HACETTEPE UNIVERSITY
FACULTY OF SCIENCE
TURKEY

# HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS

A Bimonthly Publication
Volume 44 Issue 4
2015

ISSN 1303 5010

# HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS

Volume 44 Issue 4
August 2015

**A Peer Reviewed Journal**
**Published Bimonthly by the**
**Faculty of Science of Hacettepe University**

**Abstracted/Indexed in**

SCI-EXP, Journal Citation Reports, Mathematical Reviews,
Zentralblatt MATH, Current Index to Statistics,
Statistical Theory & Method Abstracts,
SCOPUS, Tübitak-Ulakbim.

ISSN 1303 5010

This Journal is typeset using LaTeX.

HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS

**Cilt 44 Sayı 4 (2015)**

ISSN 1303 – 5010

| KÜNYE |
|---|
| YAYININ ADI:<br>HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS |
| YIL : 2015 SAYI : 44 - 4 AY : Ağustos |
| YAYIN SAHİBİNİN ADI : H. Ü. Fen Fakültesi Dekanlığı adına<br>Prof. Dr. Bekir Salih |
| SORUMLU YAZI İŞL. MD. ADI : Prof. Dr. Yücel Tıraş |
| YAYIN İDARE MERKEZİ ADRESİ : H. Ü. Fen Fakültesi Dekanlığı |
| YAYIN İDARE MERKEZİ TEL. : 0 312 297 68 50 |
| YAYININ TÜRÜ : Yaygın |
| BASIMCININ ADI : Hacettepe Üniversitesi Hastaneleri Basımevi. |
| BASIMCININ ADRESİ : 06100 Sıhhıye, ANKARA. |
| BASIMCININ TEL. : 0 312 305 1020 |
| BASIM TARİHİ - YERİ : - ANKARA |

Hacettepe Journal of Mathematics and Statistics

**A Bimonthly Publication – Volume 44 Issue 4 (2015)**
ISSN 1303 – 5010

## EDITORIAL BOARD



Published by Hacettepe University
Faculty of Science

## CONTENTS

### Mathematics

# MATHEMATICS

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 761 – 766

# The fractional airy transform

Alireza Ansari *

**Abstract**

In this note, we introduce the fractional Airy transform using the higher order derivatives of the Airy function and the Airy polynomials. Then, we show that the new integral transform is coincided to the natural Airy transform in particular case of the scaling parameter. Some properties of this transform are also given.

*2000 AMS Classification:* Primary 33C10; Secondary 65R10.

**Keywords:** Airy function, Airy polynomials, Airy transform.

Received 11/03/2014 : Accepted 21/04/2014 Doi : 10.15672/HJMS.2015449440

## 1. Introduction and Preliminaries

The general solution of the Airy differential equation

$$y'' - xy = 0, \quad x \in \mathbb{R}, \tag{1.1}$$

is given by

$$y(x) = c_1 Ai(x) + c_2 Bi(x), \tag{1.2}$$

where $Ai(x)$ and $Bi(x)$ are the Airy functions of first and second kinds, respectively, such that [2, 10]

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos(xt + \frac{t^3}{3})dt, \tag{1.3}$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left(e^{xt-\frac{t^3}{3}} + \sin(xt + \frac{t^3}{3})\right) dt. \tag{1.4}$$

In view of the Airy function of first kind $Ai(x)$, Widder in [12] introduced the Airy transform with scaling parameter $\alpha$ in terms of the convolution product of Fourier transform as follows

$$\widehat{\mathcal{A}}_\alpha\{f(\xi); x\} = \frac{1}{|\alpha|}\int_{-\infty}^\infty f(\xi) Ai(\frac{x-\xi}{\alpha}) d\xi, \quad \alpha \in \mathbb{R}. \tag{1.5}$$

*Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P. O. Box 115, Shahrekord, Iran
Email: alireza_1038@yahoo.com

The inversion formula of the above transform is easily obtained via the orthogonality relation of the Airy function

$$\frac{1}{\alpha^2}\int_{-\infty}^{\infty} Ai(\frac{x-y}{\alpha})Ai(\frac{x-z}{\alpha})dx = \delta(y-z), \tag{1.6}$$

as $\widehat{\mathcal{A}}_{\alpha}^{-1} = \widehat{\mathcal{A}}_{-\alpha}$.
Later, Hunt in [6] and Bertoncini et al. in [4] used this transform in molecular physics and in evaluation of quantum transport, respectively. For more contributions of the Airy transform, for example, Babusci et al. [3] obtained the formal solution of the third order PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}, \quad u(x,0) = f(x), \quad t > 0 \tag{1.7}$$

with respect to the Airy transform of the function $f(x)$, that is

$$u(x,t) = \frac{1}{\sqrt[3]{3t}}\int_{-\infty}^{\infty} Ai(\frac{x-\xi}{\sqrt[3]{3t}})f(\xi)d\xi. \tag{1.8}$$

For more details of the above solution in the Airy diffusion equation, see [10]. As another application of this transform, Jiang et al. [5] used the two dimensional Airy transform with kernel $w_{\alpha\beta}(xy) = \frac{1}{|\alpha\beta|}Ai(\frac{x}{\alpha})Ai(\frac{y}{\beta})$ for analyzing the Airy beams in optics and Torre [9] applied the Airy transform to derive the three-variable Hermite polynomials and their generating functions. Also, Varlamov [11] obtained the Riesz fractional derivative of the product of Airy transforms and presented this product in terms of the Bessel function of zero order $J_0(x)$ and the Riesz fractional derivative of the Airy function.
Now in this paper, among the Airy transform of the elementary functions (which can be found in [10]), we concern to the function $x^n, n \in \mathbb{N}$, which its Airy transform leads us to the Airy polynomials $\mathrm{Pi}_n(x)$,

$$\widehat{\mathcal{A}}_{\alpha}\{\xi^n; x\} = \alpha^n \mathrm{Pi}_n(\frac{x}{\alpha}) = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} \xi^n Ai(\frac{x-\xi}{\alpha})d\xi. \tag{1.9}$$

Some important properties of the Airy polynomials can be written by the following relations which leads us to the definition of fractional Airy transform in Section 2.

**Property 1:** The bi-orthogonality relation between the Airy polynomials and the higher order derivatives of Airy function is given by [1]

$$\int_{-\infty}^{\infty} \mathrm{Pi}_n(x)Ai^{(n)}(x)dx = (-1)^n n!. \tag{1.10}$$

**Property 2:** The Airy transform of the Airy polynomials is given in terms of the Airy polynomials as follows [9]

$$\widehat{\mathcal{A}}_{\alpha}\{\mathrm{Pi}_n(\xi); x\} = (1+\alpha^3)^{\frac{n}{3}} \mathrm{Pi}_n\left(\frac{x}{(1+\alpha^3)^{\frac{1}{3}}}\right). \tag{1.11}$$

**Property 3:** The higher order derivatives of the Airy function in Property 1, can be simplified into the following relation [7, 8]

$$Ai^{(n)}(x) = p_n(x)Ai(x) + q_n(x)Ai'(x), \quad n \in \mathbb{N}, \tag{1.12}$$

where the polynomials $p_n$ and $q_n$ are given by the recurrence relations

$$p_{n+2}(x) = xp_n(x) + np_n(x), \tag{1.13}$$
$$q_{n+2}(x) = xq_n(x) + nq_n(x), \tag{1.14}$$
$$p_{n+1}(x) = p'_n(x) + xq_n(x), \tag{1.15}$$
$$q_{n+1}(x) = q'_n(x) + p_n(x), \tag{1.16}$$

with the generating functions formulas

$$\pi[Bi'(x)Ai(x+t) - Ai'(x)Bi(x+t)] = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \tag{1.17}$$

$$\pi[Ai(x)Bi(x+t) - Bi(x)Ai(x+t)] = \sum_{n=0}^{\infty} q_n(x)\frac{t^n}{n!}. \tag{1.18}$$

## 2. The Fractional Airy Transform

Now in this section, using the obtained results in Section 1, we introduce the fractional Airy transform. In the following lemma, the Airy function is applied as a generating function.

**2.1. Lemma.** *For $a, b \in \mathbb{R}$, the following series holds for the Airy polynomials and the higher order derivatives of Airy function*

$$\sum_{n=0}^{\infty} \mathrm{Pi}_n(bx)Ai^{(n)}(y)\frac{a^n}{n!} = \frac{1}{(a^3-1)^{\frac{1}{3}}} Ai\left(\frac{-abx-y}{(a^3-1)^{\frac{1}{3}}}\right). \tag{2.1}$$

**Proof:** Setting the relations (1.9) and (1.12) in the left hand side of (2.1), we get the above series as

$$\begin{aligned} S &= \int_{-\infty}^{\infty}\left[Ai(y)\left(\sum_{n=0}^{\infty}\frac{(a\xi)^n}{n!}p_n(y)\right) + \left(Ai'(y)\sum_{n=0}^{\infty}\frac{(a\xi)^n}{n!}q_n(y)\right)\right]Ai(bx-\xi)d\xi \\ &= \pi Ai(y)\int_{-\infty}^{\infty}\left[Bi'(y)Ai(y+a\xi) - Ai'(y)Bi(y+a\xi)\right]Ai(bx-\xi)d\xi \\ &+ \pi Ai'(y)\int_{-\infty}^{\infty}\left[Ai(y)Bi(y+a\xi) - Bi(y)Ai(y+a\xi)\right]Ai(bx-\xi)d\xi \\ &= \pi\left(Ai(y)Bi'(y) - Ai'(y)Bi(y)\right)\int_{-\infty}^{\infty}Ai(y+a\xi)Ai(bx-\xi)d\xi. \end{aligned} \tag{2.2}$$

Since the Wronskian of functions $Ai(y)$ and $Bi(y)$ is equal to $W(Ai(y), Bi(y)) = Ai(y)Bi'(y) - Ai'(y)Bi(y) = \frac{1}{\pi}$, the last integral in (2.2) is simplified to

$$S = \frac{1}{(a^3-1)^{\frac{1}{3}}} Ai\left(\frac{-abx-y}{(a^3-1)^{\frac{1}{3}}}\right), \tag{2.3}$$

where we used the following identity for simplification [10]

$$\int_{-\infty}^{\infty} Ai\left(\frac{\xi+a}{\alpha}\right)Ai\left(\frac{\xi+b}{\beta}\right)d\xi = \frac{|\alpha\beta|}{|\beta^3-\alpha^3|^{\frac{1}{3}}} Ai\left(\frac{b-a}{(\beta^3-\alpha^3)^{\frac{1}{3}}}\right),\ \beta \neq \alpha. \tag{2.4}$$

**2.2. Theorem.** *The following relation*

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi);x\} = \frac{1}{|(1+\alpha^3)^\beta - 1|}\int_{-\infty}^{\infty} Ai\left(\frac{(1+\alpha^3)^{\frac{\beta-1}{3}}x - \xi}{(1+\alpha^3)^\beta - 1}\right)f(\xi)d\xi, \tag{2.5}$$

*is the fractional Airy transform of order $\beta \in \mathbb{R}$.*

**Proof:** According to the relation (1.11), we intend to find the fractional Airy transform such that

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{\mathrm{Pi}_n(\xi);x\} = (1+\alpha^3)^{\frac{\beta n}{3}}\,\mathrm{Pi}_n\left(\frac{x}{(1+\alpha^3)^{\frac{1}{3}}}\right), \quad \beta \in \mathbb{R}. \tag{2.6}$$

For this purpose, we suppose that the function $f(x)$ can be expanded in terms of the Airy polynomials as

$$f(x) = \sum_{n=0}^{\infty} a_n \mathrm{Pi}_n(x), \tag{2.7}$$

where the coefficients $a_n$ are found from the bi-orthogonal property of Airy polynomials (1.10)

$$a_n = \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} Ai^{(n)}(y) f(y) dy. \tag{2.8}$$

Finally, on account of equation (2.6), the effect of $\widehat{\mathcal{A}}_{\alpha_\beta}$ on $f(x)$ is

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi); x\} = \sum_{n=0}^{\infty} a_n (1+\alpha^3)^{\frac{\beta n}{3}} \mathrm{Pi}_n \left( \frac{x}{(1+\alpha^3)^{\frac{1}{3}}} \right), \tag{2.9}$$

which by substituting the coefficients (2.8) into (2.9), we get $\widehat{\mathcal{A}}_{\alpha_\beta}$ as

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi); x\} = \int_{-\infty}^{\infty} \left( \sum_{n=0}^{\infty} \mathrm{Pi}_n \left( \frac{x}{(1+\alpha^3)^{\frac{1}{3}}} \right) Ai^{(n)}(\xi) \frac{[-(1+\alpha^3)^{\frac{\beta}{3}}]^n}{n!} \right) f(\xi) d\xi. \tag{2.10}$$

Now, by using Lemma 2.1 and setting $a = |-(1+\alpha^3)^{\frac{\beta}{3}}|$, $b = |\frac{1}{(1+\alpha^3)^{\frac{1}{3}}}|$, we obtain the fractional Airy transform $\widehat{\mathcal{A}}_{\alpha_\beta}$ in the following form

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi); x\} = \frac{1}{|(1+\alpha^3)^{\beta} - 1|} \int_{-\infty}^{\infty} Ai \left( \frac{(1+\alpha^3)^{\frac{\beta-1}{3}} x - \xi}{(1+\alpha^3)^{\beta} - 1} \right) f(\xi) d\xi. \tag{2.11}$$

**2.3. Remark.** It is obvious that, by setting $\beta = 1, \alpha^3 = \lambda$ in (2.5), we obtain the natural Airy transform with scaling parameter $\lambda$.

**2.4. Remark.** The relation (2.5) shows the fractional Airy transform is a natural Airy transform with the modified scaling parameter $\frac{1}{(1+\alpha^3)^{\beta}-1}$, that is

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi); x\} = \widehat{\mathcal{A}}_{\frac{1}{(1+\alpha^3)^{\beta}-1}}\{f(\xi); (1+\alpha^3)^{\frac{\beta-1}{3}} x\}. \tag{2.12}$$

**2.5. Remark.** According to the relation (1.6), the inversion formula of fractional Airy transform (2.5) is presented by

$$f(\xi) = \frac{1}{|(1+\alpha^3)^{\beta} - 1|} \int_{-\infty}^{\infty} Ai \left( \frac{(1+\alpha^3)^{\frac{\beta-1}{3}} x - \xi}{(1+\alpha^3)^{\beta} - 1} \right) \widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi); x\} dx, \tag{2.13}$$

which implies that for the value $\gamma = \frac{1}{(1+\alpha^3)^{\beta}-1}$, $\widehat{\mathcal{A}}_{\gamma}^{-1} = \widehat{\mathcal{A}}_{-\gamma}$.

**2.6. Example.** Using the Airy transform of $f(\xi) = e^{ik\xi}$ for $k \in \mathbb{R}$, [10]

$$\begin{aligned} \widehat{\mathcal{A}}_{\alpha_\beta}\{e^{ik\xi}; x\} &= \frac{1}{(1+\alpha^3)^{\beta} - 1} \int_{-\infty}^{\infty} Ai \left( \frac{(1+\alpha^3)^{\frac{\beta-1}{3}} x - \xi}{(1+\alpha^3)^{\beta} - 1} \right) e^{ik\xi} d\xi, \\ &= e^{i\left( k(1+\alpha^3)^{\frac{\beta-1}{3}} x + \frac{1}{3} \frac{1}{((1+\alpha^3)^{\beta}-1)^3} k^3 \right)}, \end{aligned} \tag{2.14}$$

the fractional Airy transform of the trigonometric functions $\sin(k\xi)$ and $\cos(k\xi)$ are given by the following relations

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{\cos(k\xi); x\} = \cos \left( k(1+\alpha^3)^{\frac{\beta-1}{3}} x + \frac{1}{3} \frac{1}{((1+\alpha^3)^{\beta} - 1)^3} k^3 \right), \tag{2.15}$$

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{\sin(k\xi);x\} = \sin\left(k(1+\alpha^3)^{\frac{\beta-1}{3}}x + \frac{1}{3}\frac{1}{((1+\alpha^3)^\beta-1)^3}k^3\right). \tag{2.16}$$

**2.7. Corollary.** *Let $F_{\alpha_\beta}(x)$ be the fractional Airy transform of $f(\xi)$, then the fractional Airy transform of $\xi f(\xi)$ is*

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{\xi f(\xi);x\} = \delta x F_{\alpha_\beta}(x) - \frac{\gamma^3}{\delta^2}F''_{\alpha_\beta}(x), \tag{2.17}$$

*where the parameters $\gamma$ and $\delta$ are given by*

$$\gamma = \frac{1}{(1+\alpha^3)^\beta - 1}, \quad \delta = (1+\alpha^3)^{\frac{\beta-1}{3}}. \tag{2.18}$$

**Proof:** According to the definition of fractional Airy transform of $F''_{\alpha_\beta}(x)$ and relation (1.1), we easily arrive at (2.17).

**2.8. Example.** Setting $f(\xi) = \frac{1}{\xi}$ in Corollary 2.7 and using the fact that $\widehat{\mathcal{A}}_{\alpha_\beta}\{1;x\} = 1$, we get the fractional Airy transform of $f(\xi) = \frac{1}{\xi}$ in terms of the Scorer function [10]

$$Gi(x) = \frac{1}{\pi}\int_0^\infty \sin(xt + \frac{t^3}{3})dt, \tag{2.19}$$

as follows

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{\frac{1}{\xi};x\} = \frac{\pi}{\gamma}Gi(\frac{\delta}{\gamma}x). \tag{2.20}$$

**2.9. Corollary.** *The Parseval identity for the fractional Airy transform is*

$$\int_{-\infty}^{\infty} f(\xi)g(\xi)d\xi = \int_{-\infty}^{\infty} \widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi);x\}\widehat{\mathcal{A}}_{\alpha_\beta}\{g(\xi);x\}dx. \tag{2.21}$$

**Proof:** Using the orthogonality relation of the Airy function (1.6), the proof is completed.

**2.10. Example.** Setting $f(\xi) = \frac{1}{\xi}$ and $g(\xi) = \sin(k\xi)$ in Corollary 2.9, and using the relations (2.16) and (2.23), we get the value of following well-known integral in terms of the Scorer function [10]

$$\int_{-\infty}^{\infty} Gi(\frac{\delta}{\gamma}x)\sin\left(k\delta x + \frac{k^3}{3\gamma^3}\right)dx = \gamma\,\mathrm{sgn}(k), \tag{2.22}$$

where we used the following fact for computation

$$\int_{-\infty}^{\infty} \frac{\sin(k\xi)}{\xi}d\xi = \pi\mathrm{sgn}(k). \tag{2.23}$$

## 3. Concluding Remarks

This paper provides a fractionalization form of the Airy transform. On the base of the Airy polynomials, we introduced a generalized form of the natural Airy transform and named it as the fractional Airy transform. Some properties of this transform such as transformations of elementary functions and Parseval identity were also obtained. It is hope that the employed integral transform can be considered as a promising approach in a fairly wide context of applied mathematics and physics in near future.

## Acknowledgements

The author would like to thank the Center of Excellence for Mathematics, Shahrekord university for financial support.

## References


[1] A. Ansari, H. Askari, *On fractional calculus of $\mathcal{A}_{2n+1}(x)$ function*, Appl. Math. Comput., 232 (2014) 487-497.

[2] A. Ansari, M. R. Masomi, *Some results involving integral transforms of Airy functions*, Int. Transf. Spec. Func., 26(7) (2015) 539-549.

[3] D. Babusci, G. Dattoli, D. Sacchetti, *The Airy transform and associated polynomials*, Cent. Eur. J. Phys., 9(6) (2011) 1381-1386.

[4] R. Bertoncini, A.M. Kriman, D.K. Ferry, A*iry-coordinate technique for nonequilibrium Green's function approach to high-field quantum transport*, Phys. Rev. B, 41 (1990) 1390-1400.

[5] Y. Jiang, K. Huang, X. Lu, *The optical Airy transform and its application in generating and controlling the Airy beam*, Optics Commun., 285 (2012) 4840-4843.

[6] P.M. Hunt, *A continuum basis of Airy functions matrix elements and a test calculation*, Mol. Phys., 44 (1981), 653-663.

[7] B. J. Laurenzi, *Polynomials associated with the higher derivatives of the Airy functions $Ai(z)$ and $Ai'(z)$*, arXiv:1110.2025v1 [math-ph], 2011.

[8] P.A. Maurone, A.J. Phares, *On the asymptotic behavior of the derivative of Airy functions*, J. Math. Phys., 20 (1979) 2191.

[9] A. Torre, *Airy polynomials, three-variable Hermite polynomials and the paraxial wave equation*, J. Optics A: Pure Appl. Optics, 14 (2012) 045704, 24pp.

[10] O. Vallee, M. Soares, *Airy Functions and Applications to Physics,* Imperial College Press, London, 2004.

[11] V. Varlamov, *Riesz fractional derivatives of the product of Airy transforms*, Phys. Scripta, T136 (2009) 014004, 5pp.

[12] D. V. Widder, *The Airy transform*, American Math. Mon., 86(4) (1979), 271-277.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 767 – 774

# The geometry of tangent conjugate connections

Adara M. Blaga * and Mircea Crasmareanu †

**Abstract**

The notion of conjugate connection is introduced in the almost tangent geometry and its properties are studied from a global point of view. Two variants for this type of connections are also considered in order to find the linear connections making parallel a given almost tangent structure.

*2000 AMS Classification:* 53C15; 53C05; 53C10; 53C07.

**Keywords:** almost tangent structure; conjugate linear connection.

Received 22/02/2013 : Accepted 25/03/2014 Doi : 10.15672/HJMS.2015449431

## Introduction

Let $F$ be a tensor field of $(1, 1)$-type on a given smooth manifold $M$. An interesting object in the geometry of pair $(M, F)$ is provided by the class of $F$-linear connections i.e. linear connections $\nabla$ making $F$ parallel: $\nabla F = 0$. In order to determine this class, in [9] is introduced the notion of $F$-conjugate connection associated to a fixed (non-necessary $F$-connection) $\nabla$. By denoting $\nabla^{(F)}$ this $F$-conjugate connection we have studied the geometry of $(M, F, \nabla, \nabla^{(F)})$ until now for two cases: almost complex structures in [1] and almost product structures in [2].

The present work is devoted to another remarkable type of tensor fields of $(1, 1)$-type, namely *almost tangent structures*. These structures were introduced by Clark and Bruckheimer [5] and Eliopoulos [10] around 1960 and have been investigated by several authors, see [3], [6]-[8], [16], [18]. As it is well-known, the tangent bundle of a manifold carries a canonical integrable almost tangent structure, hence the name. This tangent structure plays an important rôle in the Lagrangian description of analytical mechanics, [7]-[8], [12].

Recall that we are interested in the class of $J$-linear connections since, according to [15, p. 120], the existence of a symmetric (torsion-free) one in this class implies the

*Department of Mathematics and Computer Science, West University of Timişoara, Bld. V. Pârvan nr. 4, 300223, Timişoara, România
Email: adarablaga@yahoo.com
†Faculty of Mathematics, University "Al. I. Cuza", 700506, Iaşi, România
Email:mcrasm@uaic.ro

integrability of $J$ in the sense of $G$-structures as is discussed below; for example, $J$-linear connections of Levi-Civita type are studied in [11]. An important difference between the former structures (almost complex, almost product) and the later (almost tangent) is given by the fact that an almost tangent structure $J$ is a degenerate tensor field due to its nilpotence $J^2 = 0$, see the following Section. An example where this difference is obvious is the duality property $(\nabla^{(F)})^{(F)} = \nabla$ which holds for a non-degenerate $F$ while for almost tangent structures we have ii) of our Proposition 2.1.

The content of paper is as follows. After a short survey in almost tangent geometry we introduce the tangent conjugate connection $\nabla^{(J)}$ in Section 2 following the pattern of [1]-[2]. Its properties are studied following the same way as in the cited papers; for example the difference $\nabla^{(J)} - \nabla$ is expressed again in terms of two tensor fields of $(1,2)$-types called *structural* and *virtual* tensor fields. We study also the behavior of the tangent conjugate connections for a family of anti-commuting almost tangent structures. In the last two Sections we generalize $\nabla^{(J)}$, firstly through an exponential process and secondly with a general tensor field of $(1,2)$-type.

## 1. Almost tangent geometry revisited

Let $M$ be a smooth, $m$-dimensional real manifold for which we denote: $C^{\infty}(M)$-the real algebra of smooth real functions on $M$, $\Gamma(TM)$-the Lie algebra of vector fields on $M$, $T^r_s(M)$-the $C^{\infty}(M)$-module of tensor fields of $(r,s)$-type on $M$. An element of $T^1_1(M)$ is usually called *vector* 1-*form* or *affinor*.

Recall the concept of almost tangent geometry:

**1.1. Definition.** $J \in T^1_1(M)$ is called *almost tangent structure* on $M$ if it has constant rank and:

$$ImJ = \ker J. \tag{1.1}$$

The pair $(M, J)$ is called *almost tangent manifold*.

The name is motivated by the fact that (1.1) implies the nilpotence $J^2 = 0$ exactly as the natural tangent structure of tangent bundles. Denoting $rankJ = n$ it results $m = 2n$. If in addition, we suppose that $J$ is integrable i.e.:

$$N_J(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] = 0 \tag{1.2}$$

then $J$ is called *tangent structure* and $(M, J)$ is called *tangent manifold*.

From [17, p. 3246] we get some features of tangent manifolds:
(i) the distribution $ImJ$ $(= \ker J)$ defines a foliation denoted $V(M)$ and called *the vertical distribution*.

**1.2. Example.** $M = \mathbb{R}^2$, $J_e(x,y) = (0,x)$ is a tangent structure with $\ker J_e$ the $Y$-axis, hence the name. The subscript $e$ comes from "Euclidean".
(ii) there exists an atlas on $M$ with local coordinates $(x,y) = \left(x^i, y^i\right)_{1 \leq i \leq n}$ such that $J = \frac{\partial}{\partial y^i} \otimes dx^i$ i.e.:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{1.3}$$

We call *canonical coordinates* the above $(x,y)$ and the change of canonical coordinates $(x,y) \to (\widetilde{x}, \widetilde{y})$ is given by:

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x) \\ \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^a} y^a + B^i(x). \end{cases} \tag{1.4}$$

It results an alternative description in terms of $G$-structures. Namely, a tangent structure is a $G$-structure with:

$$G = \{C = \begin{pmatrix} A & O_n \\ B & A \end{pmatrix} \in GL(2n, \mathbb{R}); \quad A \in GL(n, \mathbb{R}), B \in gl(n, \mathbb{R})\} \tag{1.5}$$

and $G$ is the invariance group of matrix $J = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$ i.e. $C \in G$ if and only if $C \cdot J = J \cdot C$.

The natural almost tangent structure $J$ of $M = TN$ is an example of tangent structure having exactly the expression (1.3) if $(x^i)$ are the coordinates on $N$ and $(y^i)$ are the coordinates in the fibers of $TN \to N$. Also, $J_e$ of Example 1.2 has the above expression (1.3) with $n = 1$, whence it is integrable. A third class of examples is obtained by duality: if $J$ is an (integrable) endomorphism with $J^2 = 0$ then its dual $J^* : \Gamma(T^*M) \to \Gamma(T^*M)$, given by $J^*\alpha := \alpha \circ J$ for $\alpha \in \Gamma(T^*M)$, is (integrable) endomorphism with $(J^*)^2 = 0$.

## 2. Basic properties of tangent conjugate connections

Let $\nabla$ be a linear connection on the almost tangent manifold $(M, J)$ and define the *tangent conjugate connection* of $\nabla$ by:

$$\nabla^{(J)} := \nabla - J \circ \nabla J. \tag{2.1}$$

Remark that $\nabla^{(J)}$ coincides with $\nabla$ if and only if $\nabla J \subseteq \ker J = ImJ$ which means the inclusion $\nabla(\Gamma(TM) \times \ker J) \subseteq \ker J = ImJ$, in particular if $\nabla$ is a $J$-linear connection; for another case see i) of Proposition 2.3. For any $X, Y \in \Gamma(TM)$ we get:

$$\nabla^{(J)}_X Y = \nabla_X Y - J(\nabla_X JY). \tag{2.2}$$

A first set of properties for this linear connection are given by:

**2.1. Proposition.** *The tangent conjugate connection $\nabla^{(J)}$ satisfies:*
*i) $\nabla^{(J)} J = \nabla J$, which means that $\nabla$ and $\nabla^{(J)}$ are simultaneous $J$-linear connections or not;*
*ii) $\nabla^{2(J)} =: (\nabla^{(J)})^{(J)} = 2\nabla^{(J)} - \nabla$; more generally $\nabla^{n(J)} = n\nabla^{(J)} - (n-1)\nabla$ for $n \in \mathbb{N}^*$;*
*iii) its torsion is $T_{\nabla^{(J)}} = T_\nabla - J \circ d^\nabla J$ where $d^\nabla$ is the exterior covariant derivative induced by $\nabla$, namely $(d^\nabla J)(X, Y) := (\nabla_X J)Y - (\nabla_Y J)X$;*
*iv) its curvature is*

$$\begin{aligned} R_{\nabla^{(J)}}(X, Y, Z) = R_\nabla(X, Y, Z) - \nabla_X J(\nabla_Y JZ) + \nabla_Y J(\nabla_X JZ) - \\ -J[\nabla_X J(\nabla_Y Z) - \nabla_Y J(\nabla_X Z) - \nabla_{[X,Y]} JZ]. \end{aligned} \tag{2.3}$$

*In particular:*

$$R_{\nabla^{(J)}}(X, Y, JZ) = R_\nabla(X, Y, JZ) - J[\nabla_X J(\nabla_Y JZ) - \nabla_Y J(\nabla_X JZ)]. \tag{2.4}$$

*Proof* The general part of ii) follows by induction while for iii) a direct calculus yields $T_{\nabla^{(J)}}(X, Y) = T_\nabla(X, Y) - J(\nabla_X JY - \nabla_Y JX)$. □

Let $f : M \to M$ be a *tangentomorphism*, that is an automorphism of the $G$-structure defined by $J$:

$$f_* \circ J = J \circ f_*. \tag{2.5}$$

Recall that $f$ is an *affine transformation for* $\nabla$ if for any $X, Y \in \Gamma(TM)$:

$$f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y. \tag{2.6}$$

These notions are connected by:

**2.2. Proposition.** *If the tangentomorphism $f$ is an affine transformation for $\nabla$ then $f$ is also affine transformation for $\nabla^{(J)}$.*

*Proof* We have:

$$f_*(\nabla^{(J)}_X Y) = f_*(\nabla_X Y) - (f_* \circ J)(\nabla_X JY) = \nabla_{f_*X} f_*Y - J(f_*(\nabla_X JY)) =$$

$$= \nabla_{f_*X} f_*Y - J((\nabla_{f_*X} f_*(JY))) = \nabla_{f_*X} f_*Y - J((\nabla_{f_*X} J(f_*Y))) = \nabla^{(J)}_{f_*X} f_*Y$$

which yields the conclusion. $\square$

A second class of properties for the tangent conjugate connection is provided by:

**2.3. Proposition.** *i) If $J$ is $\nabla$-recurrent i.e. $\nabla J = \eta \otimes J$ for $\eta$ a 1-form, then $\nabla^{(J)} = \nabla$. ii) If $\nabla$ is symmetric and $\nabla J = \eta \otimes I$ then $\nabla^{(J)} = \nabla - \eta \otimes J$ and $\nabla^{(J)}$ is a quarter-symmetric connection.*

*Proof* i) In this case we have $J \circ \nabla J = 0$.
ii) Recall after [1, p. 122] that the quarter-symmetry means the existence of a 1-form $\pi$ and a tensor field $F$ of $(1,1)$-type such that $T_{\nabla^{(J)}} = F \wedge \pi := F \otimes \pi - \pi \otimes F$. From Proposition 2.1 we have $T_{\nabla^{(J)}}(X,Y) = T_\nabla(X,Y) - \eta(X)JY + \eta(Y)JX$, and the hypothesis $T_\nabla = 0$ yields the previous equation with $F = J$ and $\pi = \eta$. $\square$

**2.4. Example.** Let $N$ be a smooth $n$-dimensional manifold and $M = TN$ its tangent bundle; hence $m = 2n$. Let $\{x^i; 1 \le i \le n\}$ be a local system of coordinates on $N$ and consider its lift to $M$ given by $\{x^i, y^i; 1 \le i \le n\}$ with $y^i$ the coordinates on the fibres of $TN$. The canonical almost tangent structure $J$ of $M$ has the local expression (1.3) and it is integrable. Fix a general linear connection $\nabla$ on $M$ with local Christoffel symbols $\Gamma$ as follows:

$$\begin{cases} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^{(1)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(2)k}_{ij} \frac{\partial}{\partial y^k} \\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \Gamma^{(3)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(4)k}_{ij} \frac{\partial}{\partial y^k} \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^j} = \Gamma^{(5)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(6)k}_{ij} \frac{\partial}{\partial y^k} \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \Gamma^{(7)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(8)k}_{ij} \frac{\partial}{\partial y^k}. \end{cases} \tag{2.7}$$

Then its tangent conjugate connection has the expression:

$$\begin{cases} \nabla^{(J)}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^{(1)k}_{ij} \frac{\partial}{\partial x^k} + \left(\Gamma^{(2)k}_{ij} - \Gamma^{(3)k}_{ij}\right) \frac{\partial}{\partial y^k} \\ \nabla^{(J)}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \Gamma^{(3)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(4)k}_{ij} \frac{\partial}{\partial y^k} \\ \nabla^{(J)}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^j} = \Gamma^{(5)k}_{ij} \frac{\partial}{\partial x^k} + \left(\Gamma^{(6)k}_{ij} - \Gamma^{(7)k}_{ij}\right) \frac{\partial}{\partial y^k} \\ \nabla^{(J)}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \Gamma^{(7)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(8)k}_{ij} \frac{\partial}{\partial y^k}. \end{cases} \tag{2.8}$$

A special case is important in applications: the initial connection $\nabla$ is called *distinguished* or *d-connection* if it preserves the linear structure of the fibres of $M$ which means that:

$$\Gamma^{(2)} = \Gamma^{(3)} = \Gamma^{(6)} = \Gamma^{(7)} = 0. \tag{2.9}$$

It results that $\nabla$ is a $J$-connection and then its tangent conjugate connection is $\nabla^{(J)} = \nabla$.

