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Dissatisfaction levels of earliness and tardiness durations by relaxing common due date on single machine scheduling problems

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Abstract —*This paper investigates single machine earliness/tardiness problem considering the decision maker's tolerances for earliness and tardiness durations in case of a restrictive common due date. In many classical or basic earliness/tardiness problems, due dates are accepted as deterministic or rigid numbers. In this paper, common due date in a single machine scheduling problem is relaxed with lower and upper bounds and these bounds are used for illustrating the decision maker's tolerances or satisfaction levels by using fuzzy sets. As a complementary set of satisfaction levels, dissatisfaction levels can be encoded with fuzzy sets. Then, this paper uses dissatisfaction levels in order to introduce a new objective criterion that minimizes the products of earliness and tardiness durations with dissatisfaction levels.*

Keywords: Earliness, tardiness, single machine, scheduling, fuzzy sets, dissatisfaction levels, common due date
Mathematics Subject Classification: 65K05, 90C70.

1 Introduction

Earliness/tardiness (E/T) problems are significant for the companies having the Just-in-Time philosophy. Determining earliness and tardiness weights or penalties may not always be an easy task. The decision maker (DM) uses E/T weights in order to show his/her biased importance factors. In some cases, DM may use real penalty costs in currencies as important factors for scheduling problems. In this paper, dynamic weights for E/T durations are introduced as decision variables in a single machine E/T problem with a common due date by using fuzzy membership functions of relaxed common due date with upper and lower bounds. Arık and Toksarı [1] considered a multi-objective fuzzy parallel machine scheduling problem under effects of fuzzy learning and deterioration where the objectives are to minimize earliness cost, to minimize tardiness cost and to minimize the cost of setting due dates. In their study, due dates are in form of fuzzy numbers as decision variables. They proposed a Local Search algorithm.

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Jayanthi et al. [2] investigated a single machine scheduling problem with trapezoidal processing times and triangular due dates. In order to solve their problem where the objective is to minimize total weighted earliness and tardiness costs, they proposed a Quantum Particle Swarm Optimization algorithm. Niroomand et al. [3] considered a single machine scheduling problem with a fuzzy common due date by proposing hybrid greedy algorithms in order to minimize fuzzy earliness/tardiness costs. Geng et al. [4] investigated flow shop scheduling problems for earliness/tardiness minimization with uncertain processing times and distinct due windows. They proposed a Scatter Search Based Particle Swarm Optimization. Kır and Yazgan [5] used Fuzzy Axiomatic Design to determine earliness and tardiness penalty costs in a single machine scheduling problem where dairy products are considered. They proposed a two-stage solution method that firstly creates an initial solution with Tabu Search and then improves that initial solution with Genetic Algorithm. Li and Zhang [6] considered single machine due date assignment problems where the objective is to minimize the possibilistic mean of total E/T cost with fuzzy processing times and precedence constraints. Behnamian and Fatemi Ghomi [7] considered a bi-objective hybrid flow shop scheduling problems with fuzzy tasks' operation times, due dates and sequence-dependent setup times. The objectives in their problem are to minimize makespan and the total sum of E/T cost simultaneously. In the study of Engin et al. [8], fuzzy sets were used to encode uncertainties in processing times and due dates in a fuzzy job shop scheduling problem with availability constraints. They proposed a Scatter Search (SS) method to solve these problems. Yan et al. [9] investigated flow shop scheduling problems with fuzzy processing times and due windows in order to minimize total weighted E/T cost by proposing a hybrid algorithm consist of quantum genetic algorithm and particle swarm optimization. Xu and Gu [10] considered a zero-wait multiproduct scheduling with due dates under uncertainty, where the total weighted earliness/tardiness penalty is to be minimized. Li et al. [11] investigated single machine scheduling problems where the objective is to minimize total weighted possibilistic mean of E/T cost with fuzzy processing time and they investigated how to predict due dates of jobs. Lu et al. [12] studied a multi-objective scheduling problem for a single batch-processing machine with non-identical job sizes with fuzzy processing times and fuzzy due dates. The objectives in their study are to minimize cost combination of makespan, earliness/tardiness penalties and processing cost. Wang and Shi [13] considered a multi-objective job shop scheduling problem with fuzzy processing times and due windows for E/T performance criterion and they proposed a genetic algorithm for their problem.

Wang et al. [14] proposed different genetic algorithms including different crossover operator a for single machine E/T problem with fuzzy processing times. Wang et al. [14] investigated a multi-objective job shop scheduling problem with fuzzy processing times and flexible due dates by proposing a genetic simulated annealing algorithm. Wu [15] considered fuzzy earliness and fuzzy tardiness in scheduling problems by using extension principle of fuzzy set theory for triangular fuzzy processing times and trapezoidal fuzzy due dates. Li et al. [16] proposed a due date assignment problem with fuzzy processing times and precedence constraints. They showed that their problem can be polynomially solvable without precedence constraints and the problem with precedence constraints is NP-hard. Lai and Wu [17] investigated fuzzy earliness and tardiness by using the concept of possibility and necessity measures in fuzzy set theory with fuzzy processing times and fuzzy due dates. They considered lots of E/T combinations in view of possibility and necessity measures and proposed a genetic algorithm approach for these different E/T combinations.

Dong [18] considered a fuzzy single machine scheduling problem with fuzzy processing times in order to minimize weighted E/T and resource costs. Dong [19] proposed a two-stage solution approach for the problem. Lam and Cai [20] considered a single machine weighted E/T problem with a fuzzy triangular common due date and they introduced job dependent weights for their objective function. Furthermore, they stated an optimal job sequence must be V-shaped in terms of weighted processing time when the problem is agreeably weighted. In another study of Lam and Cai [21], they used genetic algorithm and fuzzy distance function for solving a single machine E/T problem with fuzzy due dates. Murata et al. [22] examined the characteristic features of multi-objective scheduling problems formulated with the concept of fuzzy due-date. Ishibuchi et al. [23] investigated fuzzy scheduling problems and conventional scheduling problems with earliness and tardiness penalties. They showed the relations between fuzzy scheduling problems and conventional scheduling problems by solving them with a proposed genetic algorithm. Some of other recent papers about fuzziness in scheduling are conducted by Toksarı and Arık [24], Arık and Toksarı [25], Jia et al. [26], Golneshini and Fazlollahtabar [27], Arık [28], Saraçoğlu and Süer [29], Liao and Su [30], Liu et al. [31] and Arık and Toksarı [32].

2 Problem formulation

The earliness penalties or costs of early jobs in scheduling problems are considered as deterministic in scheduling problems. With classical set theory; if a job completed before its due date, then this job belongs to the set of early jobs, else this job is not a member of the set of early jobs. Belonging to the set of early jobs is not a desired situation and this does not satisfy DM. Equation (1) shows the classical membership function $\mu_{E_i}(C_i): \mathbb{R}^+ \to [0,1]$ of DM's satisfaction level for an early job with respect to completion time of that job considering a common due date for all jobs.

$$\mu_{E_i}(C_i) = \begin{cases} 1, & if\ C_i \geq d, \\ 0, & if\ C_i < d, \end{cases} \tag{1}$$

where C_i is the completion time of job i and d is common due date for all jobs in the scheduling environment. Figure 1 illustrates classical membership function in Equation (1).

The early job's satisfaction level in Equation (1) is a rigid number. Like the most cases of the real life, this rigid approach for earliness may be tolerated in view of DM's tolerance degree or satisfaction degree to an unacceptable situation. In order to evaluate DM's satisfaction degree, common due date d may be relaxed with a lower bound $\underline{d}_i$ of common due date. Thus, if job i is completed on the interval between $\underline{d}_i$ and d, DM may not be fully satisfied because of this earliness amount but he/she may tolerate this earliness amount.

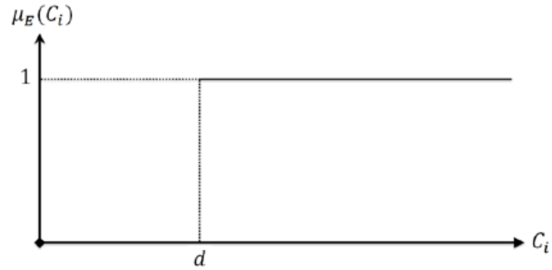

Figure 1. Classical earliness membership function

The degree of DM's satisfaction with respect to completion time of that job considering a relaxed common due date with a lower bound $\underline{d_i}$ for job i can be encoded with fuzzy sets as illustrated in Figure 2 and Equation (2).

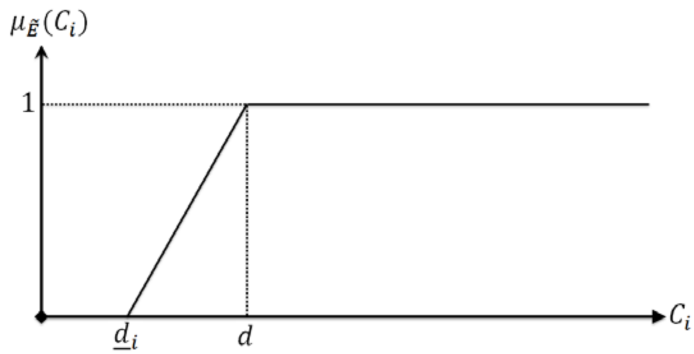

Figure 2: Fuzzy earliness membership function

$$\mu_{\tilde{E}_i}(C_i) = \begin{cases} 1, & if\ C_i \geq d, \\ \frac{C_i - \underline{d_i}}{d - \underline{d_i}}, & if\ \underline{d_i} \leq C_i < d, \\ 0, & if\ C_i < \underline{d_i}, \end{cases} \quad (2)$$

where $\mu_{\tilde{E}_i}(C_i)\colon \mathbb{R}^+ \to [0,1]$ is the membership function of DM's satisfaction level for earliness with respect to completion time of that job considering a relaxed common due date with a lower bound $\underline{d_i}$ for job i. With classical scheduling triple notation, $1|d_i = d|\sum \alpha_i E_i$ denotes a single machine scheduling problem where the objective is to minimize total weighted earliness costs for all jobs by considering jobs' weight coefficients α_i. The weight coefficients for earliness or tardiness (E/T) are mostly assumed as deterministic values. This paper proposes dynamic weight coefficients for scheduling problems, especially for E/T problems. Furthermore, dissatisfaction levels of jobs for earliness of tardiness are proposed as dynamic penalty weights in this paper. The dissatisfaction level $\tilde{\alpha}_i$ of DM for any early job i is a complementary fuzzy set of fuzzy satisfaction level $\tilde{E}_i$ such as $\mu_{\tilde{\alpha}_i}(C_i) = 1 - \mu_{\tilde{E}_i}(C_i)$. Satisfaction or dissatisfaction level is on the closed interval between 0 and 1. The complementary part of satisfaction level can be called as dissatisfaction level. Figure (3) and Equation (3) show the membership function of DM's dissatisfaction level $\mu_{\tilde{\alpha}_i}(C_i)\colon \mathbb{R}^+ \to [0,1]$ for earliness with respect to completion time of that job considering a relaxed common due date with a lower bound $\underline{d}_i$ for job i.

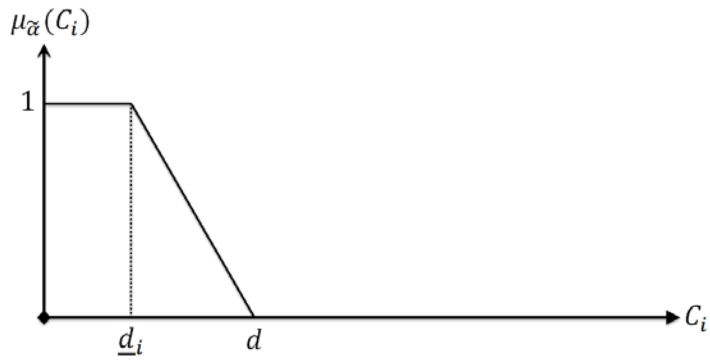

Figure 3: Fuzzy earliness weight membership function

$$\mu_{\tilde{\alpha}_i}(C_i) = \begin{cases} 0, & if\ C_i \geq d, \\ \frac{d - C_i}{d - \underline{d}_i}, & if\ \underline{d}_i \leq C_i < d, \\ 1, & if\ C_i < \underline{d}_i\ . \end{cases} \quad (3)$$

By following the same approach, the classic tardiness classical membership function $\mu_{T_i}(C_i): \mathbb{R}^+ \to [0,1]$ of DM's satisfaction level for an tardy job with respect to completion time of that job considering a common due date for all jobs can be illustrated as in Figure 4 and Equation (4).

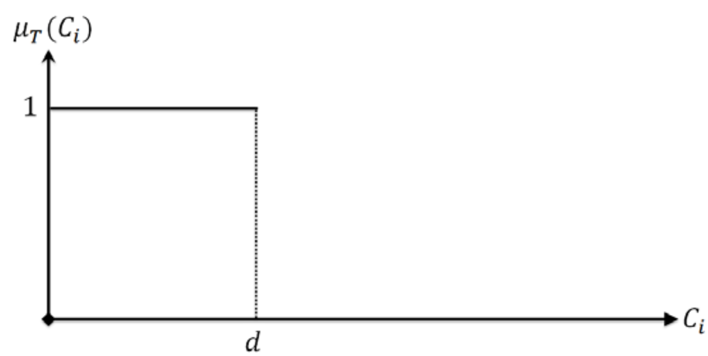

Figure 4: Classical tardiness membership function

$$\mu_{T_i}(C_i) = \begin{cases} 0, & if\ C_i > d, \\ 1, & if\ C_i \leq d, \end{cases} \quad (4)$$

equation (4) can be relaxed with an upper bound $\overline{d}_i$ of common due date for any tardy job i as seen in Figure 5 and Equation (5).

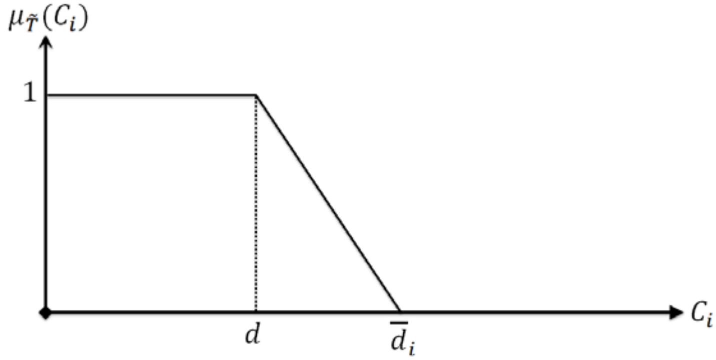

Figure 5: Fuzzy tardiness membership function

$$\mu_{\tilde{T}}(C_i) = \begin{cases} 1, & if\ d \leq C_i, \\ \frac{C_i - d}{\overline{d}_i - d}, & if\ \overline{d}_i \geq C_i > d, \\ 0, & if\ \overline{d}_i < C_i, \end{cases} \tag{5}$$

with classical scheduling triple notation, $1|d_i = d|\sum \beta_i T_i$ denotes a single machine scheduling problem where the objective is to minimize total weighted tardiness costs for all jobs by considering jobs' weight coefficients β_i. The complementary set of the satisfaction level $\tilde{T}_i$ for tardiness is dissatisfaction level $\tilde{\beta}_i$ with a membership function such as $\mu_{\tilde{\beta}_i}(C_i) = 1 - \mu_{\tilde{T}_i}(C_i)$ as shown in Figure 6 and Equation (6).

