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On the global stability of some k-order difference equations

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Abstract

We use two different techniques, one of them including fixed point tools, i.e., the Prešić type fixed point theorem, in order to study the asymptotic stability of some k-order difference equations for $k = 1$ and $k = 2$. In this way, we can study the global stability for more general initial value problems associated with particular forms of difference equations.

Keywords: Difference equation; Equilibrium point; Global attractor; Global asymptotic stability; Presic fixed point theorem.
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1. Introduction

In [12] the authors studied the dynamics and the global asymptotic stability of the second order difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-1}}, n = 0, 1, \dots \tag{1.1}$$

where $y_{-1}, y_0, A \in (0, \infty)$. They have shown that the unique positive equilibrium $\overline{y} = 1 + A$ of equation (1.1) is globally asymptotically stable. In [1], some partial answers to Conjecture 6.4.1 and Open Problem 6.4.1 in [12] where given by obtaining a sufficient condition for the global asymptotic stability of the unique positive equilibrium of the more general $(k + 1)$-order difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, n = 0, 1, \dots \tag{1.2}$$

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where $y_{-k}, \ldots, y_0, A \in (0, \infty)$ and $k \in \{2, 3, \ldots\}$. More recently, El-Owaidy et al [15], [15], and Stević [28], [29]-[31], and many other authors have studied the dynamics of the difference equations in the family

$$x_{n+1} = \alpha + \frac{x_n^p}{x_n^p}, n = 0, 1, \ldots \tag{1.3}$$

where: $\alpha \in [0, \infty)$ and $p \in [1, \infty)$ (in [15]), while, in [28], all parameters are nonnegative real numbers.

In continuation of this research work, Aloqeili [2] studied the asymptotic behaviour of the rational difference equation

$$x_{n+1} = \alpha + \frac{x_n^p}{x_{n-1}^p}, n = 0, 1, \ldots \tag{1.4}$$

where: $\alpha \in [0, \infty)$ and $p \in (0, 1)$ and $x_{-1}, x_0 \in (0, +\infty)$.

Aloqeili [2] also studied a more general difference equation, i.e.,

$$x_{n+1} = \alpha + \frac{x_n^p}{x_{n-k}^p}, n = 0, 1, \ldots \tag{1.5}$$

where: $\alpha \in [0, \infty)$ and $p \in (0, 1)$ and $x_{-k}, \ldots, x_0 \in (0, +\infty)$ and $k \in \{1, 2, \ldots\}$.

The technique of proof in the papers [1]-[3], [9]-[31] and in many others is essentially based on the linearized stability theorem, on the one hand. On the other hand, the initial conditions are in almost all cases restricted to positive values. Starting from these facts, our aim in this paper is to establish some asymptotic stability results for similar difference equations but different from those in the family of equations (1.1)-(1.5), by using an alternate technique and under more general initial conditions.

2. Basics of global asymptotic stability

In connection with the study of difference equations we shall be concerned with the particular case $X := I$, where $I \subset \mathbb{R}$. In this context, a fixed point x^* of f is also called an *equilibrium point* of the difference equation.

Remind, see for example [16], that the equilibrium point x^* of the difference equation

$$x_{n+1} = T(x_n, \ldots, x_{n-k+1}), \quad n = k-1, k, k+1, \ldots \tag{2.1}$$

is said to be *locally stable* if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x_0, x_1, \ldots, x_{k-1} \in I$ satisfying

$$|x_0 - x^*| + |x_1 - x^*| + \cdots + |x_{k-1} - x^*| < \delta, \tag{2.2}$$

one has

$$|x_n - x^*| < \epsilon, \forall n \geq 0.$$

The equilibrium point x^* of (2.1) is said to be *stable* if x^* is a locally stable solution and there exists $\delta > 0$ such that for all $x_0, x_1, \ldots, x_{k-1} \in I$ satisfying (2.2) one has

$$\lim_{n\to\infty} x_n = x^*.$$

The equilibrium point x^* of (2.1) is a *global attractor* if, for all $x_0, x_1, \ldots, x_{k-1} \in I$, one has

$$\lim_{n\to\infty} x_n = x^*.$$

The equilibrium point x^* of (2.1) is *globally asymptotically stable* if x^* is simultaneously locally stable and a global attractor of (2.1).