## 3. The structural and the virtual tensor fields

Remark that the tangent conjugate connection $\nabla^{(J)}$ of $\nabla$ can be written in another form as:

$$\nabla^{(J)} = \nabla + C^J_\nabla - B^J_\nabla \tag{3.1}$$

where:

$$\begin{cases} C^J_\nabla(X,Y) := \frac{1}{2}[(\nabla_{JX}J)Y + (\nabla_X J)JY] \\ B^J_\nabla(X,Y) := \frac{1}{2}[(\nabla_{JX}J)Y - (\nabla_X J)JY]. \end{cases} \tag{3.2}$$

which we call respectively, the *structural* and the *virtual tensor field* of $\nabla$. We obtain also the following expressions for them:

$$\begin{cases} C^J_\nabla(X,Y) = \frac{1}{2}[\nabla_{JX}JY - J(\nabla_{JX}Y + \nabla_X JY)] \\ B^J_\nabla(X,Y) = \frac{1}{2}[\nabla_{JX}JY - J(\nabla_{JX}Y - \nabla_X JY)]. \end{cases} \tag{3.3}$$

We notice that they satisfy the following properties:

$$\begin{cases} C^J_\nabla(JX,Y) = C^J_\nabla(X,JY) = -\frac{1}{2}J(\nabla_{JX}JY); \quad C^J_\nabla(JX,JY) = 0 \\ B^J_\nabla(JX,Y) = -B^J_\nabla(X,JY) = \frac{1}{2}J(\nabla_{JX}JY); \quad B^J_\nabla(JX,JY) = 0 \\ C^J_\nabla(JX,Y) = -B^J_\nabla(JX,Y) \end{cases} \tag{3.4}$$

and the skew-symmetry $(3.4_2)$ means that $B^J_\nabla(J\cdot,\cdot)$ is a vectorial 2-form. Another important property is that these tensor fields are invariant with respect to $J$-conjugation of linear connections:

$$C^J_{\nabla^{(J)}} = C^J_\nabla; \quad B^J_{\nabla^{(J)}} = B^J_\nabla. \tag{3.5}$$

With respect to the invariance of these associated tensor fields under projective changes we get that only $C^J$ is invariant:

**3.1. Proposition.** *Let $\nabla$ and $\nabla'$ be two linear projectively equivalent connections:*

$$\nabla' = \nabla + \eta \otimes I + I \otimes \eta \tag{3.6}$$

*for $\eta$ a 1-form. Then $C^J_{\nabla'} = C^J_\nabla$ and $B^J_{\nabla'} = B^J_\nabla + J \otimes (\eta \circ J)$ while the tangent conjugate connection $\nabla'^{(J)}$ of $\nabla'$ satisfies:*

$$\nabla'^{(J)} = \nabla^{(J)} + \eta \otimes I + I \otimes \eta - J \otimes (\eta \circ J) \tag{3.7}$$

*and so it is not invariant under projective equivalence.*

*Proof* Follows form a direct computation. □

## 4. Invariant distributions

Let $\mathcal{D} \subset TM$ be a fixed distribution considered as a vector subbundle of $TM$. As usually, we denote by $\Gamma(\mathcal{D})$ its $C^\infty(M)$-module of sections.

**4.1. Definition.** i) $\mathcal{D}$ is called *$J$-invariant* if $X \in \Gamma(\mathcal{D})$ implies $JX \in \Gamma(\mathcal{D})$.
ii) The linear connection $\nabla$ *restricts to* $\mathcal{D}$ if $Y \in \Gamma(\mathcal{D})$ implies $\nabla_X Y \in \Gamma(\mathcal{D})$ for any $X \in \Gamma(TM)$.

**4.2. Example.** The distribution $\mathcal{D}_J = \ker J = ImJ$ is $J$-invariant.

If $\nabla$ restricts to $\mathcal{D}$ then it may be considered as a connection in the vector bundle $\mathcal{D}$. From this fact, a connection which restricts to $\mathcal{D}$ is called sometimes *adapted to* $\mathcal{D}$.

**4.3. Proposition.** *If the distribution $\mathcal{D}$ is $J$-invariant and the linear connection $\nabla$ restricts to $\mathcal{D}$ then $\nabla^{(J)}$ also restricts to $\mathcal{D}$.*

*Proof* Fix $Y \in \Gamma(\mathcal{D})$. Then $JY \in \Gamma(\mathcal{D})$ and for any $X \in \Gamma(TM)$ we have $\nabla_X Y$, $\nabla_X JY \in \Gamma(\mathcal{D})$. Therefore, $J(\nabla_X JY) \in \Gamma(\mathcal{D})$ and so $\nabla_X^{(J)} Y = \nabla_X Y - J(\nabla_X JY) \in \Gamma(\mathcal{D})$. $\square$

**4.4. Example.** Returning to Example 4.2 we have that $\nabla_X = \nabla_X^{(J)}$ on $\mathcal{D}_J = \ker J = ImJ$.

A more general notion like restricting to a distribution is that of geodesically invariance [4, p. 118]. The distribution $\mathcal{D}$ is $\nabla$-*geodesically invariant* if for every geodesic $\gamma : [a, b] \to M$ of $\nabla$ with $\dot{\gamma}(a) \in \mathcal{D}_{\gamma(a)}$ it follows $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for any $t \in [a, b]$. The cited book gives a necessary and sufficient condition for a distribution $\mathcal{D}$ to be $\nabla$-geodesically invariant: for any $X$, $Y \in \Gamma(\mathcal{D})$, *the symmetric product* $\langle X : Y \rangle_\nabla := \nabla_X Y + \nabla_Y X$ to belong to $\Gamma(\mathcal{D})$ or equivalently, for any $X \in \Gamma(\mathcal{D})$ to have $\nabla_X X \in \Gamma(\mathcal{D})$.

A direct computation gives:

$$\langle \cdot : \cdot \rangle_{\nabla^{(J)}} = \langle \cdot : \cdot \rangle_\nabla - J \circ d^\nabla J \tag{4.1}$$

and then the $\nabla$-geodesically invariance and $\nabla^{(J)}$-geodesically invariance for $\mathcal{D}$ coincides if and only if $J \circ d^\nabla J$ is zero on $\mathcal{D} \times \mathcal{D}$. In particular, $\mathcal{D}_J$ is $\nabla$-geodesically invariant if and only if is $\nabla^{(J)}$-geodesically invariant.

## 5. Affine combination of tangent conjugate connections

In what follows we shall see what happens to the tangent conjugate connection for families of almost tangent structures. Let $J_1$, $J_2$ be two almost tangent structures; conditions for their simultaneous integrability are given in [13]-[14]. Then for any $a$, $b \in \mathbb{R}$ the tensor field $J_{ab} := aJ_1 + bJ_2$ is an almost tangent structure if and only if $J_1 J_2 = -J_2 J_1$. Then its tangent conjugate connection is given by:

$$\nabla_X^{(J_{ab})} Y = a^2 \nabla_X^{(J_1)} Y + b^2 \nabla_X^{(J_2)} Y + (1 - a^2 - b^2)\nabla_X Y - ab[J_1(\nabla_X J_2 Y) + J_2(\nabla_X J_1 Y)]. \tag{5.1}$$

**5.1. Proposition.** *Let $\nabla$ be a linear connection and $J_1$ and $J_2$ two anti-commuting almost tangent structures. If $(\nabla, J_1, J_2)$ is a mixed-recurrent structure i.e. $\nabla J_i = \eta \otimes J_j$ for $i \neq j$ then $\nabla$ is the average of the two tangent conjugate connections:*

$$\nabla = \frac{1}{2}[\nabla^{(J_1)} + \nabla^{(J_2)}] \tag{5.2}$$

*and $\nabla^{(J_{ab})}$ is an affine combination of them:*

$$\nabla^{(J_{ab})} = \frac{1 + a^2 - b^2}{2}\nabla^{(J_1)} + \frac{1 - a^2 + b^2}{2}\nabla^{(J_2)}. \tag{5.3}$$

*Proof* Applying $J_i$ to $\nabla_X J_i Y - J_i(\nabla_X Y) = \eta(X) J_j Y$ with $i \neq j$ and the anti-commuting hypothesis we obtain:

$$J_1(\nabla_X J_1 Y) = -J_2(\nabla_X J_2 Y). \tag{5.4}$$

Summing the expression of the tangent conjugate connections we get (5.2) and from a previous computation, the relation (5.3). $\square$

## 6. Exponential tangent conjugate connections

For $\theta$ a real number we define the *exponential tangent conjugate connection* of $\nabla$ as:

$$\nabla^{(J,\theta)} := \nabla - \exp(-\theta J) \circ \nabla \circ \exp(\theta J) \tag{6.1}$$

where $\exp(\pm\theta J) := \cos(\theta) \cdot I \pm \sin(\theta) \cdot J$. Explicitly we get:

$$\nabla^{(J,\theta)} = \sin^2(\theta)\nabla - \frac{1}{2}\sin(2\theta)\nabla J + \sin^2(\theta) J \circ \nabla J = 2\sin^2(\theta)\nabla - \frac{1}{2}\sin(2\theta)\nabla J - \sin^2(\theta)\nabla^{(J)} \tag{6.2}$$

and then:

$$\nabla^{(J,\theta)} J = \sin^2(\theta)\nabla J + \frac{1}{2}\sin(2\theta) J \circ \nabla J. \tag{6.3}$$

It follows:

**6.1. Proposition.** *Let $\nabla$ be a symmetric linear connection.*
*i) If $J$ is $\nabla$-recurrent with $\eta$ the 1-form of recurrence then:*

$$\nabla^{(J,\theta)} = \sin^2(\theta)\nabla - \frac{1}{2}\sin(2\theta) \cdot \eta \otimes J \tag{6.4}$$

*and $\nabla^{(J,\theta)}$ is a quarter-symmetric connection.*
*ii) If $\nabla J = \eta \otimes I$ then:*

$$\nabla^{(J,\theta)} = \sin^2(\theta)\nabla - \sin(\theta) \cdot \eta \otimes \exp(-\theta J) \tag{6.5}$$

*and:*

$$T_{\nabla^{(J,\theta)}} = \sin(\theta) \otimes \exp(-\theta J) \wedge \eta. \tag{6.6}$$

*Proof* i) Follows from the fact that the hypothesis implies $J \circ \nabla J = 0$. The quarter-symmetry elements are $F = J$ and $\pi = \sin(\theta)\cos(\theta) \cdot \eta$.
ii) From $\cos(\theta) \cdot \eta \otimes I - \sin(\theta) \cdot \eta \otimes J = \eta \otimes \exp(-\theta J)$ we get:
$T_{\nabla^{(J,\theta)}} = -\sin(\theta) \cdot [\eta \otimes \exp(-\theta J) - \exp(-\theta J) \otimes \eta]$. □

## 7. Generalized tangent conjugate connections

In this section we present a natural generalization of the tangent conjugate connection.

**7.1. Definition.** A *generalized tangent conjugate connection* of $\nabla$ is:

$$\nabla^{(J,C)} = \nabla^{(J)} + C \tag{7.1}$$

with $C \in T^1_2(M)$ an arbitrary $(1,2)$-tensor field.

Let us search for tensor fields $C$ such that the duality $(\nabla^{(J,C)})^{(J,C)} = 2\nabla^{(J,C)} - \nabla$ holds as is given by Proposition 2.1. It results that we are interested in finding solutions $C$ to the equation:

$$J(C(X, JY)) = 2C(X, Y) \tag{7.2}$$

for all $X, Y \in \Gamma(TM)$ and let us remark that: i) $C_0 = 0$ is a particular solution of (7.2); ii) applying $J$ to (7.2) gives that $ImC \subseteq \ker J = ImJ$. Then returning to (7.2) it follows from the left-hand-side that $C_0$ is the unique solution of (7.2).

Also, we have:

$$\nabla^{(J,C)} J = \nabla^{(J)} J + C(\cdot, J\cdot) - J \circ C \tag{7.3}$$

and then:
i) $\nabla^{(J,C)} J = \nabla J$ as in i) of Proposition 2.1 if and only if: $C(\cdot, J\cdot) = J \circ C(\cdot, \cdot)$,
ii) $\nabla^{(J,C)}$ is a $J$-linear connection if and only if:

$$\nabla J + C(\cdot, J\cdot) = J \circ C. \tag{7.4}$$

**Acknowledgement.** The first author acknowledges the support by the research grant PN-II-ID-PCE-2011-3-0921.

## References

[1] A. M. Blaga; M. Crasmareanu, *The geometry of complex conjugate connections*, Hacet. J. Math. Stat., 41(2012), no. 1, 119-126. MR2976917

[2] A. M. Blaga; M. Crasmareanu, *The geometry of product conjugate connections*, An. Stiint. Univ. Al. I. Cuza Iaşi Mat. (N. S.), 59(2013), no. 1, 73-84. MR3098381

[3] F. Brickell; R. S. Clark, *Integrable almost tangent structures*, J. Diff. Geom., 9(1974), 557-563. MR0348666 (50 #1163)

[4] F. Bullo; A. D. Lewis, *Geometric control of mechanical systems. Modeling, analysis, and design for simple mechanical control systems*, Texts in Applied Mathematics, 49, Springer-Verlag, New York, 2005. MR2099139 (2005h:70030)

[5] R. S. Clark; M. Bruckheimer, *Sur les structures presque tangents*, C. R. A. S. Paris, 251(1960), 627-629. MR0115181 (22 #5983)

[6] R. S. Clark; D. S. Goel, *On the geometry of an almost tangent manifold*, Tensor, 24(1972), 243-252. MR0326613 (48 #4956)

[7] M. Crampin, *Defining Euler-Lagrange fields in terms of almost tangent structures*, Phys. Lett. A, 95(1983), no. 9, 466-468. MR0708702 (84k:58072)

[8] M. Crampin; G. Thompson, *Affine bundles and integrable almost tangent structures*, Math. Proc. Camb. Phil. Soc., 98(1985), 61-71. MR0789719 (86g:53039)

[9] V. Cruceanu, *Connexions compatibles avec certaines structures sur un fibré vectoriel banachique*, Czechoslovak Math. J., 24(99)(1974), 126-142. MR0353356 (50 #5840)

[10] H. A. Eliopoulos, *Structures presque tangents sur les variétés différentiables*, C. R. A. S. Paris, 255(1962), 1563-1565. MR0142078 (25 #5472)

[11] D. S. Goel, *Selfadjoint metrics on almost tangent manifolds whose Riemannian connection is almost tangent*, Canad. Math. Bull., 17(1974/75), no. 5, 671-674. MR0383288 (52 #4169)

[12] J. Grifone, *Structure presque-tangente et connexions*, I, II. Ann. Inst. Fourier (Grenoble), 22(1972), no. 1, 3, 287-334, 291-338. MR0336636 (49 #1409), MR0341361 (49 #6112)

[13] V. Kubát, *Simultaneous integrability of two $J$-related almost tangent structures*, Comment. Math. Univ. Carolin. 20(1979), no. 3, 461-473. MR0550448 (80m:53034)

[14] V. Kubát, *On simultaneous integrability of two commuting almost tangent structures*, Comm. Math. Univ. Carolinae, 22(1981), no. 1, 149-160. MR0609943 (82e:53052)

[15] M. de León; P. R. Rodrigues, *Methods of differential geometry in analytical mechanics*, North-Holland Mathematics Studies, 158, North-Holland Publishing Co., Amsterdam, 1989. MR1021489 (91c:58041)

[16] G. Thompson; U. Schwardmann, *Almost tangent and cotangent structures in the large*, Trans. Amer. Math. Soc., 327(1991), no. 1, 313-328. MR1012509 (91m:53029)

[17] I. Vaisman, *Lagrange geometry on tangent manifolds*, Int. J. Math. Math. Sci., 51(2003), 3241-3266. MR2018588 (2004k:53116)

[18] K. Yano; E. T. Davies, *Differential geometry on almost tangent manifolds*, Ann. Mat. Pura Appl. (4), 103(1975), 131-160. MR0390958 (52 #11781)

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 775 – 787

# Functional equations related to weightable quasi-metrics

M.J. Campión [*] , E. Induráin [†] , G. Ochoa [‡] and O. Valero [§]

**Abstract**

Starting from the definition of a weightable quasi-metric we observe that several functional equations are induced in a natural way. Studying these equations we characterize weightable quasi-metrics and show that they define representable total preorders. We also analyze how to retrieve weightable quasi-metrics from real-valued functions satisfying suitable functional equations.

*2000 AMS Classification:* 39B52, 54E35, 54F05, 06A06.

**Keywords:** Functional equations in two variables; Weightable quasi-metrics; Representable total preorders.

Received 02/01/2013 : Accepted 23/04/2014 Doi : 10.15672/HJMS.2015449441

## 1. Introduction

In the last years, (weightable) quasi-metric spaces have proven to be useful in modeling many processes that arise in Theoretical Computer Science and that involve some situation of *asymmetry*. The aforementioned usefulness is due to the fact that quasi-metric spaces lack the symmetry and the Hausdorffness enjoyed by metric spaces. This fact allows to introduce techniques of measuring that, contrarily to the metric ones, reflect the asymmetry inherent to the computational process. It is then possible to develop a "metric" foundation for partial orders reasoning techniques in the spirit of D. Scott ([40, 34, 25]). Recent applications of the aforesaid metric tools based on the use

[*]Departamento de Matemáticas, Universidad Pública de Navarra. Pamplona (SPAIN). Email: mjesus.campion@unavarra.es

[†]Departamento de Matemáticas. Universidad Pública de Navarra. Pamplona (SPAIN). Email: steiner@unavarra.es

[‡]Departamento de Matemáticas. Universidad Pública de Navarra. Pamplona (SPAIN). Email: ochoa@unavarra.es

[§]Departament de Ciències Matemàtiques i Informàtica, Universitat de les Illes Balears. Palma de Mallorca (SPAIN).
Email: o.valero@uib.es

of (weightable) quasi-metrics to Complexity Analysis of Algorithms, Denotational Semantics and Program Correctnes can be found in [33, 32, 34, 17, 23, 37, 38, 26] and [25].

Inspired in part by its utility in Theoretical Computer Science, we focus our attention on the definition of a (weightable) quasi-metric ([10, 21]) and we immediately encounter some functional equation that appears in a natural way. As a matter of fact weightable quasi-metrics are characterized as the quasi-metrics that satisfy a certain functional equation that we call circuit invariance. In addition, the disymmetry function (defined in Section 2) of a weightable quasi-metric satisfies Sincov's functional equation and induces a total preorder, different, in general, from the specialization order directly induced by the given quasi-metric. To conclude, we analyze the possibility of retrieving a weightable quasi-metric from a real-valued bivariate function that satisfies Sincov's functional equation. This allows us to establish a link between apparently disparate notions, namely: i) weightable quasi-metrics, ii) real-valued bivariate functions that satisfy Sincov's functional equation, and iii) total preorders that are representable through a real-valued utility function.

This possibility of relating weightable quasi-metrics, functional equations and representable total preorders is undoubtedly an important motivation, besides their aforementioned usefulness in Theoretical Computer Science, for the study of this particular kind of quasi-metrics.

The structure of the manuscript goes as follows.

The key definitions and notations are listed and discussed in Section 2. In Section 3 we consider and analyze different functional equations in two variables that are closely associated to the concept of a quasi-metric. In Section 4 we relate those functional equations to some kinds of orderings. In Section 5 we characterize when a positively weightable quasi-metric can be retrieved from a real-valued bivariate function that satisfies either the circuit invariance functional equation or Sincov's functional equation.

## 2. Preliminaries

In what follows, $X$ will denote a nonempty set and $\mathbb{R}$ will stand for the set of real numbers.

The definition of a metric space is usually attributed to M. Fréchet (see [13]). However, asymmetric distances had already been implicitly considered by Pompeiu in [28], as mentioned in the seminal book by F. Hausdorff issued in 1914 (see [16]). Hausdorff introduced a wide sort of ideas in this direction. Having these ideas in mind, the formal definition of a quasi-metric space was issued by W. A. Wilson in 1931. (See [44, 18]). Other miscellaneous extensions, special cases and variations of the concept of a metric space (e.g. partial metric spaces, pseudo-metric spaces, etc.) are often encountered in the specialized literature [41, 25, 19].

**2.1. Definition.** Let $X$ be a nonempty set. Following the modern terminology ([20]), by a *quasi-metric* on $X$ we mean a function $d : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$ the following conditions hold:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) + d(y, z) \geq d(x, z)$.

Of course a metric on a set $X$ is a quasi-metric $d$ on $X$ satisfying, in addition, the following condition for all $x, y \in X$ :

(iii) $d(x, y) = d(y, x)$.

By a *quasi-metric space* we mean a pair $(X, d)$ such that $X$ is a nonempty set and $d$ is a quasi-metric on $X$.

If $d$ is a quasi-metric on a set $X$, then the relation $\leq_d$ on $X$ given by $x \leq_d y \Leftrightarrow d(x, y) = 0$, is an order on $X$ (see Definition 4.1 in Section 4) called the specialization order of $d$.

Given a quasi-metric $d$ on $X$, and an ordered pair $(x, y) \in X \times X$, the real number $F(x, y) = d(x, y) - d(y, x)$ is said to be the *disymmetry of the pair* $(x, y)$. The function $F : X \times X \to \mathbb{R}$ defined by $F(x, y) = d(x, y) - d(y, x) \quad (x, y \in X)$ is said to be the *disymmetry function* associated to the quasi-metric $d$ on $X$.

**2.2. Remark.** The original definition of a quasi-metric, due to Wilson ([44]) is a bit more restrictive than Definition 2.1 above. Namely, in the sense of Wilson ([44]), a *quasi-metric* $d$ on $X$ is a quasi-metric in the sense of Definition 2.1 which satisfies in addition that $d(x, y) = 0 \Leftrightarrow x = y$ for every $x, y \in X$. Obviously condition (i) in Definition 2.1 is less restrictive than the preceding condition (see also Example 2.3 below, due to Hausdorff [16]). Nowadays, according, for instance, to [29], quasi-metrics in the sense of Wilson are called $T_1$ *quasi-metrics.* As a matter of fact, any quasi-metric generates a topology in a natural way. This topology will satisfy the separation axiom $T_1$ if and only if the given quasi-metric is a quasi-metric in the sense of Wilson. This is the reason why quasi-metrics in the sense of Wilson are called $T_1$-quasi-metrics.

**2.3. Example.** ([16]) Let $\mathcal{H}$ denote the family of non-empty compact sets of the real plane $\mathbb{R}^2$. Let $d_E$ denote the usual Euclidean distance on the real plane $\mathbb{R}^2$. Given $A, B \in \mathcal{H}$, consider the non-negative real number $d_H(A, B)$ defined as follows:

$$d_H(A, B) = \max_{a \in A} \{\min_{b \in B} d_E(a, b)\}.$$

It is well-known that $d_H$ is a quasi-metric (in the sense of Definition 2.1 above) on the real plane. Moreover, it is not a metric, that is $d_H(A, B)$ could be different from $d_H(B, A) \quad (A, B \in \mathcal{H})$. This quasi-metric $d_H$ is said to be the *Hausdorff quasi-metric* on $\mathcal{H}$. By the way, note that this quasi-metric is not $T_1$, i.e., $d_H$ does not satisfy Wilson's original definition, since if $A \subsetneq B \in \mathcal{H}$, we have that $d_H(A, B) = 0$ but $d_H(B, A) \neq 0$ as well.

(A very important use in Pure Mathematics of the Hausdorff quasi-metric $d_H$ appears in the definition and study of *fractal sets*. See Chapter II, Section 6 in [6] for further details).

**2.4. Example.** Let $d_S : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be the function defined by

$$d_S(x, y) = \begin{cases} \min\{y - x, 1\} & \text{if } x \leq y \\ 1 & \text{if } x > y \end{cases}.$$

It is easy to check that $d_S$ is a quasi-metric ([29]), known as the *Sorgenfrey quasi-metric*, which is $T_1$.

**2.5. Definition.** ([10, 25, 20]) Let $X$ be a nonempty set. A quasi-metric $d$ on $X$, as well as the associated quasi-metric space $(X, d)$, are said to be *weightable* if there exists a function $w : X \to \mathbb{R}$ such that $d(x, y) + w(x) = d(y, x) + w(y)$ holds for every $x, y \in X$. The function $w$ is called a *weighting function* for $d$.

In the particular case in which there is at least one weighting function that only takes non-negative values ($w(X) \subseteq [0, +\infty)$) we say that the quasi-metric $d$ is *positively weightable*. (See Example 2.8 below).

**2.6. Example.** Let $d : \mathbb{R}\times\mathbb{R} \to [0, +\infty)$ be the function given by $d(x, y) = \max\{y-x, 0\}$ for all $x, y \in \mathbb{R}$. It is clear that $(\mathbb{R}, d)$ is a quasi-metric space which is weightable with weighting function $w : \mathbb{R} \to \mathbb{R}$ given by $w(x) = x$ for all $x \in \mathbb{R}$.

**2.7. Remarks.**

(1) Note that if $(X, d)$ is a weightable quasi-metric space with weighting function $w$, then the disymmetry function $F$ associated to $d$ is given by $F(x, y) = w(y)-w(x)$ for all $x, y \in X$.

(2) The original definition of a weightable quasi-metric ([10, 25]) does not force the weighting functions to take non-negative values. However, inspired by the applications in Theoretical Computer Science, other authors (see e.g. [21]) define a weightable quasi-metric by imposing the weighting functions to be non-negative. By this reason, we have pointed out this nuance, distinguishing accordingly between “weightable quasi-metrics" and “positively weightable quasi-metrics" in Definition 2.5.

In the next examples we provide a few weightable quasi-metrics which play a central role in several fields of Theoretical Computer Science.

**2.8. Examples.** We introduce now several well-known examples of weightable quasi-metrics.

1. The *domain of words* $\Sigma^\infty$ (see e.g. [20, 25, 31, 35]) consists of all finite and infinite sequences over a nonempty set $\Sigma$, ordered by $x \sqsubseteq y \Leftrightarrow x$ is a prefix of $y$, where we assume that the empty sequence $\phi$ is an element of $\Sigma^\infty$.

   For each $x, y \in \Sigma^\infty$ denote by $x \sqcap y$ the longest common prefix of $x$ and $y$, and for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of $x$. Thus $\ell(x) \in [1, \infty]$ whenever $x \neq \phi$, and $\ell(\phi) = 0$. Then ([20, 25]) the function $d : \Sigma^\infty \times \Sigma^\infty \to [0, +\infty)$ given by

   $$d(x, y) = 2^{-\ell(x\sqcap y)} - 2^{-\ell(x)},$$

   is a positively weightable quasi-metric on $\Sigma^\infty$ with weighting function $w$ given by $w(x) = 2^{-\ell(x)}$ for all $x \in \Sigma^\infty$. Note that the specialization order $\leq_d$ coincides with $\sqsubseteq$ . Moreover, the disymmetry function associated to $d$ is given by $F(x, y) = 2^{-\ell(y)} - 2^{-\ell(x)}$ for all $x, y \in \Sigma^\infty$.

2. The *interval domain* $I([0, 1])$ ([11, 22, 25]) consists of the nonempty closed intervals of $[0, 1]$ ordered by reverse inclusion, i.e., $[a, b] \sqsubseteq [c, d] \Leftrightarrow [a, b] \supseteq [c, d]$ . In particular, points of [0,1] are identified with the singleton intervals. Then, the function $d$ defined on $I([0, 1]) \times I([0, 1])$ by

   $$d([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\} - (b - a),$$

   is a weightable quasi-metric on $I([0, 1])$ with weighting function $w$ given by $w([a, b]) = b - a$, for all $[a, b] \in I([0, 1])$ (see e.g. [25, 31, 35]). The specialization order $\leq_d$ coincides with $\sqsubseteq$ . Moreover, the disymmetry function associated to $d$ is given by $F([a, b], [c, d]) = d + a - b - c$ for all $[a, b], [c, d] \in I([0, 1])$.

3. Denote by $\omega$ the set of non-negative integer numbers. The *complexity quasi-metric space* [34] is the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

   $$\mathcal{C} = \{f : \omega \to (0, +\infty] \mid \sum_{n=0}^{+\infty} 2^{-n} \frac{1}{f(n)} < +\infty\},$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ defined by

$$d_{\mathcal{C}}(f,g)=\sum_{n=0}^{+\infty}2^{-n}\left[\max\left(\frac{1}{g(n)}-\frac{1}{f(n)},0\right)\right].$$

Furthermore, $(\mathcal{C}, d_{\mathcal{C}})$ is weightable with weighting function $w_{\mathcal{C}}$ given by $w_{\mathcal{C}}(f)=\sum_{n=0}^{+\infty}(2^{-n}/f(n))$ for all $f\in\mathcal{C}$. The specialization order of $d_{\mathcal{C}}$ coincides with the pointwise order of $\mathcal{C}$. Moreover, the disymmetry function associated to $d_{\mathcal{C}}$ is given by $F(f,g)=\sum_{n=0}^{+\infty}2^{-n}[1/g(n)-1/f(n)]$ for all $f,g\in\mathcal{C}$.

