-dln(x+\sqrt{1+x^2})} \qquad (27)$$

respectively where $b = e^{-C_5}$

After this we get:

$$\frac{A}{y} = be^{-rx-darcsinh(x)} + 1$$

and

$$\frac{A}{y} = be^{-rx}(x + \sqrt{1+x^2})^{-d} + 1$$

And then get the hyperbolic and logarithmic logistic growth models as

$$y = \frac{A}{1 + be^{-rx-darcsinh(x)}}$$

and

$$y = \frac{A}{1 + be^{-rx}(x + \sqrt{1+x^2})^{-d}}$$

respectively.

Hyperbolic and Logarithmic Gompertz Growth Model

The origin of the hyperbolic and logarithmic Gompertz growth models suggested that in addition to constant r in the original Gompertz model and the term

$$\frac{d}{\sqrt{1+x^2}}$$

which is multiplied by the multiplication of the logarithm of the division of the maximum sustainable constant A and the growth and the growth is added where d is proportion about the hyperbolic and logarithmic Gompertz growth.

And so the hyperbolic and logarithmic Gompertz growth models start the following differential equation:

$$\frac{\partial y}{\partial x} = (r + \frac{d}{\sqrt{1+x^2}})yln(\frac{A}{y}) \qquad (28)$$

This differential equation produces two types of models in equation 29 and equation 30, respectively:

$$y = Ae^{-be^{-rx-darcsinh(x)}} \qquad (29)$$

and

$$y = Ae^{-be^{-rx}(x+\sqrt{1+x^2})^{-d}} \qquad (30)$$

The hyperbolic and logarithmic Gompertz growth model in equation 29 and equation 30 are found the following operations, respectively:

Separating the variables in equation 28 we get:

$$\frac{dy}{yln(\frac{A}{y})} = (r + \frac{d}{\sqrt{1+x^2}})dx$$

Integrating both sides we get:

$$\int \frac{Ady}{yln(\frac{A}{y})} = \int (r + \frac{d}{\sqrt{1+x^2}})dx \qquad (31)$$

Solving these integrals in equation 31 we get

$$\ln\left(\frac{A}{y}\right) = be^{-rx-darcsinh(x)} \qquad (32)$$

and

$$\ln\left(\frac{A}{y}\right) = be^{-rx}(x + \sqrt{1 + x^2})^{-d} \qquad (33)$$

where b is constant

Taking the exponential of both sides of equation 32 and equation 33, respectively we get:

$$\frac{A}{y} = e^{be^{-rx-darcsinh(x)}}$$

and

$$\frac{A}{y} = e^{be^{-rx\left(x+\sqrt{1+x^2}\right)^{-d}}}$$

respectively and then get the hyperbolic and logarithmic Gompertz growth models as

$$y = Ae^{-be^{-rx-darcsinh(x)}}$$

and

$$y = Ae^{-be^{-rx}(x+\sqrt{1+x^2})^{-d}}$$

respectively.

RESULTS AND DISCUSSION

Since the hyperbolic and logarithmic growth models in this study are exactly same models, the parameters and the results of known model selection criteria such as error sum of squares (SSE), and coefficient of determination (R^2) of them are exactly same.

By using the Tables 1 and 2 of these two sets of data, in Tables 3 and 5 the parameters of Monomolecular, Logistic and Gompertz growth models and their hyperbolic and logarithmic growth models are calculated. All calculations were made with the maple package program.

To make a comparison; error sum of squares, and coefficient of determination are used from known model selection criteria. Calculations made are tabulated.

Table 1 shows the mean fork length measurements of female and male flathead trout (*Salmo platycephalus*). While the model parameters calculated according to these values are given in Table 3, the error sum of squares and determination coefficients of the models calculated are given in Table 4. Tables 3 and Table 4 are calculated according to the values in Table 1.

Table 3 Calculated the Model Parameters According to the Gender

	MODELS	EQATION	A	b	r	d
Female	Monomolecular	$y = A(1 - be^{-rx})$	58.876	0.844	0.097	-
	Hyperbolic Monomolecular	$y = A(1 - be^{-rx - d\,arcsinh(x)})$	66.714	0.875	0.070	0.030
	Logarithmic Monomolecular	$y = A(1 - be^{-rx}\left(x + \sqrt{1+x^2}\right)^{-d})$	66.714	0.875	0.070	0.030
	Logistic	$y = \dfrac{A}{1 + be^{-rx}}$	44.050	-2.815	0.322	-
	Hyperbolic Logistic	$y = \dfrac{A}{1 + be^{-rx - d\,arcsinh(x)}}$	60.095	5.863	0.093	0.511
	Logarithmic Logistic	$y = \dfrac{A}{1 + be^{-rx}(x + \sqrt{1+x^2})^{-d}}$	60.095	5.863	0.093	0.511
	Gompertz	$y = Ae^{-be^{-rx}}$	47.833	1.504	0.210	-
	Hyperbolic Gompertz	$y = Ae^{-be^{-rx - d\,arcsinh(x)}}$	61.128	1.968	0.093	0.203
	Logarithmic Gompertz	$y = Ae^{-be^{-rx}(x + \sqrt{1+x^2})^{-d}}$	61.128	1.968	0.093	0.203
Male	Monomolecular	$y = A(1 - be^{-rx})$	62.948	0.827	0.079	-
	Hyperbolic Monomolecular	$y = A(1 - be^{-rx - darcsinh(x)})$	49.642	0.726	0.162	-0.130
	Logarithmic Monomolecular	$y = A(1 - be^{-rx}\left(x + \sqrt{1+x^2}\right)^{-d})$	49.642	0.726	0.162	-0.130
	Logistic	$y = \dfrac{A}{1 + be^{-rx}}$	44.192	2.547	0.295	-
	Hyperbolic Logistic	$y = \dfrac{A}{1 + be^{-rx - darcsinh(x)}}$	44.413	2.598	0.287	0.019
	Logarithmic Logistic	$y = \dfrac{A}{1 + be^{-rx}(x + \sqrt{1+x^2})^{-d}}$	44.413	2.598	0.287	0.019
	Gompertz	$y = Ae^{-be^{-rx}}$	48.748	1.417	0.187	-
	Hyperbolic Gompertz	$y = Ae^{-be^{-rx - darcsinh(x)}}$	46.165	1.282	0.233	-0.098
	Logarithmic Gompertz	$y = Ae^{-be^{-rx}(x + \sqrt{1+x^2})^{-d}}$	46.165	1.282	0.233	-0.098