The equilibrium point x^* of (2.1) is *unstable* if it is not locally stable.

One of the most popular method used to study the stability of equilibrium points of difference equations is based on the linearization technique, which consists in considering the linearized equation of (2.1) about the equilibrium point x^*, defined as

$$y_{n+1} = \sum_{i=0}^{k-1} \frac{\partial f(x^*, x^*, \ldots, x^*)}{\partial x_{n-i}} y_{n-i}, \tag{2.3}$$

whose characteristic equation is

$$p(\lambda) := p_1 \lambda^{k-1} + p_2 \lambda^{k-2} + \cdots + p_{k-1}\lambda + p_k = 0 \tag{2.4}$$

where

$$p_{i+1} = \frac{\partial f(x^*, x^*, \ldots, x^*)}{\partial x_{n-i}}, i = 0, 1, \ldots, k-1.$$

A generic result concerning the stability of equilibrium points of a difference equation is given by the next theorem, see [16].

Theorem 2.1. *Assume that f is a C^1 function and let x^* be an equilibrium point x^* of* (2.1).

(a) If all roots of the equation (2.4) *lie in the open unit disk $|\lambda| < 1$, then the equilibrium point x^* of* (2.1) *is asymptotically stable;*

(b) If at least one root of the equation (2.4) *has absolute value greater than one, then the equilibrium point x^* of* (2.1) *is unstable;*

Remark 2.2. It is easily seen from the way the linearized equation attached to a difference equation is constructed that the technique of linearization can be applied only to those difference equations for which f is a C^1 function. For other approaches, based on elementary arguments or on the concepts of negative and positive semicycle, see for example [1] and papers cited there.

The main aim of the next section is to illustrate how we can establish stability results for difference equations with f not necessarily a C^1 function. Most of the stability results obtained in this way are the same or at least similar to those obtained by other means in the extremely rich literature devoted to the behaviour of difference equations.

3. Results on global asymptotic stability of first order difference equations

First we study the stability of a first order difference equation similar in some sense but essentially different of (1.1).

Theorem 3.1. *Let $x_0 \in (0, \infty)$. Then the equilibrium point x^* of the difference equation*

$$x_{n+1} = 1 + \frac{2}{x_n}, n = 0, 1, \ldots, \tag{3.1}$$

is globally asymptotically stable.

Proof. It is easy to see that the unique equilibrium of equation (3.1) is the positive root of the quadratic equation $x = 1 + \dfrac{2}{x} \Leftrightarrow x^2 - x - 2 = 0$, that is $x^* = 2$.

We present three different proofs.

Proof 1. First, since $x_0 > 0$ implies $x_n > 0$ for all $n \in \mathbb{N}$, in view of $x_{n+1}x_n = x_n + 2$, we have that $x_{n+1}x_n > 2$, for all $n \in \mathbb{N}$. On the other hand, by (3.1) we get

$$|x_{n+1} - 2| \leq \left|\frac{2}{x_n} - 1\right| = \left|\frac{2 - x_n}{x_n}\right| = \frac{|2 - x_n|}{x_n} = \frac{|x_n - 2|}{x_n}$$

By induction one obtains

$$|x_{n+1}-2| \le \frac{|x_1-2|}{x_n x_{n-1}\dots x_1} < \frac{|x_1-2|}{2^{[\frac{n}{2}]}}, n \ge 2,$$

which shows that, indeed, the unique equilibrium $x^* = 2$ is globally asymptotically stable.

Proof 2. If $x_0 < 2$, then $x_1 = 1 + \frac{2}{x_0} > 1 + 1 = 2$. Then $x_2 < 2$, $x_3 > 2$,... $x_{2n} < 2$, $x_{2n+1} > 2, \dots$.