## 3. Functional equations defined through quasi-metrics

Let us see now how the definition of a weightable quasi-metric gives rise to the consideration of several functional equations. To this end, let us denote by $\mathbb{N}$ the set of positive integer numbers. The following Lemma 3.1 hangs from a well known result of the classical theory of functional equations in two variables.

**3.1. Lemma.** *Let $(X,d)$ be a quasi-metric space. Assume that the quasi-metric $d$ satisfies the functional equation of the 3-circuit, namely $d(x,y)+d(y,z)+d(z,x)=d(x,z)+d(z,y)+d(y,x)$, for every $x,y,z\in X$. Then $d$ is weightable.*

*Proof.* By hypothesis we observe that the disymmetry function $F: X\times X\to\mathbb{R}$ given by $F(x,y)=d(x,y)-d(y,x)\ \ (x,y\in X)$ satisfies $F(x,y)+F(y,z)=F(x,z),(x,y,z\in X)$. It is well known (see e.g. [4, 5, 15]) that in this case there exists a function $w: X\to\mathbb{R}$ such that $F(x,y)=w(y)-w(x)=d(x,y)-d(y,x)$, for every $x,y\in X$. Therefore $d$ is a weightable quasi-metric. □

The converse of Lemma 3.1 is also true, as well as some other equivalences stated in the following Theorem 3.2.

**3.2. Theorem.** *Let $(X,d)$ be a quasi-metric space. The following statements are equivalent:*

i) *The quasi-metric $d$ is weightable.*
ii) *The quasi-metric $d$ satisfies the functional equation of the 3-circuit, namely $d(x,y)+d(y,z)+d(z,x)=d(x,z)+d(z,y)+d(y,x)$, for every $x,y,z\in X$.*
iii) *For every $n\geq 3$, $n\in\mathbb{N}$, the quasi-metric $d$ satisfies the functional equation of the $n$-circuit, namely $d(x_1,x_2)+d(x_2,x_3)+\ldots+d(x_{n-1},x_n)+d(x_n,x_1)=d(x_1,x_n)+d(x_n,x_{n-1})+\ldots+d(x_3,x_2)+d(x_2,x_1)$, for every $x_1,x_2,x_3,\ldots,x_n\in X$.*
iv) *For some $k\geq 3$, $k\in\mathbb{N}$, the quasi-metric $d$ satisfies the functional equation of the $k$-circuit, namely $d(x_1,x_2)+d(x_2,x_3)+\ldots+d(x_{k-1},x_k)+d(x_k,x_1)=d(x_1,x_k)+d(x_k,x_{k-1})+\ldots+d(x_3,x_2)+d(x_2,x_1)$, for every $x_1,x_2,x_3,\ldots,x_k\in X$.*

*Proof.* *i) ⇒ iii)*:

Since $d$ is weightable by hypothesis, there exists a function $w: X\to\mathbb{R}$ such that $d(x,y)+w(x)=d(y,x)+w(y)$, for every $x,y\in X$. Thus, for every $x_1,x_2,x_3,\ldots,x_n\in X$ we have that $[d(x_1,x_2)+w(x_1)]+[d(x_2,x_3)+w(x_2)]+\ldots+[d(x_{n-1},x_n)+w(x_{n-1})]+[d(x_n,x_1)+w(x_n)]=[d(x_2,x_1)+w(x_2)]+[d(x_3,x_2)+w(x_3)]+\ldots+[d(x_n,x_{n-1})+w(x_n)]+[d(x_1,x_n)+w(x_1)]$. Hence $d(x_1,x_2)+d(x_2,x_3)+\ldots+d(x_{n-1},x_n)+d(x_n,x_1)=d(x_2,x_1)+d(x_3,x_2)+\ldots+d(x_n,x_{n-1})+d(x_1,x_n)=d(x_1,x_n)+d(x_n,x_{n-1})+\ldots+d(x_3,x_2)+d(x_2,x_1)$.

*iii) ⇒ ii)* and *iii) ⇒ iv)*:

These implications are obvious.

*iv) ⇒ ii)*:

It follows immediately, by taking $x_3=x_4=\ldots=x_k$.

*ii*) $\Rightarrow$ *i*): This has already been stated in Lemma 3.1.

This finishes the proof. $\square$

Now we introduce a necessary definition concerning functional equations.

**3.3. Definition.** Let $X$ be a nonempty set.

a) A bivariate function $F : X \times X \longrightarrow \mathbb{R}$ is said to satisfy *Sincov's* functional equation if $F(x, y) + F(y, z) = F(x, z)$ holds for every $x, y, z \in X$.
b) A bivariate function $F : X \times X \longrightarrow \mathbb{R}$ is said to satisfy the *separability* equation if $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ holds for every $x, y, z \in X$.

Observe that the notion of Sincov's functional equation is already involved in the previous Theorem 3.2.

The following result is well-known (see e.g. [5], pp. 122 and ff.).

**3.4. Proposition.** *Let $X$ be a nonempty set. A bivariate function $F : X \times X \longrightarrow \mathbb{R}$ satisfies the separability equation if and only if $F(x, y) = G(x) + H(y) \quad (x, y \in X)$, for some functions $G, H : X \to \mathbb{R}$ that depend of only one variable.*

**3.5. Theorem.** *Let $(X, d)$ be a quasi-metric space. Let $F$ be the dysymmetry function associated to $d$. The following statements are equivalent:*

i) *The quasi-metric $d$ is weightable.*
ii) *The disymmetry function $F$ associated to $d$ satisfies Sincov's functional equation $F(x, y) + F(y, z) = F(x, z)$, for every $x, y, z \in X$.*
iii) *The disymmetry function $F$ satisfies the functional equation of separability $F(x, y) + F(y, z) = F(x, z) + F(y, y)$, for every $x, y, z \in X$.*
iv) *The disymmetry function $F$ satisfies the functional equation $F(x, y) + F(y, z) = F(x, z) + F(t, t)$, for every $x, y, z, t \in X$.*

*Proof.* *i*) $\Leftrightarrow$ *ii*):

By Theorem 3.1, $d$ is weightable, if and only if $d(x, y) + d(y, z) + d(z, x) = d(x, z) + d(z, y) + d(y, x)$ holds for every $x, y, z \in X$. But this is equivalent to say that $[d(x, y) - d(y, x)] + [d(y, z) - d(z, y)] = [d(x, z) - d(z, x)]$, or, just changing the notation, $F(x, y) + F(y, z) = F(x, z) \quad (x, y, z \in X)$.

*ii*) $\Leftrightarrow$ *iii*) $\Leftrightarrow$ *iv*):

Just notice that, for all $x \in X$, $F(x, x) = d(x, x) - d(x, x) = 0$ holds. $\square$

## 4. Orderings induced by functional equations related to weightable quasi-metrics

In this Section 4 we study orderings that are induced in a natural way by weightable quasi-metrics. First we recall some basic definitions concerning orderings.

**4.1. Definition.** A *preorder* $\precsim$ on an arbitrary nonempty set $X$ is a binary relation on $X$ which is reflexive and transitive. If $\precsim$ is a preorder on $X$, then the pair $(X, \precsim)$ is said to be a preordered set. An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set $X$ is a preorder such that $[x \precsim y] \vee [y \precsim x]$ holds for every $x, y \in X$.

**4.2. Definition.** Let $X$ be a nonempty set. Let $\prec$ be an asymmetric binary relation defined on $X$. Associated to $\prec$ we define the reflexive and total binary relation $\precsim$ given by $x \precsim y \Leftrightarrow \neg(y \prec x) \quad (x, y \in X)$.

An *interval order* $\prec$ is an asymmetric binary relation such that $[(x \prec y) \wedge (z \prec t)] \Rightarrow [(x \prec t) \vee (z \prec y)]$ $(x, y, z, t \in X)$. An interval order $\prec$ is said to be a *semiorder* if $[(x \prec y) \wedge (y \prec z)] \Rightarrow [(x \prec w) \vee (w \prec z)]$, for every $x, y, z, w \in X$.

**4.3. Remark.** Interval orders are perhaps the best class of ordered structures to build models of uncertainty or to represent and manipulate vague or imperfectly described pieces of knowledge. The notion of an interval order was introduced[¶] by Peter C. Fishburn (see [12]), in order to study models of preference or measurement orderings whose associated indifference may fail to be transitive.

The concept of a semiorder was introduced in [24] to deal with innacuracies in measurements where a nonnegative threshold of discrimination is involved. Semiordered structures are often encountered in a wide range of applications (see e.g. [2] for further details).

We can generate total preorders, interval orders and semiorders from particular solutions of suitable functional equations, as stated in the next straightforward Proposition 4.4, whose proof is omitted for the sake of brevity. (For similar results see e.g. [3])

**4.4. Proposition.** *Let $X$ be a nonempty set.*

i) *If $F : X \times X \to \mathbb{R}$ satisfies Sincov's functional equation, then the binary relation $\precsim$ defined on $X$ by $x \precsim y \Leftrightarrow F(y,x) \leq 0 \quad (x, y \in X)$ is a total preorder.*
ii) *If $F : X \times X \to \mathbb{R}$ satisfies the separability equation and, in addition $F(t,t) \leq 0 \quad (t \in X)$, then the binary relation defined on $X$ by $x \prec y \Leftrightarrow F(x,y) > 0 \quad (x, y \in X)$ is an interval order.*
iii) *If $F : X \times X \to \mathbb{R}$ satisfies the separability equation and, in addition, there exists a non-positive real constant $K \leq 0$ such that $F(t,t) = K \quad (t \in X)$, then the binary relation defined on $X$ by $x \prec y \Leftrightarrow F(x,y) > 0 \quad (x, y \in X)$ is a semiorder.*

The following result is a direct consequence of Proposition 4.4 (part $i$), Theorem 3.2 and Theorem 3.5 in Section 3 above.

**4.5. Corollary.** *Let $X$ be a nonempty set. Let $d$ be a weightable quasi-metric defined on $X$. Let $F$ be the dysymmetry function associated to $d$. Then, the binary relation $\precsim_F$ defined on $X$ as $x \precsim_F y \Leftrightarrow F(y,x) \leq 0 \Leftrightarrow F(x,y) \geq 0 \quad (x, y \in X)$ is a total preorder.*

**4.6. Remark.** Observe that given a quasi-metric space $(X, d)$, then the following easy relationship is satisfied: If $x \leq_d y$, then $x \precsim_F y \Leftrightarrow x = y$. Moreover, note that the order relation $\precsim_F$ is total whereas that the specialization oder $\leq_d$ is not. This property could be an advantage to model certain processes in applied contexts in the sense that the order relation $\precsim_F$ allows to compare elements of $X$ that are not comparable with respect to $\leq_d$. For instance, coming back to Example 2.8 (2) we observe that $[a, b] \leq_d [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$, whereas $[a, b] \precsim_F [c, d] \Leftrightarrow b + c \leq a + d$.

**4.7. Remark.** Suppose that we want to induce interval orders or semiorders from the disymmetry function of a quasi-metric. Taking into account Proposition 4.4, we should look for disymmetry functions that satisfy the separability equation. However, if $F(x,y) + F(y,z) = F(x,z) + F(y,y)$ holds for every $x, y, z \in X$, we immediately get Sincov's functional equation $F(x,y) + F(y,z) = F(x,z) \quad (x, y, z \in X)$ because by definition of $F$, we have that $F(y,y) = d(y,y) - d(y,y) = 0$, for every $y \in X$. Thus, even if the separability equation is accomplished, we would induce an interval order or a semiorder that coincides with the asymmetric part of a total preorder. That is, if $\prec$ is the induced interval order or semiorder, in this case we have that the binary relation $\precsim$ given by

[¶]Under a different name, the concept of an interval order, as well as the concept of a semiorder, was already implicit much earlier, in the work of Norbert Wiener. (See e.g. [43]).

$x \precsim y \Leftrightarrow \neg(y \prec x)$ $(x, y \in X)$ is actually a total preorder[‖]. Notice also that this fact is the "expected one", taking into account the result stated in Theorem 3.5.

Furthermore, if the disymmetry function $F$ of a quasi-metric satisfies the separability equation $F(x,y) + F(y,z) = F(x,z) + F(y,y)$ $(x,y,z \in X)$, by Proposition 3.4 we have that $F$ can be decomposed as $F(x,y) = G(x) + H(y)$ for every $x, y \in X$. But here we have that $F(x,y) = -F(y,x)$. Hence $G(x) + H(y) = -G(y) - H(x)$, for every $x, y \in X$. Thus $G(x) + H(x) = G(y) + H(y)$, for every $x, y \in X$. But $G(x) + H(x) = F(x,x) = d(x,x) - d(x,x) = 0$, for every $x \in X$. Therefore $G(x) = -H(x)$, for every $x \in X$. Thus we have that $F(x,y) = G(x) - G(y) = H(y) - H(x) \Leftrightarrow d(x,y) - d(y,x) = H(y) - H(x) \Leftrightarrow d(x,y) + H(x) = d(y,x) + H(y)$ $(x, y \in X)$. This is an alternative argument to show that the quasi-metric $d$ must be weightable, as stated in Theorem 3.5.

Inspired by Proposition 4.4, we may pay attention to the following important detail: the total preorders associated to a weightable quasi-metric can be framed by means of the disymmetry function $F$ as well as by the weighting function $w$. Indeed, the fact $x \precsim_F y \Leftrightarrow F(x,y) \geq 0 \Leftrightarrow w(x) \leq w(y)$ $(x, y \in X)$ is crucial. This inspires the following definition.

**4.8. Definition.** Let $\precsim$ denote a total preorder defined on a nonempty set $X$. We say that $\precsim$ is *representable* if there exists a function $u : X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y)$ for every $x, y \in X$. The function $u$ is called a *numerical isotony*, or, mainly in contexts coming from Economics, a *utility function*.

The kind of numerical representation involved in Definition 4.8 is actually equivalent to a representation that uses a bivariate function accomplishing Sincov's functional equation, as the next well-known result shows. (See e.g. Theorem 1 in [7]).

**4.9. Proposition.** *Let $\precsim$ be a total preorder defined on a nonempty set $X$. Then $\precsim$ is representable if and only if there exists a bivariate function $F : X \times X \to \mathbb{R}$ such that $F$ satisfies Sincov's functional equation and, in addition, $x \precsim y \Leftrightarrow F(x,y) \geq 0$ holds for every $x, y \in X$.*

The following definition is inspired by Proposition 4.9, and it is equivalent to Definition 4.8.

**4.10. Definition.** Let $\precsim$ be a representable total preorder defined on a nonempty set $X$. A bivariate function $F : X \times X \to \mathbb{R}$ satisfying Sincov's functional equation, and such that $x \precsim y \Leftrightarrow F(x,y) \geq 0$ holds for every $x, y \in X$, is called a *bivariate numerical representation* of $\precsim$.

**4.11. Remark.** Not every total preorder is representable. A well known example is the lexicographic order $\precsim_L$ on the real plane $\mathbb{R}^2$: Given $(a,b)$, $(c,d) \in \mathbb{R}^2$, then $(a,b) \precsim_L (c,d) \Leftrightarrow [(a < c) \vee (a = c,\ b \leq d)]$. (See e.g [9] for further details).

Looking again at Corollary 4.5 we may observe that from a weightable quasi-metric we get a representable total preorder. Looking for a converse result, we may start from a representable total preorder $\precsim$ defined on a nonempty set $X$, and search for a weightable quasi-metric whose disymmetry function constitutes a bivariate numerical representation of $\precsim$. We get a positive answer to this question, as the next Proposition 4.12 states.

**4.12. Proposition.** *Let $X$ be a nonempty set. Let $\precsim$ be a representable total preorder defined on $X$. Then there exists a positively weightable quasi-metric $d : X \times X \to [0, +\infty)$*

---

‖In this case the corresponding interval order or semiorder is said to be *degenerate* or non-typical. An interval order or semiorder $\prec$ defined on a set $X$ is said to be *typical* provided that its a associated symmetric part $\precsim$ fails to be transitive.

*whose disymmetry function $F$ is a bivariate numerical representation of the given total preorder $\precsim$.*

*Proof.* Since $\precsim$ is representable, there exists a function $u : X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y)$. We may assume without loss of generality that $u$ takes strictly positive values: indeed, if $h(t) = 3 + \frac{2}{\pi} arctan(t)$ for $t \in \mathbb{R}$, we have that $h$ is a strictly increasing function whose range is $(2, 4)$, and such that $x \precsim y \Leftrightarrow h(u(x)) \leq h(u(y))$, for every $x, y \in X$, so that the composition $h \circ u$ is another utility function that represents the total preorder $\precsim$. Thus, already assuming that $u : X \to (0, +\infty)$ is a strictly positive utility representation for $\precsim$, given $x, y \in X$ we may define $d(x, y) = 0$ if $x = y$ and $d(x, y) = u(y)$ otherwise. It is straightforward to see that $d$ is a $(T_1)$ quasi-metric. Moreover, it is positively weightable: just observe that $d(x, y) + u(x) = d(y, x) + u(y)$ for every $x, y \in X$. Finally, its disymmetry function $F$ satisfies that $F(x, y) = d(x, y) - d(y, x) = u(y) - u(x)$ for every $x, y \in X$, so that $x \precsim y \Leftrightarrow u(x) \leq u(y) \Leftrightarrow u(y) - u(x) \geq 0 \Leftrightarrow F(x, y) \geq 0$. Therefore $F$ is a bivariate numerical representation for $\precsim$. □

**4.13. Remark.** Notice that the quasi-metric $d$ that appears in the statement of Proposition 4.12 is not unique, in general. As a matter of fact, to get $d$ we may use *any* strictly positive utility function $u$ that represents the total preorder $\precsim$.

To summarize this Section 4, we may notice that in Corollary 4.5 we get a representable total preorder from a weightable quasi-metric, whereas in Proposition 4.12 we retrieve a positively weightable quasi-metric from a representable total preorder. By Theorem 3.5, in both results the disymmetry functions associated to the weightable quasi-metrics involved satisfy Sincov's functional equation.

To complete the panorama, we may wonder if it is also possible to retrieve a (positively) weightable quasi-metric directly from a bivariate function that satisfies Sincov's functional equation. We answer this question throughout the next Section 5.

## 5. Retrieving positively weightable quasi-metrics from functional equations

The main questions to be analyzed throughout this Section 5 are the following:

i) Suppose that $X$ is a nonempty set and $D : X \times X \to \mathbb{R}$ is a bivariate function that satisfies the 3-circuit invariance functional equation. Can we induce from $D$, in a natural way, a positively weightable quasi-metric** on $X$?

ii) Suppose that $F : X \times X \to \mathbb{R}$ is a bivariate function that satisfies Sincov's functional equation. Can we induce from $F$ a positively weightable quasi-metric on $X$ whose disymmetry function is $F$?

To study the former question, let $X$ denote a nonempty set, and let $D : X \times X \to \mathbb{R}$ be a bivariate function such that $D(a, b) + D(b, c) + D(c, a) = D(a, c) + D(c, b) + D(b, a)$ holds for every $a, b, c \in X$. Define $F : X \times X \to \mathbb{R}$ by declaring that $F(x, y) = D(x, y) - D(y, x)$ for every $x, y \in X$. We immediately realize that $F$ satisfies Sincov's functional equation $F(a, b) + F(b, c) = F(a, c)$, for every $a, b, c \in X$. Therefore, we pass to consider the latter question, since its solution would immediately lead to a solution for the former one.

Next Lemma 5.1, Theorem 5.2 and their subsequent corollaries provide us with a positive answer.

**5.1. Lemma.** *Let $X$ be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a bivariate function that satisfies Sincov's functional equation. Suppose, in addition, that there exists a strictly positive function $G : X \to (0, +\infty)$ such that $F(x, y) = G(y) - G(x)$, for every $x, y \in$*

**Notice that we are not imposing $D$ to be the solution, not even to be a quasi-metric.

*$X$. Then there exists a positively weightable quasi-metric $d : X \times X \to [0, +\infty)$ whose disymmetry fuction is $F$.*

*Proof.* Given $x, y \in X$, we define $d(x, y) = 0$ if $x = y$ and $d(x, y) = G(y)$ otherwise. Notice that both $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ hold by definition of $d$. To check the triangle inequality, given $x, y, z \in X$ we distinguish the following cases:

Case 1: $x = y$. In this case we have that $0 = d(x, y) \leq d(x, z) + d(z, y)$ because, by definition of $d$, we have that $d(x, z) \geq 0$ and $d(z, y) \geq 0$.
Case 2: $x \neq y$; $y = z$. In this case we have that $G(y) = d(x, y) \leq d(x, z) + d(z, y) = d(x, y) + d(y, y)$ since $d(y, y) = 0$ and $d(x, y) = G(y)$ by definition of $d$.
Case 3: $x \neq y$; $x = z$. In this case the proof runs as in Case 2.
Case 4: $x \neq y$; $y \neq z$. In this case we have that $G(y) = d(x, y) \leq G(y) + d(x, z) = d(z, y) + d(x, z) = d(x, z) + d(z, y)$ since $d(x, z) \geq 0$ and $d(x, y) = d(z, y) = G(y)$ by definition of $d$.

Therefore $d$ is a quasi-metric on $X$.

It is straightforward to check that $F$ is the disymmetry function associated to $d$, so that $G$ is a weighting function for $d$. Hence $d$ is positively weightable. □

From Lemma 5.1, we finally reach the main result in this Section 5, namely Theorem 5.2, which is a characterization of real-valued bivariate functions that can be identified to the disymmetry function of some positively weightable quasi-metric.

**5.2. Theorem.** *Let $X$ be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a bivariate function that satisfies Sincov's functional equation. The following statements are equivalent:*

i) *For every $a \in X$, the trace function $F_a : X \to \mathbb{R}$ defined by $F_a(t) = F(a, t)$ for every $t \in X$, is bounded by below (i.e.: there exists a constant $A \in \mathbb{R}$ such that $F(a, t) > A$ for every $t \in X$).*
ii) *There exists an element $a \in X$ such that $F_a$ is bounded by below.*
iii) *There exists a positively weightable quasi-metric $d : X \times X \to [0, +\infty)$ whose disymmetry function is $F$.*

*Proof.* The implication $i) \Rightarrow ii)$ is trivial.

To prove the fact $ii) \Rightarrow iii)$, let $A \in \mathbb{R}$ be such that $F(a, t) > A$ for every $t \in X$. Given $x, y \in X$, we define the function $w : X \to \mathbb{R}$ as $w(t) = F(a, t) + |A|$ for every $t \in X$. Notice that $w(t) > 0$ holds for every $t \in X$, since $A + |A| \geq 0$. Moreover, for every $x, y \in X$ we have that $F(x, y) = F(x, a) + F(a, y) = F(a, y) - F(a, x) = (F(a, y) + |A|) - (F(a, x) + |A|) = w(y) - w(x)$, so that by Lemma 5.1 there exists a positively weightable quasi-metric $d$ whose disymmetry function is $F$.

To conclude, we prove the implication $iii) \Rightarrow i)$. To do so, suppose that $d : X \times X \to [0, +\infty)$ is a positively weightable quasi-metric whose disymmetry function is $F$. By Remark 2.7, we have that $F(x, y) = w(y) - w(x)$ for every $x, y \in X$, where $w$ stands for the weighting function associated to $d$. Fix any element $a \in X$. Now, given $t \in X$, we have that $F(a, t) = w(t) - w(a) > -w(a)$. In other words: $F_a$ is bounded by below. □

**5.3. Example.** Accordingly to Theorem 5.2, it is now easy to find an example of a nonempty set $X$ and a function $F : X \times X \to \mathbb{R}$ such that $F$ satisfies Sincov's functional equation, but there is *no* positively weightable quasi-metric on $X$ whose disymmetry function is $F$. Consider for instance $X = \mathbb{R}$ and $F(x, y) = x - y$ for every $(x, y) \in \mathbb{R}^2$. As a matter of fact, we may notice that no trace of $F$ is bounded by below.

**5.4. Corollary.** *Let $X$ be a finite nonempty set. Let $F : X \times X \to \mathbb{R}$ be a bivariate function that satisfies Sincov's functional equation. Then there exists a positively weightable quasi-metric $d : X \times X \to \mathbb{R}$ whose disymmetry function is $F$.*

*Proof.* This is an immediate consequence of Theorem 5.2, because $F$ is bounded since $X$ is finite. $\square$

**5.5. Corollary.** *Let $X$ be a nonempty set, endowed with a topology $\tau$ for which $X$ is a compact set. Let $F : X \times X \to \mathbb{R}$ be a continuous*[††] *bivariate function that satisfies Sincov's functional equation. Then there exists a positively weightable quasi-metric $d : X \times X \to \mathbb{R}$ whose disymmetry function is $F$.*

*Proof.* Again, this is an immediate consequence of Theorem 5.2, because $F$ is a continuous real-valued function defined on a compact set, so it is bounded (see e.g. [41], p. 20). $\square$

To finish this Section 5 we analyze the posibility of retrieving a weightable quasi-metric (in this case, *not necessarily a positively weightable one*) from a bivariate function satisfying Sincov's functional equation. Unlike Theorem 5.2 and Example 5.3, the answer is always positive, as next Theorem 5.6 proves.

**5.6. Theorem.** *Let $X$ be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a bivariate function that satisfies Sincov's functional equation. Then there exists a weightable ($T_1$) quasi-metric $d : X \times X \to [0, +\infty)$ whose disymmetry function is $F$.*

*Proof.* Since $F$ satisfies Sincov's functional equation, there exists a function $G : X \to \mathbb{R}$ such that $F$ can be decomposed as $F(x, y) = G(y) - G(x)$, for every $x, y \in X$.

Define $d : X \times X \to \mathbb{R}$ as follows:

i) $d(x, y) = 0$ if $x = y \in X$.
ii) $d(x, y) = 1 + G(y) - G(x)$ if $x \neq y \in X$ are such that $G(x) \leq G(y)$.
iii) $d(x, y) = 1$ if $x \neq y \in X$ are such that $G(x) > G(y)$.

By definition, $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$, for every $x, y \in X$.

Let us see now that $d$ satisfies the triangle inequality. To see this, given $x, y, z \in X$ we consider the following cases:

Case 1: If there is at least a coincidence between $x, y, z$ then $d(x, z) \leq d(x, y) + d(y, z)$ trivially holds.
Case 2: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(x) \leq G(y) \leq G(z)$ then we have that $d(x, z) = 1 + G(z) - G(x) < 2 + G(z) - G(x) = (1 + G(y) - G(x)) + (1 + G(z) - G(y)) = d(x, y) + d(y, z)$.
Case 3: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(x) \leq G(z) < G(y)$ then we have that $d(x, z) = 1 + G(z) - G(x) < 1 + G(y) - G(x) < (1 + G(y) - G(x)) + 1 = d(x, y) + d(y, z)$.
Case 4: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(y) < G(x) \leq G(z)$ then we have that $d(x, z) = 1 + G(z) - G(x) < 1 + G(z) - G(y) < 1 + (1 + G(z) - G(y)) = d(x, y) + d(y, z)$.
Case 5: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(y) \leq G(z) < G(x)$ then we have that $d(x, z) = 1 < 1 + d(y, z) = d(x, y) + d(y, z)$.
Case 6: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(z) < G(x) \leq G(y)$ then we have that $d(x, z) = 1 < d(x, y) + 1 = d(x, y) + d(y, z)$.
Case 7: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(z) < G(y) < G(x)$ then we have that $d(x, z) = 1 < 2 = 1 + 1 = d(x, y) + d(y, z)$.

Therefore $d$ is a quasi-metric.

A final checking shows that $d(x, y) - d(y, x) = G(y) - G(x)$ for every $x, y \in X$, so that $d$ is indeed weightable ($T_1$).

$\square$

---

[††] Here we consider that $X \times X$ is endowed with the product topology $\tau \times \tau$, whereas the real line $\mathbb{R}$ is given the usual Euclidean topology.

## 6. Final comments and some suggestions for further research

Weightable quasi-metrics are closely related to several functional equations stated for real-valued bivariate functions on a nonempty set.

As shown in the main results stated in Section 4 and Section 5, there is a close relationship between the concepts of weightable quasi-metrics, representable total preorders and solutions of Sincov's functional equation. Each of these concepts gives rise to any of the two other ones.

We leave as an open question the study of similar functional equations in the framework of generalized metric spaces of any kind (see e.g. [39]), as, in particular, cone metric spaces (see e.g. [1]), pseudo-metrics, quasi-pseudo metrics (see e.g. [19]), probabilistic and statistical metric and quasi-metric spaces (see e.g. [27, 42, 36, 14]), and/or partial metrics, as well as to extend some results arising in the classical crisp context to the fuzzy setting (see e.g. [30]).

## Acknowledgements

This work has been partially supported by the research projects MTM2009-10962, MTM2012-37894-C02-01, MTM2012-37894-C02-02, TIN2013-40765-P and TIN2013-42795-P (Spain). Thanks are given to Prof. Dr. R. G. Catalán (Pamplona, Spain) and Prof. Dr. S. Romaguera (Valencia, Spain) for their valuable suggestions and comments.

## References

[1] Abdeljawad, T. *Completion of cone metric spaces*, Hacet. J. Math. Stat. **39** (1), 67-74, 2010.

[2] Abrísqueta, F.J., Campión, M.J., Catalán, R.G., De Miguel, J.R., Estevan, A., Induráin, E., Zudaire M., Agud, L., Candeal, J.C., Díaz, S., Martinetti, D., Montes, S. and Gutiérrez García, J. *New trends on the numerical representability of semiordered structures*, Mathware Soft Comput. **19**, 25-37, 2012.

[3] Abrísqueta, F.J., Candeal, J.C., Catalán, R.G., De Miguel, J.R. and Induráin, E. *Generalized Abel functional equations and numerical representability of semiorders*, Publ. Math. Debrecen **78**, 557-568, 2011.

[4] Aczél, J. *Lectures on Functional Equations and their Applications*, (Academic Press, New York and San Diego, 1966).

[5] Aczél, J. *A Short Course on Functional Equations Based upon Recent Applications to the Social and Behavioral Sciences*, (D. Reidel, Dordrecht, 1987)

[6] Barnsley, M. *Fractals everywhere*, (Academic Press, New York, 1985).

[7] Bosi, B., Candeal, J.C., Induráin, E., Olóriz, E. and Zudaire M. *Numerical Representations of Interval Orders*, Order **18**, 171-190, 2001.

[8] Bukatin, M., Kopperman, R., Matthews, S.G., and Pajoohesh H. *Partial metric spaces*, Amer. Math. Monthly **116**, 708-718, 2009.

[9] Candeal, J.C. and Induráin E., *Lexicographic behaviour of chains*, Arch. Math. **72**, 145-152, 1999.

[10] Deza, M.M. and Deza E. *Encyclopedia of Distances*, (Springer-Verlag, Berlin, 2009).

[11] Escardo M., *Introduction to Real PCF,* Notes for an invited speach at the 3rd Real Numbers and Computers Conference (RNC3), (Université Pierre et Marie Curie, Paris, 1998). Available at: http://www.cs.bham.ac.uk/ mhe/papers.html.

[12] Fishburn, P.C. *Intransitive indifference with unequal indifference intervals*, J. Math. Psych. **7**, 144-149, 1970.

[13] Fréchet, M. *Sur quelques points du calcul fonctionnel*, Rend. Circ. Mat. Palermo **22**, 1-74, 1906.

[14] Grabiec, M.T., Cho, Y.J. and Saadati, R. *Families of quasi-pseudo metrics generated by probabilistic quasi-pseudo-metric spaces*, Surv. Math. Appl. **2**, 123-143, 2007.

[15] Gronau, D. *A remark on Sincov's functional equation*, Not. S. Afr. Math. Soc. **31**, 1-8, 2000.