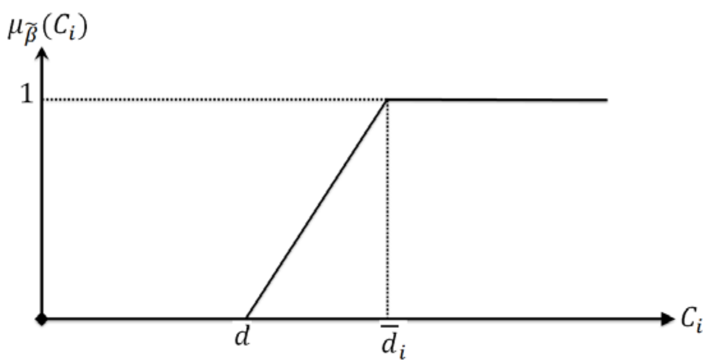

Figure 6: Fuzzy tardiness weight membership function

$$\mu_{\tilde{\beta}_i}(C_i) = \begin{cases} 0, & if\ d \leq C_i, \\ \frac{\overline{d}_i - C_i}{\overline{d}_i - d}, & if\ \overline{d}_i \geq C_i > d, \\ 1, & if\ \overline{d}_i < C_i. \end{cases} \tag{6}$$

The weighted single machine E/T problem $1|d_i = d|\sum \alpha_i E_i + \sum \beta_i T_i$ can be expressed as $1|\underline{d_i} < d < \overline{d_i}|\sum \mu_{\tilde{\alpha}_i}(C_i) E_i + \sum \mu_{\tilde{\beta}_i}(C_i) T_i$ for minimizing earliness and tardiness amounts and dissatisfaction of DM, simultaneously. This new performance criterion aims to minimize the sum of the products of earliness/tardiness durations and dissatisfaction levels of them in view of DM.

3 Mixed integer non-linear mathematical model

In this section of the paper, a mixed integer non-linear mathematical programming (MINLP) model is proposed. $\mu_{\tilde{\alpha}_i}(C_i)$ and $\sum \mu_{\tilde{\beta}_i}(C_i)$ are piecewise linear functions, $T_i = \max(0,\ C_i - d)$ and $E_i = \max(0,\ d - C_i)$. $\sum \mu_{\tilde{\alpha}}(C_i)E_i + \sum \mu_{\tilde{\beta}_i}(C_i)T_i$ is a non-linear objective function. Each of $\mu_{\tilde{\alpha}_i}(C_i)$ and $\sum \mu_{\tilde{\beta}_i}(C_i)$ functions has three intervals on the real axis as shown in Figures 7 and 8. In order to simplify mathematical model, these intervals are used in the proposed MINLP. The completion time C_i can be placed on any of these intervals in Figure 7 and Figure 8.

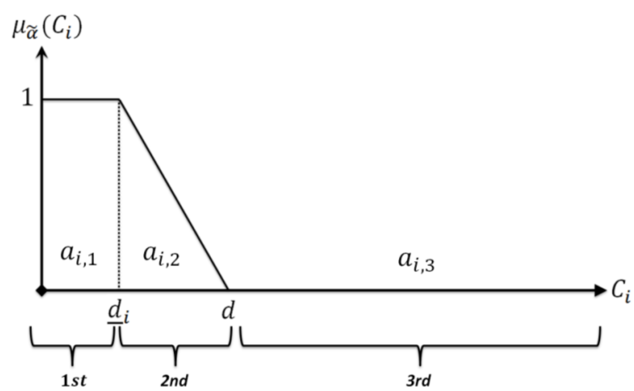

Figure 7: Intervals of $\mu_{\tilde{\alpha}}(C_i)$ functions on the real axis

In case of earliness, C_i can be represented with $A_{i,k}$, where k =1,2,3 and k is index for intervals in Figure 7. In order to determine the interval that C_i is on, assignment variables $a_{i,k}$ can be used as follows:

$$C_i = \sum_{k=1}^{3} a_{i,k} A_{i,k} \ \ \forall i \qquad (7)$$

$$\sum_{k=1}^{k=3} a_{i,k} = 1 \ \forall i \qquad (8)$$

$$A_{i,1} \le \underline{d}_i a_{i,1} \ \forall i \qquad (9)$$

$$\underline{d}_i a_{i,2} \le A_{i,2} \le d a_{i,2} \ \forall i \qquad (10)$$

$$A_{i,3} \ge d a_{i,3} \ \forall i \qquad (11)$$

where $a_{i,k} \in \{0,1\}$ and $A_{i,k} \ge 0 \ \forall\ i, k$. $\mu_{\tilde{\alpha}_i}(C_i)$ value is simply obtained by using $a_{i,k}$ decision variables as follows:

$$\mu_{\tilde{\alpha}_i}(C_i) = a_{i,1} * 1 + a_{i,2} * \frac{d - C_i}{d - \underline{d}_i} + a_{i,3} * 0 \ \forall i. \quad (12)$$

Equation (12) can be written as follows:

$$\mu_{\tilde{\alpha}_i}(C_i) = a_{i,1} + a_{i,2} * \frac{d - C_i}{d - \underline{d}_i} \ \forall i. \quad (13)$$

Equation (13) can be simply regulated as $\mu_{\tilde{\alpha}_i}(C_i) = \min(1, E_i/P_i h_i)$.

In case of tardiness, C_i can be represented with $B_{i,t}$, where t =1,2,3 and t is index for intervals in Figure 8. In order to determine the interval that C_i is on, assignment variables $b_{i,t}$ can be used as follows:

$$C_i = \sum_{t=1}^{3} b_{i,t} B_{i,t} \ \forall i \quad (14)$$

$$\sum_{t=1}^{t=3} b_{i,t} = 1 \ \forall i \quad (15)$$

$$B_{i,1} \le d b_{i,1} \ \forall i \quad (16)$$

$$d b_{i,2} \le B_{i,2} \le \overline{d_i} b_{i,2} \ \forall i \quad (17)$$

$$B_{i,3} \ge \overline{d_i} b_{i,3} \ \forall i \quad (18)$$

where $b_{i,t} \in \{0,1\}$ and $B_{i,t} \ge 0 \ \forall \ i, t$. $\mu_{\tilde{\beta}_i}(C_i)$ value is simply obtained by using $b_{i,t}$ decision variables as follows:

$$\mu_{\tilde{\beta}_i}(C_i) = b_{i,1} * 0 + b_{i,2} * \frac{\overline{d}_i - C_i}{\overline{d}_i - d} + b_{i,3} * 1 \ \forall i. \quad (19)$$

Equation (19) can be written as follows:

$$\mu_{\tilde{\beta}_i}(C_i) = b_{i,2} * \frac{\overline{d}_i - C_i}{\overline{d}_i - d} + b_{i,3} \ \forall i. \quad (20)$$

Equation (20) can be simply regulated as $\mu_{\tilde{\beta}_i}(C_i) = \min(1, T_i/P_i h_i)$.

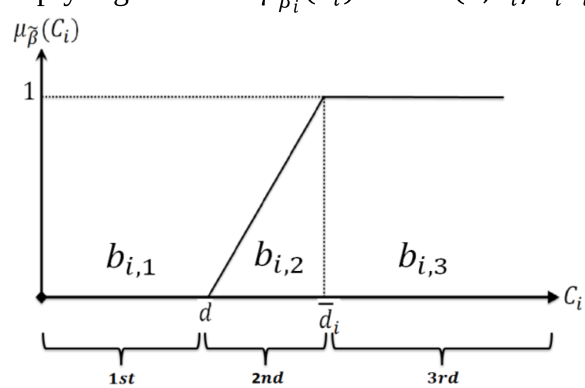

Figure 8: Intervals of $\mu_{\tilde{\beta}_i}(C_i)$ functions on the real axis

The mathematical model can be structured by using Equations (7-11) and Equations (14-18) as follows:

Indices:
i*: index for jobs (* $i = 1,2,\dots,n$*)*
r*: index for position numbers (* $i = 1,2,\dots,n$*)*
k*: index for earliness time interval (* $k = 1,2,3$*)*
t*: index for tardiness time interval (* $t = 1,2,3$*)*

Parameters:
P_i*: processing time of job* i
h_i*: relaxing factor for upper and lower bounds of common due date for job* i
h*: restrictive factor for common due date*

Decision Variables:
C_i*: completion time of job* i
E_i*: earliness time of job* i
T_i*: tardiness time of job* i
C_r*: completion time of the job assigned on position* r
P_r*: processing time of the job assigned on position* r
$A_{i,k}$*: completion time of job* i *on* k^{th} *earliness interval*
$B_{i,t}$*: completion time of job* i *on* t^{th} *tardiness interval*
d*: common due date for all jobs*
$\underline{d}_i$*:lower bound of common due date for job* i
$\overline{d}_i$*: upper bound of common due date for job* i
$X_{i,r}$*: if job* i *is assigned on position* r *on the machine,* $X_{i,r} = 1$*; otherwise,* $X_{i,r} = 0$
$a_{i,k}$*: if* C_i *is on* k^{th} *earliness interval,* $a_{i,k} = 1$*; otherwise,* $a_{i,k} = 0$
$b_{i,t}$*: if* C_i *is on* t^{th} *tardiness interval,* $b_{i,k} = 1$*; otherwise,* $b_{i,k} = 0$
$\mu_{\tilde{\beta}_i}(C_i)$*: dissatisfaction level of DM in case of tardiness for job* i
$\mu_{\tilde{\alpha}_i}(C_i)$*: dissatisfaction level of DM in case of earliness for job* i

Objective Function:

$$Min \sum_i^n \mu_{\tilde{\alpha}_i}(C_i)E_i + \sum_i^n \mu_{\tilde{\beta}_i}(C_i)T_i \quad (21)$$

Subject to:

$$d = h\sum_{i=1}^n P_i \quad (22)$$

$$\underline{d}_i = d - P_i h_i \quad (23)$$

$$\overline{d}_i = d + P_i h_i \quad (24)$$

$$\mu_{\tilde{\alpha}_i}(C_i) = a_{i,1} + a_{i,2}\frac{d-C_i}{d-\underline{d}_i} \quad \forall i \quad (25)$$

$$\mu_{\widetilde{\beta}_i}(C_i) = b_{i,2}\frac{\overline{d}_i - C_i}{\overline{d}_i - d} + b_{i,3} \; \forall i \quad (26)$$

$$C_i \geq \sum_{k=1}^{3} a_{i,k} A_{i,k} \; \forall i \quad (27)$$

$$\sum_{k=1}^{k=3} a_{i,k} = 1 \; \forall i \quad (28)$$

$$A_{i,1} \leq \underline{d_i} a_{i,1} \; \forall i \quad (29)$$

$$\underline{d_i} a_{i,2} \leq A_{i,2} \; \forall i \quad (30)$$

$$A_{i,2} \leq d a_{i,2} \; \forall i \quad (31)$$

$$A_{i,3} \geq d a_{i,3} \; \forall i \quad (32)$$

$$C_i \geq \sum_{t=1}^{3} b_{i,t} B_{i,t} \; \forall i \quad (33)$$

$$\sum_{t=1}^{t=3} b_{i,t} = 1 \; \forall i \quad (34)$$

$$B_{i,1} \leq d b_{i,1} \; \forall i \quad (35)$$

$$d b_{i,2} \leq B_{i,2} \; \forall i \quad (36)$$

$$B_{i,2} \leq \overline{d_i} b_{i,2} \; \forall i \quad (37)$$

$$B_{i,3} \geq \overline{d_i} b_{i,3} \; \forall i \quad (38)$$

$$\sum_{r=1}^{n} X_{i,r} = 1 \; \forall i \quad (39)$$

$$\sum_{i=1}^{n} X_{i,r} = 1 \; \forall r \quad (40)$$

$$C_i + E_i - T_i = d \; \forall i \quad (41)$$

$$C_i = \sum_{r=1}^{n} X_{i,r} \, C_r \; \forall i \quad (42)$$

$$C_r = C_{r-1} + P_r \; \forall r \quad (43)$$

$$P_r = \sum_{i=1}^{n} X_{i,r} \, P_i \; \forall r \quad (44)$$

$$C_{r=0} = 0 \quad (45)$$

$$\mu_{\widetilde{\alpha}_i}(C_i) \leq 1 \quad (46)$$

$$\mu_{\tilde{\beta}_i}(C_i) \leq 1 \tag{47}$$

$$C_r, P_r \geq 0 \ \forall r \tag{48}$$

$$C_i \geq 0 \ \forall i \tag{49}$$

$$A_{i,k} \geq 0 \ \forall \ i,k \tag{50}$$

$$B_{i,t} \geq 0 \ \forall \ i,t \tag{51}$$

$$a_{i,k} \ \in \{0,1\} \ \forall \ i,k \tag{52}$$

$$b_{i,t} \ \in \{0,1\} \ \forall \ i,t \tag{53}$$

$$X_{i,r} \ \in \{0,1\} \ \forall \ i,r \tag{54}$$

Objective function (21) is to minimize the products of earliness/tardiness with dissatisfaction levels of DM simultaneously. Constraint (22) shows that common due date d is equal to the product of the sum of processing times with a restrictive factor that is predetermined by DM. Constraints (23-24) shows that upper and lower bounds of common due date for job i are relaxed with the same amount that is equal to the product of processing time of job i with a predetermined relaxing factor h_i. Constraints (25-26) are dissatisfaction levels of DM for earliness and tardiness, respectively. These constraints are introduced in Equations (13-20) previously. Constraints (27-32) are to determine the earliness interval where C_i is placed on and these constraints are introduced in Equations (7-11). Constraints (33-38) are to determine the tardiness interval where C_i is placed on and these constraints are introduced in Equations (14-18). Constraint (39) assures that only one job can be assigned to any position of the machine. Constraint (40) guarantees that only one position number can be used to assign a job. Constraint (41) shows that completion time, earliness duration and tardiness duration must be balanced with the common due date. Constraint (42) is decision variable transformation between completion times that are dependent on job index and position index, respectively. Constraint (43) shows that completion time of the job on position r is equal to sum of previous position's completion time and processing time of the job on position r. Constraint (44) is to determine which job is assigned to position r. Constraint (45) assures that the machine is ready to process jobs at the beginning and all jobs have same release date. Constraints (46-47) assure that dissatisfaction levels are not more than 1. Constraints (48-54) define domains of decision variables.