Similarly, if $x_0 > 2$, then $x_1 = 1 + \frac{2}{x_0} < 2$, $x_2 > 2$, $x_3 < 2$,... $x_{2n} > 2$, $x_{2n+1} < 2, \dots$. Consider in the following the first case (the second case is similar). Then the subsequence $\{x_{2n}\}$ is increasing, the subsequence $\{x_{2n+1}\}$ is decreasing. Indeed, since

$$x_{2n+1} = 1 + \frac{2}{x_{2n}} = 1 + \frac{2}{1+\frac{2}{x_{2n-1}}} = 1 + \frac{2x_{2n-1}}{x_{2n-1}+2} = \frac{3x_{2n-1}+2}{x_{2n-1}+2}, n \ge 1,$$

$$x_{2n+1} < x_{2n-1} \Leftrightarrow x_{2n-1}^2 - x_{2n-1} - 2 > 0,$$

which is true, since $x_{2n-1} > 2$. Similarly, one proves that $\{x_{2n}\}$ is increasing. Therefore

$$x_2 \le x_n \le x_1, \forall n \ge 1,$$

which shows that the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are both monotone and bounded, hence convergent. Denote

$$u = \lim_{n\to\infty} x_{2n+1}; \quad v = \lim_{n\to\infty} x_{2n}.$$

By letting $n \to \infty$ in the recurrence

$$x_{2n+1} = 1 + \frac{2}{x_{2n}},$$

we get $u = 1 + \frac{2}{v} \Leftrightarrow uv = v + 2$, while by letting $n \to \infty$ in the recurrence

$$x_{2n} = 1 + \frac{2}{x_{2n-1}},$$

we get $v = 1 + \frac{2}{u} \Leftrightarrow uv = u + 2$. This yields $u = v = 2$ and hence the unique equilibrium $x^* = 2$ of (3.1) is globally asymptotically stable.

Proof 3. We denote

$$y_n = \frac{1}{x_n + 1},$$

and by some calculations one obtains that the sequence $\{y_n\}$ satisfies the linear recurrence relation

$$2y_{n+1} = -y_n + 1, n \ge 1.$$

Now, one finds $\{y_n\}$ and then $\{x_n\}$ in closed form (we omit the details) and then we show directly that the unique equilibrium $x^* = 2$ of (3.1) is globally asymptotically stable. □

Remark 3.2. Note that in Theorem 3.1 above we assumed only positive values for the initial point x_0, like in the case of the difference equations (1.1), (1.3), (1.4) and (1.5). It is still possible to allow initial value problems with negative values for difference equations of the form (2.1), but only with an appropriate change in the difference equation itself, as shown by the next result.

Theorem 3.3. *Let* $x_0 \in (-\infty, -\frac{3}{2}) \cup (-\frac{6}{7}, 0) \cup (0, \infty)$. *Then the equilibrium point* $x^* = 3$ *of the difference equation*

$$x_{n+1} = 2 + \frac{3}{x_n}, n = 0, 1, \dots, \tag{3.2}$$

is globally asymptotically stable.

Proof. We can use any of the three variants of proof presented to Theorem 3.1 by noting that, if $x_0 \in (-\frac{6}{7}, 0)$, then $x_1 \in (-\infty, -\frac{3}{2})$ and therefore $x_2 \in (0, +\infty)$ and so all inequalities similar to those in the proofs of Theorem 3.1 hold for $n \geq 2$. □

Remark 3.4. It is interesting to note that, despite the fact that the difference equation (3.2) has two equilibrium points, -1 and 3, only $x^* = 3$ is stable, while $x^* = -1$ is unstable (being a *repelling fixed point* of the dynamical system $\{x_n\}$).

The best result we can prove for the difference equation (3.2) is the following one (for a complete proof, see Chapter 13 in [4]).

Theorem 3.5. *Let $x_0 \in \mathbb{R}$. Then the equilibrium point $x^* = 3$ of the difference equation*

$$x_{n+1} = 2 + \frac{3}{x_n}, n = 0, 1, \ldots, \tag{3.3}$$

is globally asymptotically stable.

Proof. We first use Problem 13f in [4], which shows that none of the terms of $\{x_n\}$ satisfying (3.3) can belong to the set

$$E = \left\{ e_n : e_n = \frac{3(-1)^{n+1} + 3^{n+1}}{(-1)^{n+1} - 3^{n+1}}, n = 0, 1, 2, \ldots \right\}$$

and then prove that for any $x_0 \notin E$, the equilibrium point x^* of the difference equation is globally asymptotically stable. □

Remark 3.6. Note that $e_0 = 0$, $e_1 = -\frac{3}{2}$, $e_2 = -\frac{6}{7}, \ldots$, which motivates the choice of the values of x_0 in Theorem 3.3.