[16] Hausdorff, F. *Grundzüge der Mengenlehre*, (Viet, Leipzig,1914).
[17] Hitzler, P. and Seda, A.K. *Generalized distance functions in the theory of computation*, The Computer Journal **53**, 443-464, 2010.
[18] Kofner, J. *On quasi-metrizability*, Topology Proc. **5**, 111-138, 1980.
[19] Künzi, H.P.A. *Complete quasi-pseudo-metric spaces*, Acta Math. Hungar. **59**, 121-146, 1992.
[20] Künzi, H.P.A. *Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology*, in: Handbook of the History of General Topology, Aull, C.E. and Lowen R. (Eds.), vol. **3**, (Kluwer, Dordrecht, 2001), 853-968.
[21] Künzi, H.P.A. and Vajner V. *Weighted quasi-metrics*, Ann. NY Acad. Sci. **728**, 64-77, 1994.
[22] Lawson, J. *Computation on metric spaces via domain theory*, Topology Appl. **85**, 247-263, 1998.
[23] Llull-Chavarría, J. and Valero, O. *An application of generalized complexity spaces to denotational semantics via domain of words*, LNCS **5457**, 530-541, 2009.
[24] Luce, R.D. *Semiorders and a theory of utility discrimination*, Econometrica **24**, 178-191, 1956.
[25] Matthews, S.G. *Partial metric topology*, Ann. NY Acad. Sci. **728**, 183-197, 1994.
[26] Matthews, S.G. *An extensional treatment of lazy data flow deadlock*, Theoret. Comput. Sci. **151**, 195-205, 1995.
[27] Menger, K. *Statistical metrics*, Proc. Nat. Acad. Sci. USA **28**, 535-537, 1942.
[28] Pompeiu, D. *Sur la continuité des fonctions de variables complexes*, Ann. Fac. Sci. Toulouse **7**, 264-315, 1905.
[29] Romaguera, S., Sánchez-Pérez, E.A. and Valero, O. *Quasi-normed monoids and quasi-metrics*, Publ. Math. Debrecen **62**, 53-69, 2003.
[30] Romaguera, S., Sapena, A. and Valero, O. *Quasi-uniform isomorphisms in fuzzy quasi-metric spaces, bicompletion and D-completion*, Acta Math. Hungar. **114**, 49-60, 2007.
[31] Romaguera, S. and Schellekens, M. *Partial metric monoids and semivaluation spaces*, Topology Appl. **153**, 948-962, 2005.
[32] Romaguera, S., Tirado, P. and Valero, O. *New results on mathematical foundations of asymptotic complexity analysis of algorithms via complexity spaces*, Int. J. Comput. Math. **89**, 1728-1741, 2012.
[33] Romaguera, S. and Valero, O. *Asymptotic complexity analysis and denotational semantics for recursive programs based on complexity spaces*, in: Semantics-Advances in Theories and Mathematical Models, Afzal M.T. (Ed.), (InTech Open Science, Rijeka, 2012), 99-120.
[34] Schellekens, M. *The Smyth completion: A common foundation for denotational semantics and complexity analysis*, Electronic Notes Theoret. Comput. Sci. **1**, 535-556, 1995.
[35] Schellekens, M. *A characterization of partial metrizability. Domains are quantifiable*, Theoret. Comput. Sci. **305**, 409-432, 2003.
[36] Schweizer, B. and Sklar, A. *Probabilistic Metric Spaces*, (North-Holland, New York 1983).
[37] Seda, A.K. *Quasi-metrics and the Semantics of Logic Programs*, Fund. Inform. **29**, 97-117, 1997.
[38] Seda, A.K. *Some Issues Concerning Fixed Points in Computational Logic: Quasi-Metrics, Multivalued Mappings and the Knaster-Tarski Theorem*, Topology Proc. **24**, 223-250, 1999.
[39] Shantawi, W. *Coupled fixed point theorems in generalized metric spaces*, Hacet. J. Math. Stat. **40** (3), 441-441, 2011.
[40] Smyth, M.B. *Quasi Uniformities: Reconciling Domains with Metric Spaces*, In: Main M., Melton A., Mislove, M. and Schmidt, D. (Eds.), Mathematical Foundations of Programming Language Semantics, Lect. Notes Comput. Sci. **298**, (Springer-Verlag, Berlin,1987), 236-253.
[41] Steen L.A. and , Seebach (Jr.) J.A. *Counterexamples in topology*, (Holt, Rinehart and Winston, Inc., New York,1970).
[42] Wald, A. *On a statistical generalization of metric spaces*, Proc. Nat. Acad. Sci. USA **29**, 196-197, 1943.
[43] Wiener, N. *Contribution to the theory of relative position*, Proc. Camb. Philos. Soc. **17**, 441-449, 1914.
[44] Wilson,W.A. *On quasi-metric spaces*, Amer. J. Math. **53**, 675-684, 1931.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 789 – 799

# On Gini mean difference bounds via generalised Iyengar results

Pietro Cerone *

**Abstract**

A variety of mathematical inequalities have been utilised to obtain approximation and bounds of the Gini mean difference.The Gini mean difference or the related index is a widely used measure of inequality in numerous areas such as in health, finance and population attributes arenas.The paper extends the Iyengar inequality to a Riemann-Stieltjes setting and obtains new results relating to the Gini mean difference.

*2000 AMS Classification:* 62H20, 62G10, 62P20, 26D15.

**Keywords:** Gini mean difference, Gini index, Lorenz curve, Variance, Coefficient of variation,Iyengar inequality and Riemann-Stieltjes.

Received 09/04/2013 : Accepted 21/03/2014 Doi : 10.15672 /HJMS.2015449430

## 1. Introduction

Let $f : \mathbb{R} \to [0, \infty)$ be a *probability density function* (pdf), meaning that $f$ is integrable on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(t)\,dt = 1$, and define

$$F(x) := \int_{-\infty}^{x} f(t)\,dt,\ x \in \mathbb{R} \quad \text{and} \quad E(f) := \int_{-\infty}^{\infty} x f(x)\,dx, \tag{1.1}$$

to be its *cumulative distribution function* and the *expectation* or *mean* provided that the integrals exist and are finite.

The *mean difference*

$$R_G(f) := \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|\,dF(x)\,dF(y) \tag{1.2}$$

was proposed by Gini in 1912 [14], after whom it is usually named, but it was discussed by Helmert and other German writers in the 1870's (cf. H.A. David [12]). The mean difference has a certain theoretical attraction, being dependent on the spread of the variate values among themselves rather than on the deviations from some central value

*Department of Mathematics and Statistics,La Trobe University,Bundoora VIC 3086, Australia.
Email: p.cerone@latrobe.edu.au

([21, p. 48]). Further, as noted by Kendall and Stuart ([21, p. 48]), its defining integral (1.2) may converge when the *variance* $\sigma^2(f)$,

$$(1.3) \qquad \sigma^2(f) := \int_{-\infty}^{\infty} (x - E(f))^2 \, dF(x),$$

does not. It can, however, be more difficult to compute than (1.3).

Another useful concept is the *mean deviation* $M_D(f)$, defined by [21, p. 48]

$$(1.4) \qquad M_D(f) := \int_{-\infty}^{\infty} |x - E(f)| \, dF(x) = 2 \int_{\mu}^{\infty} (x - E(f)) \, dF(x).$$

As G.M. Giorgi noted in [15], some of the many reasons for the success and the relevance of the Gini mean difference or *Gini index* $I_G(f)$,

$$(1.5) \qquad I_G(f) = \frac{R_G(f)}{E(f)},$$

are their simplicity, certain interesting properties and useful decomposition possibilities, and these attributes have been analysed in an earlier work by Giorgi [16]. For a bibliographic portrait of the Gini index, see [15] where numerous references are given.

The Gini index given by (1.5) is a measure of relative inequality since it is a ratio of the Gini mean difference, a measure of dispersion, to the average value $\mu = E(f)$. Other measures are the coefficient of variation $V = \frac{\sigma}{\mu}$ and half the relative mean deviation $\frac{M_D(f)}{2\mu}$ where $M_D(f)$ is as defined in (1.4).

From (1.1), $F(x)$ is assumed to strictly increase on its support and its mean $\mu = E(f)$ exist. These assumptions imply that $F^{-1}(p)$ is well defined and is the population's $p^{\text{th}}$ quantile. The theoretical Lorenz curve (Gastwirth [13]) corresponding to a given $F(x)$ is defined by

$$(1.6) \qquad L(p) = \frac{1}{\mu} \int_0^p F^{-1}(x) \, dx, \qquad 0 \le p \le 1.$$

Now $F^{-1}(x)$ is non decreasing and so from (1.6) $L(p)$ is convex and $L'(p) = 1$ at $p = F(\mu)$.

The area between the Lorenz curve and the line $p$, is known as the area of concentration.

The most common measure of inequality is the Gini index defined by (1.5) which may be shown to be equivalent to twice the area of concentration ([13])

$$(1.7) \qquad C = \int_0^1 c(p) \, dp, \quad c(p) = p - L(p).$$

$c(p)$ vanishes at $p = 0$ or 1 and is concave since $L(p)$ is convex. Further, there is a *point of maximum discrepancy* $p^*$ between the Lorenz curve and the line of equality which satisfies

$$(1.8) \qquad c(p^*) \ge c(p) \qquad \text{for all } \; p \in [0, 1].$$

The point $p^* = F(\mu)$ and $c(p^*) = \frac{M_D(f)}{2\mu}$ where $M_D(f)$ is given by (1.4).

The study of income inequality has gained considerable importance and the the Lorenz curve and the associated Gini mean or Gini index are certainly the most popular meausres of income inequality. These have also however found application in many other problems within the health, finance and population arenas.

In a sequence of four papers, Cerone and Dragomir ([6] – [10]) developed approximation and bounds from identities involving the Gini mean difference $R_G(f)$. Some of these results involved using the well known Sonin and Korkine identities. Cerone [3] procured some approximations and bounds utilising the well known Steffensen and Karamata inequalities. Further, the characteristics of the Lorenz curve, $L(p)$ and its connection to the Gini index via (1.7) to obtain upper and lower bounds for both $L(p)$ and $I_G(f)$ was analysed by the author in [4]. This was accomplished by utilising the well known Young's integral inequality and some less well known reverse inequalities.

The main aim of the current paper is to develop generalisations and extensions of the Iyengar inequality to allow the approximation and bounds of Riemann-Stieltjes integrals and weighted integrals in a less restrictive framework.These developments are then used to procure novel results for the approximation and bounds of the Gini mean difference.

## 2. Some identities Associated with the Gini mean difference

Some identities for the Gini mean difference, $R_G(f)$ through which results for the Gini index $I_G(f)$ may be procured via the relationship (1.5) will be stated here. These have been used in [6] – [10] to obtain approximations and bounds. The reader is referred to the book [21], Exercise 2.9, p. 94 or [6].

The following results hold (see for instance [21, p. 54] or [6];[7],using the well known Sonin identity; and [8] using the Korkine identity respectively.

**2.1. Theorem.** *With the above notation, the identities*

$$R_G(f)=\int_{-\infty}^{\infty}(1-F(y))F(y)\,dy=2\int_{-\infty}^{\infty}xf(x)F(x)\,dx-E(f); \tag{2.1}$$

$$\begin{aligned} R_G(f) &= 2\int_{-\infty}^{\infty}(x-E(f))(F(x)-\gamma)f(x)\,dx \\ &= 2\int_{-\infty}^{\infty}(x-\delta)\left(F(x)-\frac{1}{2}\right)f(x)\,dx \end{aligned} \tag{2.2}$$

*for any $\gamma,\delta\in\mathbb{R}$; and*

$$R_G(f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-y)(F(x)-F(y))f(x)f(y)\,dxdy, \tag{2.3}$$

*hold.*

The following lemma was proven in [4] bounding the Gini index via the Lorenz curve and the area of concentration $C$. The identity is also proven in [21, p. 49] in a different way.

**2.2. Lemma.** *The following identity holds*

$$R_G(f)=\mu I_G(f)=2\mu C, \tag{2.4}$$

*where the quantities are defined by (1.2), (1.5), (1.6) – (1.7).*

## 3. Iyengar Inequality for Riemann-Stieltjes Integrals

In 1938 Iyegar using geometric arguments developed the following result in the paper [18] .

**3.1. Theorem.** *Let $h : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ and for some $M > 0$ we have $\left|h'(x)\right| \leq M$ then,*

$$\left|\int_a^b h(x)\,dx - \frac{h(a)+h(b)}{2}(b-a)\right| \leq \frac{M}{4}(b-a)^2 - \frac{(h(b)-h(a))^2}{4M}. \tag{3.1}$$

**Remark**

It should be noted that for $m \leq h'(x) \leq M$ then $\left|h'(x) - \frac{m+M}{2}\right| \leq \frac{M-m}{2}$ so that the Iyengar's result may be extended by applying it to $k(x) = h(x) - \frac{m+M}{2}x$ with bound $M_k = \frac{M-m}{2}$.

The following result extends the Iyengar inequality to involve Riemann-Stieltjes integrals while also relaxing the differentiability condition.

**3.2. Theorem.** *Let $h, g : [a,b] \to \mathbb{R}$ be such that $g$ is non decreasing function and for all $x \in [a,b]$ and $M > 0$ the following conditions hold,*

$$|h(x) - h(a)| \leq M \cdot (x-a) \text{ and } |h(x) - h(b)| \leq M \cdot (b-x)\ . \tag{3.2}$$

*Then for any $t \in [a,b]$*

$$\left|\int_a^b h(x)\,dg(x) - \{[g(t)-g(a)]\,h(a) + [g(b)-g(t)]\,h(b)\}\right| \tag{3.3}$$

$$\leq M\left[\int_a^t (x-a)dg(x) + \int_t^b (b-x)dg(x)\right]. \tag{3.4}$$

*Proof.* We have from (3.2)

$$\begin{aligned} h(a) - M(x-a) &\leq h(x) \leq h(a) + M(x-a) \text{ and} \\ h(b) - M(b-x) &\leq h(x) \leq h(b) + M(b-x) \end{aligned}$$

so that since $g(x)$ is non decreasing on $[a,b]$ it follows that

$$h(a)\int_a^t dg(x) - M\int_a^t (x-a)dg(x) \leq \int_a^t h(x)dg(x) \leq h(a)\int_a^t dg(x) + M\int_a^t (x-a)dg(x)$$

and

$$h(b)\int_t^b dg(x) - M\int_t^b (b-x)dg(x) \leq \int_t^b h(x)dg(x) \leq h(b)\int_t^b dg(x) + M\int_t^b (b-x)dg(x).$$

Combining the last two results produces

$$\begin{aligned} &-M\left[\int_a^t (x-a)dg(x) + \int_t^b (b-x)dg(x)\right] \\ &\leq \int_a^b h(x)dg(x) - \left\{h(a)\int_a^t dg(x) + h(b)\int_t^b dg(x)\right\} \\ &\leq M\left[\int_a^t (x-a)dg(x) + \int_t^b (b-x)dg(x)\right]. \end{aligned} \tag{3.5}$$

Simplifying and using the properties of the modulus produces (3.3). □

**3.3. Corollary.** *Let the conditions of Theorem 3.2 persist then the coarser but simpler bound is given by,*

$$\begin{aligned} &\left|\int_a^b h(x)\,dg(x) - \{[g(t)-g(a)]\,h(a) + [g(b)-g(t)]\,h(b)\}\right| \\ &\leq M\left[\frac{b-a}{2} + \left|t - \frac{a+b}{2}\right|\right] (g(b)-g(a)), \end{aligned} \tag{3.6}$$

*with the smallest bound occuring at* $t = \frac{a+b}{2}$.

*Proof.* Let the bound from (3.3) be denoted by

$$B(t) := \int_a^t (x-a)dg(x) \ + \ \int_t^b (b-x)dg(x)$$

so that

$$\begin{aligned} |B(t)| &= \left| \int_a^b K(x,t)dg(x) \right| \leq \int_a^b |K(x,t)|\, dg(x) \\ &\leq \sup_{x\in[a,b]} |K(x,t)| \int_a^b dg(x) = \max\{t-a, b-t\}\,(g(b)-g(a)). \end{aligned}$$

Now, using the fact that $\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|$ produces (3.6) and the fact that the best of these occurs at $t = \frac{a+b}{2}$ is obvious. □

**3.4. Theorem.** *Let* $h, g : [a,b] \to \mathbb{R}$ *be such that* $g$ *is non decreasing for all* $x \in [a,b]$ *and for* $M > 0$ *the following conditions hold,*

$$|h(x) - h(a)| \leq M \cdot (x-a) \ \text{and} \ |h(x) - h(b)| \leq M \cdot (b-x) \ . \tag{3.7}$$

*Then for* $t \in [a,b]$ *the tightest bound is given by*

$$-MD(t^*) \leq \int_a^b h(x)\, dg(x) \ - [h(b)g(b) - h(a)g(a)] \leq MD(t_*), \tag{3.8}$$

*or*

$$-2M\delta(t_m)g(t_m) \leq \int_a^b h(x)\, dg(x) \ - [h(b)g(b) - h(a)g(a)] \leq 2M\Delta(t_m)g(t_m), \tag{3.9}$$

*where for* $\alpha = \frac{a+b}{2}$ *and* $\beta = \frac{h(b)-h(a)}{2}$; $t^* = \alpha - \frac{\beta}{M}$ *and* $t_* = \alpha + \frac{\beta}{M}$ *or* $D(t_m) = 0$ *with*

$$\begin{aligned} D(t) &= \int_t^b g(x)dx - \int_a^t g(x)dx \quad \text{and} \\ \delta(t) &= \frac{h(b)-h(a)}{2M} - \left(t - \frac{a+b}{2}\right), \\ \Delta(t) &= -\left[\frac{h(b)-h(a)}{2M} + \left(t - \frac{a+b}{2}\right)\right]. \end{aligned}$$

*Here,* $t^* \in [a, \frac{a+b}{2}]$ *and* $t_* \in [\frac{a+b}{2}, b]$.

*Proof.* From (3.5) we have on integration by parts of the Riemann-Stieltjes integrals $\int_t^b (b-x)dg(x)$ and $\int_a^t (x-a)dg(x)$,

$$L(t) \leq \int_a^b h(x)\, dg(x) \ - [h(b)g(b) - h(a)g(a)] \leq R(t) \tag{3.10}$$

where

$$\begin{aligned} L(t,-M) &= -2M\left[\frac{h(b)-h(a)}{2M} - \left(t - \frac{a+b}{2}\right)\right] g(t) \\ &\quad -M\left[\int_t^b g(x)dx - \int_a^t g(x)dx\right] \end{aligned} \tag{3.11}$$

and $R(t) = L(t,M)$.

We notice that (3.11) may be simplified by choosing $t = t^* = \alpha - \frac{\beta}{M}$ or $t = t_m$ where $D(t_m) = 0$ to produce the two lower bounds in (3.8) and (3.9). A similar reasoning provides the two upper bounds where $t = t_* = \alpha + \frac{\beta}{M}$.

The best bounds may be procured from the supremum of the lower bounds and the infimum of the upper bounds for $t \in [a, b]$ .Further, using the conditions in (3.7) it may be demonstrated that $t^* \in [a, \frac{a+b}{2}]$ and $t_* \in [\frac{a+b}{2}, b]$. □

The following theorem develops a weighted Iyengar inequality.

**3.5. Theorem.** *Let $h, w : [a, b] \to \mathbb{R}$ be such that $w(x) > 0$ for $x \in (a, b)$ and for $M > 0$ the following conditions hold,*

$$|h(x) - h(a)| \leq M \cdot (x - a) \text{ and } |h(x) - h(b)| \leq M \cdot (b - x) \ . \tag{3.12}$$

*Then for $t \in (a, b)$ the tightest bound is given by*

$$\begin{aligned} &\left| \int_a^b w(x)h(x)\, dx \ - \{h(b)W(b) + M\,[I(t_*) - \ I(t^*)]\} \right| \\ \leq \quad & M \left\{ \int_a^b (b - x)w(x)dx \ - [I(t^*) + \ I(t_*)] \right\}, \end{aligned} \tag{3.13}$$

*where for $\alpha = \frac{a+b}{2}$ and $\beta = \frac{h(b)-h(a)}{2}$; $t^* = \alpha - \frac{\beta}{M}$ and $t_* = \alpha + \frac{\beta}{M}$ with*

$$I(t) = \int_a^t (t - x)w(x)dx \ . \tag{3.14}$$

*If $w(a) = 0$ then the bounds at $t = a$ need to be compared with $L(t^*)$ and $R(t_*)$ and similarly for $w(b) = 0$.*

*Proof.* Let $g(x) = \int_a^x w(u)du$ in (3.5) then

$$H(t) - M \cdot K(t) \leq \int_a^b w(x)h(x)\, dx \ \leq H(t) + M \cdot K(t) \tag{3.15}$$

where

$$W(t) = \int_a^t w(x)dx \ , \tag{3.16}$$

$$H(t) = h(a)W(t) + h(b)\,[W(b) - W(t)] \text{ and,} \tag{3.17}$$

$$K(t) = \int_a^t (x - a)w(x)dx \ + \int_t^b (b - x)w(x)dx.$$

If we now let $L(t, -M)$ represent the lower bound (3.15) $L(t)$, namely

$$L(t) = H(t) - M \cdot K(t) \tag{3.18}$$

and $R(t) = L(t, M)$ represent the upper bound,

$$R(t) = H(t) + M \cdot K(t). \tag{3.19}$$

so that (3.15) may be written in the form

$$\left| \int_a^b w(x)h(x)\, dx - \frac{R(t) + L(t)}{2} \right| \leq \frac{R(t) - L(t)}{2}. \tag{3.20}$$

Then we have that

$$L'(t) = \{[h(a) - h(b)] - M \cdot [2t - (a + b)]\}\, w(t)$$

and so the largest lower bound occurs at $t^* = \frac{a+b}{2} - \frac{h(b)-h(a)}{2M}$ since $w(t) > 0$ for $t \in (a, b)$. In a similar fashion we have that the smallest upper bound occurs at $t_* = \frac{a+b}{2} + \frac{h(b)-h(a)}{2M}$.

Thus we have from (3.15) that

$$L(t^*) \leq \int_a^b w(x)h(x)\, dx \ \leq R(t_*) \tag{3.21}$$

and so

$$(3.22)\qquad \left|\int_a^b w(x)h\,(x)\,dx - \frac{R(t_*)+L(t^*)}{2}\right| \le \frac{R(t_*)-L(t^*)}{2}.$$

where after some simplification,

$$(3.23)\qquad L(t^*) = h(b)W(b) - M\int_a^b (b-x)w(x)dx + 2M\int_a^{t^*}(t^*-x)w(x)dx$$

and,

$$(3.24)\qquad R(t_*) = h(b)W(b) + M\int_a^b (b-x)w(x)dx - 2M\int_a^{t_*}(t_*-x)w(x)dx.$$

The result (3.13) is procured following some straight forward algebra from (3.22). □

**Remark**

It should be noted that taking $w(x) = 1$ in Theorem 3.5 recaptures the Iyengar result of Theorem 3.1 under less restrictive conditions (3.12) rather than $\left|h^{'}(x)\right| < M$. It should be further emphasised that for $m \le \frac{h(x)-h(a)}{x-a} \le M$ and $m \le \frac{h(b)-h(x)}{b-x} \le M$ the above results may be extended by taking $k(x) = h(x) - \frac{M+m}{2}x$ to produce the conditions of the above results for $|k(x)-k(a)| \le \frac{M-m}{2}\cdot(x-a)$ and $|k(x)-k(b)| \le \frac{M-m}{2}\cdot(b-x)$.

## 4. Application of Extended Iyengar Results to Gini Mean Difference

We are now in a positition to obtain bounds utilising the Iyengar type inequalities developed above to obtain approximation and bounds for the Gini mean difference. We shall make use of the following identities, where $f$ is the pdf and $F$ its corresponding distribution,

$$(4.1)\qquad R_G\,(f) = \int_a^b (1-F\,(x))\,F\,(x)\,dx = 2\int_a^b xf\,(x)\,F\,(x)\,dx - E\,(f)\,.$$

**4.1. Theorem.** *Let $f\,(x)$ be a pdf on $[a,b]$, $f(x) \le M$ and $F(x) = \int_a^x f(u)du$ then the Gini Mean Difference $R_G\,(f)$ satisfies*

$$(4.2)\qquad \begin{aligned} &|R_G\,(f) + E(f) - 2\left\{bf(b)E(f) + M\left[I(t^*) - I(t_*)\right]\right\}| \\ \le\quad & 2M\left\{\int_a^b (b-x)xf(x)dx \;-\left[I(t_*)+I(t^*)\right]\right\}\end{aligned}$$

*where $t^* = \frac{a+b}{2} - \frac{bf(b)-af(a)}{2M}$ and $t_* = \frac{a+b}{2} + \frac{bf(b)-af(a)}{2M}$ with,*

$$I(t) = \int_a^t (t-x)xf(x)dx\ .$$

*For $f(a) = 0$ we have*

$$(4.3)\qquad |R_G\,(f) - [2bf(b)-1]\,E(f)| \le 2M\int_a^b (b-x)xf(x)dx.$$

*For $f(b) = 0$ we have*

$$(4.4)\qquad |R_G\,(f) - [2af(a)-1]\,E(f)| \le 2M\int_a^b (x-a)xf(x)dx.$$

*Finally, for $f(a) = f(b) = 0$ we have*

$$(4.5)\qquad |R_G\,(f) + E(f)| \le 2M\left\{\frac{b-a}{2}E(f) - \left|\mathcal{M}_2 - \frac{b+a}{2}E(f)\right|\right\}$$

*where* $\mathcal{M}_2 = \int_a^b x^2 f(x)dx$, *the second moment of* $f(x)$.

*Proof.* In Theorem 3.5 let $w(x) = xf(x)$ and $h(x) = F(x)$ so that $\left|h^{'}(x)\right| = f(x) \leq M$.

Now from (4.1) we have

$$\frac{R_G\left(f\right)+E\left(f\right)}{2} = \int_a^b xf\left(x\right)F\left(x\right)dx, \tag{4.6}$$

and so we have three possible cases to consider.

The first is that $w(x) = xf(x) > 0$ for $x \in [a,b]$ which we have from (3.20) that

$$\begin{aligned}&\left|\int_a^b xf(x)F\left(x\right)dx \ - \left\{bf(b)E(f) + M\left[I(t^*) - I(t_*)\right]\ \right\}\right| \\ \leq\ & M\left\{\int_a^b (b-x)xf(x)dx \ - \left[I(t_*) + I(t^*)\right]\right\}\end{aligned}$$

where $t^* = \frac{a+b}{2} - \frac{bf(b)-af(a)}{2M}$ and $t_* = \frac{a+b}{2} + \frac{bf(b)-af(a)}{2M}$ with,

$$I(t) = \int_a^t (t-x)xf(x)dx\ .$$

Using the identity (4.6) produces the result as stated in (4.2).

Now for $w(a) = af(a) = 0$ we have from (3.15)- (3.20),

$$\begin{aligned}L(a) &= bf(b)E(f) - M\int_a^b (b-x)xf(x)dx \text{ and,} \\ R(a) &= bf(b)E(f) + M\int_a^b (b-x)xf(x)dx\end{aligned}$$

and so $\frac{R(a)\ +\ L(a)}{2} = bf(b)E(f)$ and $\frac{R(a)\ -\ L(a)}{2} = M\int_a^b (b-x)xf(x)dx$ which results in (4.3) on using (3.20) .

For $w(b) = bf(b) = 0$ we have from (3.15)- (3.20),

$$\begin{aligned}L(b) &= af(a)E(f) - M\int_a^b (x-a)xf(x)dx \text{ and,} \\ R(b) &= af(a)E(f) + M\int_a^b (x-a)xf(x)dx\end{aligned}$$

and so $\frac{R(b)\ +\ L(b)}{2} = af(a)E(f)$ and $\frac{R(b)\ -\ L(b)}{2} = M\int_a^b (x-a)xf(x)dx$ from which we obtain (4.4) on using (3.20) .

Finally, for $w(a) = w(b) = 0$ so that $f(a) = f(b) = 0$ then from (4.3) and (4.4) on choosing the minimum of the bounds produces the stated result. □

**4.2. Theorem.** *Let* $f(x)$ *be a pdf on* $[a,b]$, $f(x) \leq M$ *and* $F(x) = \int_a^x f(u)du$ *then the Gini Mean Difference* $R_G(f)$ *satisfies*

$$\begin{aligned}&\left|R_G\left(f\right) - \left\{E(f) + M\left[J(t_*) - J(t^*)\right]\right\}\right| \qquad (4.7)\\ \leq\ & M\left\{\frac{1}{2}\left[(b-a)^2 - (t_*-a)^2 - (t^*-a)^2\right]\right. \\ &\left. -J(b) + \left[\ J(t_*) + J(t^*)\right]\right\} \qquad (4.8)\end{aligned}$$

*where* $t^* = \frac{a+b}{2} - \frac{1}{2M}$ *and* $t_* = \frac{a+b}{2} + \frac{1}{2M}$ *with*

$$J(t) = \int_a^t (t-x)F(x)dx\ .$$

*Further, for $F(b) = 1$ we have*

$$(4.9) \qquad |R_G(f)| \leq \frac{M}{2}\int_a^b (x-a)^2 f(x)dx = \frac{M}{2}\{\mathcal{M}_2 - a[2E(f) - a]\}.$$

where $\mathcal{M}_2 = \int_a^b x^2 f(x)dx$.

*Proof.* In Theorem 3.5 let $w(x) = 1 - F(x)$ and $h(x) = F(x)$ so that $\left|h'(x)\right| = f(x) \leq M$.Now from (4.1) we have

$$(4.10) \qquad R_G(f) = \int_a^b (1 - F(x))F(x)dx;$$

and so we have two possible cases to consider namely, that $w(x) = 1 - F(x) > 0$ for $x \in [a, b)$ and $w(b) = 0$.

Now for $t \in [a, b)$ we have from (3.22) that

$$\begin{aligned} & |R_G(f) - \{E(f) + M[I(t^*) - I(t_*)]\}| \\ \leq \quad & M\left\{\int_a^b (b-x)(1-F(x))dx \; - [\, I(t_*) + I(t^*)]\right\} \end{aligned}$$

where $t^* = \frac{a+b}{2} - \frac{1}{2M}$ and $t_* = \frac{a+b}{2} + \frac{1}{2M}$ with,

$$I(t) = \int_a^t (t-x)(1-F(x))dx \; .$$

After some algebraic simplification the results as depicted in (4.7) are establisted.

Now, for $w(b) = 1 - F(b) = 0$ we have from (3.18) - (3.20),

$$L(b) = -M\int_a^b (x-a)(1-F(x))dx \text{ and } R(b) = M\int_a^b (x-a)(1-F(x))dx$$

and so $\frac{R(b)+L(b)}{2} = 0$ and $\frac{R(b)-L(b)}{2} = M\int_a^b (x-a)(1-F(x))dx$ from which we obtain (4.9) on using (3.20) and some simplification. □

An investigation of bounds for the Gini mean difference from the Iyengar inequality (3.1) and the identity depicted in Lemma 2.2 reproduces a the result

$$0 \leq R_G(f) \leq \frac{1}{b-a}(b - E(f))(E(f) - a),$$

obtained by Gastwirth [13, p. 308] by a different approach.

**Conclusion**

The paper has extended results relating to the Ingear inequality to less restrictive conditions and involving Reimann-Stieltjes integrals. This in turn has let to a weighted version in form of Theorem 3.5 which recaptures the Iyengar result when the weight function is 1. The generalised Iyengar results are then used in the final section to obtain approximation and bounds for the Gini Mean Difference.The novel bounds for realistic pdfs such as those contained in (4.5) and (4.9) involve the first and second moments.

**Acknowledgement**

Some of the work for this article was undertaken while at Victoria University and some of the initial ideas were developed while on sabbatical at the University of Melbourne.

The author would also like to thank the referee's thorough contribution.