4 Numerical example

In this section, a numerical example for the proposed problem is given for the readers. The numerical example in this section has 10 jobs that are ready to be processed on a single machine. Processing times of jobs are in Table 1. Preemption is not allowed and ready times of all jobs are equal to zero. Each job has same relaxing factor h_i=0.5. The restrictive factor h for common due date are predetermined by DM. For different restrictive factors between 0.1 and 1.5 by incrementing h with 0.1, the problem is solved and solutions of the problem for these restrictive factors are given in Table 2 and Figure 9. Solutions were obtained via Dicopt Solver in Gams 21.6 software.

Table 1: Processing times of the numerical example

i	1	2	3	4	5	6	7	8	9	10
P_i	5	6	16	8	9	16	10	12	23	11

Table 2: Solutions of the numerical example for different h levels

Restrictive factor h	**Common due date d**	**The optimal sequence**	**Objective Function**
0.1	11.6	1,2,4,5,7,10,8,3,6,9	394.920
0.2	23.2	8,5,2,4,7,1,10,6,3,9	330.676
0.3	34.8	3,6,1,5,2,10,7,4,8,9	302.116
0.4	46.4	3,1,5,2,4,7,10,8,6,9	257.040
0.5	58.0	6,1,3,8,10,2,7,4,9,5	268.727
0.6	69.6	9,3,5,6,4,2,1,8,7,10	233.160
0.7	81.2	9,2,7,10,8,3,6,5,4,1	303.480
0.8	92.8	9,3,6,7,5,10,8,2,4,1	270.340
0.9	104.4	9,3,6,8,10,7,5,2,1,4	299.653
1.0	116.0	9,3,6,8,10,7,5,4,2,1	381.000
1.1	127.6	9,3,6,8,10,7,5,4,2,1	497.000
1.2	139.2	9,3,6,8,10,7,5,4,2,1	613.000
1.3	150.8	9,3,6,8,10,7,5,4,2,1	719.000
1.4	162.4	9,3,6,8,10,7,5,4,2,1	845.000
1.5	174.0	9,3,6,8,10,7,5,4,2,1	961.000

As seen in Table 2, while restrictive factor h is increasing and the problem is still restricted ($d < \sum P_i$), the sequence is changing and objective function values fluctuate because the common due date is increasing with restrictive factor. Increasing the common due date leads the schedule is changed because there is a similar v-shaped property for the problem. The v-shaped property presents a sequence where jobs are ordered in decreasing order of their weighted processing times until the common due date and then the remaining jobs are ordered in increasing order of their weighted processing times. This property is common for classical single machine weighted earliness/tardiness scheduling problems and as seen from Table 2, this property can be seen for $1|\underline{d_i} < d < \overline{d_i}|\sum \mu_{\tilde{\alpha}_i}(C_i)E_i + \sum \mu_{\tilde{\beta}_i}(C_i)T_i$ problem. While problem is a non-restricted ($d \geq \sum P_i$) and h is increasing, the sequence stays same and objective function values are increasing because of earliness.

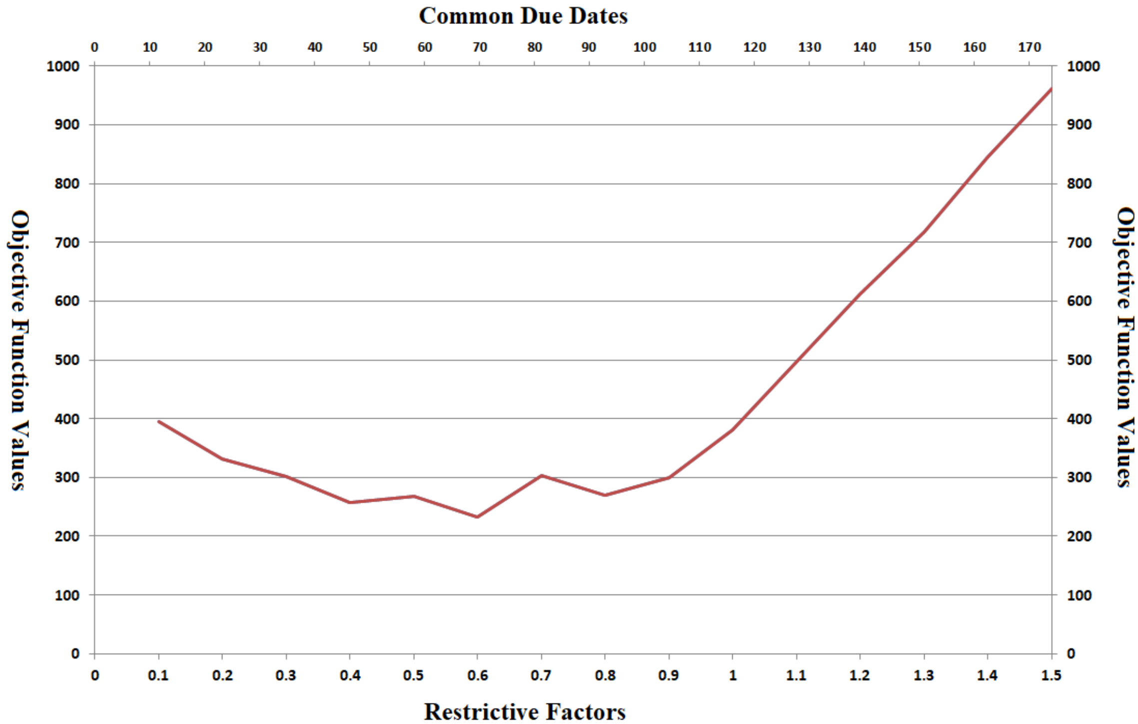

Figure 9: Solutions of the numerical example for different h levels and common due dates

5 Conclusion

In this paper, a new performance criterion $1|\underline{d_i} < d < \overline{d_i}|\sum \mu_{\tilde{\alpha}_i}(C_i)E_i + \sum \mu_{\tilde{\beta}_i}(C_i)T_i$ that minimizes the sum of the products of earliness/tardiness durations and dissatisfaction levels of them in view of DM is introduced. Dissatisfaction levels denote tolerances for earliness and tardiness durations considering a common due date. This approach may be used for different due dates of jobs. A numerical example for different restrictive levels is given in this paper. Single machine scheduling problems are basic of scheduling problems. Therefore, this approach can be used in more complex production systems that are mainly considered as a part of the companies having Just-in-time philosophy. The extending of this performance criterion for more complex scheduling environments and fuzzification of other parameters such as processing times can be considered in future researches.

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Three-Term Conjugate Gradient (TTCG) methods

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***Abstract* — *In this study, a comprehensive hybrid formula was developed for some known algorithms of "Three-Term Conjugate Gradient (TTCG) methods" for solving problems of unconstrained optimization by combining the three most important vectors (y_k, d_k, g_{k+1}) in an exceedingly new vector denoted by z_k defined in section two. The proposed vector z_k can also be considered as a special case or modified variant of the vector p_k within the general versions of Yasushi, Yabe, and Ford. As a theoretical aspect, global convergence, sufficient descend and conjugacy were studied in the presence of strong Wolfe condition. On the practical side, the proposed formula was compared with its counterpart to the researchers Yasushi, Yabe, and Ford. Where the results were encouraging and proved the efficiency of proposed algorithms than comparative algorithms using 35 nonlinear functions.*

Keywords Unconstrained optimization, Conjugate gradient, Global convergence.
Mathematics Subject Classification: 80C50, 30A40, 90C26.

1 Introduction

The discus problem is "unconstrained optimization":

$$\min f(x), \quad x \in R^n$$

(1)

where $f : R^n \to R$ is continuously differentiable and *f* gradient at *x*, which is represented by $g(x) = \nabla f(x)$ is existing. There exist many types of numerical methods to solve equation (1) including Steepest Descent (SD), Newton, CG and Quasi-Newton (QN) methods. Because of being simple and having requirement of very low memory, CG method plays a significant role, particularly when there is a large scale, the method of CG is very effec-

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tive. Let $x_0 \in R^n$ be a primary solution for problem (1). The nonlinear method of CG is generally planned by this iterative formula: [12]

$$x_{k+1} = x_k + \alpha_k d_k,$$
(2)

where x_k the current is iterate point, $\alpha_k > 0$ represents a step length that is determined by a line search, and d_k refers to the search direction, defined by:

$$d_k = \begin{cases} -g_k & \text{if} \quad k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if} \quad k > 0, \end{cases}$$
(3)

where $\beta_k \in R$ is a parameter ($0 < \beta_k < 1$) and g_{k+1} denotes $g(x_{k+1})$. Some familiar formulas for β_k exist. They are shown below: [5]

$$\beta_k^{FR} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}, \qquad \text{(Fletcher-Reeves (FR), 1964)}$$

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k}, \qquad \text{(Hestenes -Stiefel (HS), 1952)}$$

$$\beta_k^{PR} = \frac{g_{k+1}^T y_k}{g_k^T g_k}, \qquad \text{(Polak- Ribiere (PR), 1969)}$$

where $\| \, . \, \|$ stands for the Euclidean norm of vectors and $y_k = g_{k+1} - g_k$. In this paper, three term CG methods were proposed, which are based on Yasushi, Yabe and Ford (2009) [10]. Generally, in the convergence analysis of CG methods, one hopes the ILS, such as the Strong Wolfe Conditions (SWC), which is shown as follows [6]:

Definition 1. Strong Wolfe Conditions (SWC) aims at finding line search α_k where:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k, \quad 0 \le \delta \le \frac{1}{2}$$
$$\left| d_k^T g(x_k + \alpha_k d_k) \right| \le -\sigma d_k^T g_k, \quad \delta \le \sigma \le 1.$$
(4)

Definition 2. Convex combination gives a finite number of points (which can be vectors, scalars) where all coefficients are non-negative and sum to 1 such that [13]

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots\ldots\ldots + \alpha_n x_n \text{ and } \alpha_1 + \alpha_2 + \ldots\ldots\ldots\ldots + \alpha_n = 1.$$

This article is organized as follows: in the second section, a general review is presented on three term CG algorithms. In section 3, new hybrid TTCG techniques are presented. In section 4, the properties of global convergence for the proposed new methods of CG are analysed. In section 5, some numerical comparisons were reported against general formula of Yasushi, Yabe and Ford 3TCG by substituting (y_k, d_k, g_k) instead of the vector

p_k in each case by using 35-test problems in the CUTE [7]; in addition, general conclusions are given in section 5.

2 Three-term CG methods

Recently, researchers widely examined methods of three-term conjugate gradient for improving the classical conjugate gradient method efficiency. Beale presented the initial three-term nonlinear method of CG in [8], in which the following formula determines search direction:

$$d_{k+1} = -g_{k+1} + \beta_k + \gamma_k d_t.$$

In Beale's algorithm [8], the parameter $\beta_k = \beta_k^{FR}$ or $\{\beta_k^{HS}, \beta_k^{PR}$ …, etc.$\}$. In [9], another method of "three-term conjugate gradient was proposed by Nazareth", in which the computation of search direction is done using this formula:

$$d_{k+1} = -y_k + \frac{y_k^T y_k}{y_k^T d_k} d_k + \frac{y_{k-1}^T y_k}{y_{k-1}^T d_{k-1}} d_{k-1}.$$

In [4], a descent modified algorithm of PRP conjugate gradient was developed, in which the following formula of three-term is used to obtain the search direction:

$$d_{k+1} = -g_{k+1} + \frac{g_{k+1}^T y_k}{g_k^T g_k} d_k - \frac{g_{k+1}^T d_k}{g_k^T g_k} y_k.$$

In [3], the modification of method of HS conjugate gradient was done by employing a method of descent three-term conjugate gradient, which is read

$$d_{k+1} = -g_{k+1} + \frac{g_{k+1}^T y_k}{s_k^T g_k} s_k - \frac{g_{k+1}^T s_k}{s_k^T g_k} y_k.$$

More recently in [10], a general form of three-term conjugate gradient methods, which always generate a sufficient descent direction by formula:

$$d_k = -g_k + \beta_k d_k - \beta_k \frac{g_k^T d_{k-1}}{g_k^T p_k} p_k.$$

The parameter β_k like Beale's form.

3 New Direction for TTCG

`The general idea of the hybrid was built by combining theoretical and reasonable advantages within completely different methods. As shown below, the motivation behind this hybrid is to select a set of good qualities for some known three-term conjugate gradient methods and generate new generic formulas for TTCG methods. Installing a method with high specifications is achieved by picking two parameters ϕ_1 *and* ϕ_2 with the created convex combination between typically used vectors in optimization (y_k, g_k, s_k) and place it rather than of P_k in Yasushi, Yabe and Ford and attached to the form of the most important features as following:

$$d_k = \begin{cases} -g_k, & \text{if } k=0 \text{ or } g_k^T z = 0, \\ -g_k + \beta_k d_{k-1} - \beta_k \dfrac{g_k^T d_{k-1}}{g_k^T z} z, & \text{otherwise}, \end{cases} \tag{5}$$

$$z = \phi_1 y_{k-1} + \phi_2 g_k + (1 - \phi_1 - \phi_2) d_{k-1}, \tag{6}$$

where ϕ_1 and ϕ_2 are scalars that take values in the interval [0, 1] and $\phi_2 < \phi_1$. The vector $z \in R^n$ as defined above is convex combination from type three which join the vectors y_{k-1} , g_k *and* d_{k-1}.This proposed method is reduced to the standard HS or PRP method in the case of exact line search since $g_k^T d_{k-1} = 0$. In this case, also it should be noticed that the proposed method includes "the three-term conjugate gradient methods proposed by Zhang et al." [1-3]. The methods (5),(6) with $\beta_k = \beta_k^{FR}, \phi_1 = \phi_2 = 0$ and $z = g_k$ becomes the method by [2] and if $g_k^T y_k \neq o$, the methods (5),(6) with $\beta_k = \beta_k^{PR}$, $\phi_1 = 1,\ \phi_2 = 0$ and $z = y_k$ becomes the method by [1]. If $g_k^T y_k \neq o$, the method (5)-(6) with $\beta_k = \beta_k^{HS}$ $\phi_1 = 1,\ \phi_2 = 0$ and $z = y_k$ becomes the method by [3]. In addition, the method (5)-(6) with $\beta_k = \beta_k^{PR}$ and $z = g_k$ becomes the method by [4]. More important, it can be considered as a key when selecting vectors y_{k-1} , g_k *and* d_{k-1} as switch from formula to formula and considered a special instance of the formula three-term conjugate gradient algorithm given by Narushima et al. [10], when putting $z = y_k$, it means $\phi_1 = 1,\ \phi_2 = 0$, as mentioned above.