4. Results on global asymptotic stability of second order difference equations

In this section we illustrate a technique based on Prešić fixed point theorem for the study of stability of a second order difference equation. The same technique works in the case of a k-order difference equation.

Theorem 4.1. *Let x_0, $x_1 \in [0, +\infty)$, $x_0 - x_1 \leq 40$. Prove that the unique equilibrium of the difference equation*

$$x_{n+1} = \sqrt{x_n + 45} - \sqrt{x_{n-1} + 5}, n \geq 1. \tag{4.1}$$

is globally asymptotically stable.

Proof. Observe that the sequence $\{x_n\}_{n \geq 0}$ defined by (4.1) is actually of the form

$$x_{n+1} = f(x_n, x_{n-1})$$

where $f : I^2 \to I$, with $I = [0, +\infty)$, is given by

$$f(x_1, x_2) = \sqrt{x_1 + 45} - \sqrt{x_2 + 5},\ x_1, x_2 \in [0, +\infty).$$

It is easy to prove that for all $x_0, x_1, x_2 \in [0, +\infty)$ f satisfies the inequality

$$|f(x_0, x_1) - f(x_1, x_2)| \leq \alpha_1 |x_0 - x_1| + \alpha_2 |x_1 - x_2|, \tag{4.2}$$

where $\alpha_1 = \dfrac{1}{6\sqrt{5}}$, $\alpha_2 = \dfrac{1}{2\sqrt{5}}$ and $\alpha_1 + \alpha_2 = \dfrac{2}{3\sqrt{5}} < 1$.

By applying Prešić fixed point theorem, see for example [18], we obtain that $\{x_n\}_{n \geq 0}$ converges to 4, the unique fixed point of f, for all $x_0, x_1, x_2 \in [0, +\infty)$, which proves the theorem. □

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References

[1] Abu-Saris, R. M., DeVault, R., *Global stability of* $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. Appl. Math. Lett. 16 (2003), no. 2, 173–178.

[2] Aloqeili, M., *On the difference equation* $x_{n+1} = \alpha + \frac{x_n^p}{x_{n-1}^p}$. J. Appl. Math. Comput. 25 (2007), no. 1-2, 375–382.

[3] Amleh, A. M., Grove, E. A., Ladas, G., Georgiou, D. A., *On the recursive sequence* $x_{n+1} = \alpha + x_{n-1}/x_n$. J. Math. Anal. Appl. 233 (1999), no. 2, 790–798.

[4] Berinde, V., *Exploring, Investigating and Discovering in Mathematics*, Birkhäuser, Basel, 2004.

[5] Berinde, V., *Iterative approximation of fixed points*, Second edition. Lecture Notes in Mathematics, 1912. Springer, Berlin, 2007.

[6] Berinde, V., Păcurar, M., *Stability of k-step fixed point iterative methods for some Prešić type contractive mappings.* J. Inequal. Appl. 2014, 2014:149, 12 pp.

[7] Berinde, V., Păcurar, M., *Two elementary applications of some Prešić type fixed point theorems.* Creat. Math. Inform. 20 (2011), no. 1, 32–42

[8] Berinde, V., Păcurar, M., *O metodă de tip punct fix pentru rezolvarea sistemelor ciclice*, Gazeta Matematică, Seria B, 116 (2011), No. 3, 113-123

[9] Camouzis, E., DeVault, R., Ladas, G., *On the recursive sequence* $x_{n+1} = -1 + (x_{n-1}/x_n)$. J. Differ. Equations Appl. 7 (2001), no. 3, 477–482.

[10] Cirić, L.B., Prešić, S.B., *On Prešić type generalization of the Banach contraction mapping principle*, Acta Math. Univ. Comenianae, 76 (2007), No. 2, 143-147

[11] Chen, Y.-Z., *A Prešić type contractive condition and its applications*, Nonlinear Anal. 71 (2009), no. 12, e2012–e2017

[12] DeVault, R., Ladas, G., Schultz, S. W., *On the recursive sequence* $x_{n+1} = A/x_n + 1/x_{n-2}$. Proc. Amer. Math. Soc. 126 (1998), no. 11, 3257–3261.