## References

[1] P. CERONE, On an identity for the Chebychev functional and some ramifications, *J. Ineq. Pure and Appl. Math.,* **3**(1) Art. 4, (2002). [ONLINE `http://jipam.vu.edu.au/v3n1/`].

[2] P. CERONE, On some generalisatons of Steffensen's inequality and related results, *J. Ineq. Pure and Appl. Math.,* **2**(3) Art. 28, (2001). [ONLINE `http://jipam.vu.edu.au/v2n3/`].

[3] P. CERONE, Bounding the Gini mean difference, *Inequalities and Applications.* International Series of Numerical Mathematics. Vol. 157. C. Bandle, A. Losonczi, Zs. Pales and M. Plum, Eds.,Birkhüser Verlag Basel (2008), 77-89.

[4] P. CERONE, On Young's inequality and its reverse for bounding the Lorenz curve and Gini mean*, Journal of Mathematical Inequalities* **3(3)**(2009), 369-381.

[5] P. CERONE and S.S. DRAGOMIR, A refinement of the Grüss inequality and applications, *Tamkang J. Math.*, **38**(1) (2007), 37-49. (See also *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 14. [ONLINE: `http://rgmia.vu.edu.au/v5n2.html`]

[6] P. CERONE and S.S. DRAGOMIR, A survey on bounds for the Gini mean difference, *Advances in Inequalities from Probability Theory and Statistics*, N.S. Barnett and S.S. Dragomir (Eds.), Nova Science Publishers, (2008), 81-111.

[7] P. CERONE and S.S. DRAGOMIR, Bounds for the Gini mean difference via the Sonin identity, *Comp. Math. Modelling*, **50** (2005), 599-609.

[8] P. CERONE and S.S. DRAGOMIR, Bounds for the Gini mean difference via the Korkine identity, *J. Appl. Math. & Computing* (Korea), **22**(3) (2006), 305-315.

[9] P. CERONE and S.S. DRAGOMIR, Bounds for the Gini mean difference of continuous distributions defined on finite intervals (I), *Applied Mathematics Letters,* **20** (2007), 782-789.

[10] P. CERONE and S.S. DRAGOMIR, Bounds for the Gini mean difference of continuous distributions defined on finite intervals (II), *Comput. Math. Appl.*, **52**(10-11) (2006), 1555-1562.

[11] X.-L. CHENG and J. SUN, A note on the perturbed trapezoid inequality, *J. Inequal. Pure & Appl. Math.*, 3(2) (2002), Article. 29,
[ONLINE `http://jipam.vu.edu.au/article.php?sid=181`]

[12] H.A. DAVID, Gini's mean difference rediscovered, *Biometrika*, **55** (1968), 573.

[13] J.L. GASTWIRTH, The estimation of the Lorentz curve and Gini index, *Rev. Econom. Statist.*, **54** (1972), 305-316.

[14] C. GINI, Variabilità e Metabilità, contributo allo studia della distribuzioni e relationi statistiche, *Studi Economica-Gicenitrici dell' Univ. di Coglani,* **3** (1912), art 2, 1-158.

[15] G.M. GIORGI, Bibliographic portrait of the Gini concentration ratio, *Metron,* **XLVIII**(1-4) (1990), 103–221.

[16] G.M. GIORGI, Alcune considerazioni teoriche su di un vecchio ma per sempre attuale indice: il rapporto di concentrazione del Gini, *Metron,* **XLII**(3-4) (1984), 25–40.

[17] G.H. HARDY, J.E. LITTLEWOOD and G. POLYA, *Inequalities*, Cambridge Univ. Press., Cambridge,1952

[18] K.S.K.IYENGAR, Note on an inequality.*Math. Student,***6(**1938), 75-76.

[19] J. KARAMATA, O prvom stavu srednjih vrednosti odredenih integrala, *Glas Srpske Kraljevske Akademje*, **CLIV** (1933), 119-144.

[20] J. KARAMATA, Sur certaines inégalités relatives aux quotients et à la différence de $\int fg$ et $\int f \cdot \int g$, *Acad. Serbe Sci. Publ. Inst. Math.,* **2** (1948), 131-145.

[21] M. KENDALL and A. STUART, *The Advanced Theory of Statistics*, Volume 1, Distribution Theory, Fourth Edition, Charles Griffin & Comp. Ltd., London, 1977.

[22] A. LUPAŞ, On two inequalities of Karamata, *Univ. Beograd. Publ. Elektrotechn. Fak. Ser. Mat. Fiz.,* No. 602-633 (1978), 119-123.

[23] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[24] J.F. STEFFENSEN, On certain inequalities between mean values and their application to actuarial problems, *Skandinavisk Aktuarietidskrift,* (1918), 82-97.

[25] A. WITKOWSKI, On Young's inequality, *J. Ineq. Pure and Appl. Math.,* **7**(5) Art. 164, (2007). [ONLINE `http://jipam.vu.edu.au/article.php?sid=782`].

[26] W.H. YOUNG, On classes of summable functions and their Fourier series, *Proc. Roy. Soc. London* (A), **87** (1912), 225-229.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 801 – 811

# Quasi-primary submodules satisfying the primeful property II

Hosein Fazaeli Moghimi * and Mahdi Samiei †

**Abstract**

In this paper we continue our study about quasi-primary submodules (probably satisfying the primeful property), that was defined and studied in Part I (see [8]). We define a quasi-primary decomposition for submodules of a module over a commutative ring with identity and study various types of the corresponding minimal forms. In particular, we discuss these decompositions for submodules of multiplication modules and also arbitrary modules over Noetherian rings.

*2000 AMS Classification:* 13C13, 13C99.

**Keywords:** Quasi-primary submodule, Primeful property, Radical of a submodule, Multiplication module, Reduced quasi-primary decomposition, Module-reduced quasi-primary decomposition, Shortest quasi-primary decomposition.

Received 28/01/2013 : Accepted 14/04/2014 Doi : 10.15672 /HJMS.2015449436

## 1. INTRODUCTION

Throughout this paper all rings are commutative with identity, and all modules are unitary. Recently, the decomposition theory associated with various generalizations of prime and primary ideals has been the domain of concerns of many researches (see for example [18, 21, 24]). Here we follow this topic in the context of quasi-primary submodules; the recent generalization of quasi-primary ideals. Some concepts which are used frequently in this paper have been gathered in the following definition.

**1.1. Definition.** Let $N$ be a proper submodule of an $R$-module $M$.

(1) $N$ is prime(resp. primary) if $rx \in N$ for $r \in R$ and $x \in M$ implies either $r \in (N : M)$ (resp. $r \in \sqrt{(N : M)}$) or $x \in N$ (see [5, 14, 22, 15, 17]).

(2) The intersection of all prime submodules of $M$ containing $N$, denoted $radN$, is called prime radical of $N$ (see [3, 10, 13, 16, 19, 26]).

*Department of Mathematics, University of Birjand, P.O.Box 97175-615, Birjand, Iran. Email: hfazaeli@birjand.ac.ir

†Department of Mathematics, University of Birjand, P.O.Box 97175-615, Birjand, Iran. Email:mahdisamiei@birjand.ac.ir

(3) $N$ is quasi-primary if $rx \in N$ for $r \in R$ and $x \in M$, then either $r \in \sqrt{(N:M)}$ or $x \in radN$. Clearly every primary submodule is quasi-primary, but not conversely in general (see Example 1.2 and Example 2.3).
(4) $N$ satisfies the primeful property provided that for every prime ideal $p$ containing $(N:M)$ there exists a prime submodule $P$ contains $N$ such that $(P:M)=p$. In particular, $M$ is primeful if the zero submodule of $M$ satisfies the primeful property. Every submodule of a finitely generated module satisfies the primeful property (see [8, 12]).
(5) $N$ has a quasi-primary decomposition if $N = N_1 \cap N_2 \cap \cdots \cap N_s$, where each $N_i$ is a quasi-primary submodule of $M$. If $N_i \not\supseteq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_s$, then the above quasi-primary decomposition is called
 (5.1) reduced, if the ideals $\sqrt{(N_i:M)}$ are distinct primes.
 (5.2) module-reduced, if the submodules $radN_i$ are distinct primes.
 (5.3) shortest, if none of the intersection $(N_{i_1}:M) \cap (N_{i_2}:M) \cap \cdots \cap (N_{i_t}:M)$ $(t>1)$ is a quasi-primary ideal.
(6) An $R$-module $M$ is said to be a multiplication module, if every submodule of $M$ has the form $IM$ for some ideal $I$ of $R$. For example any cyclic module is a multiplication module. However, there is a multiplication module which is not finitely generated [7, p.770]. Also, free modules with finite rank greater than one are finitely generated modules which are not multiplication modules [15, Corollary 2.5 and Theorem 3.5]. It is well-known that $M$ is a multiplication $R$-module if and only if for each submodule $N$ of $M$, $N = (N:M)M$. (see for more study [1, 7, 23]).
(7) The support of M, written $Supp(M)$, is defined to be the set of prime ideals $p$ of $R$ such that $M_p \neq 0$ (see [6, 20]).
(8) A prime ideal $p$ of $R$ is associated to $M$ if $p$ is the annihilator of an element of $M$. The set of all primes associated to $M$ is denoted by $Ass(M)$ (see [6, 20]).

**1.2. Example.** Indeed, every power of a prime ideal as well as that of a primary or a quasi-primary ideal is quasi-primary; but a power of a prime ideal is not necessarily primary (for example see [2, Example after proposition 4.1, part 3]). Now we follow this fact to give an example in the module setting. It is well-known that if $F$ is a free $R$-module and $I$ is an ideal of $R$, then $(IF:F)=I$ and $rad(IF)=\sqrt{I}F$ [25, Proposition 2.2]. It is routine to verify that $q$ is a quasi-primary (resp. primary, prime) ideal of $R$ if and only if $qF$ is a quasi-primary (resp. primary, prime) submodule of $F$ [8, Theorem 2.19]. These show that there is a rich supply of quasi-primary submodules which are not primary.

Recall that a proper ideal $q$ of $R$ is quasi-primary if $rs \in q$ for $r, s \in R$ implies $r \in \sqrt{q}$ or $s \in \sqrt{q}$ (see [8, 9]). It is well-known that $q$ is a quasi-primary ideal of $R$ if and only if $\sqrt{q}$ is a prime ideal of $R$ [9, p.176]. For a submodule $N$ of a multiplication $R$-module $M$ which satisfies the primeful property, we prove that $N$ is a quasi-primary submodule of $M$ if and only if $(N:M)$ is a quasi-primary ideal of $R$ if and only if $radN$ is a prime submodule of $M$ if and only if $N = qM$ for some quasi-primary ideal $q$ of $R$ with $ann(M) \subseteq q$ (Theorem 2.2). We use this fact to investigate the relationships between reduced and module-reduced and shortest quasi-primary decompositions of submodules of multiplication modules (Corollary 2.6 and Proposition 2.11 and Theorem 2.13). Also we give some uniqueness theorems as follow:
**Theorem 2.13.** Let $M$ be a multiplication $R$-module and $N$ a submodule of $M$. Let $N = N_1 \cap N_2 \cap \cdots \cap N_s = N'_1 \cap N'_2 \cap \cdots \cap N'_t$ be two reduced quasi-primary decompositions of $N$ as intersection of quasi-primary submodules satisfying the primeful property. Then $s=t$ and the prime ideals $p_i = \sqrt{(N_i:M)}$ must be, without regard to their order,

identical to the prime ideals $p'_j = \sqrt{(N'_j : M)}$.

**Theorem 3.5.** Let $R$ be a Noetherian ring and $M$ an $R$-module. Let $N$ be a submodule of $M$ such that $N = N_1 \cap N_2 \cap \cdots \cap N_s = N'_1 \cap N'_2 \cap \cdots \cap N'_t$ be two reduced quasi-primary decompositions of $N$ where $N_i$(resp. $N'_j$) is $\mathrm{p}_i$-quasi-primary (resp. $\mathfrak{p}_j$-quasi-primary). Then $s = t$ and (after reordering if necessary) $\mathrm{p}_i = \mathfrak{p}_i$ and $radN_i = radN'_i$ for $1 \leq i \leq s$.

**Theorem 3.7.** Let $N$ be a proper submodule of a module $M$ over a Noetherian ring $R$. If $N = \cap_{i=1}^s N_i$ is a module-reduced quasi-primary decomposition and $N_i$ $(1 \leq i \leq s)$ satisfies the primeful property such that $radN = \cap_{i=1}^s radN_i$, then $Ass(M/radN) \subseteq \{p_1, \cdots, p_s\} \subseteq Supp(M/radN)$. In particular, $Ass(M/radN) = \{p_{i_1}, p_{i_2}, \cdots, p_{i_t}\}$ where $p_{i_j}$ $1 \leq j \leq t$ are minimal elements of $\{p_1, \cdots, p_s\}$.

**Theorem 3.11.** Let $M$ be a module over a Noetherian ring $R$. Let $N$ be a proper submodule of $M$ satisfying the primful property. If $N = \cap_{i=1}^s N_i$ is a module-reduced quasi-primary decomposition and $N_i$ satisfies the primeful property, $1 \leq i \leq s$, such that $radN = \cap_{i=1}^s radN_i$. If $p_j = \sqrt{(N_j : M)}$ is a minimal element of $\{p_1, \cdots, p_s\}$, then $radN_j$ is uniquely determined by $N$.

## 2. QUASI-PRIMARY SUBMODULES OF MULTIPLICATION MODULES

Let $M$ be a multiplication $R$-module. If $p$ is a prime ideal containing $ann(M)$, then $(pM : M) = p$ [7, Lemma 2.10]. In particular a proper submodule $pM$ is a prime submodule of $M$ if and only if $p$ is a prime ideal containing $ann(M)$ [7, Corollary 2.11]. Now we have the following result:

**2.1. Lemma.** *Let $R$ be a ring and $I$ an ideal of $R$. Let $M$ be a multiplication $R$-module. If $IM$ satisfies the primeful property, then so does $\sqrt{I}M$. In this case $\sqrt{(IM : M)} = \sqrt{(\sqrt{I}M : M)}$.*

*Proof.* Let $p$ be a prime ideal containing $(\sqrt{I}M : M)$. Since $IM$ satisfies the primeful property, there exists a prime submodule $P$ containing $IM$ such that $(P : M) = p$. By [7, Corollary 2.11], $P = p'M$ for some prime ideal $p'$ containing $ann(M)$. Since $IM \subseteq p'M$, by [7, Lemma 2.10] $I \subseteq p'$. Hence $\sqrt{I}M \subseteq P$, as required. Also the similar argument follows that $rad(IM) = rad(\sqrt{I}M)$ and so we have the second part. □

**2.2. Theorem.** *Let $N$ be a submodule of a multiplication $R$-module $M$ which satisfies the primeful property. Then the following statements are equivalent:*

(i) *$N$ is a quasi-primary submodule of $M$;*
(ii) *$(N : M)$ is a quasi-primary ideal of $R$;*
(iii) *$radN$ is a prime submodule of $M$;*
(iv) *$N = qM$ for some quasi-primary ideal $q$ of $R$ with $ann(M) \subseteq q$.*

*Proof.* (i)⇒ (ii) is clear, since $\sqrt{(N : M)} = (radN : M)$.
(ii)⇒ (iii). It is easy to check that $radN$ is a proper submodule of $M$, since $N$ satisfies the primeful property. Now the proof is completed by [7, Corollary 2.11 and Theorem 2.12].
(iii)⇒ (i) is obtained by a direct application of the definition of quasi-primary submodules.
(ii)⇒ (iv) is clear.
(iv)⇒ (iii). Let $q$ be a quasi-primary ideal of $R$ containing $(0 : M)$ and $N = qM$. By [7, Theorem 2.12] and Lemma 2.1, we have $radN = \sqrt{(N : M)}M = \sqrt{(qM : M)}M = \sqrt{(\sqrt{q}M : M)}M = \sqrt{\sqrt{q}}M = \sqrt{q}M$. Thus by [7, Corollary 2.11], $radN$ is a prime submodule of $M$. □

**2.3. Example.** Let $M$ be a finitely generated faithful multiplication $R$-module (for example $M$ can be considered as a non-zero ideal of a principal ideal domain $R$). Then for each ideal $I$ of $R$, $(IM : M) = I$ [7, Theorem 3.1]. Thus if $q$ is a quasi-primary ideal of $R$ which is not primary, then $qM$ is a quasi-primary submodule of $M$ which is not primary (see Theorem 2.2 above.)

**2.4. Proposition.** *Let $M$ be a non-zero multiplication $R$-module. If $ann(x) = 0$ for some $x \in M$, then every submodule of $M$ satisfies the primeful property.*

*Proof.* Assume $N$ is a submodule of $M$ and $p$ a prime ideal of $R$ containing $(N : M)$. It suffices to show that $pM$ is a prime submodule of $M$. By [7, Corollary 2.11], we must prove that $pM \neq M$. Assume on the contrary that $pM = M$. Suppose $x \in M$ and $ann(x) = 0$. Since $M$ is multiplication, there exists an ideal $J$ of $R$ such that $Rx = JM$. Thus $Rx = JM = JpM = pJM = px$ and so $1 - r \in ann(x)$ for some $r \in p$, a contradiction. □

It is well-known that if $M$ is a finitely generated multiplication $R$-module, then $M$ is weak cancellation, i.e. $IM \subseteq JM$, for ideals $I, J$ of $R$, implies $I \subseteq J + ann(M)$ ([1, Theorem 3] and [22, Corollary to Theorem 9]). By combining this fact and Theorem 2.2, we have the following immediate result.

**2.5. Corollary.** *Let $N$ be a submodule of a finitely generated multiplication $R$-module $M$. Then*

(i) *$N$ is a minimal quasi-primary submodule of $M$ if and only if there exists a minimal quasi-primary ideal* $\mathrm{q}$ *of $R$ containing $ann(M)$ such that $N = \mathrm{q}M \neq M$.*
(ii) *Every quasi-primary submodule of $M$ contains a minimal quasi-primary submodule.*

*Proof.* (i) is clear.
(ii). It suffices to show that every quasi-primary ideal of $R$ contains a minimal quasi-primary ideal. Let $\mathrm{q}$ be a quasi-primary ideal of $R$ and $\Lambda = \{\mathfrak{q} : \mathfrak{q}$ is a quasi-primary ideal of $R$ with $\mathfrak{q} \subseteq \mathrm{q}\}$. Since $\mathrm{q} \in \Lambda$, we have $\Lambda \neq \emptyset$. We define a partially order by reverse inclusion, that is, for $\mathfrak{q}_\mathfrak{i}, \mathfrak{q}_\mathfrak{j} \in \Lambda$, $\mathfrak{q}_\mathfrak{i} \preceq \mathfrak{q}_\mathfrak{j}$ if and only if $\mathfrak{q}_\mathfrak{i} \supseteq \mathfrak{q}_\mathfrak{j}$, so that a maximal member of this partially ordered set is just a minimal member of $\Lambda$ with respect to inclusion. Let $\Omega$ be a non-empty subset of $\Lambda$ which is totally ordered with respect to the above partial order. It is easy to verify that $Q = \cap_{\mathfrak{q} \in \Omega} \mathfrak{q}$ is an upper bound for $\Omega$ in $\Lambda$. Now Zorn's lemma completes the proof. □

In [7, Corollary 1.7], it has shown that if $M$ is a multiplication module, then $\cap_{\lambda \in \Lambda}(I_\lambda M) = (\cap_{\lambda \in \Lambda}[I_\lambda + annM])M$ for every non-empty collection of ideals $I_\lambda$ $(\lambda \in \Lambda)$ of $R$. Using this fact, we have the following result:

**2.6. Corollary.** *Let $M$ be a multiplication $R$-module and $N$ a submodule of $M$. Let $N_i$ $(1 \leq i \leq s)$ be a collection of submodules of $M$ satisfying the primeful property. Then the following statements are equivalent:*

(i) *$(N : M) = (N_1 : M) \cap \cdots \cap (N_s : M)$ is a reduced quasi-primary decomposition of $I$;*
(ii) *$N = N_1 \cap \cdots \cap N_s$ is a reduced quasi-primary decomposition of $N$;*
(iii) *$N = N_1 \cap \cdots \cap N_s$ is a module-reduced quasi-primary decomposition of $N$.*

**2.7. Corollary.** *Let $I$ be an ideal of $R$ containing $ann(M)$. Let $M$ be a multiplication $R$-module. If $I = q_1 \cap \cdots \cap q_s$ is a reduced quasi-primary decomposition of $I$, then $IM = q_1M \cap \cdots \cap q_sM$ is a reduced and module-reduced quasi-primary decomposition of $IM$.*

The following is an immediate consequence of Theorem 2.2 and [9, Theorem 1].

**2.8. Corollary.** *Let $M$ be a multiplication $R$-module and $N$ a submodule $M$. Let $N_i = q_iM$, $(1 \leq i \leq s)$ be a collection of quasi-primary submodules of $M$ satisfying the primeful property. Then $N_1 \cap \cdots \cap N_s$ is a quasi-primary submodule of $M$ if and only if among the prime ideals $\sqrt{(N_i : M)}$ there is a $\sqrt{(N_k : M)}$ such that $\sqrt{(N_k : M)} \subseteq \sqrt{(N_i : M)}$.*

Recall that a representation $N = N_1 \cap N_2 \cdots \cap N_s$ of a submodule $N$ of an $R$-module $M$ is shortest, if none of the $N_i$ can be omitted and none of the intersection $(N_{i_1} : M) \cap (N_{i_2} : M) \cap \cdots \cap (N_{i_t} : M)$ $(t > 1)$ is a quasi-primary ideal.

**2.9. Proposition.** *Let $M$ be a multiplication $R$-module and $N$ a submodule $M$. Let $N_i = q_iM$ $(1 \leq i \leq s)$ be a collection of submodules of $M$ satisfying the primeful property. Then every quasi-primary decomposition $N = N_1 \cap N_2 \cap \cdots \cap N_s$ has a shortest quasi-primary decomposition.*

*Proof.* First we omit every superfluous term $N_i$. Second, assume there exist submodules $N_{i_1}, N_{i_2}, \cdots, N_{i_t}$ such that $\sqrt{(N_{i_1} : M)} \subseteq \sqrt{(N_{i_2} : M)} \subseteq \cdots \subseteq \sqrt{(N_{i_t} : M)}$. Put $N'_i = N_{i_1} \cap N_{i_2} \cap \cdots \cap N_{i_t}$. Then by Corollary 2.8, $N'_i$ is a quasi-primary submodule of $M$. Thus $N = N'_1 \cap N'_2 \cap \cdots \cap N'_r$ is a shortest quasi-primary decomposition of $N$. □

**2.10. Corollary.** *Let $M$ be a multiplication module with a submodule $N$. If $N = N_1 \cap N_2 \cap \cdots \cap N_s$ is a shortest quasi-primary decomposition such that each $N_i$ $(1 \leq i \leq s)$ satisfies the primeful property, then all the prime ideals belonging to the quasi-primary submodules which occur in a shortest quasi-primary decomposition of $N$ are isolated.*

**2.11. Proposition.** *Let $M$ be a multiplication $R$-module and $N$ a submodule of $M$. Let $N_i$ $(1 \leq i \leq s)$ be a collection of submodules of $M$ satisfying the primeful property. If $N = N_1 \cap N_2 \cap \cdots \cap N_s$ is a shortest quasi-primary decomposition, then it is a reduced and module-reduced quasi-primary decomposition of $N$.*

*Proof.* It is clear that the ideals $\sqrt{(N_i : M)}$ are prime for every $i$ $(1 \leq i \leq t)$. Assume, on the contrary, there exists $j \neq i$ such that $\sqrt{(N_j : M)} = \sqrt{(N_i : M)}$. Then $(N_i : M) \cap (N_j : M)$ is a quasi-primary ideal of $R$, since $\sqrt{(N_i \cap N_j : M)} = \sqrt{(N_i : M)}$ is a prime ideal of $R$, a contradiction. Therefor $N = N_1 \cap N_2 \cap \cdots \cap N_t$ is a reduced quasi-primary decomposition and by Corollary 2.6 is also a module-reduced quasi-primary decomposition. □

In general, the converse of the above proposition is not true. For instance, let $R = K[x, y]$ be the ring of polynomials in $x, y$ with coefficients in a field $K$. Consider the ideal $I = (x^2y, xy^2)$ of $R$. It is clear that $radI = (xy)$ is not a prime ideal and so $I$ is not quasi-primary. $I = (x) \cap (y) \cap (x^2, y^2)$ is a reduced quasi-primary decomposition that is not shortest [9, p. 181].

The following is an immediate result of Proposition 2.6 and Proposition 2.11.

**2.12. Corollary.** *Let $M$ be a multiplication $R$-module and $N$ a submodule of $M$. Let $N_i$ $(1 \leq i \leq s)$ be a collection of submodules of $M$ satisfying the primeful property. If $N$ has a quasi-primary decomposition, then it has both reduced and module-reduced quasi-primary decompositions.*

**2.13. Theorem.** *Let $M$ be a multiplication $R$-module and $N$ a submodule of $M$. Let $N_i = q_iM$, $(1 \leq i \leq s)$ be a collection of submodules of $M$ satisfying the primeful property. Then $N = N_1 \cap N_2 \cap ... \cap N_s$ is a shortest quasi-primary decomposition of $N$ if and only if $(N : M) = (N_1 : M) \cap (N_2 : M) \cap ... \cap (N_s : M)$ is a shortest quasi-primary decomposition of the ideal $(N : M)$.*

*Proof.* $\Rightarrow$) Assume, on the contrary, that $(N:M) = (N_1:M)\cap(N_2:M)\cap\cdots\cap(N_s:M)$ is not shortest. Then either $(N_t:M)$ may be omitted for some $1\le t\le s$ or $(N_{i_1}:M)\cap(N_{i_2}:M)\cap\cdots\cap(N_{i_r}:M)$ is a quasi-primary ideal for some $r>1$. Firstly, assume $(N_t:M)\supseteq(N_1:M)\cap\cdots\cap(N_{t-1}:M)\cap(N_{t+1}:M)\cap\cdots\cap(N_s:M)$. Therefor $\sqrt{(N_t:M)}\supseteq\sqrt{(N_1:M)}\cap\cdots\cap\sqrt{(N_{t-1}:M)}\cap\sqrt{(N_{t+1}:M)}\cap\cdots\cap\sqrt{(N_m:M)}$. Since $\sqrt{(N_t:M)}$ is a prime ideal, there exists $k\neq t$ such that $\sqrt{(N_k:M)}\subseteq\sqrt{(N_t:M)}$. Now Corollary 2.10 shows that $\sqrt{(N_k:M)}=\sqrt{(N_t:M)}$. Thus $N=N_1\cap N_2\cap...\cap N_s$ is not a reduced quasi-primary decomposition, which contradicts the Proposition 2.11. Secondly, if $(N_{i_1}:M)\cap(N_{i_2}:M)\cap\cdots\cap(N_{i_r}:M)$ is a quasi-primary ideal for some $r>1$, then there is a minimal prime ideal $\sqrt{(N_{i_k}:M)}$ among the prime ideals $\sqrt{(N_{i_j}:M)}$ $(1\le j\le r)$, which contradicts the Corollary 2.10.
$\Leftarrow$) Suppose $(N:M)=(N_1:M)\cap(N_2:M)\cap...\cap(N_s:M)$ is a shortest quasi-primary decomposition of the ideal $(N:M)$ in $R$. Multiplying by $M$, we get $N=N_1\cap N_2\cap...\cap N_s$. It is easy to check that the above representation is a shortest quasi-primary decomposition of $N$. □

**2.14. Theorem.** *Let $M$ be a multiplication $R$-module and $N$ a submodule of $M$. Let $N=N_1\cap N_2\cap\cdots\cap N_s=N'_1\cap N'_2\cap\cdots\cap N'_t$ be two reduced quasi-primary decompositions of $N$ as intersection of quasi-primary submodules satisfying the primeful property. Then $s=t$ and the prime ideals $p_i=\sqrt{(N_i:M)}$ must be, without regard to their order, identical to the prime ideals $p'_j=\sqrt{(N'_j:M)}$.*

*Proof.* Let $N=N_1\cap N_2\cap\cdots\cap N_s=N'_1\cap N'_2\cap\cdots\cap N'_t$ be two shortest quasi-primary decompositions of $N$. By Theorem 2.13, we have two shortest quasi-primary decompositions $(N:M)=(N_1:M)\cap(N_2:M)\cap\cdots\cap(N_s:M)=(N'_1:M)\cap(N'_2:M)\cap\cdots\cap(N'_t:M)$ of the ideal $(N:M)$. Now the proof is completed by [9, Theorem 6]. □

**2.15. Proposition.** *Let $N$ and $K$ be quasi-primary submodules of a multiplication $R$-module $M$ satisfying the primeful property. Then $N\cap K$ is quasi-primary if and only if $radN\subseteq radK$ or $radK\subseteq radN$.*

*Proof.* Since $N\cap K$ is a quasi-primary submodule, $\sqrt{(N\cap K:M)}=\sqrt{(N:M)}\cap\sqrt{(K:M)}$ is a prime ideal of $R$ and so $\sqrt{(N:M)}\subseteq\sqrt{(K:M)}$ or $\sqrt{(K:M)}\subseteq\sqrt{(N:M)}$. Equivalently $(radN:M)\subseteq(radK:M)$ or $(radK:M)\subseteq(radN:M)$. Therefore $radN\subseteq radK$ or $radK\subseteq radN$, since $M$ is a multiplication module. The reverse argument implies that $(N\cap K:M)$ is a quasi-primary ideal and so by Theorem 2.2, $N\cap K$ is a quasi-primary submodule of $M$. □

## 3. QUASI-PRIMARY DECOMPOSITION OF SUBMODULES OF MODULES OVER NOETHERIAN RINGS

In [6, Theorem 3.10], it has been shown that every proper submodule of a Noetherian module has a primary decomposition and so a fortiori quasi-primary decomposition. In particular, every submodule of finitely generated modules or faithful multiplication modules over Noetherian rings has a quasi-primary decomposition [7, p.764]. This gives rise to the question: is there a submodule of a module which has a quasi-primary decomposition, but has not any primary decomposition. Let us now present positive answer to this question below.

**3.1. Example.** Since the set of ideals of a valuation domain is totally ordered under inclusion, we conclude that every proper ideal of a valuation domain is quasi-primary [11, Theorem 5.10]. On the other hand, it is proved that for a local domain $R$, every proper ideal of $R$ is primary if and only if $dimR = 1$ [4, Theorem2.4]. Now let $R$ be a valuation domain with $dimR > 1$. Then there exists a quasi-primary ideal $q$ of $R$ which is not primary. Now if $q = q_1 \cap q_2 \cap \cdots \cap q_n$ is a reduced primary decomposition of $q$, then there is $1 \leq j \leq n$ such that $q_j \subseteq \sqrt{q} \subseteq \sqrt{q_j}$. Thus $\sqrt{q_j}$ is a minimal element of the set $\{\sqrt{q_1}, \sqrt{q_2}, \cdots, \sqrt{q_n}\}$. We claim that $q_j$ is minimal among the ideals $q_1, q_2, \cdots, q_n$ and so $q = q_j$. This contradicts the choice of $q$. Let $q_i \subseteq q_j$ for some $i \neq j$. By minimality of $\sqrt{q_j}$ we must have $\sqrt{q_i} = \sqrt{q_j}$, which contradicts the fact that $q = q_1 \cap q_2 \cap \cdots \cap q_n$ is a reduced primary decomposition of $q$. Thus $q_i \nsubseteq q_j$ for every $i \neq j$. Now since the set of ideals of $R$ is totally ordered under inclusion, we must have $q_j \subseteq q_i$ for every $i \neq j$, as required.