2.1 The New Three-Term CG-Algorithms:

Step1 (Initializing): Given an initial point $x_0 \in R^n$ and positive parameters, $0 < \phi_2 < \phi_1 \leq 1$, $\psi = 0.2$, $0 \leq \delta \leq 0.5$ and $\delta \leq \sigma \leq 1$. Set the initial search direction $d_0 = -g_0$ and let k = 0.

Step2 (Criterion of Termination): When $\|g_k\| \leq \varepsilon$, after that stop.

Step 3 (Line search): Determine step length $\alpha_k > 0$ satisfies "the Strong Wolfe condition" (4) with Acceleration scheme [5]: "compute $z = x_k + \alpha_k d_k$, $y_k = g_k - g_z$, $g_z = \nabla f(z)$, and Compute $a_k = \alpha_k g_k^T d_k$, $b_k = -\alpha_k y_k^T d_k$, if $b_k \neq 0$, then compute $\varphi_k = -\dfrac{a_k}{b_k}$ and update the variables as $x_{k+1} = x_k + \varphi_k \alpha_k d_k$; otherwise update the variables as $x_{k+1} = x_k + \alpha_k d_k$.

Step4 (Finding the direction): Compute the new search direction (5),(6), where the scalar parameter β_k is indecently chosen (in practice, β_k^{FR} is substituted).

Step5 (Restart procedure): If $|g_{k+1}^T g_k| \geq \psi \|g_{k+1}\|^2$, then go to Step (1) else continue (this is Powell restart).

Step6 (Loop): Let k = k +1 and go to Step (2).

4 Convergence Analysis

Now, the basic global convergence property of the new three-term CG-Algorithms must be proved under the condition that the following assumption is held.

Assumption (A):

(i) The level set $S = \{x : x \in R^n, f(x) \le f(x_0)\}$ is bounded, where x_0 is the starting point, and there exists a positive constant such that, for all: $B > 0$ and defined below.

(ii) In a neighbourhood Ω of S, f is differentiable continuously and its gradient g is continuously Lipchitz, namely, a constant $L \ge 0$ exists, where

$$\| g(x) - g(x_k) \| \le L \| x - x_k \|, \forall x, x_k \in \Omega \tag{7}$$

Obviously, Assumption (A, i) results in “a positive constant D, where:

$$B = \max\{\|x - x_k\|, \forall x, x_k \in S\}. \tag{8}$$

Here B refers to Ω diameter. From Assumption (A, ii), it is also known that a constant $\gamma \ge 0$ exists, where:

$$\|g(x)\| \le \gamma, \forall x \in S \tag{9}$$

In a number of studies on methods of CG, the descent condition or sufficient descent has a significant role; however, this condition is sometimes difficult to be achieved [1]

Theorem 4.1. (Descent property)[1: Suppose that the assumption (A) is held, independently of choice the parameter β_k and line search, consider the search directions d_k generated from (5-6), it is proved that the search direction easily satisfies the sufficient method with $c = 1$,

$$d_k^T g_k \le -c\|g_k\|^2.$$

Proof. Start with multiplying the direction d_k in (5-6) by the gradient g_k

$$d_k{}^T g_k = -\|g_k\|^2 + \beta_k\, d_{k-1}^T g_k - \beta_k \frac{g_k{}^T d_{k-1}}{g_k{}^T z_k} z_k^T g_k, \tag{10}$$

$$d_k{}^T g_k = -\|g_k\|^2 + \beta_k\, d_{k-1}^T g_k - \beta_k \frac{g_k{}^T d_{k-1}}{g_k{}^T z_k} g_k^T z_k, \tag{11}$$

$$d_k{}^T g_k = -\|g_k\|^2.$$

Hence, by comparing the result with standard sufficiently descent condition, the proposed direction held this condition by the value of $c = 1$.

Theorem 4.2. (Conjugacy Property): Suppose that the step-size α_k satisfies the standard Wolfe conditions, consider the search directions d_k generated from (5-6), then the search directions d_{k+1} are conjugate for all k that is

$$d_{k+1}^T y_k = -c_0\, g_{k+1}^T s_k ,$$

where c_0 positive constant.

Proof. Begin by multiplying the proposed direction by the vector y_k

$$y_k^T d_{k+1} = -y_k^T g_{k+1} + \beta_k y_k^T d_k - \beta_k \frac{g_{k+1}^T d_k}{g_{k+1}^T z} y_k^T z \qquad (12)$$

By using the following reality to get

$y_k = g_{k+1} - g_k \;\Rightarrow\; y_k^T z = g_{k+1}^T z - g_k^T z \;\Rightarrow\; y_k^T z \le g_{k+1}^T z$

Taking the last one and put it in (12)

$$\begin{aligned}
&\le -y_k^T g_{k+1} + \beta_k y_k^T d_k - \beta_k \frac{g_{k+1}^T d_k}{g_{k+1}^T z} g_{k+1}^T z \\
&\le -y_k^T g_{k+1} + \beta_k y_k^T d_k - \beta_k g_{k+1}^T d_k \\
&= -y_k^T g_{k+1} + \beta_k (g_{k+1}^T d_k - g_k^T d_k - g_{k+1}^T d_k) \\
&\le -y_k^T g_{k+1} - \beta_k g_k^T d_k \\
&= -(\|g_{k+1}\|^2 - g_{k+1}^T g_k) - \beta_k g_k^T d_k \\
&= -\|g_{k+1}\|^2 + g_{k+1}^T g_k - \beta_k g_k^T d_k \\
&\le -\|g_{k+1}\|^2 + \|g_{k+1}\|^2 - \beta_k g_k^T d_k
\end{aligned}$$

using curvature inequality in (4)

$$\le -\beta_k g_k^T d_k \quad \le -\frac{\beta_k}{\sigma\alpha} g_{k+1}^T s_k .$$

Hence the conjugacy condition $d_{k+1}^T y_k = -c_0\, g_{k+1}^T s_k$ is done with $c_0 = \beta/\sigma\alpha$.

Property 4.1. Consider a general CG method and suppose that [11]

$$0 < \varsigma \le \|g_k\| \le \gamma, \quad \forall k \ge 0$$

(13)

It can be said that a CG method has the property (4.1) if there exists two constants b>1 and $\lambda > 0$ such that for all k,

$$|\beta_k| \le b \qquad (14)$$

$$\text{If } \|s_k\| \le \lambda \text{ then } |\beta_k| \le \frac{1}{2b} \text{ for all } \lambda > 0. \qquad (15)$$

Lemma 4.2. Assume that d_{k+1} is a descent direction and g_k satisfies the Lipchitz condition $\|g(x) - g(x_k)\| \le L\|x - x_k\|$ for all x on the line segment connecting x and x_k, where L is constant if the line search direction satisfies Strong Wolfe condition, then[6]:

$$\alpha_k \geq \frac{(1-\sigma)\left|d_k{}^T g_k\right|}{L\|d_k\|^2}. \tag{16}$$

Proof. Using curvature inequality in (4)

$$\sigma d_k{}^T g_k \leq d_k{}^T g_{k+1} \leq -\sigma d_k{}^T g_k$$
$$\Rightarrow \sigma d_k{}^T g_k \leq d_k{}^T g_{k+1}. \tag{17}$$

Subtracting $d_k{}^T g_k$ from both sides of (35) and using Lipchitz condition yields:

$$(1-\sigma) d_k{}^T g_k \leq d_k{}^T (g_{k+1} - g_k) \leq L\alpha_k \|d_k\|^2 \tag{18}$$

As d_k is a descent direction" and $\sigma \leq 1$, so (16) holds:

$$\alpha_k \geq \frac{(1-\sigma)\left|d_k{}^T g_k\right|}{L\|d_k\|^2}.$$

The conclusion of the following Lemma, often called the Zoutendijk condition, is used to prove the global convergence of any nonlinear CG method. Zoutendijk [18] originally gave it under the Strong Wolfe line search (4). In the following Lemma, this condition will be proved.

Lemma 4.3. Suppose Assumption (A) holds. Consider the iteration process of the form (5),(6), where d_{k+1} satisfies the descent condition ($d_k{}^T g_k \leq 0$) for all $k \geq 1$ and α_k satisfies (4). Then

$$\sum_{k\geq 1} \frac{(g_k{}^T d_k)^2}{\|d_k\|^2} < +\infty. \tag{19}$$

Proof: From the first inequality in (4), the following equation can be obtained:

$$f_{k+1} - f_k \leq \delta \alpha_k g_k{}^T d_k.$$

Combining this with the results in Lemma (4.2), yields

$$f_{k+1} - f_k \leq \frac{\delta(1-\sigma)}{L} \frac{(g_k{}^T d_k)^2}{\|d_k\|^2}. \tag{20}$$

Using the bound-ness of function f in Assumption (A), hence

$$\sum_{k\geq 1} \frac{(g_k{}^T d_k)^2}{\|d_k\|^2} < +\infty. \tag{21}$$

Theorem 4.3. Suppose that assumption A holds and consider the new algorithm obtained by (3-1,3-2) where α_k is computed by Wolf Line Search , then

$$\underset{k\to\infty}{Lim} \inf \|g_k\| = 0.$$

Proof. The proof is well done by contradiction, so it is supposed that the conclusion is not true, then $\|g_k\| \neq 0$, as mentioned above, there exists a constants $\varsigma, \gamma > 0$ such that

$$0 < \varsigma \leq \|g_k\| \leq \gamma, \quad \forall k \geq 0$$

Now by taking the square norm of both sides of the proposed new direction

$$\|d_{k+1}\| = \left\| -g_{k+1} + \beta_{k+1} d_k - \beta_{k+1} \frac{g_{k+1}^T d_k}{g_{k+1}^T z} z \right\|$$

$$\leq \|g_{k+1}\| + \beta_{k+1}\|d_k\| + \beta_{k+1} \frac{|(g_{k+1}^T d_k)|}{|(g_{k+1}^T z)|}\|z\|$$

$$\leq \|g_{k+1}\| + \beta_{k+1}\|d_k\| + \beta_{k+1} \frac{\|g_{k+1}\|\|d_k\|}{\|g_{k+1}\|\|z\|}\|z\| \qquad \text{(By Cauchy Schwarz)}$$

$$= \|g_{k+1}\| + 2\ \beta_{k+1}\|d_k\| \ < \ \gamma + 2\ \beta_{k+1} B \qquad \{ C = \gamma + 2\ \beta_{k+1} B \}$$

So that $\|d_{k+1}\|^2 < C^2$, dividing by the quality $\|g_{k+1}\|^4$ to get

$$\frac{\|d_{k+1}\|^2}{\|g_{k+1}\|^4} < \frac{C^2}{\|g_{k+1}\|^4}$$

$$\sum_{k=1}^{\infty} \frac{\|d_{k+1}\|^2}{\|g_{k+1}\|^4} > C^2 \gamma^{-2} = \infty.$$

Which is in contrary to Lemma 4.3, then $Lim \inf \|g_k\| = 0$.

5 Numerical Results

To evaluate the reliability of the proposed methods, they were tested against the Yasushi, Yabe and Ford 3TCG methods with different options of p_k such as (y_k, d_k, g_{k+1}) using the same test problems as shown in Table (1). The comparison includes some of the known test functions that contributed to CUTE [7] in different dimensions (100, 400, 600, 1000). The program was written with a double precision account using Fortran 6.6. The comparative performance of the algorithm is evaluated by considering both the total number of function evaluations that is normally assumed as the usually factor in each iteration, total number of iterations and the time. The standard of convergence criterion was

$$\|g_{k+1}\| \leq 1 \times 10^{-6}. \qquad (22)$$

Percentage Performance of each New algorithm was against 100% Yasushi,Yabe and Ford 3TCG algorithms with different choice of p_k by y_k, d_k, g_{k+1} respectively, as shown in Table 4.2.

Based on the above tables, it can be concluded that the new algorithm beats Yasushi, Yabe and Ford 3TCG methods in all NOI; NOFG and Time. NOI is about (35-70) % percentages. However, the new algorithm also beats Yasushi, Yabe and Ford 3TCG methods in all NOFG about (8-52) % percentages. In addition, the new algorithm also beats Yasushi, Yabe and Ford 3TCG methods time about (27-73) % percentages.

Table 1: Show the Comparison between New and Yasushi,Yabe and Ford 3TCG methods for the total of n different dimensions n= 100, 400, 700,1000, for each test problem ($\phi_1 = 0.3, \phi_2 = 0.4$, $\varepsilon = 1x10^{6}$).

Number of Problem	New method by Z vectors NOI/NOFG/TIME	3TCG by $\mathbf{y_k}$ vectors **(Yasushi,Yabe, Ford)** NOI/NOFG/TIME	3TCG by $\mathbf{g_{k+1}}$vectors **(Yasushi,Yabe, Ford)** NOI/NOFG/TIME	3TCG by d_k vectors **(Yasushi,Yabe, Ford)** NOI/NOFG/TIME
1	235 365 0.17	265 388 0.22	348 480 0.28	425 556 0.30
2	84 161 0.02	84 161 0.01	84 161 0.02	84 161 0.00
3	35 78 0.02	35 78 0.02	35 78 0.01	35 78 0.01
4	348 389 0.19	388 429 0.22	494 536 0.26	2305 2351 1.73
5	228 265 0.04	302 339 0.05	236 273 0.03	373 406 0.04
6	176 188 0.14	214 226 0.17	75 87 0.06	179 189 0.13
7	250 260 0.03	60 70 0.02	2176 2188 0.06	6016 6107 0.46
8	66 79 0.08	66 79 0.06	66 79 0.08	66 79 0.08
9	126 174 0.01	141 187 0.02	90 138 0.01	151 203 0.02
10	75 116 0.05	83 118 0.05	75 94 0.05	75 94 0.04
11	178 233 0.05	228 283 0.06	165 220 0.03	248 300 0.07
12	95 135 0.01	2069 2113 0.17	95 135 0.01	2069 2113 0.18
13	14 28 0.00	14 28 0.01	14 28 0.00	14 28 0.02
14	516 548 0.05	450 493 0.07	478 514 0.06	2909 2970 0.55
15	373 402 0.07	348 377 0.06	1420 1449 0.28	435 464 0.07
16	49 117 0.03	43 112 0.01	47 55 0.01	2033 2122 0.35
17	36 48 0.00	36 47 0.02	36 48 0.00	36 48 0.02
18	43 52 0.00	43 54 0.01	43 52 0.00	43 52 0.02
19	353 388 0.06	43 54 0.01	294 329 0.03	424 459 0.06
20	29 38 0.02	29 38 0.01	29 38 0.00	29 38 0.01
21	274 307 0.03	320 353 0.05	247 277 0.03	364 393 0.05
22	174 209 0.03	183 198 0.06	251 273 0.10	191 219 0.04
23	220 263 0.08	219 261 0.08	183 209 0.05	308 387 0.09
24	49 77 0.00	45 82 0.02	51 79 0.02	47 84 0.00
25	101 127 0.02	101 127 0.01	101 127 0.01	101 127 0.03
26	216 227 0.06	207 218 0.06	234 245 0.08	333 352 0.08
27	49 80 0.00	49 80 0.02	55 84 0.00	55 84 0.00
28	49 80 0.00	32 66 0.02	32 66 0.01	**32 66 0.02**
29	35 43 0.01	35 43 0.02	35 43 0.03	35 43 0.03
30	18 30 0.00	18 30 0.00	18 30 0.02	18 30 0.02
31	75 94 0.03	75 94 0.03	75 94 0.03	75 94 0.03
32	285 333 0.04	287 331 0.03	294 342 0.02	364 448 0.06
33	8 28 0.00	11 34 0.00	11 34 0.00	11 34 0.00
34	32 44 0.00	32 44 0.00	32 44 0.01	32 44 0.02
35	2064 2100 0.04	4082 4135 0.46	4036 4064 0.19	2138 2245 0.29
Total	6958 8106 2:18	10637 11770 3:33	11955 12993 3:08	22053 23468 8:18

Table 2: Performance of the new algorithm against 100% of Yasushi, Yabe and Ford algorithm, as followed in Table 1.