[13] Elabbasy, E. M., El-Metwally, H., Elsayed, E. M., *On the difference equation* $x_{n+1} = ax_n - bx_n/(cx_n - dx_{n-1})$. Adv. Difference Equ. 2006, Art. ID 82579, 10 pp.

[14] El-Owaidy, H. M., Ahmed, A. M., Mousa, M. S., *On asymptotic behaviour of the difference equation* $x_{n+1} = \alpha + \frac{{x_{n-1}}^p}{{x_n}^p}$. J. Appl. Math. Comput. 12 (2003), no. 1-2, 31–37.

[15] El-Owaidy, H. M., Ahmed, A. M., Mousa, M. S., *On asymptotic behaviour of the difference equation* $x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}$. Appl. Math. Comput. 147 (2004), no. 1, 163–167.

[16] Kocić, V. L., Ladas, G., *Global behavior of nonlinear difference equations of higher order with applications.* Mathematics and its Applications, 256. Kluwer Academic Publishers Group, Dordrecht, 1993.

[17] Păcurar, M., *Approximating common fixed points of Prešić-Kannan type operators by a multi-step iterative method*, An. Ştiinţ,. Univ. "Ovidius" Constanţa Ser. Mat. 17 (2009), no. 1, 153–168

[18] Păcurar, M., *Iterative Methods for Fixed Point Approximation*, Risoprint, Cluj-Napoca, 2010

[19] Păcurar, M., *A multi-step iterative method for approximating fixed points of Prešić-Kannan operators*, Acta Math. Univ. Comen. New Ser., 79 (2010), No. 1, 77-88

[20] Păcurar, M., *A multi-step iterative method for approximating common fixed points of Prešić-Rus type operators on metric spaces*, Stud. Univ. Babeş-Bolyai Math. 55 (2010), no. 1, 149–162.

[21] Păcurar, M., *Fixed points of almost Prešić operators by a k-step iterative method*, An. Ştiint,. Univ. Al. I. Cuza Iaşi, Ser. Noua, Mat. 57 (2011), Supliment 199–210

[22] Păvăloiu, I., *Rezolvarea ecuaţiilor prin interpolare*, Editura Dacia, Cluj-Napoca, 1981

[23] Păvăloiu, I. and Pop, N., *Interpolare şi aplicaţii*, Risoprint, Cluj-Napoca, 2005

[24] Prešić, S.B., *Sur une classe d' inéquations aux différences finites et sur la convergence de certaines suites*, Publ. Inst. Math. (Beograd)(N.S.), 5(19) (1965), 75–78

[25] Rus, I.A., *An iterative method for the solution of the equation* $x = f(x, \ldots, x)$, Rev. Anal. Numer. Théor. Approx., 10 (1981), No.1, 95–100

[26] Saleh, M.; Aloqeili, M., *On the rational difference equation* $y_{n+1} = A + \frac{y_{n-k}}{y_n}$. Appl. Math. Comput. 171 (2005), no. 2, 862–869.

[27] Saleh, M., Aloqeili, M., *On the rational difference equation* $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. Appl. Math. Comput. 177 (2006), no. 1, 189–193.

[28] Stević, S., *On the recursive sequence* $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}$. J. Appl. Math. Comput. 18 (2005), no. 1-2, 229–234.

[29] Stević, S., *On the recursive sequence* $x_{n+1} = A + x_n^p/x_{n-1}^p$. Discrete Dyn. Nat. Soc. 2007, Art. ID 34517, 9 pp.

[30] Stević, S., *On the recursive sequence* $x_{n+1} = A + x_n^p/x_{n-1}^r$. Discrete Dyn. Nat. Soc. 2007, Art. ID 40963, 9 pp.

[31] Stević, S., *On the recursive sequence* $x_{n+1} = \max\left\{c, \frac{x_n^p}{x_{n-1}^p}\right\}$. Appl. Math. Lett. 21 (2008), no. 8, 791–796.