It has been shown that a reduced primary decomposition is unique in the sense of the set of prime ideals belonging to primary submodules of two primary decompositions are the same and the set of primary submodules with isolated associated primes are also identical [6, Theorem 3.10]. In this section we study quasi-primary submodules of modules over Noetherian rings. In particular, we give some uniqueness theorems for reduced and module-reduced quasi-primary decomposition (Theorem 3.6, Theorem 3.8 and Theorem 3.12).

**3.2. Lemma.** *Let $R$ be a Noetherian ring and $N$ a $p$-quasi-primary submodule of an $R$-module $M$. Then there exists a positive integer $n$ such that $p^n \subseteq (N : M)$.*

*Proof.* Taking $p = (r_1, \cdots, r_t)$. For each generator $r_i$, there is a positive integer $n_i$ such that $r_i^{n_i} \in (N : M)$. Let $n$ has the value $n = \sum_{i=1}^{t}(n_i - 1) + 1$. Now $p^n$ is generated by monomials $r_1^{m_1} \cdots r_t^{m_t}$ with $\sum_{j=1}^{t} m_j = n$, because at least for one of the subscripts $j$ we have $s_j \geq n$. Hence $p^n \subseteq (N : M)$. □

Since a faithful multiplication module $M$ over a Noetherian ring $R$ is Noetherian ([7, p.764]), then every submodule of $M$ satisfies the primeful property. Thus we can replace "satisfying the primeful property" for these submodules of $M$ with "faithfulness" for $M$ in Theorem 3.3 and and Theorem 3.5.

**3.3. Theorem.** *Let $R$ be a Noetherian ring and $M$ a multiplication $R$-module. Let $N$ be a submodule of $M$ which satisfies the primeful property. Then $N$ is quasi-primary if and only if there exists a unique prime ideal $p$ of $R$ such that $p^t \subseteq (N : M) \subseteq p$ for some positive integer $t$.*

*Proof.* ($\Rightarrow$) By Theorem 2.2, $(N : M)$ is a quasi primary ideal. If $p = \sqrt{(N : M)}$, then by Lemma 3.2 $p^t \subseteq (N : M) \subseteq p$ for some positive integer $t$. If $p'$ is a prime ideal of $R$ and $p'^s \subseteq (N : M) \subseteq p'$, then $p' = \sqrt{(N : M)} = p$.
($\Leftarrow$) It is clear that $(N : M)$ is quasi-primary ideal. Now the proof is completed by Theorem 2.2. □

**3.4. Lemma.** *Let $M$ be a multiplication $R$-module and $N_1$ a submodule of $M$. Let $N_2$ be a quasi-primary submodule of $M$ satisfying the primeful property such that $\mathrm{p} = \sqrt{(N_1 : M)} = \sqrt{(N_2 : M)}$ and $N_1 \subseteq N \subseteq N_2$. Then $N$ is a $\mathrm{p}$-quasi-primary submodule of $M$.*

*Proof.* It is clear that $\sqrt{(N_1 : M)} = \sqrt{(N : M)} = \sqrt{(N_2 : M)} = p$ and so $(N : M)$ is a $p$-quasi-primary ideal of $R$. Now if p is a prime ideal containing $(N : M)$, then $(N_2 : M) \subseteq \mathrm{p}$. Since $N_2$ satisfies the primeful property, there exists a prime submodule $P$ containing $N_2$ and so $N$ such that $(P : M) = \mathrm{p}$. Thus $N$ satisfies the primeful property. Now by Theorem 2.2, $N$ is a $p$-quasi-primary submodule of $M$. □

**3.5. Theorem.** *Let $R$ be a Noetherian ring and $M$ a multiplication $R$-module. Let $N_i$ $(1 \leq i \leq t)$ be a collection of quasi-primary submodules of $M$ with $\sqrt{(N_i : M)} = p_i$. If $N_1$ satisfies the primeful property and $p_1 \subseteq p_i$ for each $1 \leq i \leq t$, then $N = (\Pi_{i=1}^{t}(N_i : M))M$ is also $p_1$-quasi-primary.*

*Proof.* Since $R$ is a Noetherian ring there are positive integers $s_i$ $(1 \leq i \leq t)$ such that $p_1^{s_1+s_2\cdots+s_t}M \subseteq p_1^{s_1}p_2^{s_2}\cdots p_t^{s_t}M \subseteq (N_1 : M)(N_2 : M)\cdots(N_t : M)M \subseteq p_1p_2\cdots p_tM \subseteq p_1M$. Thus

$$p_1 \subseteq \sqrt{(p_1^{s_1+s_2\cdots+s_t}M : M)} \subseteq \sqrt{(p_1p_2\cdots p_tM : M)} \subseteq p_1$$

and so $\sqrt{(p_1p_2\cdots p_tM : M)} = p_1$. Now by a similar consideration of Lemma 3.4, it can be shown that $p_1p_2\cdots p_tM$ satisfies the primeful property. Hence by Theorem 2.2, $N = (\Pi_{i=1}^{t}(N_i : M))M$ is $p_1$-quasi-primary. □

.

**3.6. Theorem.** *Let $R$ be a Noetherian ring and $M$ an $R$-module. Let $N$ be a submodule of $M$ such that $N = N_1 \cap N_2 \cap \cdots \cap N_s = N_1' \cap N_2' \cap \cdots \cap N_t'$ be two reduced quasi-primary decompositions of $N$ where $N_i$(resp. $N_j'$) is $\mathrm{p}_i$-quasi-primary (resp. $\mathfrak{p}_j$-quasi-primary). Then $s = t$ and (after reordering if necessary) $\mathrm{p}_i = \mathfrak{p}_i$ and $radN_i = radN_i'$ for $1 \leq i \leq s$.*

*Proof.* Without loss of generality we may assume that $\mathfrak{p}_1$ is one of the minimal elements of the set $\{\mathrm{p}_1, \cdots \mathrm{p}_s, \mathfrak{p}_1, \cdots \mathfrak{p}_t\}$. Since $N_1$ is $\mathrm{p}_1$-quasi-primary, there exists a positive integer $t$ such that $\mathrm{p}_1^tM \subseteq N_1$ and hence

$$\mathrm{p}_1^t(N_2 \cap N_3 \cap \cdots \cap N_s) \subseteq N = N_1' \cap N_2' \cap \cdots \cap N_t'.$$

If $N_2 \cap N_3 \cap \cdots \cap N_s \subseteq radN_1'$, then we have $\cap_{i=2}^{s}\mathrm{p}_i \subseteq \mathfrak{p}_1$ and so $\mathrm{p}_i \subseteq \mathfrak{p}_1$ for some $2 \leq i \leq s$. Thus by assumption $\mathrm{p}_i = \mathfrak{p}_1$ for some $2 \leq i \leq s$. In the other case, suppose $N_2 \cap N_3 \cap \cdots \cap N_s \nsubseteq radN_1'$. Since $N_1'$ is quasi-primary, we have $\mathrm{p}_1^t \subseteq \mathfrak{p}_1$ and hence $\mathrm{p}_1 \subseteq \mathfrak{p}_1$. Now by minimality of $\mathfrak{p}_1$, we conclude that $\mathrm{p}_1 = \mathfrak{p}_1$. Since $\{\mathrm{p}_1, \mathrm{p}_2 \cdots, \mathrm{p}_s\}$ and $\{\mathfrak{p}_1, \mathfrak{p}_2 \cdots, \mathfrak{p}_t\}$ are sets of distinct prime ideals, with a similar argument we have $s = t$ and $\mathrm{p}_i = \mathfrak{p}_i$ for $1 \leq i \leq s$.

For the second part, since $\mathrm{p}_i$ are all distinct, there exists $r_i \in \mathrm{p}_i \backslash \mathrm{p}_1$ for each $2 \leq i \leq s$. Then $r = r_2r_3\cdots r_s \in \mathrm{p}_i$ for $i > 1$, but $r \notin \mathrm{p}_1$. Since $N_i$(resp. $N_i'$) is $\mathrm{p}_i$-quasi-primary, there exists an integer $n_i$(resp. $m_i$) such that $r^{n_i} \in (N_i : M)$(resp. $r^{m_i} \in (N_i' : M)$) for each $2 \leq i \leq s$. Let $n = max\{n_2, \cdots, n_s, m_2 \cdots, m_s\}$. Then $r^n \in (N_i : M)$ and $r^n \in (N_i' : M)$ for each $2 \leq i \leq s$. Now if $x \in N_1$, then $r^nx \in N$ whence $r^nx \in N_1'$. It follows from the definition that $x \in radN_1'$. Therefore $N_1 \subseteq radN_1'$. A similar argument shows that $N_1' \subseteq radN_1$ and hence $radN_1 = radN_1'$. □

**3.7. Lemma.** *Let $M$ be an $R$-module. If $\{N_i : 1 \leq i \leq t\}$ is a finite collection of submodules of $M$ which satisfy the primeful property, then so does $\cap_{i=1}^{t}N_i$.*

*Proof.* Clear. □

**3.8. Theorem.** *Let $N$ be a proper submodule of a module $M$ over a Noetherian ring $R$. If $N = \cap_{i=1}^{t}N_i$ is a module-reduced quasi-primary decomposition and $N_i$ $(1 \leq i \leq t)$ satisfies the primeful property such that $radN = \cap_{i=1}^{t}radN_i$, then $Ass(M/radN) \subseteq \{p_1, \cdots, p_t\} \subseteq Supp(M/radN)$. In particular, $Ass(M/radN) = \{p_{i_1}, p_{i_2}, \cdots, p_{i_s}\}$ where $p_{i_j}$ $(1 \leq j \leq s)$ are minimal elements of $\{p_1, \cdots, p_t\}$.*

*Proof.* Let $p$ be an associated prime of $M/radN$, so that $p = ann(x + radN)$, $0 \neq x + radN \in M/radN$. Renumber the $N_i$ so that $x \notin radN_i$ for $1 \leq i \leq j$ and $x \in radN_i$ for $j+1 \leq i \leq t$. Since $N_i$ is a quasi-primary submodule satisfying the primeful property, $p_i = \sqrt{(N_i : M)}$ is a prime ideal of $R$ $(1 \leq i \leq t)$. Since $p_i$ is finitely generated, $p_i^{n_i} M \subseteq N_i$ for some $n_i \geq 1$. Therefore $(\cap_{i=1}^{j} p_i^{n_i})x \subseteq \cap_{i=1}^{t} radN_i = radN$, so $\cap_{i=1}^{j} p_i^{n_i} \subseteq ann(x + radN) = p$. Since $p$ is prime, $p_i \subseteq p$ for some $i \leq j$. We claim that $p_i = p$, so that every associated prime must be one of the $p_i$'s. To verify this, let $r \in p$. Then $r(x + radN) = radN$ and $x \notin radN_i$ and since $radN_i$ is prime we have $r \in \sqrt{(N_i : M)} = p_i$, as claimed. By [8, Lemma 3.4], $M/radN_i$ is a primeful $R$-module. Now since $p_i \supseteq (radN : M)$ for each $1 \leq i \leq t$, we have $Ass(M/radN) \subseteq \{p_1, p_2, \cdots, p_t\} \subseteq Supp(M/radN)$, by [12, Proposition 3.4]. For the second part, we show that minimal elements of $\{p_1, \cdots, p_t\}$ are equal to minimal elements of $Supp(M/radN)$. Let $p_j$ be a minimal element of $\{p_1, \cdots, p_t\}$ and $p \subseteq p_j$ for some $p \in Supp(M/radN)$. By [8, Lemma 3.4] and Lemma 3.7 $radN$ satisfies the primeful property and hence by [12, Proposition 3.4] $p \supseteq (radN : M)$. Thus $\cap_{i=1}^{t} p_i \subseteq p \subseteq p_j$. Since $p$ is prime, there exists $p_i$ $(1 \leq i \leq t)$ such that $p_i \subseteq p \subseteq p_j$ and so $p_i = p = p_j$, by minimality of $p_j$. Now the proof is completed by [20, Theorem 9.39]. □

Noth that, by the proof of Theorem 3.8, the minimal prime ideals of the set $\{p_1, \cdots, p_t\}$ are uniquely determined by $N$, as follows.

**3.9. Corollary.** *Let $N$ be a proper submodule of a module $M$ over a Noetherian ring $R$. Let $N = \cap_{i=1}^{t} N_i$ be a module-reduced quasi-primary decomposition and $N_i$ satisfies the primeful property, $1 \leq i \leq t$, such that $radN = \cap_{i=1}^{t} radN_i$. Let $p_i = \sqrt{(N_i : M)}$ for $1 \leq i \leq t$. Then the minimal primes which occur in the set $\{p_1, \cdots, p_t\}$ are uniquely determined by $N$.*

**3.10. Corollary.** *Let $N$ be a proper submodule of a module $M$ over a Noetherian ring $R$ which satisfies the primeful property. Then $N$ is $p$-quasi-primary if and only if $Ass(M/radN) = p$.*

**3.11. Lemma.** *Let $M$ be a module over a Noetherian ring $R$, and $N$ a quasi-primary submodule of $M$ satisfying the primeful property with $p = \sqrt{(N : M)}$. Let $p'$ be any prime ideal of $R$.*

(i) *If $p \nsubseteq p'$, then $M_{p'} = (radN)_{p'}$.*
(ii) *If $p \subseteq p'$, then $radN = f^{-1}((radN)_{p'})$ where $f$ is the mapping $x \mapsto x/1$ from $M$ into $M_{p'}$.*

*Proof.* (i). It is easy to verify that there is a bijection between $Ass_{R_{p'}}(M/radN)_{p'}$ (which coincide with $Ass_{R_{p'}}(M_{p'}/(radN)_{p'})$) and the intersection $Ass_R(M/radN) \cap S$, where $S$ is the set of prime ideals contained in $p'$. By Corollary 3.10, there is only one associated prime of $M/radN$ over $R$, namely $p$, which is not contained in $p'$ by hypothesis. Thus $Ass_R(M/radN) \cap S$ is empty, so by [20, Corollary 9.35], $M_{p'}/(radN)_{p'} = 0$, and the result follows.

(ii). As in Corollary 3.10, $Ass_R(M/radN) = \{p\}$. Since $p \subseteq p'$, we have $R \backslash p' \subseteq R \backslash p$. By [20, Corollary 9.36], $R \backslash p'$ contains no zero-divisors of $M/radN$, because all such zero-divisors belong to $p$. Thus the natural map $g : x \to x/1$ from $M/radN$ to $(M/radN)_{p'} \cong (M_{p'}/(radN)_{p'})$ is injective. Assume $x \in f^{-1}((radN)_{p'})$. Then $f(x) \in (radN)_{p'}$, so $f(x) + (radN)_{p'}$ is 0 in $M_{p'}/(radN)_{p'}$. By injectivity of the natural map $M/radN \to (M/radN)_{p'}$, $x + radN$ is 0 in $M/radN$, in other words, $x \in radN$. Thus $f^{-1}((radN)_{p'}) \subseteq radN$ and the reverse inclusion is clear. □

**3.12. Theorem.** *Let $N$ be a proper submodule of a module $M$ over a Noetherian ring $R$ satisfying the primeful property. If $N = \cap_{i=1}^{t} N_i$ is a module-reduced quasi-primary decomposition and $N_i$ satisfies the primeful property, $1 \leq i \leq t$, such that $radN = \cap_{i=1}^{t} radN_i$. If $p_j = \sqrt{(N_j : M)}$ is a minimal element of $\{p_1, \cdots, p_t\}$, then $radN_j$ is uniquely determined by $N$.*

*Proof.* Suppose that $p_j$ is minimal, so that $p_j \not\supseteq p_i$, $i \neq j$. By Lemma 3.11(i) with $p = p_i$, $p' = p_j$, we have $(radN_i)_{p_j} = M_{p_j}$ for $i \neq j$. By Lemma 3.11(ii), we have $radN_j = f^{-1}((radN_j)_{p_j})$, where $f$ is the natural map from $M$ to $M_{p_j}$. Hence we have

$$\begin{aligned} (radN)_{p_j} &= (radN_j)_{p_j} \cap (\cap_{i \neq j} (radN_i)_{p_j}) \\ &= (radN_j)_{p_j} \cap M_{p_j} = (radN_j)_{p_j}. \end{aligned}$$

Thus $radN_j = f^{-1}((radN_j)_{p_j}) = f^{-1}((radN)_{p_j})$ depends only on $N$ and $p_j$, and since $p_j$ is the minimal prime associated with $N$, it follows that $radN_j$ depends only on $N$. $\square$

Acknowledgements. The author would like to thank the referee/referees for a number of constructive comments and valuable suggestions.

## References


[1] D. D. Anderson, *Multiplication Ideals, Multiplication Rings and the Ring $R[X]$*. Canad. J. Math., 28 (1976), 760-768.

[2] M. F. Atiyah, I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley publishing company, 1969.

[3] M. Behboodi, *On the Prime Radical and Bear's Lower Nilradical of Modules*, Acta Math. Hungar. (3) (2009), 293-306.

[4] R. Chaudhuri, *A Note on Generalized Primary Rings*, Mat. Vesnik 13(28)(1976), 375-377.

[5] J. Dauns, *Prime Modules*, J. Reine Angew. Math., 298 (1978), 156-181.

[6] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, 1994.

[7] Z. A. EL-Bast, P. F. Smith, *Multiplication Modules*, Comm. Algebra, 16(4) (1988), 755-779.

[8] H. Fazaeli Moghimi, M. Samiei, *Quasi-primary Submodules Satisfying the Primeful Property I*, Hacet. J. Math. Stat., Submitted.

[9] L. Fuchs, *On Quasi-primary Ideals*, Acta Sci. Math. (Szeged), 11 (1947) 174-183.

[10] J. Jenkins, P. F. Smith, *On the Prime Radical of a Module over a Commutative Ring*, Comm. Algebra, 20 (l992), 3593-3602.

[11] M. D. Larsen, P. J. McCarthy, *Multiplicative Ideal Theory*, Academic press, 1971.

[12] C. P. Lu, *A Module Whose Prime Spectrum Has the Surjective Natural Map*, Houston J. Math., 33(1) (2007),127-143.

[13] C. P. Lu, *M-radicals of Submodules in Modules II*, Math. Japonica., 35 (1990), 991-1001.

[14] R. McCasland, M. Moore, *Prime Submodules*. Comm. Algebra, 20 (1992), 1803-1817.

[15] R. L. McCasland, M. E. Moore, P. F. Smith, *On the Spectrum of a Module over a Commutative Ring*, Comm. Algebra, 25(1) (1997), 79-103.

[16] R. L. McCasland, P. F. Smith, *Generalised Associated Primes and Radicals of Submodules*, Int. Electron. J. Algebra, 4 (2008), 159-176.

[17] M. E. Moor, S. J. Smith, *Prime and Radical Submodules of Modules over Commutative Rings*, Comm. Alegbra, 30 (2002), 5073-5064.

[18] R. Naghipour, M. Sedghi, *Weakly Associated Primes and Primary Decomposition of Modules over Commutative Rings*, Acta Math. Hungar. 110 (1-2) (2006), 1-12.

[19] H. Sharif, Y. Sharifi, S. Namazi, *Rings Satisfying the Radical Formula*, Acta Math. Hungar., 71(1-2) (1996), 103-108.

[20] R. Y. Sharp, *Steps in Commutative Algebra*, Cambridge University Press, Cambridge, 1990.

[21] P. F. Smith, *Uniqueness of Primary Decompositions*, Turk. J. Math. 27 (2003), 425-434.

[22] P. F. Smith, *Primary Modules over Commutative Rings*, Comm. Algebra, 43 (2001), 103-111.

[23] P. F. Smith, *Some Remarks on Multiplication Modules*, Arch. der Math., 50 (1988), 223-235.

[24] A. Soleyman Jahan, *Prime Filtrations and Primary Decompositions of Modules*, Comm. Algebra, 39 (2011), 116-124.

[25] D. Pusat-Yilmaz, P.F. Smith, *Radicals of Submodules of Free Modules*, Comm. Algebra, 27(5) (1999), 2253-2266.

[26] D. Pusat-Yilmaz, P. F. Smith, *Modules Which Satisfy the Radical Formula*, Acta Math. Hungar. 95 (1-2) (2002), 155-167.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 813 – 821

# Convergence processes of approximating operators

Özlem Girgin ATLIHAN *

**Abstract**

The aim of this paper is to present Korovkin type theorems on approximatin of continuous functions with the use of $A-$statistical convergence and matrix summability method which includes both convergence and almost convergence. Since statistical convergence and almost convergence methods are incompatible, we conclude that these methods can be used alternatively to get some approximation results.

*2000 AMS Classification:* Primary 41A25, 41A36; Secondary 47B38.

**Keywords:** Matrix summability method, sequence of positive linear operators, the Korovkin approximation theorem, A-statistical convergence.

Received 22/01/2014 : Accepted 08/04/2014 Doi : 10.15672/HJMS.2015449435

## 1. Introduction

The so-called Bohman-Korovkin theorem on approximation of continuous functions on a compact interval provides conditions in order to make a decision whether a sequence of positive linear operators converges to the identity operator [2],[7],[14], and so on many proofs have appeared in a variety of settings of this result (see[15],[18], [20],[27]). In [27], Uchiyama have given an alternate proof of it by using inequalities related to variance. If the sequence of positive linear operators does not convergence to the identity operator then it might be benefical to use summability methods ([1],[13],[16],[22],[26],[28]).

The main point of using summability theory has always been to make a nonconvergent sequence to converge. This was the motivation behind Fèjer's famous theorem showing Cesàro method being effective in making the Fourier series of a continuous periodic function to converge [29]. In this paper, using Uchiyama's idea [27], we give quite simple proofs of the Korovkin type approximation theorems studied in ([9],[13],[23]). And also we develop some Korovkin type results with the use of summation process and statistical convergence methods respectively.

We pause to collect some notation.

*Pamukkale University, Faculty of Science and Letters, Department of Mathematics, Kınıklı 20070, Denizli, TURKEY.
Email: oatlihan@pau.edu.tr

Let $C[a,b]$ be the vector space of all real-valued continuous functions on $[a,b]$ and let $L$ be a linear operator on $C[a,b]$. We say that $L$ is positive if $Lf \geq 0$ whenever $f \geq 0$ on $[a,b]$. Note that $C[a,b]$ is a Banach space with norm $\|f\| = \max_{x\in[a,b]} |f(x)|$ and we denote norm of $L$ operator by $\|L\| = \max\{\|Lf\| : \|f\| \leq 1\}$.

A subsequence $\mathcal{B}$ of $C[a,b]$ is called a subalgebra if $f.g$ belongs to $\mathcal{B}$ whenever $f$ and $g$ are members of $\mathcal{B}$.

We first recall the following lemma introduced in [27], which is useful in proving our results.

[A] **Lemma.** Let $\mathcal{B}$ be a norm-closed subalgebra of $C[a,b]$ that contains 1. If $L$ is a positive linear operator on $\mathcal{B}$ with $L(1) \leq 1$, then

$$V(h) := L(h^2) - (L(h))^2 \geq 0$$

for every $h$ in $\mathcal{B}$. Morever, for $f, g$ and $k$ in $\mathcal{B}$:

$$(1.1)\qquad |L(fg) - L(f)L(g)|^2 \leq V(f)V(g)$$

$$(1.2)\qquad \|L(fg) - L(f)L(g)\| \leq \|V(f)\|^{\frac{1}{2}} \|V(g)\|^{\frac{1}{2}}$$

$$(1.3)\qquad \|L(fg) - L(f)L(g)\| \leq \|V(f)\|^{\frac{1}{2}} \|V(g) + V(k)\|^{\frac{1}{2}}$$

We now turn our attention to matrix summability method.

Let $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. A sequence $\{L_j\}$ of positive linear operators of $C[a,b]$ into $C[a,b]$ is called an $\mathcal{A}$ - summation process on $C[a,b]$ if $\{L_j(f)\}$ is $\mathcal{A}$- convergent to $f$ for every $f \in C[a,b]$, i.e.,

$$(1.4)\qquad \lim_k \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\| = 0, \text{ uniformly in } n,$$

where it is assumed that the series in (1.4) converges for each $k, n$ and $f$. Recall that a sequence of real numbers $\{x_j\}$ is said to be $\mathcal{A}-$convergent (or $\mathcal{A}-$summable) to $L$ if $\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j = L$, (uniformly in $n$), ([19],[25]).

If $A^{(n)} = A$ for some matrix $A$, then $\mathcal{A}-$summability is the ordinary matrix summability by $A$. If $a_{kj}^{(n)} = 1/k$ for $n \leq j < k+n$, $(n = 1, 2, ...)$, and $a_{kj}^{(n)} = 0$ otherwise, then $\mathcal{A}-$summability reduces to almost convergence method [18]. Let $\{L_j\}$ be a sequence of positive linear operators of $C\ [a,b]$ into $C\ [a,b]$ such that for each $k, n \in \mathbb{N}$

$$(1.5)\qquad \sum_{j=1}^{\infty} a_{kj}^{(n)} \|L_j(1)\| < \infty.$$

Furthermore, for each $k, n \in \mathbb{N}$ and $f \in C[a,b]$, let

$$B_k^{(n)}(f;x) = \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x)$$

which is well defined by (1.5), and belongs to $B[a,b]$. Observe that $\left\|B_k^{(n)}\right\| = maks\left\{\left\|B_k^{(n)}(f)\right\| : \|f\| \leq 1\right\}$.Hence $\left\|B_k^{(n)}\right\| = \left\|B_k^{(n)}(1)\right\|$.Some unification on Korovkin-type results through the use of a summability method may be found in ([3],[4],[5],[6],[8]).

## 2. Korovkin Type Approximation Theorems via Summation Process

This section is motivated by that of Uchiyama [27]. We give quite simple proof of a Korovkin type theorem which has been developed by Nishishiraho via $\mathcal{A}-$ summation process, with the use of inequalities related to variance. And also we obtain Korovkin type results for positive linear operators over $C_{2\pi}$ and $C(D)$ respectively with the use of $\mathcal{A}-$ summation process which includes both convergence and almost convergence.

**2.1. Theorem.** Let $\mathcal{A} := \left\{A^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. Assume that $\{L_j\}$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ for which (1.5) holds. If,

$$\lim_k \left\| B_k^{(n)} h - h \right\| = 0, \text{ uniformly in } n, \tag{2.1}$$

for all $h = 1,\ x,\ x^2$ then $\{L_j\}$ is $\mathcal{A}-$ summation process on $C[a,b]$ i.e., for every $f \in C[a,b]$,

$$\lim_k \left\| B_k^{(n)} f - f \right\| = 0, \text{ uniformly in } n.$$

*Proof.* We proceed as in [27]. Since $\lim_k \left\| B_k^{(n)} 1 - 1 \right\| = 0$, (uniformly in $n$), we have $\lim_k \left\| B_k^{(n)} 1 \right\| = 1$, (uniformly in $n$). Without loss of generality we may assume that $\left\| B_k^{(n)} \right\| \neq 0$ for all $n$ and $k$.By considering $\frac{B_k^{(n)}}{\left\| B_k^{(n)} \right\|}$ in place of $B_k^{(n)}$, without loss of generality we assume that $B_k^{(n)}(1) \leq 1$ for all $n, k$. This implies that$\left\| B_k^{(n)} \right\| \leq 1$ for all $n, k$. Using (1.2), for every $f$ in $C[a,b]$ and for all $n, k$,

we can write

$$\begin{aligned} & \left\| B_k^{(n)}(xf) - B_k^{(n)}(x).B_k^{(n)}(f) \right\|^2 \\ \leq & \left\| B_k^{(n)}(x^2) - (B_k^{(n)}(x))^2 \right\| \left\| B_k^{(n)}(f^2) - (B_k^{(n)}(f))^2 \right\|. \end{aligned} \tag{2.2}$$

Since $\left\| B_k^{(n)} \right\| \leq 1$, we get

$$\left\| B_k^{(n)}(f^2) - (B_k^{(n)}(f))^2 \right\| \leq \left\| B_k^{(n)} \right\| \left\{ \left\| f^2 \right\| + \|f\|^2 \right\} \leq 2\|f\|^2.$$

Considering hypothesis we conclude that

$$\lim_k B_k^{(n)}(x^2) = x^2 = \lim_k (B_k^{(n)}(x))^2, \text{ uniformly in } n,$$

this implies that the right-hand side of (2.2) tends to zero (uniformly in $n$). We see that

$$\left\| B_k^{(n)}(xf) - xf \right\| \leq \left\| B_k^{(n)}(xf) - B_k^{(n)}(x).B_k^{(n)}(f) \right\| \left\| B_k^{(n)}(x).B_k^{(n)}(f) - xf \right\|.$$

If $\lim_k \left\| B_k^{(n)} f - f \right\| = 0$, uniformly in $n$, then it follows from (2.2) that

$$\lim_k \left\| B_k^{(n)}(xf) - xf \right\| = 0, \text{ uniformly in } n.$$

Here by taking $h = 1, x, x^2$ instead of $f$. We obtain (2.1) holds for $h = x^m$ for $m = 0, 1, 2, ....$Since $B_k^{(n)}$ is linear, (2.1) holds for every polynomial $p$. From the Weierstrass theorem asserting the norm-density of polynomials in $C\,[a,b]$ (see [24],p.159), we have for every $f \in C\,[a,b]$

$$\begin{aligned}\left\| B_k^{(n)} f - f \right\| &\leq \left\| B_k^{(n)} f - B_k^{(n)} p \right\| + \left\| B_k^{(n)} p - p \right\| + \|f - p\| \\ &\leq 2\,\|f - p\| + \left\| B_k^{(n)} p - p \right\|.\end{aligned}$$

Taking supremum over $n$ and letting $k \to \infty$, result follows. □

Let $C_{2\pi}$ be the space of real-valued continuous functions $f$ on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$. Then $C_{2\pi}$ is closed subalgebra of $C\,[-\pi, \pi]$ and 1 belongs to $C_{2\pi}$.

In the following theorem, we extend Korovkin type approximation theorem for a sequence of positive linear operators over $C_{2\pi}$ via $\mathcal{A}-$ summation process.

**2.2. Theorem**. Let $\mathcal{A} := \left\{ A^{(n)} \right\}$ be a sequence of infinite matrices with non-negative real entries. Assume that $\{L_j\}$ be a sequence of positive linear operators from $C_{2\pi}$ into $C_{2\pi}$ for which (1.5) holds. If,

$$\lim_k \left\| B_k^{(n)} h - h \right\| = 0, \text{ uniformly in } n,$$

for all $h = 1,\ \sin x,\ \cos x$, then $\{L_j\}$ is $\mathcal{A}-$ summation process on $C_{2\pi}$ i.e., for every $f \in C_{2\pi}$,

$$\lim_k \left\| B_k^{(n)} f - f \right\| = 0, \text{ uniformly in } n.$$

*Proof.* As in the proof of Theorem 2.1, without loss generality we assume that $B_k^{(n)}(1) \leq 1$. By (1.3), for every $f$ in $C_{2\pi}$,we have

$$\begin{aligned}&\left\| B_k^{(n)}(f \sin x) - B_k^{(n)}(f).B_k^{(n)}(\sin x) \right\|^2 \\ &\leq \left\| B_k^{(n)}(f^2) - (B_k^{(n)}(f))^2 \right\| \left\| B_k^{(n)}(\sin^2 x) - (B_k^{(n)}(\sin x))^2 + B_k^{(n)}(\cos^2 x) - (B_k^{(n)}(\cos x))^2 \right\| \\ &= \left\| B_k^{(n)}(f^2) - (B_k^{(n)}(f))^2 \right\| \left\| B_k^{(n)}(1) - (B_k^{(n)}(\sin x))^2 - (B_k^{(n)}(\cos x))^2 \right\| \\ &\leq 2\,\|f\|^2 . \left\| B_k^{(n)}(1) - (B_k^{(n)}(\sin x))^2 - (B_k^{(n)}(\cos x))^2 \right\|.\end{aligned}$$

Considering hypothesis we conclude that

$$\lim_k \left\| B_k^{(n)}(f \sin x) - B_k^{(n)}(f).B_k^{(n)}(\sin x) \right\| = 0, \text{ uniformly in } n.$$

Observe now that

$$\begin{aligned}&\left\| B_k^{(n)}(f \sin x) - f \sin x \right\| \\ &\leq \left\| B_k^{(n)}(f \sin x) - B_k^{(n)}(f).B_k^{(n)}(\sin x) \right\| \left\| B_k^{(n)}(f).B_k^{(n)}(\sin x) - f \sin x \right\|.\end{aligned}$$

If $\lim_k \left\| B_k^{(n)} f - f \right\| = 0$,(uniformly in $n$) then $\lim_k \left\| B_k^{(n)}(f \sin x) - f \sin x \right\| = 0$, (uniformly in $n$). We obtain similarly that $\lim_k \left\| B_k^{(n)}(f \cos x) - f \cos x \right\| = 0$, (uniformly in $n$). By taking $h = 1, \sin x, \cos x$ instead of $f$ then (2.1) holds for $h = \sin^m x \cos^t x$ for all nonnegative integers $m$ and $t$, which ensures that it is valid for $h = \sin mx. \cos tx$ for all such $m$ and $t$. Thus (2.1) holds for every trigonometric polynomial $p$, and since the latter functions are dense in $C_{2\pi}$. (see [24],p:190) we have for every $f$ in $C_{2\pi}$ that $\lim_k \left\| B_k^{(n)} f - f \right\| = 0$, uniformly in $n$. □

We next consider the space $C(D)$ of complex-valued continuous functions $f$ on the closed unit disk $D = \{z : |z| \leq 1\}$ in the complex plane.