Tools	3TCG By y	New	3TCG By g	New	3TCG By s	New
NOI	100%	65.41%	100%	58.20%	100%	31.55%
NOFG	100%	68.87%	100%	62.38%	100%	34.54%
Time	100%	64.78%	100%	73.40%	100%	28.04%

6 Conclusions

In this paper, a three-term conjugate gradient method was projected. The good property of these methods is that this algorithm can produce sufficient descent with conjugancy direction, under a few assumptions. The proposed CG methods are shown to be globally convergent for uniformly convex and general functions, respectively. Some numerical results are reported against Yasushi, Yabe and Ford 3TCG algorithm which demonstrated the viability of the new proposed CG algorithms with the scalars ϕ_1 and ϕ_2.

6 Appendix

The details of the 35-test functions used are:

1-Extended Trigonometric Function. 2-Extended Penalty Function. 3-Raydan2 Function.

4-Hager Function. 5-Generalized Tridiagonal-1 Function. 6-Extended Three Exponential Function. 7-Diagonal 4 Function. 8-Diagonal5 Function. 9-Extended Himmelblau Function. 10-Generalized PSC1 Function. 11- Extended Block Diagonal BD1 Function. 12-Extended Quadratic Penalty QP1 Function. 13-Extended Quadratic QF2 Function. 14-Extended EP1 Function.15-Extended Tri-diagonal 2 Function. 16- DIXMAANA Function. 17-DIXMAANB Function. 18- DIXMAANC Function. 19-EDENSCH Function. 20-DIAGONAL 6 Function. 21-ENGVALI Function. 22-DENSCHNA Function. 23-DENSCHNC Function. 24-DENSCHNB Function. 25-DENSCHNF Function. 26-Extended Block–Diagonal BD2 Function. 27-Generalized quadratic GQ1 Function. 28-DIAGONAL 7 Function. 29- DIAGONAL 8 Function. 30- Full Hessian Function.
31-SINCOS Function. 32- Generalized quadratic GQ2 Function. 33-ARGLINB Function. 34-HIMMELBG Function. 35-HIMMELBH Function

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On Positive Solution for a Non-Local Fractional Boundary Value Problem

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Abstract — *We discuss the existence and uniqueness of positive solution for a fractional boundary value problem by using some fixed point theorems and the upper and lower solutions method. An example is also given to illustrate the obtained results.*

Keywords: Fractional boundary value problem, Positivity of solution, Fixed point theorem, Upper and lower solutions.
Mathematics Subject Classification: 34A08, 34B15.

1 Introduction

Recently, fractional differential equations have revealed to be great worth in the modeling of many phenomena in several fields of sciences, economics and engineering. For this purpose, we find many applications in electrochemistry, viscoelasticity, control theory, electrical networks, signal process. see [10-13]. Significant developments in fractional differential equations can be found in the monographs [11,12,13,15]. Different methods are introduced in the investigation of fractional differential equations, such as the theory of fixed points, see [1-11,15-17].

In [7], the authors proved the existence of at least one or three positive solutions of the following problem, by applying the Guo-Krasnosel'skii and Avery-Peterson fixed-point theorems and under growing conditions on the nonlinear term f:

$$\begin{cases} D_{0^+}^q u(t) = a(t) f(u(t)), \quad 0 \le t \le 1, 2 < q \le 3 \\ u(0) = u'(0) = 0, \quad u''(0) = \alpha u(1), \end{cases}$$

here $D_{0^+}^q$ denotes the fractional derivative of Riemann-Liouville type, f is a given real function and the function a is continuous on $[0,1]$.

In [14], Matar studied the positivity of solution for the following boundary value problem:

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$$\begin{aligned} D_{0^+}^q u(t) &= f(t, u(t)), \quad 0 < t < 1, 1 < q \leq 2 \\ u(0) &= 0, \quad u'(0) = \theta > 0, \end{aligned}$$

here the function f is continuous on $[0,1] \times \mathbb{R}$. By introducing the so-called upper and lower control functions and applying a fixed-point theorem on a cone, the author was able to establish the existence and uniqueness of the positive solution.

The purpose of this work is to establish sufficient conditions for the existence and uniqueness of the positive solution of the following fractional boundary value problem (P)

$$\begin{cases} {}^cD_{0^+}^q u(t) = f(t, u(t)), \quad 0 \leq t \leq T, \;\; 2 < q < 3 \\ u(0) = u'(0) = 0, \quad u''(0) = \alpha > 0, \end{cases}$$

where the function f is continuous and nonnegative on $[0,T] \times \mathbb{R}$. We denote by ${}^cD_{0^+}^q$ the fractional derivative of Caputo type.

This work is organized as follows. We expose the tools that will be used later in the next section. The third section is devoted to the study of the existence of at least one positive solution of the problem (P) by the help of Schauder's theorem fixed on the cone, then we prove the uniqueness of positive solutions of the problem (P) by using Banach's contraction principle. We end this section with an example that elucidates the results obtained.

2 Preliminaries

In this section, we present some definitions and lemmas from fractional calculus theory, which will be needed later.

Definition 2.1. *For a continuous function g on $[a,b]$, we define the Riemann-Liouville fractional integral of order α by*

$$I_{a^+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s) ds, \; \alpha > 0$$

Definition 2.2. *The Caputo fractional derivative of order α of a function f is defined by*

$${}^cD_{a^+}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^n(s)}{(t-s)^{\alpha-n+1}} ds$$

where $n = [\alpha] + 1$, $([\alpha]$ is the entire part of $\alpha)$.

Lemma 2.3. *The solution of the homogenous differential equation ${}^cD_{a^+}^\alpha g(t) = 0$ is given by $g(t) = c_1 + c_2 t + c_3 t^2 + \ldots + c_n t^{n-1}$, with $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, if $g \in C([0,1])$.*

Lemma 2.4. *We have $I_{0^+}^p I_{0^+}^q f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^q I_{0^+}^p f(t)$ and ${}^cD_{a^+}^q I_{0^+}^q f(t) = f(t)$, for all $t \in [a,b]$, $p, q \geq 0$ and $f \in L_1[a,b]$.*

Now, we transform the problem (P) to an equivalent integral equation.

Lemma 2.5. *u is a solution of the problem (P) if and only if u is a solution of the integral equation*

$$u(t) = \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds.$$

Proof. The proof is standard, then we omit it. □

Define $E = C[0,T]$ equipped with the norm $\|u\| = \max_{t\in[0,T]} |u(t)|$. Define the subspace K of E as the set of nonnegative functions. Let a and b be two nonnegative real such that $b > a$. Define the upper control function and the lower control function of a function $u \in [a,b]$, respectively by

$$U(t,u) = \sup_{\lambda\in[a,u]} f(t,\lambda), \; L(t,u) = \inf_{\lambda\in[u,b]} f(t,\lambda)$$

Obviously, $U(t,u)$ and $L(t,u)$ are nondecreasing according to u, monotonous and satisfy $L(t,u) \le f(t,u) \le U(t,u)$.

We make the following hypotheses:

(H_1) There exist u_*, u^* two elements in K, verifying $a \le u_*(t) \le u^*(t) \le b$ and

$$\begin{cases} u^*(t) \ge \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} U(s, u^*(s))\, ds + \frac{\alpha}{2}t^2 \\ u_*(t) \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L(s, u_*(s))\, ds + \frac{\alpha}{2}t^2. \end{cases}$$

(H_2) For any x, y belonging to E and $t \in [0,T]$, we can find a number $0 < \eta < 1$ such that

$$|f(t,y) - f(t,x)| \le \eta \|y - x\|.$$

The function u_* is called lower solution for problem (P) and u^* is called upper solutions.

Define the integral operator A on E as

$$Au(t) = \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds. \tag{1}$$

Definition 2.6. *We say that u is a positive solution of problem (P) if $u(t) > 0$, for all $t \in [0,T]$ and the boundary conditions in (P) are satisfied.*

Theorem 2.7. *Under the hypothesis (H_1) the fractional boundary value problem (P) has at least one positive solution u belonging to E and satisfying $u_*(t) \le u(t) \le u^*(t)$.*

Proof. Let

$$C = \{u \in K, u_*(t) \le u(t) \le u^*(t), 0 \le t \le T\},$$

remark that if $u \in C$, then $\|u\| \le b$. Hence, C is bounded, convex and closed subset of E.

Claim 1. A is uniformly bounded on C.

The operator A is continuous on C since f is continuous. Set

$$M = \max\{f(t, u(t)), t \in [0,T], \|u\| \le b\}.$$

Let $u \in C$, then $\|u\| \leq b$ and we have

$$|Au(t)| \leq \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s))\, ds,$$
$$\leq \frac{\alpha}{2}T^2 + \frac{MT^q}{\Gamma(q+1)}.$$

Thus

$$\|Au\| \leq \frac{\alpha}{2}T^2 + \frac{MT^q}{\Gamma(q+1)}.$$

Hence A is uniformly bounded.

Claim 2. Au is equicontinuous. In fact, for $0 \leq t_1 < t_2 \leq T$, it yields

$$\begin{aligned}
|Au(t_2) - Au(t_1)| &\leq \frac{\alpha}{2}\left(t_2^2 - t_1^2\right) + \\
&\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}f(s,u(s)ds - \frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}f(s,u(s)ds\right| \\
&\leq \alpha T(t_2-t_1) + \frac{1}{\Gamma(q)}\int_0^{t_1}\left((t_2-s)^{q-1} - (t_1-s)^{q-1}\right)f(s,u(s)ds \\
&+ \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,u(s)ds \\
&\leq \alpha T(t_2-t_1) + \frac{MT(t_2-t_1)}{\Gamma(q-1)} + \frac{(t_2-t_1)^q}{\Gamma(q+1)} \to 0, as\ t_1 \to t_2.
\end{aligned}$$

Thanks to Arzela-Ascoli Theorem we deduce the compacity of A.

Let $u \in C$, then by the definition of the control functions and the hypothesis (H1), it yields

$$\begin{aligned}
Au(t) &= \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s))\, ds \\
&\leq \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} U(t, u^*(t))\, ds \\
&\leq u^*(t),
\end{aligned}$$

and

$$\begin{aligned}
Au(t) &= \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s))\, ds \\
&\geq \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} L(s, u_*(s))\, ds \\
&\geq u_*(t).
\end{aligned}$$

Hence, $u_*(t) \leq Au(t) \leq u^*(t)$, $0 \in t \leq T$, from which we deduce $A(C) \subseteq C$. Finally, we conclude by Schauder fixed point theorem, that A has at least one fixed point and consequently, the problem (P) has at least one positive solution u in E between the lower and upper solutions. □

The uniqueness of the positive solution of (P) is given in the following theorem.

Theorem 2.8. *The problem (P) has a unique positive solution $u \in E$, if the hypotheses (H_1) and (H_2) and the inequality*

$$\frac{\eta T^q}{\Gamma(q+1)} < 1, \tag{2}$$

are satisfied.

Proof. Since the hypothesis (H_1) is satisfied then, we conclude by Theorem 3.2 that the problem (P) has at least one positive solution in E. We claim that the operator A is a contraction on E. In fact, for any $u, v \in E$, we have

$$\begin{aligned} |Au(t) - Av(t)| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left| f(s, u(s)) - f(s, v(s)) \right| ds \\ &\leq \frac{\eta T^q}{\Gamma(q+1)} \| u - v \|, \end{aligned}$$

finally, taking (2) into account, then A is a contraction and thus the problem (P) has a unique positive solution $u \in C$. □

Example 2.9. *Let us choose in the problem (P), $q = \frac{8}{3}$, $T = 1$, $f(t,u) = 1 + \frac{t}{2(u+1)}$, $0 \leq t \leq 1$, $u \geq 0$, $[a,b] = [0,1]$ and $\alpha = 1$. Since f is decreasing according to u, then*

$$U(t,u) = 1 + \frac{t}{2},\ L(t,u) = 1 + \frac{t}{4},$$

If we set

$$\begin{aligned} u^*(t) &= \frac{t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \frac{t^{\frac{11}{3}}}{\Gamma\left(\frac{14}{3}\right)} + \frac{1}{2}t^2 \\ &\geq \frac{1}{\Gamma\left(\frac{8}{3}\right)} \int_0^t (t-s)^{\frac{5}{3}} U(s, u^*(s))\, ds + \frac{1}{2}t^2 \\ &= \frac{t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \frac{t^{\frac{11}{3}}}{2\Gamma\left(\frac{14}{3}\right)} + \frac{1}{2}t^2 \end{aligned}$$

and

$$\begin{aligned} u_*(t) &= \frac{t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \frac{t^{\frac{11}{3}}}{8\Gamma\left(\frac{14}{3}\right)} + \frac{1}{2}t^2 \\ &\leq \frac{1}{\Gamma\left(\frac{8}{3}\right)} \int_0^t (t-s)^{\frac{5}{3}} L(s, u^*(s))\, ds + \frac{1}{2}t^2 \\ &= \frac{t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \frac{t^{\frac{11}{3}}}{4\Gamma\left(\frac{14}{3}\right)} + \frac{1}{2}t^2. \\ 0 &\leq u_*(t) \leq u^*(t) \leq 1. \end{aligned}$$

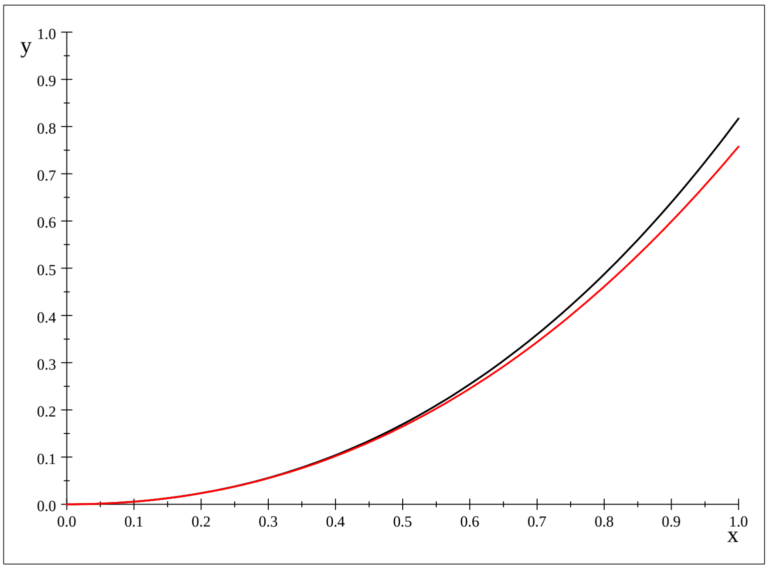

Figure 1: u_* in red, u^* in black.

hence assumption (H_1) *holds, then the problem* (P) *has at least one positive solution. Moreover, there exists* $\eta = \frac{1}{4}$, *such that hypothesis* (H_2) *is satisfied and*

$$\frac{\eta T^q}{\Gamma(q+1)} = \frac{1}{4\Gamma\left(\frac{11}{3}\right)} = 6.231\,0 \times 10^{-2} < 1.$$

We conclude by Theorem 3.3, the uniqueness of positive solution u *satisfying* $u_*(t) \leq u(t) \leq u^*(t)$, $0 \leq t \leq 1$.