According to Table 4, the hyperbolic and logarithmic models of the models used in this study has approximately the same least error sum of squares and the highest determination coefficient. Therefore, it can be said that the most suitable models for the data set are the hyperbolic and logarithmic ones.

Table 2 shows the growth values of the eucalyptus plant as a whole. While the model parameters calculated according to these values are given in Table 5, the error sum of squares and the determination coefficients of the models calculated are given in Table 6.
 Table 5 and Table 6 are calculated according to the values in Table 2.

Table 4 Error Sum of Squares and Determination Coefficients of the Models Calculated

MODELS	Female		Male	
	Error Sum Squares	Determination Coefficients of the models (R^2)	Error Sum Squares	Determination Coefficients of the models (R^2)
Monomolecular	0.368	0.9995	1.774	0.9971
Hyperbolic Monomolecular	0.303	0.9996	1.154	0.9981
Logarithmic Monomolecular	0.303	0.9996	1.154	0.9981
Logistic	2.284	0.9967	1.089	0.9982
Hyperbolic Logistic	0.301	0.9996	1.087	0.9982
Logarithmic Logistic	0.301	0.9996	1.087	0.9982
Gompertz	1.059	0.9985	1.222	0.9980
Hyperbolic Gompertz	0.305	0.9996	1.090	0.9982
Logarithmic Gompertz	0.305	0.9996	1.090	0.9982

Table 5 Model Parameters Calculated

MODELS	EQATION	A	b	r	d
Monomolecular	$y = A(1 - be^{-rx})$	29.999	1.014	0.166	-
Hyperbolic Monomolecular	$y = A(1 - be^{-rx - darcsinh(x)})$	24.609	0.986	0.357	-0.253
Logarithmic Monomolecular	$y = A(1 - be^{-rx}\left(x + \sqrt{1 + x^2}\right)^{-d})$	24.609	0.986	0.357	-0.253
Logistic	$y = \dfrac{A}{1 + be^{-rx}}$	21.887	10.685	0.781	-
Hyperbolic Logistic	$y = \dfrac{A}{1 + be^{-rx - darcsinh(x)}}$	29.104	39.955	-0.061	1.894
Logarithmic Logistic	$y = \dfrac{A}{1 + be^{-rx}(x + \sqrt{1 + x^2})^{-d}}$	29.104	39.955	-0.061	1.894
Gompertz	$y = Ae^{-be^{-rx}}$	23.071	3.118	0.488	-
Hyperbolic Gompertz	$y = Ae^{-be^{-rx - darcsinh(x)}}$	30.054	4.926	0.062	0.804
Logarithmic Gompertz	$y = Ae^{-be^{-rx}(x + \sqrt{1 + x^2})^{-d}}$	30.054	4.926	0.062	0.804

It is seen in Table 6 that the model with the lowest sum of error squares and the highest determination coefficient is the hyperbolic Gompertz growth model.

Table 6 Error Sum of Squares and Determination Coefficients of the Models Calculated

MODELS	Error Sum Squares	Determination Coefficients of the models (R^2)
Monomolecular	5.266	0.9908
Hyperbolic Monomolecular	2.608	0.9954
Logarithmic Monomolecular	2.608	0.9954
Logistic	8.713	0.9847
Hyperbolic Logistic	2.417	0.9958
Logarithmic Logistic	2.417	0.9958
Gompertz	3.917	0.9931
Hyperbolic Gompertz	2.301	0.9960
Logarithmic Gompertz	2.301	0.9960

CONCLUSION

In this study, three different sigmoidal growth models, which are frequently found in scientific studies, and their hyperbolic growth models have been used. These models have been applied to two different data sets. It has been determined which model is suitable according to the known model selection criteria. It is shown that the results are compatible with each other. It is shown that the appropriate models for both sets of data are their hyperbolic and Logarithmic ones according to the model selection criteria used in this study. That's instead of the growth models used in this study, the use of their hyperbolic and Logarithmic growth models provided the least error sum of squares and the maximal determination coefficients for two data sets. In other words, hyperbolic and Logarithmic growth models have the lowest sum of error squares and the highest determination coefficient according to their classical growth models. In addition, the lowest sum of error squares and the highest determination coefficient of hyperbolic and Logarithmic growth models are the same. Therefore, the researchers can use their hyperbolic and Logarithmic growth models in addition to the growth models for getting better results. In this study, instead of comparing the models with each other, hyperbolic and logarithmic models were compared with the same model. In addition, for the other growth models not used in this study, their hyperbolic and logarithmic growth models can be investigated for getting the best model choice.

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