In what follows we require the following

[B] **Lemma.** [27] If $L$ is a positive linear operators on $C(D)$ with $L(1) \leq 1$, then

$$V(h) := L(|h|^2) - |L(h)|^2 \geq 0$$

for every $h$ in $C(D)$. Morever, for $f$ and $g$ in $C(D)$ it is the case that

$$|L(fg) - L(f)L(g)|^2 \leq V(f)V(g)$$

$$\|L(fg) - L(f)L(g)\| \leq \|V(f)\|^{\frac{1}{2}} \|V(g)\|^{\frac{1}{2}} \tag{2.4}$$

We now give a Korovkin type approximation theorem for a sequence of positive linear operators defined on $C(D)$ via $\mathcal{A}-$ summation process.

**2.3. Theorem**. Let $\mathcal{A} := \left\{A^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. Assume that $\{L_j\}$ be a sequence of positive linear operators from $C(D)$ into $C(D)$ for which (1.5) holds. If,

$$\lim_k \left\|B_k^{(n)} h - h\right\| = 0, \text{ uniformly in } n,$$

for all $h = 1,\ z,\ |z|^2$, then $\{L_j\}$ is $\mathcal{A}-$ summation process on $C(D)$ i.e., for every $f \in C(D)$,

$$\lim_k \left\|B_k^{(n)} f - f\right\| = 0, \text{uniformly in } n.$$

*Proof.* We may assume that $B_k^{(n)}(1) \leq 1$ for all $n, k$. Since (2.4), we have

$$\begin{aligned}\left\|B_k^{(n)}(zf) - B_k^{(n)}(z).B_k^{(n)}(f)\right\|^2 &\leq \left\|B_k^{(n)}(\,|z|^2\,) - \left|B_k^{(n)}(z)\right|^2\right\| \left\|B_k^{(n)}(\,|f|^2\,) - \left|B_k^{(n)}(f)\right|^2\right\| \\ &\leq 2\,\|f\|^2 \left\|B_k^{(n)}(\,|z|^2\,) - \left|B_k^{(n)}(z)\right|^2\right\|.\end{aligned}$$

By the hypothesis we get

$$\lim_k B_k^{(n)}(|z|^2) = |z|^2 = \lim_k \left|B_k^{(n)}(z)\right|^2, \text{ uniformly in } n,$$

this implies that

$$\lim_k \left\|B_k^{(n)}(zf) - B_k^{(n)}(z).B_k^{(n)}(f)\right\| = 0, \text{ uniformly in } n.$$

We can write

$$\left\|B_k^{(n)}(zf) - zf\right\| \leq \left\|B_k^{(n)}(zf) - B_k^{(n)}(z).B_k^{(n)}(f)\right\| \left\|B_k^{(n)}(z).B_k^{(n)}(f) - zf\right\|$$

if $\lim_k \left\|B_k^{(n)} f - f\right\| = 0$, (uniformly in $n$) then $\lim_k \left\|B_k^{(n)}(zf) - zf\right\| = 0$, (uniformly in $n$). We obtain that (2.1) holds for $\overline{h}$ whenever it holds for $h$. Here by taking $h = 1, z, |z|^2$ instead of $f$, (2.1) holds for $h = \overline{z}^m.z^k$ for all non-negative integers $m$ and $k$, hence for every polynomial in $z$ and $\overline{z}$. By Stone's theorem (see [24],p:165) the set of all such polynomials is dense in $C(D)$, so (2.1) holds for every $f$ in $C(D)$.

□

## 3. A Korovkin Type Approximation Theorem via Statistical Convergence

In this section we give simple proofs for statistical analog of Korovkin's theorems considered in [9] and [13], also using $A-$statistically convergence, we extend a Korovkin type result for positive linear operators over the space $C(D)$.
First we recall the concept of $A-$statistical convergence. Let $A := (a_{jn})$, $j, n = 1, 2, ...$, be an infinite summability matrix. For a given sequence $x := (x_n)$, the $A-$transform of $x$, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j := \sum_n a_{jn}x_n$, provided the series converges for each $j$. The matrix $A$ is said to be regular if $\lim_j (Ax)_j = L$ whenever $\lim x = L$ [12].Suppose that $A$ is a non-negative regular summability matrix. Then $x$ is $A-$statistically convergent to $L$ if for every $\varepsilon > 0$

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

In this case we write $st_A - \lim x = L$ ([11],[17]). The case in which $A = C_1$, the Cesaro matrix of order one, reduces to the statistical convergence ([10],[11]). Also if $A = I$, the identity matrix, then it reduces to the ordinary convergence.
Note that, if $A = (a_{jn})$ is a non-negative regular matrix such that $\lim_j \max_n \{a_{jn}\} = 0$, then $A-$statistical convergence is stonger than convergence [17].

**3.1. Theorem**.Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $\{L_n\}$ be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. If,

$$st_A - \lim_n \|L_n h - h\| = 0. \tag{3.1}$$

for all $h = 1$, $x$, $x^2$, then, for every $f \in C[a, b]$

$$st_A - \lim_n \|L_n f - f\| = 0.$$

*Proof.* As in the proof of Theorem 2.1 , without loss of generality we assume that in $L_n(1) \leq 1$ for all $n$. By (1.2) , for every $f$ in $C[a, b]$ and for all $n$ , we can write

$$\|L_n(xf) - L_n(x).L_n(f)\|^2 \leq \left\|L_n(x^2) - (L_n(x))^2\right\| \left\|L_n(f^2) - (L_n(f))^2\right\|. \tag{3.2}$$

Since $\|L_n\| \leq 1$, we get

$$\left\|L_n(f^2) - (L_n(f))^2\right\| \leq \|L_n\| \left\{\left\|f^2\right\| + \|f\|^2\right\} \leq 2\|f\|^2$$

by hypothesis we obtain that

$$st_A - \lim_n L_n(x^2) = x^2 = st_A - \lim_n (L_n(x))^2$$

this implies that the right-hand side of (3.2) is $A-$statistically convergent to zero. Observe that

$$\|L_n(xf) - xf\| \leq \|L_n(xf) - L_n(x).L_n(f)\| . \|L_n(x).L_n(f) - xf\| .$$

If $st_A - \lim_n \|L_n f - f\| = 0$, then it follows from (3.2) that

$$st_A - \lim_n \|L_n(xf) - xf\| = 0.$$

Here by taking $h = 1, x, x^2$ instead of $f$. We see that (3.1) holds for $h = x^m$ for $m = 0, 1, 2, ....$Since $L_n$ is linear, (3.1) holds for every polynomial $p$. Since $\|L_n\| \leq 1$ for every $n$, theorem follows from the Weierstrass theorem asserting the norm-density of polynomials in $C[a, b]$ . $\square$

**3.2. Theorem**. Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $\{L_n\}$ be a sequence of positive linear operators from $C_{2\pi}$ into $C_{2\pi}$. If,

$$st_A - \lim_n \|L_n h - h\| = 0.$$

for all $h = 1, \sin x, \cos x$, then, for every $f \in C_{2\pi}$

$$st_A - \lim_n \|L_n f - f\| = 0.$$

*Proof.* As in the proof of Theorem 2.2, there is no loss of generality in assuming that $L_n(1) \leq 1$. By (1.3), we have for every $f$ in $C_{2\pi}$,

$$\begin{aligned} &\|L_n(f \sin x) - L_n(f).L_n(\sin x)\|^2 \\ &\leq \left\|L_n(f^2) - (L_n(f))^2\right\| \left\|L_n(\sin^2 x) - (L_n(\sin x))^2 + L_n(\cos^2 x) - (L_n(\cos x))^2\right\| \\ &= \left\|L_n(f^2) - (L_n(f))^2\right\| \left\|L_n(1) - (L_n(\sin x))^2 - (L_n(\cos x))^2\right\| \\ &\leq 2\|f\|^2 . \left\|L_n(1) - (L_n(\sin x))^2 - (L_n(\cos x))^2\right\| \end{aligned}$$

By the hypothesis we have

$$st_A - \lim_n \|L_n(f \sin x) - L_n(f)L_n(\sin x)\| = 0.$$

This implies that $st_A - \lim_n \|L_n(f \sin x) - f \sin x\| = 0$ whenever $st_A - \lim\limits_n \|L_n f - f\| = 0$.We see similarly that $st_A - \lim_n \|L_n(f \cos x) - f \cos x\| = 0$ in this situtation. Thus (3.1) holds for $h = \sin^m x \cos^t x$ for all nonnegative integers $m$ and $t$, which ensures that it is valid for $h = \sin mx. \cos tx$ for all such $m$ and $t$.Thus (3.1) holds for every trigonometric polynomial $p$, and since the latter functions are dense in $C_{2\pi}$.(see [24],p:190) we have for every $f \in C_{2\pi}$ that

$$st_A - \lim_n \|L_n f - f\| = 0.$$

□

**3.3. Theorem.** Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $\{L_n\}$ be a sequence of positive linear operators from $C(D)$ into $C(D)$. If,

$$st_A - \lim_n \|L_n h - h\| = 0.$$

for all $h = 1, z, |z|^2$, then, for every $f \in C(D)$

$$st_A - \lim_n \|L_n f - f\| = 0.$$

*Proof.* We may assume that $L_n(1) \leq 1$ for all $n$. It is evident that (3.1) holds for $\overline{h}$ whenever it holds for $h$. The estimate (2.4) guarantees that (3.1) holds for $h = \overline{z}^m.z^k$ for all nonnegative integers $m$ and $k$, hence for every polynomial in $z$ and $\overline{z}$. By Stone's theorem the set of all such polynomials is dense in $C(D)$, so (3.1) holds for every $f$ in $C(D)$.

□

Note that if we replace $A$ by the identity matrix we get the complex Korovkin theorem.

**3.1. Remark.** Now we exhibit two examples of sequences of positive linear operators. The first one shows that Theorem 3.3 does not work, so the classical Korovkin theorem does not work either; but Theorem 2.3 works. The second one gives that Theorem 2.3 does not work but Theorem 3.3 does work. In order to see this let $\{L_j\}$ be a sequence of positive linear operators from $C(D)$ into $C(D)$ satisfying the hypothesis of the classical complex Korovkin theorem. Assume now that $\mathcal{A} = \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ is a sequence of infinite matrices defined by $a_{kj}^{(n)} = 1/k$ if $n \leq j < n + k$, and $a_{kj}^{(n)} = 0$ otherwise. In this case $\mathcal{A}-$summability method reduces to almost convergence.. We also take $A = C_1$ in Theorem 3.3. In this case $A-$ statistical convergence reduces to the statistical convergence. Then consider the following two examples.

**(a)** Take $(u_j) = \{(-1)^j\}$. Note that $u$ is almost convergent to zero [19], but it is not statistically convergent [11]. Now define

$$T_j(f;x) = (1 + u_j)L_j(f;x) \text{ for all } f \in C(D).$$

Then observe that $\{T_j\}$ satisfies Theorem 2.3, but it satisfies neither the classical Korovkin theorem nor the present Theorem 3.3.

**(b)** Consider a non-negative sequence $(u_j)$ which is statistically convergent to zero but not almost convergent. Such an example may be found in [21]. Proceeding exactly as in the case (a) we can construct a sequence of positive linear operators so that it is statistically convergent to the identity operator but not almost convergent. These two methods are incompatible [21].

The examples given above suggest that if the sequence of positive linear operators does not converge then we can use alternatively either almost convergence method or statistical convergence method to get some Korovkin type approximation results.

## References


[1] O.Agratini, A-Statistical Convergence of a Class of İntegral Operators. Appl. Math. Inf. Sci. 6, (2), 325-328, 2012.

[2] F. Altomare and M. Campiti, Korovkin type Approximation Theory and its Application. (Walter de Gruyter Studies in Math. 17, de Gruyter&Co., Berlin, 1994).

[3] G. A. Anastassiou and O. Duman, Towards Intelligent Modelling: Statistical Approximation Theory,( Intelligent Systems Reference Library, Vol.14, Springer, Berlin, 2011).

[4] Ö. G. Atlihan and C. Orhan, Summation process of positive linear operators, Comp.Math. Appl., 56, 1188-1195, 2008.

[5] Ö.G.Atlihan, H.Gül İnce, C. Orhan, Some variations of the Bohman-Korovkin Theorem, Math.Comp. Modelling. 50, 1205-1210, 2009.

[6] H. T. Bell, Order summability and almost convergence, Proc. Amer. Math. Soc. 38, 548-552.,1973.

[7] H. Bohman, On approximation of continuous and analytic functions. Ark. Mat. 2, 43-56, 1952.

[8] J. Boos, Classical and Modern Methods in Summability. (Oxford Sci. Publ., London. 2000).

[9] O. Duman, Statistical approximation for periodic functions, Demonstratio Math. 36, 873–878 , 2003.

[10] H. Fast, Sur la convergence statistique, Colloq. Math. 2, 241-244 ,1951.

[11] J. A. Fridy, On statistical convergence, Analysis. 5, 301-313., 1985.

[12] A. R. Freedman, J. J. Sember, Densities and summability, Pacific J. Math. 95, 293-305 ,1981.

[13] A. D. Gadjiev and C.Orhan, Some aprproximation theorems via statistical convergence, Rocky Mountain Journal of Math. 32 (1) 129-137 ,2002.

[14] P. P. Korovkin, On convergence of linear positive operators in the space of continuos functions, Doklady Akad. NaukSSSR. 90, 961-964 ,1953.

[15] P.P. Korovkin, Linear operators and Theory of Approximation.( Hindustan Publ. Co., Delhi, 1960).

[16] J. P. King and J. J. Swetits, Positive linear operators and summability, J. Austral. Math. Soc. 11, 281-290, 1970.

[17] E. Kolk, Matrix summability of statistically convergent sequences, Analysis. 13, 77-83 ,1993.

[18] G.G. Lorentz, Approximation of Functions. (Chelsa Publ. Company. New York, 1986).

[19] G.G. Lorentz, A contribution to the theory of divergent sequence, Acta Math. 80, 167-190 ,1948.

[20] H. E. Lomelí and C. L. García, Variations on a Theorem of Korovkin, The Amer. Math. Monthly, 113 (8) 744-750 ,2006.
[21] H. I. Miller and C. Orhan, On almost convergent and statistically convergent subsequences, Acta Math. Hungar. 93, 135–151 ,2001.
[22] R. N. Mohapatra, Quantitative results on almost convergence of a sequence of positive linear operators, J.of Approx. Theory, 20, 239-250 ,1977.
[23] T. Nishishiraho, Convergence of positive linear approximation processes, Tôhoku Math. J., 35, 441-458 ,1983.
[24] W. Rudin, Principles of Mathematical Analysis, ( McGraw-Hill, New York, 1976).
[25] M. Stieglitz, Eine verallgenmeinerung des begriffs festkonvergenz, Math. Japonica. 18, 53-70 ,1973.
[26] J. J. Swetits, On summability and positive linear operators, J. Approx. Theory. 25, 186-188 ,1979.
[27] M. Uchiyama, Proofs of Korovkin's theorems via inequalities, The Amer. Math. Monthly, 110, 334-336 ,2003.
[28] T.Yurdakadim, E.Tas, and İ.Sakaoğlu, Approximation of functions by the sequence of integral operators. Appl. Math. Comput., Vol. 219(8), 3863-3871., 2012.
[29] A. Zygmund, Trigonometric Series. (Cambridge University Press. 1979).

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 823 – 837

# Restricted hom-Lie algebras

Baoling Guan * and Liangyun Chen †

**Abstract**

The paper studies the structure of restricted hom-Lie algebras. More specifically speaking, we first give the equivalent definition of restricted hom-Lie algebras. Second, we obtain some properties of $p$-mappings and restrictable hom-Lie algebras. Finally, the cohomology of restricted hom-Lie algebras is researched.

*2000 AMS Classification:* 17A60, 17B56, 17B50

**Keywords:** Restricted hom-Lie algebras; restrictable hom-Lie algebras; $p$-mapping; cohomology.

Received 25/09/2013 : Accepted 21/04/2014 Doi : 10.15672 /HJMS.2015449439

## 1. Introduction

The concept of a restricted Lie algebra is attributable to N. Jacobson in 1943. It is well known that the Lie algebras associated with algebraic groups over a field of characteristic $p$ are restricted Lie algebras [14]. Now, restricted theories attract more and more attentions. For example: restricted Lie superalgebras[6], restricted Lie color algebras[2], restricted Leibniz algebras[4], restricted Lie triple systems[8] and restricted Lie algebras [5] were studied, respectively.

However, The notion of hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov in [7] as part of a study of deformations of the Witt and the Virasoro algebras. In a hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the hom-Jacobi identity. Some q-deformations of the Witt and the Virasoro algebras have the structure of a hom-Lie algebra [7]. Because of close relation to discrete and deformed vector fields and differential calculus [7, 9, 10], hom-Lie algebras are widely studied recently [1, 3, 11, 12, 16, 17, 18]. As a natural generalization of a restricted Lie algebra, it seems

*School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
Email:
†College of Sciences, Qiqihar University, Qiqihar 161006, China
Email:chenly640@nenu.edu.cn Corresponding author.
Supported by NNSF of China (No.11171055), Natural Science Foundation of Jilin province (No. 201115006), Scientific Research Foundation for Returned Scholars, Ministry of Education of China and the Fundamental Research Funds for the Central Universities(No. 11SSXT146).

desirable to investigate the possibility of establishing a parallel theory for restricted hom-Lie algebras. As is well known, restricted Lie algebras play predominant roles in the theories of modular Lie algebras [15]. Analogously, the study of restricted hom-Lie algebras will play an important role in the classification of the finite-dimensional modular simple hom-Lie algebras.

The paper study the structure of restricted hom-Lie algebras. Let us briefly describe the content and setup of the present article. In Sec. 2, the equivalent definition of restricted hom-Lie algebras is given. In Sec. 3, we obtain some properties of $p$-mappings and restrictable hom-Lie algebras. In Sec. 4, we research the cohomology of restricted hom-Lie algebras.

In the paper, $\mathbb{F}$ is a field of prime characteristic. Let $L$ denote a finite-dimensional restricted hom-Lie algebra over $\mathbb{F}$.

**1.1. Definition.** [14] Let $L$ be a Lie algebra over $\mathbb{F}$. A mapping $[p]: L \to L, a \mapsto a^{[p]}$ is called a $p$-mapping, if

(1) $\mathrm{ad}a^{[p]} = (\mathrm{ad}a)^p,\ \forall a \in L$,

(2) $(ka)^{[p]} = k^p a^{[p]},\ \forall a \in L, k \in \mathbb{F}$,

(3) $(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum\limits_{i=1}^{p-1} s_i(a,b)$,

where $(\mathrm{ad}(a \otimes X + b \otimes 1))^{p-1}(a \otimes 1) = \sum\limits_{i=1}^{p-1} i s_i(a,b) \otimes X^{i-1}$ in $L \otimes_{\mathbb{F}} \mathbb{F}[X], \forall a, b \in L$, The pair $(L, [p])$ is referred to as a restricted Lie algebra.

**1.2. Definition.** [13] (1) A hom-Lie algebra is a triple $(L, [\cdot,\cdot]_L, \alpha)$ consisting of a linear space $L$, a skew-symmetric bilinear map $[\cdot,\cdot]_L : \Lambda^2 L \to L$ and a linear map $\alpha : L \to L$ satisfying the following hom-Jacobi identity:

$$[\alpha(x), [y,z]_L]_L + [\alpha(y), [z,x]_L]_L + [\alpha(z), [x,y]_L]_L = 0$$

for all $x, y, z \in L$;

(2) A hom-Lie algebra is called a multiplicative hom-Lie algebra if $\alpha$ is an algebraic morphism, i.e., for any $x, y \in L$, we have $\alpha([x,y]_L) = [\alpha(x), \alpha(y)]_L$;

(3) A sub-vector space $\eta \subset L$ is called a hom-Lie subalgebra of $(L, [\cdot,\cdot]_L, \alpha)$ if $\alpha(\eta) \subset \eta$ and $\eta$ is closed under the bracket operation $[\cdot,\cdot]_L$, i.e., $[x,y]_L \in \eta$ for all $x, y \in \eta$;

(4) A sub-vector space $\eta \subset L$ is called a hom-Lie ideal of $(L, [\cdot,\cdot]_L, \alpha)$ if $\alpha(\eta) \subset \eta$ and $[x,y]_L \in \eta$ for all $x \in \eta, y \in L$.

## 2. The equivalent definition of restricted hom-Lie algebras

Let $(L, [\cdot,\cdot]_L, \alpha)$ be a multiplicative hom-Lie algebra over $\mathbb{F}$. For $c \in L$ satisfying $\alpha(c) = c$, we define $\mathrm{ad}c(a) := [\alpha(a), c]$. Put $L_0 := \{x | \alpha(x) \neq x\} \cup \{0\}$ and $L_1 := \{x | \alpha(x) = x\}$. Then $L = L_0 \cup L_1$ and $L_1$ is a hom-Lie subalgebra of $L$.

**2.1. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha)$ be a multiplicative hom-Lie algebra over $\mathbb{F}$. A mapping $[p]: L_1 \to L_1, a \mapsto a^{[p]}$ is called a $p$-mapping, if

(1) $[\alpha(y), x^{[p]}] = (\mathrm{ad}x)^p(y),\ \forall x \in L_1, y \in L$,

(2) $(kx)^{[p]} = k^p x^{[p]},\ \forall x \in L_1, k \in \mathbb{F}$,

(3) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum\limits_{i=1}^{p-1} s_i(x,y)$,

where $(\mathrm{ad}(x \otimes X + y \otimes 1))^{p-1}(x \otimes 1) = \sum\limits_{i=1}^{p-1} i s_i(x,y) \otimes X^{i-1}$ in $L \otimes_{\mathbb{F}} \mathbb{F}[X], \forall x, y \in L_1, \alpha(x \otimes X) = \alpha(x) \otimes X$. The pair $(L, [,]_L, \alpha, [p])$ is referred to as a restricted hom-Lie algebra.

From the above definition, we may see that (i) $\alpha(x^{[p]}) = (\alpha(x))^{[p]}$ for all $x \in L_1$, i.e., $\alpha \circ [p] = [p] \circ \alpha$; (ii) By (1) of the definition, one gets $\mathrm{ad}x^{[p]} = (\mathrm{ad}x)^p$ for all $x \in L_1$.

Let $(L, \alpha)$ be a hom-Lie algebra over $\mathbb{F}$ and $f : L \to L$ be a mapping. $f$ is called a $p$-semilinear mapping, if $f(kx+y) = k^p f(x) + f(y)$, $\forall x, y \in L$, $\forall k \in \mathbb{F}$. Let $S$ be a subset of a hom-Lie algebra $(L, \alpha)$. We put $C_L(S) := \{x \in L | \ [\alpha(y), x] = 0, \forall y \in S\}$. $C_L(S)$ is called the centralizer of $S$ in $L$. Put $C(L) := \{x \in L | \ [\alpha(y), x] = 0, \forall y \in L\}$. $C(L)$ is called the center of $L$.

**2.2. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha)$ be a restricted hom-Lie algebra over $\mathbb{F}$. A hom-Lie subalgebra $H$ of $L$ is called a $p$-subalgebra, if $x^{[p]} \in H_1$ for all $x \in H_1$, where $H_1 = \{x \in H | \alpha(x) = x\}$.

**2.3. Proposition.** *Let $L$ be a hom-Lie subalgebra of a restricted hom-Lie algebra $(G, [\cdot,\cdot]_G, \alpha, [p])$ and $[p]_1 : L_1 \to L_1$ a mapping. Then the following statements are equivalent:*

(1) *$[p]_1$ is a $p$-mapping on $L_1$.*

(2) *There exists a $p$-semilinear mapping $f : L_1 \to C_G(L)$ such that $[p]_1 = [p] + f$.*

*Proof.* (1)$\Rightarrow$(2). Consider $f : L_1 \to G$, $f(x) = x^{[p]_1} - x^{[p]}$. Since $\mathrm{ad}f(x)(y) = [\alpha(y), f(x)] = 0, \forall x \in L_1, y \in L$, $f$ actually maps $L_1$ into $C_G(L)$. For $x, y \in L_1, k \in \mathbb{F}$, we obtain

$$\begin{aligned}
&f(kx+y)\\
&= k^p x^{[p]_1} + y^{[p]_1} + \sum_{i=1}^{p-1} s_i(kx, y) - k^p x^{[p]} - y^{[p]} - \sum_{i=1}^{p-1} s_i(kx, y)\\
&= k^p f(x) + f(y),
\end{aligned}$$

which proves that $f$ is $p$-semilinear.

(2)$\Rightarrow$(1). We next will check three conditions of the definition step and step. For $x, y \in L_1$, we have

$$\begin{aligned}
&(x+y)^{[p]_1} = (x+y)^{[p]} + f(x+y)\\
&= x^{[p]} + f(x) + y^{[p]} + f(y) + \sum_{i=1}^{p-1} s_i(x, y)\\
&= x^{[p]_1} + y^{[p]_1} + \sum_{i=1}^{p-1} s_i(x, y)
\end{aligned}$$

and

$$\begin{aligned}
&(kx)^{[p]_1} = (kx)^{[p]} + f(kx)\\
&= k^p x^{[p]} + k^p f(x)\\
&= k^p (x^{[p]} + f(x))\\
&= k^p x^{[p]_1}.
\end{aligned}$$

For $x \in L_1, z \in L$, one gets

$$\begin{aligned}
&\mathrm{ad}x^{[p]_1}(z) = \mathrm{ad}(x^{[p]} + f(x))(z)\\
&= \mathrm{ad}x^{[p]}(z) + \mathrm{ad}f(x)(z)\\
&= \mathrm{ad}x^{[p]}(z)\\
&= (\mathrm{ad}x)^p(z).
\end{aligned}$$

The proof is complete. $\square$

**2.4. Corollary.** *The following statements hold.*

(1) *If $C(L)=0$, then $L$ admits at most one $p$-mapping.*

(2) *If two $p$-mappings coincide on a basis, then they are equal.*

(3) *If $(L,[\cdot,\cdot]_L,\alpha,[p])$ is restricted, then there exists a $p$-mapping $[p]^{'}$ of $L$ such that $x^{[p]^{'}}=0,\ \forall x\in C(L_1)$.*

*Proof.* (1) We set $G=L$. Then $C_G(L)=C(L)$, the only $p$-semilinear mapping occurring in Proposition 2.3 is the zero mapping.

(2) If two $p$-mappings coincide on a basis, their difference vanishes since it is $p$-semilinear.

(3) $[p]|_{C(L_1)}$ defines a $p$-mapping on $C(L_1)$. Since $C(L_1)$ is abelian, it is $p$-semilinear. Extend this to a $p$-semilinear mapping $f:L_1\to C(L_1)$. Then $[p]^{'}:=[p]-f$ is a $p$-mapping of $L$, vanishing on $C(L_1)$. □

From the proof of Theorem 2 in [18], we see the following definition:

**2.5. Definition.** Let $(L,[\cdot,\cdot]_L,\alpha_L)$ be a hom-Lie algebra, and let $j:L\to U_{HLie}(L)$ be the composition of the maps $L\hookrightarrow F_{HNAs}(L)\twoheadrightarrow U_{HLie}(L)$. The pair $(U_{HLie}(L),j)$ is called a universal enveloping algebra of $L$ if for every hom-associative algebra $(A,\mu_A,\alpha_A)$ and every morphism $f:L\to HLie(A)$ of hom-Lie algebras, there exists a unique morphism $h:U_{HLie}(L)\to A$ of hom-associative algebras such that $f=h\circ j$ (as morphisms of $\mathbb{F}$-modules).

In the special case of $G=U_{HLie}(L)^{-}\supset L$, where $U_{HLie}(L)$ is the universal enveloping algebra of hom-Lie algebra $L$ (see [18]) and $U_{HLie}(L)^{-}$ denotes a hom-Lie algebra given by hom-associative algebra $U_{HLie}(L)$ via the commutator bracket. We have the following theorem:

**2.6. Theorem.** *Let $(e_j)_{j\in J}$ be a basis of $L_1$ such that there are $y_j\in L_1$ with $(\mathrm{ad}e_j)^p=\mathrm{ad}y_j$. Then there exists exactly one $p$-mapping $[p]:L_1\to L$ such that $e_j^{[p]}=y_j,\forall j\in J$.*

*Proof.* For $z\in L_1$, we have $0=((\mathrm{ad}e_j)^p-\mathrm{ad}y_j)(z)=[\alpha(z),e_j^p-y_j]$. Then $e_j^p-y_j\in C_{U_{HLie}(L_1)}(L_1),\forall j\in J$. We define a $p$-semilinear mapping $f:L_1\to C_{U_{HLie}(L_1)}(L_1)$ by means of

$$f(\sum\alpha_je_j):=\sum\alpha_j^p(y_j-e_j^p).$$

Consider $V:=\{x\in L_1|x^p+f(x)\in L_1\}$. The equation

$$(kx+y)^p+f(kx+y)=k^px^p+y^p+\sum_{i=1}^{p-1}s_i(kx,y)+k^pf(x)+f(y)$$

ensures that $V$ is a subspace of $L_1$. Since it contains the basis $(e_j)_{j\in J}$, we conclude that $x^p+f(x)\in L_1,\ \forall x\in L_1$. By virtue of Proposition 2.3, $[p]:L_1\to L, x^{[p]}:=x^p+f(x)$ is a $p$-mapping on $L_1$. In addition, we obtain $e_j^{[p]}=e_j^p+f(e_j)=y_j$, as asserted. The uniqueness of $[p]$ follows from Corollary 2.4. □

**2.7. Definition.** A multiplicative hom-Lie algebra $(L,[\cdot,\cdot]_L,\alpha_L)$ is called restrictable, if $(\mathrm{ad}x)^p\in\mathrm{ad}L_1$ for all $x\in L_1$, where $\mathrm{ad}L_1=\{\mathrm{ad}x|x\in L_1\}$.

**2.8. Theorem.** *$L$ is a restrictable hom-Lie algebra if and only if there is a $p$-mapping $[p]:L_1\to L_1$ which makes $L$ a restricted hom-Lie algebra.*

*Proof.* ($\Leftarrow$) By the definition of $p$-mapping $[p]$, for $x\in L_1$, there exists $x^{[p]}\in L_1$ such that $(\mathrm{ad}x)^p=\mathrm{ad}x^{[p]}\in\mathrm{ad}L_1$. Hence $L$ is restrictable.