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A new filled function method with two parameters in a directional search

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Abstract — *In this paper, we investigate the solution of the problem of finding the global minimizer for the unconstrained objective function, for that, a new algorithm developed, this algorithm based on two steps. First, we transform the problem into a one-dimensional according to the number of directions. Second, we construct a new filled function at each direction in order to minimize the one-dimensional problem and then to find the global minimizer of the multi-dimensional function. We present the results of numerical experiments using test problems taken from literature studies. The experiment results indicate the effectiveness and accuracy of the purposed filled function methods.*

Keywords: global optimization, dimensional search, filled function method.
Mathematics Subject Classification: 80C50, 30A40, 90C26.

1 Introduction

Global optimizations are important tools for examining complicated function spaces such as these located in modern high-fidelity engineering models. Those models present increasingly accurate insights within system behaviours but are usually costly to estimate and difficult to search. While methods exist for determining global optimization problems there is yet room for improving faster, more reliable, and easier to implement algorithms [1]. The filled function method that firstly introduced via Ge(1987) [2,3], and then reviewed in various searches, is an efficient method for determining the global optimization approaches. It modifies the objective function as a filled function and then obtains a best local minimizer frequently by optimizing the filled function formed on the minimizer found previously [4]. The main purpose of this paper is to introduce and formalize a new filled function in two parameters. Firstly, we provide formal definitions and assumptions. Next, we offer and investigate the theoretical prosperities of the filled function and

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propose the solution algorithm. Finally, we report experimental results by applying the algorithm on several test problems to confirm the effectiveness of the new method.

2 Basic Concepts

Suppose the unconstrained problem:

$$min f(x) \quad s.t \quad x \in S, \quad S \subset \mathbb{R}^n \tag{1}$$

where $f : S \longrightarrow R$ is a continuously differentiable function. Now, we introduce the following definitions.

Definition 2.1. [5] A point $x^* \in S$ is said to be a global minimizer of the function f on S if:

$$f(x^*) \leq f(x) \, \forall x \in S,$$

and it is called a strict global minimizer point of f on S if:

$$f(x^*) < f(x) \qquad \forall x \in S, \quad x \neq x^*.$$

Definition 2.2. [5] A point $x_k^* \in S$ is said to be a local minimizer of f on S if there exists a neighborhood $B(x_k^*; \epsilon)$, with $\epsilon > 0$ such that

$$f(x_k^*) \leq f(x) \quad \forall x \in S \cap B(x_k^*; \epsilon)$$

and it is called a strict local minimizer of f on S if there exists a neighborhood $B(x_k^*; \epsilon)$, with $\epsilon > 0$ such that

$$f(x_k^*) < f(x) \qquad \forall x \in S \cap B(x_k^*; \epsilon), \quad x \neq x_k^*.$$

Definition 2.3. [5] A basin of $f(x)$ at an isolated minimizer x_k^* is a connected domain $B(x_k^*)$ which contains x_k^* and in which starting from any point the steepest descent trajectory of $f(x)$ converges to x_k^*. but outside which the steepest descent trajectory of $f(x)$ does not converge to x_k^*. A hill of $f(x)$ at x_k^* is the basin of $-f(x)$ at its minimizer x_k^*, if x_k^* is a maximizer of $f(x)$.

Definition 2.4. [2] Let x_k^* is a current minimizer of f. Let $B(x_k^*)$ is the basin of f at x_k^* over S. A function $F : S \to \mathbb{R}$ is said to be a filled function of f at x_k^* if it satisfies the following properties:

- x_k^* is a maximizer of F and whole basin $B(x_k^*)$ of f at x_k^* over S becomes a part of a hill of F;
- F has no stationary points in any basin of f higher than $B(x_k^*)$;
- If f has a basin $B(x_{k+1}^*)$ at x_{k+1}^* lower than $B(x_k^*)$, then there exists a point $x' \in B(x_{k+1}^*)$ is a minimizer of F.

The evolution of the filled functions supports the subsequent periods. The typical models of the filled functions as a first creation are the function (2) and (3) [6] which offered as following

$$F(x, a, \beta) = exp(-\frac{\|x - x_k^*\|}{\beta^2})\frac{1}{(a + f(x))} \tag{2}$$

$$D(x, a, \beta) = -[\beta^2 \ln(a + f(x)) + \|x - x_k^*\|^p] \tag{3}$$

These functions have a common feature, there are two flexible parameters a and β. However, the task of modifying these parameters is extremely challenging. Due to this restriction, the next creation filled functions were introduced which have only a single parameter. For instance, the function introduced in (4)[6] is performed by

$$E(x, r) = -(f(x) - f(x_k^*))exp(r\|x - x_k^*\|^2). \tag{4}$$

The function given in (4) is much simpler than the those presented in the prior generation. Furthermore, as the parameter r grows larger and larger, the swiftly growing of the exponential function value could lead to an influx of the computation [7]. To beat this lack, another filled function suggested as follows:

$$Q(x, r) = \frac{1}{ln(1 + f(x) - f(x_k^*))} - r\|x - x_k^*\|^2, \tag{5}$$

this filled function still holds the feature of function (5) with one parameter, in addition to that, it has no exponential terms. It can be considered as the third generation filled functions(for more samples see [9-11]).

Throughout the rest of this paper, we assume that the following assumptions are satisfied:

Assumption 1. The function $f(x)$ is differential in R^n and the number of minimizers can be infinite, but the number of the different value of minimizers is finite.

Assumption 2. $f(x) : R^n \to R$ is coercive, i.e., $f(x) \to +\infty$ as $\|x\| \to +\infty$.

3 Transforming the problem into one-dimensional

Directional search method is based on the directions d_k, $k = 1, ..., m$. If we have an objective function $f(x)$ with n-dimensions, we can use the line $l_\alpha = x_0 + \alpha d_k, \alpha \in R$ to construct a one-dimensional problem $L(\alpha)$. Moreover, we might want to choose α_k^* as the answer of

$$min_\alpha L(\alpha) = f(x_0 + \alpha d_k), \tag{6}$$

that means α_k^* at the direction d_k can be a result of a one-dimensional minimization problem (for more information see [8]). We obtain a local minimizer α_k^1 of $L(\alpha)$ then we construct the filled function on $L(\alpha)$, next, we take an initial point as a starting to find the second minimizer α_k^2 of $L(\alpha)$. By repeat the above process we will obtain the global minimizer α_k^* at the direction d_k as a solution of one-dimensional problem $L(\alpha)$, and by using $\hat{x}_k = x_0 + \alpha_k^* d_k$ we can minimize $f(x)$ when we use $\hat{x}_k$ as a starting point. Consequently, by comparing all minimizer points $\hat{x}_k$, $k = 1, ..., m$ with each other we will obtain the global minimizer of the problem f.

In the next section, we introduce a new filled function to minimize the one-dimensional problem $L(\alpha)$.

4 A new filled function

We suppose that the point α_k^1 is a local minimizer of the function $L(\alpha)$ that can be determined by any efficient method.

Reducing the objective function from a multi-dimensional as a one-dimensional function making the minimization process easier and more efficient. For this purpose, we offer a new filled function as follows:

$$I_\alpha(\alpha, \alpha_k^1) = G(L(\alpha) - L(\alpha_k^1))U(|\alpha - \alpha_k^1|^2), \tag{7}$$

and

$$G(u) = \begin{cases} cos(\beta u), & u < 0; \\ 1, & \text{othewise,} \end{cases}$$

where $u = L(\alpha) - L(\alpha_k^1)$, $\beta > 1$, and $U(\kappa)$ is an escape function. The private form of $U(\kappa)$ presented in literature into several forms, for instance,κ^ρ ,$exp(\kappa^\rho)$ and $arctan(\kappa^\rho)$, where ρ is a positive integer. In the proposed paper the function U will be selected as $-\rho|\alpha - \alpha_k^1|^2$,$\rho > 0$, that is the final form of the filled function will be as following:

$$I_\alpha(\alpha, \alpha_k^1) = -\rho|\alpha - \alpha_k^1|^2 G(L(\alpha) - L(\alpha_k^1)), \tag{8}$$

where the parameters β and ρ require to be adjusted appropriately.
The proposed filled function is continuously differentiable with two parameters. the new idea and advantages of the proposed algorithm are: First, this algorithm converts the objective function from multi-dimensional as a one-dimensional function this allows us to obtain the global minimizer easier. Second, the trigonometric function $cos(\beta u)$ allows to add many stationary points in the lower basin, this idea has many advantages, for example, it helps to reduce the time and the function evaluations which are very important in cases like this as we see can clearly in the experimental results. Now, let α_k^1 be the current local minimizer of $L(\alpha)$, then we can define:

$$L_{S_1} = \{\alpha | L(\alpha) \geq L(\alpha_k^1), \alpha \in R, \alpha \neq \alpha_k^1\},\ and\ L_{S_2} = \{\alpha | L(\alpha) < L(\alpha_k^1), \alpha \in R\}.$$

The next theorems show that the function $I_\alpha(\alpha, \alpha_k^1)$ achive Definition 2.4.

Theorem 4.1. *Let α_k^1 be a local minimizer of $I_\alpha(\alpha, \alpha_k^1)$, then α_k^1 is a strictly local maximizer of $I_\alpha(\alpha, \alpha_k^1)$.*

Proof. Since α_k^1 is a local minimizer of $L(\alpha)$, there exists a neighborhood $N(\alpha_k^1, \epsilon^*)$ of α_k^1, $\epsilon^* > 0$ such that $L(\alpha) \geq L(\alpha_k^1)$ for all $\alpha \in N(\alpha_k^1, \epsilon^*)$. Then, for all $\alpha \in N(\alpha_k^1, \epsilon^*)$, $\alpha \neq \alpha_k^1$, we have:
$I_\alpha(\alpha, \alpha_k^1) = -\rho|\alpha - \alpha_k^1|^2 < 0 = I_\alpha(\alpha_k^1, \alpha_k^1)$.
Thus, α_k^1 is a strict local maximizer of $I_\alpha(\alpha, \alpha_k^1)$. □

Theorem 4.2. *Assume that α_k^1 is a local minimizer of $L(\alpha)$ and α is any point in L_{S_1} then $I_\alpha(\alpha, \alpha_k^1)$ has no a stationary point on L_{S_1} .*

Proof. Since $L(\alpha) \geq L(\alpha_k^1)$ and $\alpha \neq \alpha_k^1$, we have:
$I_\alpha(\alpha, \alpha_k^1) = -\rho|\alpha - \alpha_k^1|^2$,$\nabla I_\alpha(\alpha, \alpha_k^1) = -2\rho(\alpha - \alpha_k^1)$.
This means that $\nabla I_\alpha(\alpha, \alpha_k^1) \neq 0$, i.e. α is not a stationary point of $I_\alpha(\alpha, \alpha_k^1)$. □

Theorem 4.3. *Suppose α_k^1 is a local minimizer of $L(\alpha)$ but not a global minimizer, and $L_{S_2} = \{\alpha | L(\alpha) < L(\alpha_k^1), \alpha \in R\}$ is not empty, then there exists a point $\alpha' \in L_{S_2}$ is a local minimizer of $I_\alpha(\alpha, \alpha_k^1)$.*

Proof. Let $L_{S_3} = \{\alpha | L(\alpha) \leq L(\alpha_k^1), \alpha \in R\}$ and $\partial L_{S_2} = \{\alpha | L(\alpha) = L(\alpha_k^1), \alpha \in R\}$, then $L_{S_3} = L_{S_2} \bigcup \partial L_{S_2}$, that means ∂L_{S_2} is the boundary of the sets L_{S_2} and L_{S_3}. Since $L(\alpha)$ is continuous, then ∂L_{S_2} and L_{S_3} are bounded and closed sets.

Now for any $\alpha \in \partial L_{S_2}$ we have

$$I_\alpha(\alpha, \alpha_k^1) = -\rho |\alpha - \alpha_k^1|^2,$$

also, for any $\alpha \in L_{S_2}$ we have

$$I_\alpha(\alpha, \alpha_k^1) = -\rho |\alpha - \alpha_k^1|^2 \cos(\beta(L(\alpha) - L(\alpha_k^1))).$$

Since $I_\alpha(\alpha, \alpha_k^1)$ is continuously differentiable and has the term $\cos(\beta(L(\alpha) - L(\alpha_k^1)))$, $\beta > 1$ then there is at least one point exists $\alpha' \in L_{S_2}$ is a minimizer of the function $I_\alpha(\alpha, \alpha_k^1)$. □

Algorithm

According to the investigation and hypotheses in the earlier section, a new algorithm to obtaining the global minimizer of the function $f(x)$ will be proposed, and the experimental results will be provided as follows.

Step 1 (Initialization) Determine the parameters $\beta > 1$ and $\rho > 0$, choose a starting point $x_0 \in S$, generate direction d_k,$k = 1, 2, ..., m$, and set $\epsilon = 10^{-2}$;

Step 2 Create $L(\alpha) = f(x_0 + \alpha d_k)$ as a one-dimensional function;

Step 3
1. Obtain the local minimizer α_k^i of $L(\alpha)$ starting from α_0 and then choose $\varrho = -1$.
2. Construct the filled function $I_\alpha(\alpha, \alpha_k^i)$ at α_k^i;
3. Start from $\alpha_0 = \alpha_k^i + \varrho\epsilon$ to find a minimizer υ_1 of $I_\alpha(\alpha, \alpha_k^i)$;
4. If υ_1 in S go to (5) otherwise go to (7);
5. Minimize $L(\alpha)$ start from υ_1 to obtain α_{k+1}^i and then, go to (6);
6. If the point α_{k+1}^i in S let $\alpha_k^i = \alpha_{k+1}^i$, and go to (2).
7. If $\varrho = 1$ terminate the iteration and give $\alpha_k^* = \alpha_k^i$ otherwise; let $\varrho = 1$ go to(3).

Step 4 Calculate $\hat{x}_k$ using $\hat{x}_k = x_0 + \alpha_k^* d_k$, and consequently, find x_k^* of $f(x)$ by using $\hat{x}_k$ as the initial point.

Step 5 If $k < m$, let $k = k + 1$ and produce d_{k+1} as a new search direction and go to (Step 2) otherwise; go to (Step 6).

Step 6 Pick out the global minimizer of $f(x)$ using :

$$x^* = \min\{f(x_1^*), f(x_2^*), ..., f(x_m^*)\}.$$

5 The Experimental Results

In order to achieve the merit of the proposed algorithm in this paper, we selected test functions taken from literature. The algorithm is examined on all proposed problems and the results are submitted in Tables 1. and a comparative with the algorithm in [8] submitted in Tables 2. The following symbols are used in this paper:

x_0 The starting point.

f_{eval} total number of functions evaluations $f(x)$, $L(\alpha)$ and $I_\alpha(\alpha, \alpha_k^1)$.

T the mean of sum running time.

f_{mean} the mean of the best value in the 10 runs.

f_{best} the best value in 10 runs.

$ratio$ the rate of successfully obtaining true optimal solution among 10 runs.

Problem 1. *(Two-dimensional function)*

$$minf(x) = (1 - 2x_2 + csin(4\pi x_2) - x_1)^2 + (x_2 + 0.5sin(2\pi x_1))^2$$

$$s.t \quad 0 \leq x_1 \leq 10, \quad -10 \leq x_2 \leq 0,$$

where $c = 0.2, 0.5, 0.05$. The global minimum function value $f(x^*) = 0$ for all c.

Problem 2. *(Three-hump back camel function)*

$$minf(x) = 2x_1^2 - 1.05x_1^4 + \frac{1}{6}x_1^6 - x_1x_2 + x_2^2$$

$$s.t \quad |x_i| \leq 3, \quad i = 1, 2.$$

The global minimizer is $x^* = (0, 0)^T$.

Problem 3. *(Six-hump back camel function)*

$$minf(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 - x_1x_2 - 4x_2^2 + 4x_2^4$$

$$s.t \quad |x_i| \leq 3, \quad i = 1, 2.$$

The global minimizer is $x^* = (-0.0898, -0.7127)^T$ or $x^* = (0.0898, 0.7127)^T$.

Problem 4. *(Treccani function)*

$$minf(x) = x_1^4 + 4x_1^3 + 4x_1^2 + x_2^2$$

$$s.t \quad |x_i| \leq 3, \quad i = 1, 2.$$

The global minimizers are $x^* = (0, 0)^T$ and $x^* = (-2, 0)^T$.

Table 1: The results obtained by our algorithm

No	n	T	f_{eval}	x^*	f_{mean}	f_{best}	$ratio$
1.	2(c=0.2)	0.1658	128	(0.4091; 0.2703)	1.0004e-15	4.6810e-16	100
	2(c=0.5)	0.2742	56	(1.0000; 0.0000)	8.8150e-13	7.3282e-31	100
	2(c=0.05)	0.2464	56	(1.0000; 0.0000)	2.9313e-14	2.4652e-31	100
2.	2	0.2699	28	(0.0000; 0.0000)	3.9199e-15	1.0793e-32	100
3.	2	0.2786	364	(0.0898; 0.7127)	-1.0316	-1.0316	100
4.	2	0.2382	280	(0.0000; 0.000)	3.0507e-16	1.5866e-32	100
5.	2	0.2942	224	(0.0000; -1.0000)	3.0000	3.0000	100
6.	2	0.7117	414	(-1.4251; -0.8003)	-186.7309	-186.7309	100
7.	2	0.2927	336	(1.0000; 1.0000)	1.1647e-14	1.0980e-15	100
	3	0.4143	216	(1.000; 1.000; 1.000)	3.7328e-08	7.0755e-16	100

Table 2: The results obtained by algorithm [8] and our algorithm on the problems 1-7

No	n	The algorithm in [8]			The proposed algorithm		
		T	f_{eval}	f_{best}	T	f_{eval}	f_{best}
1(c=0.2)	2	0.648842	518	1.0707e-30	0.1658	128	4.6810e-16
1(c=0.5)	2	0.721799	522	1.0707e-30	0.2742	56	7.3282e-31
1(c=0.05)	2	0.644013	306	1.4252e-18	0.2464	56	2.4652e-31
2	2	0.762039	360	2.1294e-16	0.2699	28	1.0793e-32
3	2	0.900348	384	-1.0316	0.2786	364	-1.0316
4	2	0.920637	364	2.7399e-17	0.2382	280	1.5866e-32
5	2	0.996568	400	3.0000	0.2942	224	3.0000
6	2	2.003763	480	-186.7309	0.7117	414	-186.7309
7	2	0.856628	244	2.3558e-31	0.2927	336	1.0980e-15
7	3	1.31539	244	1.5705e-31	0.4143	216	7.0755e-16

Problem 5. *(Goldstein and Price function)*

$$min f(x) = g(x)h(x)$$

$$s.t \quad |x_i| \leq 3, \quad i = 1, 2.$$

where

$$g(x) = 1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)$$

and

$$h(x) = 30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 - 48x_2 - 36x_1x_2 + 27x_2^2)$$

$x^* = (0, -1)^T$.

Problem 6. *(Two-dimensional Shubert function)*

$$minf(x) = (\sum_{i=1}^{5} icos[(i+1)x_1] + i)(\sum_{i=1}^{5} icos[(i+1)x_2] + i)$$

$$s.t \quad 0 \le x_i \le 10, \quad i = 1, 2, 3, 4.$$

This problem has 760 local minimizers in total. The global minimum value is$f(x^*) = -186.7309$.

Problem 7. *(n-dimensional function)*

$$minf(x) = \frac{\pi}{n}[10sin^2(\pi x_1) + g(x) + (x_n - 1)^2]$$

$$s.t \quad |x_i| \le 10, \quad i = 1, \cdots, 10$$

where

$$g(x) = \sum_{i=1}^{n-1}[(x_i - 1)^2(1 + 10sin^2(\pi x_{i+1}))].$$

The global minimizer of this problem is $x^* = (1, \cdots, 1)$ for all n.

It is seen from Tables 1 and 2 that the introduced algorithm has many advantages, for instance, the global minimizers of all test problems listed above can be found, this implies the effectiveness of the introduced algorithm. Moreover, from column ratio in Table 1, the ratio of the successful runs are 100%, which confirms that the introduced algorithm is stable. In addition, the difference between f_{mean} and f_{best} is small this implies that the introduced algorithm is stable and robust to the initial points and parameter variation.

6 Conclusion

In this paper, a new filled function introduced for global optimization. The main approach was to transform the objective function into one-dimensional function depending on the directional search and minimize it in each direction. The computational results confirm that this algorithm is actually effective and reliable and the comparison with an actual algorithm confirmed that the introduced method was more efficient and relevant.

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Exact Travelling Wave Solutions of the Nonlinear Gardner-Kawahara Equation by the Standard $\left(\frac{G'}{G}\right)$ – Expansion Method

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Abstract — *The basic idea of this article is to study the solution of the Gardner-Kawahara equation which is modelled to investigate the waves in magnetized plasma. To follow standard* $\left(\frac{G'}{G}\right)$*-expansion method more common forms of solutions are obtained, if the parameters were taken at special values, periodic, solitary, and rational results will obtained.*

Keywords: $\left(\frac{G'}{G}\right)$–expansion method, Gardner-Kawahara equation, Solitary waves.
Mathematics Subject Classification: 74J35, 76B25.

1 Introduction

Here in this research, we will gain the solitary wave solution of nonlinear Gardner – Kawahara equation (1.1) in the shape [1,2]

$$\frac{\partial \psi}{\partial t} + a\frac{\partial \psi}{\partial x} + \lambda\psi\frac{\partial \psi}{\partial x} - \alpha\psi^2\frac{\partial \psi}{\partial x} + \mu\frac{\partial^3 \psi}{\partial x^3} + \beta\frac{\partial^5 \psi}{\partial x^5} = 0 \tag{1.1}$$

It is one more particular case of equation extended KdV equation

$$\frac{\partial \psi}{\partial t} + a\frac{\partial \psi}{\partial x} + \lambda\psi\frac{\partial \psi}{\partial x} + \mu\frac{\partial^3 \psi}{\partial x^3} + \beta\frac{\partial^5 \psi}{\partial x^5} - \alpha\psi^2\frac{\partial \psi}{\partial x} + \gamma_1\psi\,\frac{\partial^3 \psi}{\partial x^3} + \gamma_2\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x^2} = 0$$

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when $\gamma_1 = \gamma_2 = 0$. The extended KdV equation leads to the Kawahara equation, when $\alpha = \gamma_1 = \gamma_2 = 0$.

Eq. (1.1) happens in the notion in plasmas and in notion of shallow water waves with surface tension and notion of magneto-acoustic waves. Eq. (1.1) describing solitary-wave propagation in media, was first proposed by Kawahara in 1972 [3]. Lately, Wang et al. [4] introduced that the traveling wave results can be explained by a polynomial in $\left(\frac{G'}{G}\right)$, where G=G(η) satisfies the following second-order ordinary linear differential equation $G''(\eta) + \gamma G'(\eta) + \delta G(\eta) = 0$, where $\eta = x - kt$, and $\gamma, \delta,$ and k are constants. Actually, $\left(\frac{G'}{G}\right)$–standard method has been successfully stratified to acquire exact solution for an assortment of nonlinear evolution equations, see [5,6,7,8,9,10,11,12,13,14,15]. This report is systematized as follows: In part 2, we offer the synopsis of the $\left(\frac{G'}{G}\right)$ –expansion technique. In part 3, we explain the applications of the $\left(\frac{G'}{G}\right)$–standard method. Finally, the conclusions are present in part 4.

2 The Synopsis of the $\left(\frac{G'}{G}\right)$ –Standard Method

In this part, we explain the $\left(\frac{G'}{G}\right)$–standard method to explore traveling wave results of nonlinear equations, let us consider a nonlinear evolution equation in two variables xt in the form:

$$\mathcal{L}\,(\psi, \psi_t, \psi_x, \psi_{xt}, \psi_{tt}, \psi_{xx}, \dots \dots \dots) = 0, \tag{2.1}$$

where $\psi = \psi(x,t)$ is an unknown function and $\mathcal{L}$ is a polynomial in $\psi = \psi(x,t)$, in which highest order derivatives and nonlinear terms are involved. The major procedures of this method are presented in this research as follows:

Step 1. Collecting the separate variables x and t into one variable $\eta = x - kt$, we assume that

$$\psi(x,t) = \psi(\eta), \ \ \eta = x - kt \tag{2.2}$$

When k is constant. Replacing (2.2) into (2.1), then we will gain the following differential ordinary equation (ODE):

$$\mathcal{O}\,(\psi, k\psi', \psi', k\psi'', k^2\psi'', \psi'', \dots \dots \dots) = 0 \tag{2.3}$$

Step 2. In case of need, we integrate (2.3) as many times as possible and assume the solution of (2.3) which can be expressed of the form

$$\psi(\eta) = \sum_{i=0}^{N} b_i \left(\frac{G'}{G}\right)^i , \tag{2.4}$$

where $G = G(\eta)$ satisfies the second order linear equation (ODE)

$$G''(\eta) + \gamma G'(\eta) + \delta G(\eta) = 0, \tag{2.5}$$

Where b_i, γ and δ are real constant with $b_N \neq 0$. Next, the prime denotes the derivative respective to η .Using the general solutions of (2.5), we get

$$\left(\frac{G'}{G}\right) = \begin{cases} -\frac{\gamma}{2} + \frac{\sqrt{\gamma^2 - 4\delta}}{2}\left(\frac{r_1 \sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\} + r_2 \cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}{r_1 \cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\} + r_2 \sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}\right), & \text{when } \gamma^2 - 4\delta > 0 \\ -\frac{\gamma}{2} + \frac{\sqrt{4\delta - \gamma^2}}{2}\left(\frac{-r_1 \sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\} + r_2 \cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}{r_1 \cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\} + r_2 \sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}\right), & \text{when } \gamma^2 - 4\delta < 0 \\ \left(\frac{r_2}{r_1 + r_2\eta}\right) - \frac{\gamma}{2}, & \text{when } \gamma^2 - 4\delta = 0 \end{cases}$$

Step 3. We determine the positive integer N by considering the homogeneous balance between nonlinear terms and the highest order derivatives showing in ODE (2.3) and replacing (2.4) into (2.3), then we use the general solutions of (2.5), and summation all terms with the similar order of $\left(\frac{G'}{G}\right)$ together, next setting each coefficient of this polynomial to zero yields a group of algebraic equations for $b_i, k, y,$ and δ.

Step 4. We solve the nonlinear algebraic equations of step3 by maple to find the constants $b_i, k, \gamma,$ and δ. Substituting these values into (2.4) and using the general solutions of (2.5).

3 Applications of the Method

In this part, the $\left(\frac{G'}{G}\right)$–standard method has been used it to check the results leading to solitary wave solutions to the Gardner –Kawahara equation. In order to explore the solitary wave result of (1.1), we are applying the transformations

$$\psi(x,t) = \psi(\eta) \qquad , \qquad \eta = x - kt$$

Then, (1.1) for $\psi(x,t) = \psi(\eta)$ become

$$-k\psi' + a\psi' + \lambda\psi\psi' - \alpha\psi^2\psi' + \mu\psi''' - \beta\psi''''' = 0 \tag{3.1}$$

Integrating (3.1) with respect to η once and putting the integration constant equal to zero, we obtain

$$-k\psi + a\psi + \frac{\lambda}{2}\psi^2 - \frac{\alpha}{3}\psi^3 + \mu\psi'' - \beta\psi'''' = 0 \tag{3.2}$$

By balancing between $\psi^3$and ψ'''' we get N=2, then Eq. (3.2) has the following solution:

$$\psi(\eta) = b_0 + b_1\left(\frac{G'}{G}\right) + b_2\left(\frac{G'}{G}\right)^2, \qquad b_2 \neq 0 \tag{3.3}$$

where $b_0, b_1, and\ b_2$ are unknown constants. Substituting (3.3) along with (2.5) into (3.2) and summation each terms with the similar power of $\left(\frac{G'}{G}\right)$, the left side of (3.2) is transmuted into a polynomial in $\left(\frac{G'}{G}\right)$. Putting the coefficients of all powers of $\left(\frac{G'}{G}\right)$ to zero yields a group of nonlinear equations (3.4) for $b_0, b_1, b_2, a, \lambda, \alpha, \mu, \beta$ & γ as follows:

$$\left(\frac{G'}{G}\right)^6 : \frac{-1}{3}\alpha {b_2}^3 + 120\beta b_2 = 0$$

$$\left(\frac{G'}{G}\right)^5 : -\alpha b_1 {b_2}^2 + \beta(336 b_2\gamma + 24 b_1) = 0$$

$$\left(\frac{G'}{G}\right)^4 : \frac{-1}{3}\alpha\left(b_0 {b_2}^2 + 2{b_1}^2 b_2 + b_2\left(2b_0b_2 + {b_1}^2\right)\right) + \frac{1}{2}\lambda {b_2}^2 + 6\mu b_2 + \beta(330 b_2\gamma^2 + 60 b_1\gamma + 240 b_2\delta) = 0$$

$$\left(\frac{G'}{G}\right)^3 : \lambda b_1 b_2 - \frac{1}{3}\alpha\left(4b_0b_1b_2 + b_1\left(2b_0b_2 + {b_1}^2\right)\right) + \mu(10b_2\gamma + 2b_1) + \beta(130 b_2\gamma^3 + 50 b_1\gamma^2 + 440 b_2\gamma\delta + 40 b_1\delta) = 0$$

$$\left(\frac{G'}{G}\right)^2 : \frac{-1}{3}\alpha\left(b_0\left(2b_0b_2 + {b_1}^2\right) + 2{b_1}^2 b_2 + b_2{b_0}^2\right) - kb_2 + \beta(16b_2\gamma^4 + 15b_1\gamma^3 + b_2\gamma^2\delta + 60b_1\gamma\delta + 136b_2\delta^2) + ab_2 + \frac{1}{2}\lambda\left(2b_0b_2 + {b_1}^2\right) + \mu(4b_2\gamma^2 + 3b_1\gamma + 8b_2\delta) = 0$$

$$\left(\frac{G'}{G}\right)^1 : ab_1 - kb_1 - \alpha {b_0}^2 b_1 + \lambda b_0 b_1 + \beta(b_1\gamma^4 + 30b_2\gamma^3\delta + 22b_1\gamma^2\delta + 120b_2\gamma\delta^2 + 16b_1\delta^2) + \mu(b_1\gamma^2 + 6b_2\gamma\delta + 2b_1\delta) = 0$$

$\left(\frac{G'}{G}\right)^0 : -b_0k + \mu(b_1\gamma\delta + 2b_2\delta^2) + \frac{1}{2}{b_0}^2\lambda + b_0a + \beta(b_1\gamma^3\delta + 14b_2\gamma^2\delta^2 + 8b_1\gamma\delta^2 +$ *16b2δ3+232−13b03α=0.*

Solving the above system of algebraic equations by Maple, we get the following result:

$$b_1=\pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma \quad , \; b_2=\pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}} \quad , \qquad \delta=0 \; , \; k=k$$

$$\lambda=\frac{2}{3}\frac{3{b_0}^2\alpha\gamma^2\sqrt{10}\sqrt{\frac{\beta}{\alpha}}-12\beta\gamma^4 b_0-2{b_0}^3\alpha-696\beta}{b_0\left(\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2-b_0\right)} \; ,$$

$$\lambda=\frac{2}{3}\frac{-3{b_0}^2\alpha\gamma^2\sqrt{10}\sqrt{\frac{\beta}{\alpha}}-12\beta\gamma^4 b_0-2{b_0}^3\alpha-696\beta}{b_0\left(-\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2-b_0\right)}$$

$$\mu=\frac{-1}{3}\;\frac{3\sqrt{10}\sqrt{\frac{\beta}{\alpha}}b_0\beta\gamma^4+\sqrt{10}\sqrt{\frac{\beta}{\alpha}}{b_0}^3\alpha-15{b_0}^2\beta\gamma^2-696\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\beta}{b_0\left(\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2-b_0\right)}$$

$$\mu=\frac{-1}{3}\;\frac{-3\sqrt{10}\sqrt{\frac{\beta}{\alpha}}b_0\beta\gamma^4-\sqrt{10}\sqrt{\frac{\beta}{\alpha}}{b_0}^3\alpha-15{b_0}^2\beta\gamma^2+696\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\beta}{b_0\left(-\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2-b_0\right)}$$

$$\mathrm{a}=\frac{-1}{3}\;\frac{2\sqrt{10}\sqrt{\frac{\beta}{\alpha}}{b_0}^3\alpha\gamma^2-12{b_0}^2\beta\gamma^4-3\sqrt{10}\sqrt{\frac{\beta}{\alpha}}b_0\gamma^2k-{b_0}^4\alpha+696\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2+3{b_0}^2k-1392b_0\beta}{b_0\left(\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2-b_0\right)}$$

$$\mathrm{a}=\frac{-1}{3}\;\frac{-2\sqrt{10}\sqrt{\frac{\beta}{\alpha}}{b_0}^3\alpha\gamma^2-12{b_0}^2\beta\gamma^4+3\sqrt{10}\sqrt{\frac{\beta}{\alpha}}b_0\gamma^2k-{b_0}^4\alpha-696\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2+3{b_0}^2k-1392b_0\beta}{b_0\left(-\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2-b_0\right)}$$

Now, Eq. (3.3) becomes

$$\psi(\eta)=b_0\pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma\left(\frac{G'}{G}\right)\pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(\frac{G'}{G}\right)^2 \; , \qquad (3.5)$$

Using the public solutions of Eq. (2.5) into Eq. (3.5), we have three kinds of traveling wave solutions. When $\gamma^2-4\delta>0$, we get the hyperbolic function solution of Eq. (1.1)

$$\psi_{1,2}(\eta) = b_0 \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma\left(-\frac{\gamma}{2}+\frac{\sqrt{\gamma^2-4\delta}}{2}\left(\frac{r_1 sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}+r_2 cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}{r_1 cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}+r_2 sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}\right)\right) \pm$$

$$6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(-\frac{\gamma}{2}+\frac{\sqrt{\gamma^2-4\delta}}{2}\left(\frac{r_1 sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}+r_2 cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}{r_1 cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}+r_2 sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}\right)\right)^2 , \quad (3.6)$$

In particular, if $r_1 \neq 0$, $r_2 = 0$, $\gamma > 0$, $\delta = 0$, then Eq. (3.6) becomes

$$\psi_{1,2}(\eta) = b_0 \pm \frac{3}{2}\sqrt{\frac{10\beta}{\alpha}}\gamma^2 \pm \frac{3}{2}\sqrt{\frac{10\beta}{\alpha}}(\gamma^2-4\delta)tanh^2\left(\frac{1}{2}\gamma\eta\right), \quad (3.7)$$

When $\gamma^2 - 4\delta < 0$, we get the trigonometric function solution of Eq. (1.1)

$$\psi_{3,4}(\eta) = b_0 \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma\left(-\frac{\gamma}{2}+\frac{\sqrt{4\delta-\gamma^2}}{2}\left(\frac{-r_1 sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}+r_2 cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}{r_1 cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}+r_2 sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}\right)\right) \pm$$

$$6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(-\frac{\gamma}{2}+\frac{\sqrt{4\delta-\gamma^2}}{2}\left(\frac{r_1 sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}+r_2 cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}{r_1 cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}+r_2 sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}\right)\right)^2 , \quad (3.8)$$

If $r_1 = 0$, $r_2 \neq 0$, $\gamma = 0$, $\delta > 0$, then Eq. (3.8) becomes

$$\psi_{3,4}(\eta) = b_0 \pm 6\sqrt{\frac{10\beta}{\alpha}}\delta\cot^2(\eta) \quad , \quad (3.9)$$

when $\gamma^2 - 4\delta = 0$, we get the rational function solution of Eq. (1.1)

$$\psi_{5,6}(\eta) = b_0 \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma\left(\left(\frac{r_2}{r_1+r_2\eta}\right)-\frac{\gamma}{2}\right) \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(\left(\frac{r_2}{r_1+r_2\eta}\right)-\frac{\gamma}{2}\right)^2 , \quad (3.10)$$

If $r_1 = 0$, $r_2 \neq 0$, $\gamma = 2$, $\delta = 1$, then Eq. (3.10) becomes

$$\psi_{5,6}(\eta) = b_0 \pm 12\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(\frac{1}{\eta} - 1\right) \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(\frac{1}{\eta} - 1\right)^2, \qquad (3.11)$$

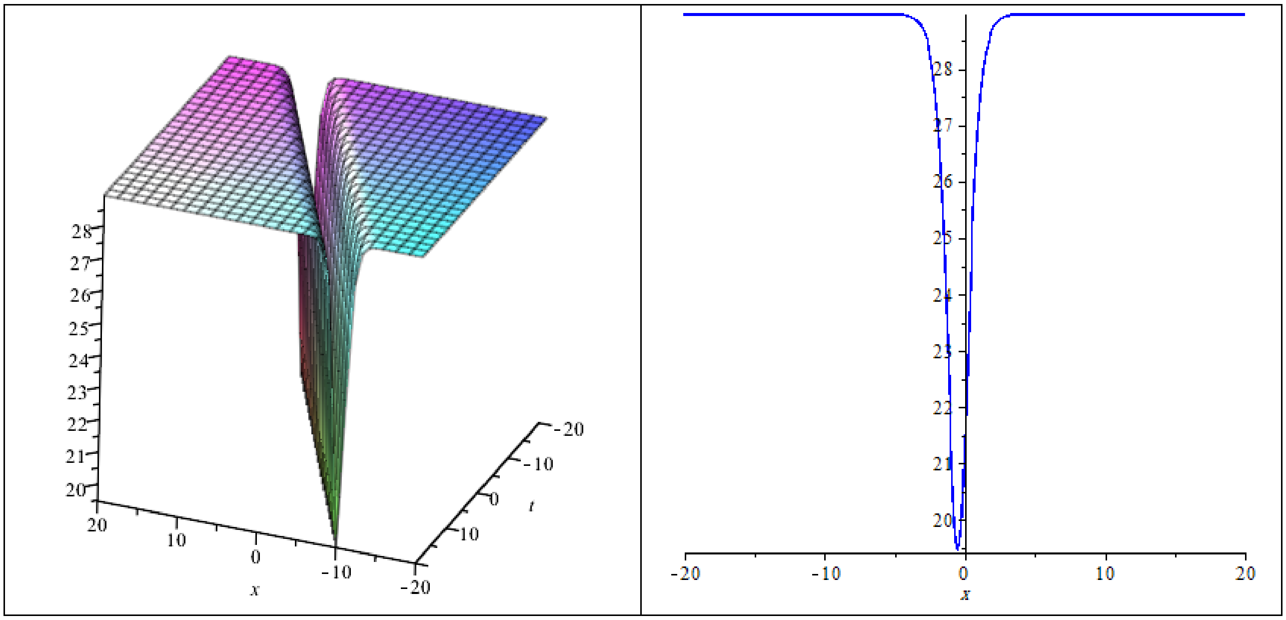

Figure 1. The wave solution given by (3.7) in 3D- and 2D-plots, when $k = -0.5$, $\beta = 1$ $\gamma = 2$, $\delta = 0.5$, $b_0 = 0.5$, $\alpha = 1$.

Remark 1. We have verified all the gained solutions by setting them back into the equations (3.4) with the aid of Maple.

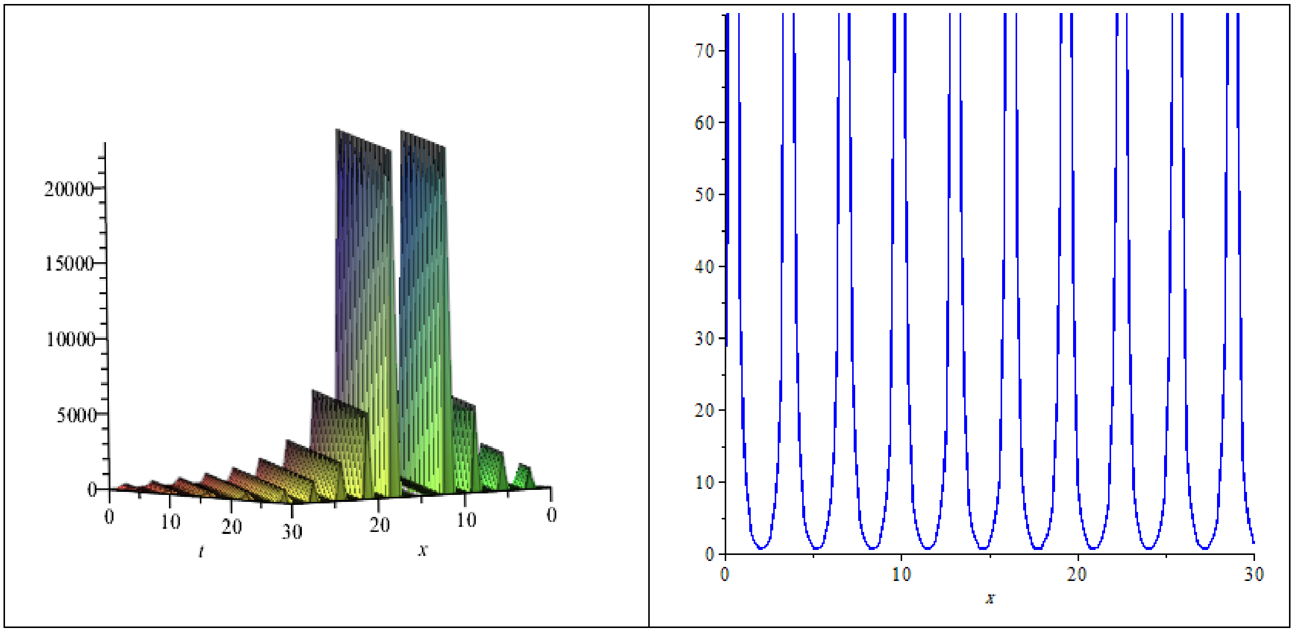

Figure 2. The wave solution given by (3.9) in 3D- and 2D-plots, when $k = 0.5, \beta = 1$ $\gamma = 2, \delta = \frac{1}{3}, b_0 = 0.5, \alpha = 1$.

4 Conclusion

In this research, the $\left(\frac{G'}{G}\right)$–standard method is efficiently and successfully utilized on the Gardner-Kawahara to find new solitary waves solutions. The kind of accurate solitary wave result is variety along with different value of appropriate choice of parameters (r_1 and r_2) . We note that the special case contains the trigonometric functions, the hyperbolic functions, and the rational functions. It is also a beneficial technique to solve other nonlinear evolution equations.

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