($\Rightarrow$) Let $L$ be restrictable. Then for $x\in L_1$, we have $(\mathrm{ad}x)^p\in\mathrm{ad}L_1$, that is, there exists $y\in L_1$ such that $(\mathrm{ad}x)^p=\mathrm{ad}y$. Let $(e_j)_{j\in J}$ be a basis of $L_1$. Then there exists

$y_j \in L_1$ such that $(\mathrm{ad}e_j)^p = \mathrm{ad}y_j (j \in J)$. By Theorem 2.6, there exists exactly one $p$-mapping $[p] : L_1 \to L_1$ such that $e_j^{[p]} = y_j, \forall j \in J$, which makes $L$ a restricted hom-Lie algebra. $\square$

## 3. Properties of $p$-mappings and restrictable hom-Lie algebras

In the section, we will discuss some properties of $p$-mappings and restrictable hom-Lie algebras.

**3.1. Definition.** [13] Let $(L, [\cdot,\cdot]_L, \alpha)$ and $(\Gamma, [\cdot,\cdot]_\Gamma, \beta)$ be two hom-Lie algebras. A linear map $\phi : L \to \Gamma$ is said to be a morphism of hom-Lie algebras if

$$\phi[u,v]_L = [\phi(u), \phi(v)]_\Gamma, \qquad \forall u,v \in L, \tag{3.1}$$

$$\phi \circ \alpha = \beta \circ \phi. \tag{3.2}$$

Denote by $\mathfrak{G}_\phi = \{(x, \phi(x)) | x \in L\} \subseteq L \oplus \Gamma$ the graph of a linear map $\phi : L \to \Gamma$.

**3.2. Definition.** A morphism of hom-Lie algebras $\phi : (L, [\cdot,\cdot]_L, \alpha, [p]_1) \to (\Gamma, [\cdot,\cdot]_\Gamma, \beta, [p]_2)$ is said to be restricted if $\phi(x^{[p]_1}) = (\phi(x))^{[p]_2}$ for all $x \in L$.

**3.3. Proposition.** *Given two restricted hom-Lie algebras $(L, [\cdot,\cdot]_L, \alpha, [p]_1)$ and $(\Gamma, [\cdot,\cdot]_\Gamma, \beta, [p]_2)$, there is a restricted hom-Lie algebra $(L \oplus \Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$, where the bilinear map $[\cdot,\cdot]_{L\oplus\Gamma} : \wedge^2(L \oplus \Gamma) \to L \oplus \Gamma$ is given by*

$$[u_1 + v_1, u_2 + v_2]_{L\oplus\Gamma} = [u_1, u_2]_L + [v_1, v_2]_\Gamma, \qquad \forall u_1, u_2 \in L, v_1, v_2 \in \Gamma,$$

*and the linear map $(\alpha+\beta) : L \oplus \Gamma \to L \oplus \Gamma$ is given by*

$$(\alpha+\beta)(u+v) = \alpha(u) + \beta(v), \qquad \forall u \in L, v \in \Gamma,$$

*the $p$-mapping $[p] : L \oplus \Gamma \to L \oplus \Gamma$ is given by*

$$(u+v)^{[p]} = u^{[p]_1} + v^{[p]_2}, \qquad \forall u \in L, v \in \Gamma.$$

*Proof.* Recall that $L_1 = \{x \in L | \alpha(x) = x\}$ and $\Gamma_1 = \{x \in \Gamma | \beta(x) = x\}$. For any $u_1, u_2 \in L, v_1, v_2 \in \Gamma$, we have

$$\begin{aligned} [u_2 + v_2, u_1 + v_1]_{L\oplus\Gamma} &= [u_2, u_1]_L + [v_2, v_1]_\Gamma \\ &= -[u_1, u_2]_L - [v_1, v_2]_\Gamma \\ &= -[u_1 + v_1, u_2 + v_2]_{L\oplus\Gamma}. \end{aligned}$$

The bracket is obviously skew-symmetric. By a direct computation we have

$$\begin{aligned} & [(\alpha+\beta)(u_1+v_1), [u_2+v_2, u_3+v_3]_{L\oplus\Gamma}]_{L\oplus\Gamma} \\ & +c.p.((u_1+v_1), (u_2+v_2), (u_3+v_3)) \\ = \; & [\alpha(u_1) + \beta(v_1), [u_2, u_3]_L + [v_2, v_3]_\Gamma]_{L\oplus\Gamma} + c.p. \\ = \; & [\alpha(u_1), [u_2, u_3]_L]_L + c.p.(u_1, u_2, u_3) + [\beta(v_1), [v_2, v_3]_\Gamma]_\Gamma \\ & +c.p.(v_1, v_2, v_3) \\ = \; & 0, \end{aligned}$$

where $c.p.(a,b,c)$ means the cyclic permutations of $a, b, c$. For any $u_1 \in L_1, v_1 \in \Gamma_1, u_2 \in L, v_2 \in \Gamma$, we obtain

$$\begin{aligned} & \mathrm{ad}(u_1+v_1)^{[p]}(u_2+v_2) = [(\alpha+\beta)(u_2+v_2), (u_1+v_1)^{[p]}]_{L\oplus\Gamma} \\ & = [\alpha(u_2) + \beta(v_2), u_1^{[p]_1} + v_1^{[p]_2}]_{L\oplus\Gamma} \\ & = [\alpha(u_2), u_1^{[p]_1}]_L + [\beta(v_2), v_1^{[p]_2}]_\Gamma \\ & = \mathrm{ad}u_1^{[p]_1}(u_2) + \mathrm{ad}v_1^{[p]_2}(v_2) \end{aligned}$$

$$= (\mathrm{adu}_1)^p(u_2) + (\mathrm{adv}_1)^p(v_2)$$

and

$$\begin{aligned}
&(\mathrm{ad}(u_1+v_1))^p(u_2+v_2)\\
&= [[[\alpha^p(u_2)+\beta^p(v_2), \overbrace{u_1+v_1], u_1+v_1], \cdots, u_1+v_1}^{p}]_{L\oplus\Gamma}\\
&= [[[\alpha^p(u_2), \overbrace{u_1], u_1], \cdots, u_1}^{p}]_L + [[[\beta^p(v_2), \overbrace{v_1], v_1], \cdots, v_1}^{p}]_\Gamma\\
&= (\mathrm{adu}_1)^p(u_2) + (\mathrm{adv}_1)^p(v_2).
\end{aligned}$$

Hence $\mathrm{ad}(u_1+v_1)^{[p]}(u_2+v_2) = (\mathrm{ad}(u_1+v_1))^p(u_2+v_2)$, thus $\mathrm{ad}(u_1+v_1)^{[p]} = (\mathrm{ad}(u_1+v_1))^p$. Moreover, for any $u_1, u_2 \in L_1, v_1, v_2 \in \Gamma_1$, one gets

$$\begin{aligned}
&((u_1+v_1)+(u_2+v_2))^{[p]} = ((u_1+u_2)+(v_1+v_2))^{[p]} = (u_1+u_2)^{[p]_1} + (v_1+v_2)^{[p]_2}\\
&= u_1^{[p]} + u_2^{[p]} + \sum_{i=1}^{p-1} s_i(u_1,u_2) + v_1^{[p]} + v_2^{[p]} + \sum_{i=1}^{p-1} s_i(v_1,v_2)\\
&= (u_1^{[p]} + v_1^{[p]}) + (u_2^{[p]} + v_2^{[p]}) + (\sum_{i=1}^{p-1} s_i(u_1,u_2) + \sum_{i=1}^{p-1} s_i(v_1,v_2))\\
&= (u_1+v_1)^{[p]} + (u_2+v_2)^{[p]} + \sum_{i=1}^{p-1}(s_i(u_1,u_2)) + s_i(v_1,v_2))\\
&= (u_1+v_1)^{[p]} + (u_2+v_2)^{[p]} + \sum_{i=1}^{p-1} s_i((u_1,v_1)+(u_2,v_2))
\end{aligned}$$

and

$$\begin{aligned}
&(k(u_1+v_1))^{[p]} = (ku_1+kv_1)^{[p]} = (ku_1)^{[p]_1} + (kv_1)^{[p]_2}\\
&= k^p u_1^{[p]_1} + k^p v_1^{[p]_2} = k^p(u_1^{[p]_1} + v_1^{[p]_2})\\
&= k^p(u_1+v_1)^{[p]}.
\end{aligned}$$

Therefore, $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$ is a restricted hom-Lie algebra. $\square$

**3.4. Proposition.** *A linear map $\phi : (L, [\cdot,\cdot]_L, \alpha, [p]_1) \to (\Gamma, [\cdot,\cdot]_\Gamma, \beta, [p]_2)$ is a restricted morphism of restricted hom-Lie algebras if and only if the graph $\mathfrak{G}_\phi \subseteq L\oplus\Gamma$ is a restricted hom-Lie subalgebra of $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$.*

*Proof.* Let $\phi : (L, [\cdot,\cdot]_L, \alpha) \to (\Gamma, [\cdot,\cdot]_\Gamma, \beta)$ be a restricted morphism of restricted hom-Lie algebras. By (3.1), we have

$$[u+\phi(u), v+\phi(v)]_{L\oplus\Gamma} = [u,v]_L + [\phi(u),\phi(v)]_\Gamma = [u,v]_L + \phi[u,v]_L.$$

Then the graph $\mathfrak{G}_\phi$ is closed under the bracket operation $[\cdot,\cdot]_{L\oplus\Gamma}$. Furthermore, by (3.2), we have

$$(\alpha+\beta)(u+\phi(u)) = \alpha(u) + \beta\circ\phi(u) = \alpha(u) + \phi\circ\alpha(u),$$

which implies that $(\alpha+\beta)(\mathfrak{G}_\phi) \subseteq \mathfrak{G}_\phi$. Thus, $\mathfrak{G}_\phi$ is a hom-Lie subalgebra of $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta)$. Moreover, for $u+\phi(u) \in \mathfrak{G}_\phi$, one gets

$$(u+\phi(u))^{[p]} = u^{[p]_1} + (\phi(u))^{[p]_2} = u^{[p]_1} + \phi(u^{[p]_1}) \in \mathfrak{G}_\phi.$$

Thereby, the graph $\mathfrak{G}_\phi \subseteq L\oplus\Gamma$ is a restricted hom-Lie subalgebra of $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$.

Conversely, if the graph $\mathfrak{G}_\phi \subseteq L \oplus \Gamma$ is a restricted hom-Lie subalgebra of $(L \oplus \Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$, then we have

$$[u+\phi(u), v+\phi(v)]_{L\oplus\Gamma} = [u,v]_L + [\phi(u),\phi(v)]_\Gamma \in \mathfrak{G}_\phi,$$

which implies that

$$[\phi(u),\phi(v)]_\Gamma = \phi[u,v]_L.$$

Furthermore, $(\alpha+\beta)(\mathfrak{G}_\phi) \subset \mathfrak{G}_\phi$ yields that

$$(\alpha+\beta)(u+\phi(u)) = \alpha(u) + \beta\circ\phi(u) \in \mathfrak{G}_\phi,$$

which is equivalent to the condition $\beta\circ\phi(u) = \phi\circ\alpha(u)$, i.e. $\beta\circ\phi = \phi\circ\alpha$. Therefore, $\phi$ is a morphism of restricted hom-Lie algebras. Since $\mathfrak{G}_\phi$ is a restricted hom-Lie subalgebra of $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$, we have

$$(u+\phi(u))^{[p]} = u^{[p]_1} + (\phi(u))^{[p]_2} \in \mathfrak{G}_\phi.$$

Thus, $(\phi(u))^{[p]_2} = \phi(u^{[p]_1})$ for $u \in L$, i.e., $\phi$ is a restricted morphism. □

One advantage in considering restrictable hom-Lie algebras instead of restricted ones rests on the following theorem.

**3.5. Theorem.** *Let $f : (L, [\cdot,\cdot]_L, \alpha, [p]_1) \to (L^{'}, [\cdot,\cdot]_{L^{'}}, \beta, [p]_2)$ be a surjective restricted morphism of hom-Lie algebras. If $L$ is restrictable, so is $L^{'}$.*

*Proof.* It follows from $f$ is a surjective mapping that $L^{'} = f(L)$. Then for $x \in L_1$, we have $\beta(f(x)) = f(\alpha(x)) = f(x)$ and $f(x) \in L_1^{'}$, where $L_1 = \{x \in L | \alpha(x) = x\}$ and $L_1^{'} = \{x \in L^{'} | \beta(x) = x\}$. For $y \in L$, one gets

$$\begin{aligned}
&(\mathrm{ad}f(x))^p(f(y)) = (\mathrm{ad}f(x))^{p-1}[\beta(f(y)), f(x)]\\
&= (\mathrm{ad}f(x))^{p-2}[[\beta^2(f(y)), \beta(f(x))], f(x)]\\
&= [[[\beta^p f(y), \underbrace{f(x)], f(x)], \cdots, f(x)]}_{p}\\
&= \beta^p[[[f(y), \underbrace{f(x)], f(x)], \cdots, f(x)]}_{p}\\
&= \beta^p \circ f[[[y, \underbrace{x], x], \cdots, x]}_{p} = f[[[\alpha^p(y), \underbrace{x], x], \cdots, x]}_{p}\\
&= f((\mathrm{ad}x)^p(y)) = f((\mathrm{ad}x^{[p]_1})(y)) = f[\alpha(y), x^{[p]_1}]\\
&= f[\alpha(y), \alpha(x^{[p]_1})] = f\circ\alpha[y, x^{[p]_1}] = \beta\circ f[y, x^{[p]_1}]\\
&= \beta[f(y), f(x^{[p]_1})] = [\beta(f(y)), \beta(f(x^{[p]_1}))]\\
&= [\beta(f(y)), f(x^{[p]_1})] = \mathrm{ad}f(x^{[p]_1})(f(y))\\
&= \mathrm{ad}(f(x))^{[p]_2}(f(y)).
\end{aligned}$$

We have $(\mathrm{ad}f(x))^p = \mathrm{ad}(f(x))^{[p]_2} \in \mathrm{ad}{L_1}^{'}$. Hence $L^{'}$ is restrictable. □

**3.6. Theorem.** *Let $A$ and $B$ be hom-Lie ideals of hom-Lie algebra $(L, [\cdot,\cdot]_L, \alpha)$ such that $L = A \oplus B$. Then $L$ is restrictable if and only if $A, B$ are restrictable.*

*Proof.* ($\Leftarrow$) If $A, B$ are restrictable, for $x \in L_1$ with $\alpha(x) = x$, we may suppose that $x = x_1 + x_2$, where $x_1 \in A, x_2 \in B$. Then $\alpha(x_1+x_2) = \alpha(x_1) + \alpha(x_2) = x_1 + x_2$. Since $A$ and $B$ are hom-Lie ideals, one gets $\alpha(x_1) \in A, \alpha(x_2) \in B$. we obtain $\alpha(x_1) = x_1$ and

$\alpha(x_2) = x_2$. As $A, B$ are restrictable, then there exists $y_1 \in A_1, y_2 \in B_1$ with $\alpha(y_1) = y_1$ and $\alpha(y_2) = y_2$, such that $(\mathrm{ad}x_1)^p = \mathrm{ad}y_1$ and $(\mathrm{ad}x_2)^p = \mathrm{ad}y_2$. Thus,

$$\begin{aligned}&(\mathrm{ad}(x_1 + x_2))^p = (\mathrm{ad}x_1 + \mathrm{ad}x_2)^p\\ &= (\mathrm{ad}x_1)^p + (\mathrm{ad}x_2)^p = \mathrm{ad}y_1 + \mathrm{ad}y_2\\ &= \mathrm{ad}(y_1 + y_2).\end{aligned}$$

Therefore, $L$ is restrictable.

($\Rightarrow$) If $L$ is restrictable , so are $A \cong L/B$, $B \cong L/A$ by Theorem 3.5. □

**3.7. Corollary.** *Let $A, B$ be restrictable hom-Lie ideals of a restricted hom-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, [p])$ such that $L = A + B$ and $[A, B] = 0$. Then $L$ is restrictable.*

*Proof.* Define a mapping $f : A \oplus B \to L, (x, y) \mapsto x + y$. Clearly, $f$ is a surjection. For $(x_1, y_1), (x_2, y_2) \in A \oplus B$, by $[A, B] = 0$, one gets $[x_1, y_2] = [y_1, x_2] = 0$. We have

$$\begin{aligned}&f[(x_1, y_1), (x_2, y_2)] = f([x_1, x_2], [y_1, y_2])\\ &= [x_1, x_2] + [y_1, y_2] = [x_1, x_2] + [x_1, y_2] + [y_1, x_2] + [y_1, y_2]\\ &= [x_1 + y_1, x_2 + y_2] = [f(x_1, y_1), f(x_2, y_2)].\end{aligned}$$

Moreover, one gets

$$\begin{aligned}&\alpha \circ f(x, y) = \alpha(x + y)\\ &= \alpha(x) + \alpha(y) = f((\alpha(x), \alpha(y)))\\ &= f \circ \alpha(x, y).\end{aligned}$$

Therefore, $\alpha \circ f = f \circ \alpha$. For $x \in A, y \in B$, $\alpha(x, y) = (x, y)$, we have

$$\begin{aligned}&f((x, y)^{[p]}) = f((x^{[p]_1}, y^{[p]_2}))\\ &= x^{[p]_1} + y^{[p]_2} = (x + y)^{[p]}\\ &= (f(x, y))^{[p]}.\end{aligned}$$

Thus, $f$ is a restricted morphism. By Theorem 3.6, we have $A \oplus B$ is restrictable. By Theorem 3.5, one gets $L$ is restrictable. □

**3.8. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha)$ be a hom-Lie algebra and $\psi$ be a symmetric bilinear form on $L$. $\psi$ is called associative, if $\psi(x, [z, y]) = \psi([\alpha(z), x], y)$.

**3.9. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha)$ be a hom-Lie algebra and $\psi$ a symmetric bilinear form on $L$. Set $L^\perp = \{x \in L | \psi(x, y) = 0, \forall\ y \in L\}$. $L$ is called nondegenerate, if $L^\perp = 0$.

**3.10. Theorem.** *Let $L$ be a subalgebra of the restricted hom-Lie algebra $(G, [\cdot,\cdot]_G, \alpha, [p])$ with $C(L) = \{0\}$. Assume $\lambda : G \times G \to \mathbb{F}$ to be an associative symmetric bilinear form, which is nondegenerate on $L \times L$. Then $L$ is restrictable.*

*Proof.* Since $\lambda$ is nondegenerate on $L \times L$, every linear form $f$ on $L$ is determined by a suitably chosen element $y \in L : f(z) = \lambda(y, z), \forall z \in L$. Let $x \in L_1$. Then there exists $y \in L$ such that

$$\lambda(x^{[p]}, z) = \lambda(y, z), \forall z \in L.$$

This implies that $0 = \lambda(x^{[p]} - y, [L, L]) = \lambda([\alpha(L), x^{[p]} - y], L)$ and $[\alpha(L), x^{[p]} - y] = 0$. Therefore, $x^{[p]} - y \in C(L) = \{0\}$ and $y = x^{[p]} \in L_1$. Moreover, we obtain

$$(\mathrm{ad}x|_L)^p = \mathrm{ad}x^{[p]}|_L = \mathrm{ad}y|_L,$$

which proves that $L$ is restrictable. □

**3.11. Proposition.** *Let $(L, [\cdot,\cdot]_L, \alpha)$ be a restrictable hom-Lie algebra and $H$ a subalgebra of $L$. Then $H$ is a $p$-subalgebra for some mapping $[p]$ on $L$ if and only if $(\mathrm{ad}H_1|_L)^p \subseteq \mathrm{ad}H_1|_L$.*

*Proof.* ($\Rightarrow$) If $H$ is a $p$-subalgebra, then for $x \in H_1$, $x^{[p]} \in H_1$, and $(\mathrm{ad}x)^p = \mathrm{ad}x^{[p]} \subseteq \mathrm{ad}H_1|_L$. Hence, $(\mathrm{ad}H_1|_L)^p \subseteq \mathrm{ad}H_1|_L$.

($\Leftarrow$) If $(\mathrm{ad}H_1|_L)^p \subseteq \mathrm{ad}H_1|_L$, then $H$ is restrictable. By Theorem 2.8, $H$ is restricted. Thereby, $H$ is a $p$-subalgebra of $L$. □

## 4. Cohomology of restricted hom-Lie algebras

In this section, we will discuss the cohomology of restricted hom-Lie algebras in the abelian case, which is similar to the reference [5].

**4.1. Definition.** [12] A hom-associative algebra is a triple $(V, \mu, \alpha)$ consisting of a linear space $V$, a bilinear map $\mu : V \times V \longrightarrow V$ and a linear space homomorphism $\alpha : V \longrightarrow V$ satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)).$$

There is a functor from the category of hom-associative algebras in the category of hom-Lie algebras.

**4.2. Proposition.** [12] *Let $(A, \mu, \alpha)$ be a hom-associative algebra defined on the linear space $A$ by the multiplication $\mu$ and a homomorphism $\alpha$. Then the triple $(A, [\cdot,\cdot], \alpha)$ where the bracket is defined for $x, y \in A$ by $[x, y] = \mu(x, y) - \mu(y, x)$, is a hom-Lie algebra. We also denote it by $(A^-, [\cdot,\cdot], \alpha)$.*

The following definition is analogous to that of the restricted universal enveloping algebra in the reference [14].

**4.3. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha, [p])$ be a restricted hom-Lie algebra. The $(u(L), \mu^{'}, \alpha^{'}, i)$ consisting of a hom-associative algebra $(u(L), \mu^{'}, \alpha^{'})$ with unity and a restricted hom-morphism $i : (L, [\cdot,\cdot]_L, \alpha, [p]) \to (u(L)^-, \mu^{'}, \alpha^{'})$ is called a restricted hom-universal enveloping algebra of $L$ if given any hom-associative algebra $(A, \mu^{''}, \alpha^{''})$ with unity and any restricted hom-morphism $f : (L, [\cdot,\cdot]_L, \alpha, [p]) \to (A^-, \mu^{''}, \alpha^{''})$, there exists a unique morphism $\bar{f} : (u(L), \mu^{'}, \alpha^{'}) \to (A, \mu^{''}, \alpha^{''})$ of hom-associative algebras such that $\bar{f} \circ i = f$.

**4.4. Definition.** [11] Let $A = (V, \mu, \alpha)$ be a hom-associative $\mathbb{F}$-algebra. An $A$-module is a triple $(M, f, \gamma)$ where $M$ is $\mathbb{F}$-vector space and $f, \gamma$ are $\mathbb{F}$-linear maps, $f : M \longrightarrow M$ and $\gamma : V \otimes M \longrightarrow M$, such that the following diagram commutes:

$$\begin{tikzcd} V \otimes M \arrow[r, "\gamma"] & M \\ V \otimes V \otimes M \arrow[u, "\alpha\otimes\gamma"] \arrow[r, "\mu\otimes f"] & V \otimes M \arrow[u, "\gamma"] \end{tikzcd}$$

We let $S^*(L)$ and $\Lambda^*(L)$ denote the symmetric and alternating algebras of restricted hom-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, [p])$, respectively. Bases for the homogeneous subspaces of degree $k$ for these spaces consist of monomials $e^{\mu} = e_1^{\mu_1} \cdots e_n^{\mu_n}$ and $e_{\vec{i}} = e_{i_1} \wedge \cdots \wedge e_{i_k}$, respectively, where

$\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}^n$ satisfies $\mu_j \geq 0, |\mu| = \sum_j \mu_j = k$;

$\vec{i} = (i_1, \cdots, i_k) \in \mathbb{Z}^k$ satisfies $1 \leq i_1 < \cdots < i_k \leq n$.

Let $\gamma : \lambda \mapsto \lambda^p$ denote the Frobenius automorphism of $\mathbb{F}$. If $V$ is an abelian group with an $\mathbb{F}$-vector space structure given by $\mathbb{F} \to \mathrm{End}(V)$, then the composition

$$\mathbb{F} \xrightarrow{\gamma^{-1}} \mathbb{F} \to \mathrm{End}(V)$$

gives another vector space structure on $V$ which we will denote by $\overline{V}$. Of course $\overline{V}$ is isomorphic to $V$ as an $\mathbb{F}$-vector space (they have the same dimension). We note that if $W$ is any other $\mathbb{F}$-vector space, then a $p$-semilinear map $V \to W$ is a linear map $\overline{V} \to W$ and vice versa.

In sequel, $(L, [\cdot,\cdot]_L, \alpha, [p])$ denotes a finite-dimensional restricted hom-Lie algebra over $\mathbb{F}$ such that $[g_i, g_j] = 0$ for all $g_i, g_j \in L$ and $(u(L), \alpha^{'}, i)$ denotes the restricted hom-universal enveloping algebra of $L$. Here we take $\alpha = \alpha^{'}$ and $\alpha$ satisfies $\alpha(u_1u_2) = \alpha(u_1)\alpha(u_2)$ for $u_1, u_2 \in u(L)$. For $s, t \geq 0$, we define

$$C_{s,t} = S^t\overline{L}_1 \otimes \Lambda^s L \otimes u(L)$$

with the $u(L)$-module structure given by

$$u(h_1 \cdots h_t \otimes g_1 \wedge \cdots \wedge g_s \otimes x) = h_1 \cdots h_t \otimes g_1 \wedge \cdots \wedge g_s \otimes \alpha(u)x,$$

where $h_i, g_j \in L$ and $u, x \in u(L)$. If either $s < 0$ or $t < 0$, we put $C_{s,t} = 0$ and define

$$C_k = \bigoplus_{2t+s=k} C_{s,t}$$

for all $k \in \mathbb{Z}$. Note that each $C_k$ is a free $u(L)$-module. If not both $t = 0$ and $s = 0$, we then define a map

$$d_{s,t} : C_{s,t} \to C_{t,s-1} \oplus C_{t-1,s+1}$$

by the formulas

$$d_{t,s}(h_1 \cdots h_t \otimes g_1 \wedge \cdots \wedge g_s \otimes x)$$

$$= \sum_{i=1}^{s} (-1)^{i-1} h_1 \cdots h_t \otimes \alpha(g_1) \wedge \cdots \widehat{\alpha(g_i)} \cdots \wedge \alpha(g_s) \otimes \alpha(g_i)x \tag{4.1}$$

$$+ \sum_{j=1}^{t} h_1 \cdots \widehat{h_j} \cdots h_t \otimes h_j^{[p]} \wedge \alpha(g_1) \wedge \cdots \wedge \alpha(g_s) \otimes \alpha(x) \tag{4.2}$$

$$- \sum_{j=1}^{t} h_1 \cdots \widehat{h_j} \cdots h_t \otimes h_j \wedge \alpha(g_1) \wedge \cdots \wedge \alpha(g_s) \otimes h_j^{p-1}x. \tag{4.3}$$

For $k \geq 1$, we define the map $d_k : C_k \to C_{k-1}$ by $d_k = \bigoplus_{2t+s=k} d_{s,t}$. Then we obtain the following theorem.

**4.5. Theorem.** *The maps $d_k$ defined above satisfy $d_{k-1}d_k = 0$ for $k \geq 1$, so that $C = (C_k, d_k)$ is an augmented complex of free $u(g)$-modules.*

*Proof.* The terms in the sum (4.1) are elements of $C_{t,s-1}$ whereas the terms in the sums (4.2) and (4.3) lie in $C_{t-1,s+1}$. Therefore, in order to compute $d_{k-1}d_k = 0$, we must apply $d_{t,s-1}$ to (4.1) and $d_{t-1,s+1}$ to (4.2) and (4.3). Applying $d_{t,s-1}$ to (5), we have

$$\begin{aligned}
& d_{t,s}\big(\sum_{i=1}^{s}(-1)^{i-1} h_1 \cdots h_t \otimes \alpha(g_1) \wedge \cdots \widehat{\alpha(g_i)} \cdots \wedge \alpha(g_s) \otimes \alpha(g_i)x\big) \\
= \quad & \sum_{i=1}^{s}(-1)^{i-1}\big(\sum_{\sigma<i}(-1)^{\sigma-1} h_1 \cdots h_t \otimes \alpha^2(g_1) \wedge \cdots \widehat{\alpha^2(g_\sigma)} \cdots \widehat{\alpha^2(g_i)} \cdots \wedge \alpha^2(g_s) \\
& \otimes \alpha^2(g_\sigma)(\alpha(g_i)x) \\
& + \sum_{\sigma>i}(-1)^{\sigma} h_1 \cdots h_t \otimes \alpha^2(g_1) \wedge \cdots \widehat{\alpha^2(g_i)} \cdots \widehat{\alpha^2(g_\sigma)} \cdots \wedge \alpha^2(g_s) \otimes \alpha^2(g_\sigma)(\alpha(g_i)x)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t \otimes h_j^{[p]} \wedge \alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\otimes\alpha(\alpha(g_i)x)\\
&-\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t \otimes h_j \wedge \alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\otimes h_j^{p-1}(\alpha(g_i)x)\big)\\
=\quad&\sum_{i=1}^{s}(-1)^{i-1}\big(\sum_{\sigma<i}(-1)^{\sigma-1}h_1\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_\sigma)}\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\\
&\otimes(\alpha(g_\sigma)\alpha(g_i))\alpha(x)\\
&+\sum_{\sigma>i}(-1)^{\sigma}h_1\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\widehat{\alpha^2(g_\sigma)}\cdots\wedge\alpha^2(g_s)\otimes(\alpha(g_\sigma)\alpha(g_i))\alpha(x)
\end{aligned}
$$

$$
\begin{aligned}
(4.4)\ +\quad&\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t \otimes h_j^{[p]} \wedge \alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\otimes\alpha(\alpha(g_i)x)\\
(4.5)\ -\quad&\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t \otimes h_j \wedge \alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\otimes(h_j^{p-1}\alpha(g_i))\alpha(x)\big).
\end{aligned}
$$

Since $\alpha(g_i)\alpha(g_j)=\alpha(g_j)\alpha(g_i)$ in $u(g)$, the terms in the first two sums in the parentheses cancel in pairs when summed over all $i$. This leaves the sum over $i$ of (4.4) and (4.5). Now we apply $d_{t-1,s+1}$ to (4.2).

$$
d_{t-1,s+1}\big(\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j^{[p]}\wedge\alpha(g_1)\wedge\cdots\wedge\alpha(g_s)\otimes\alpha(x)\big)=\sum_{j=1}^{t}
$$

$$
\begin{aligned}
(4.6)\quad&\big(\sum_{\sigma=1}^{s}(-1)^{\sigma}h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j^{[p]}\wedge\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_\sigma)}\cdots\wedge\alpha^2(g_s)\otimes\alpha^2(g_\sigma)\alpha(x)\\
(4.7)\quad&+h_1\cdots\widehat{h_j}\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha(h_j^{[p]})\alpha(x)\\
(4.8)\quad&+\sum_{\tau\neq j}h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau^{[p]}\wedge h_j^{[p]}\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha^2(x)\\
(4.9)\quad&-\sum_{\tau\neq j}h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau\wedge h_j^{[p]}\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes h_\tau^{p-1}\alpha(x)\big).
\end{aligned}
$$

We note that the terms in (4.8) cancel in pairs since interchanging the first two terms in the alternating product multiplies the term by $-1$. Finally, we apply $d_{t-1,s+1}$ to (4.3) to get

$$
\begin{aligned}
&d_{t-1,s+1}\big(-\sum_{j=1}^{t}h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha(g_s)\otimes h_j^{p-1}x\big)\\
&=-\sum_{j=1}^{t}\big(\sum_{\sigma=1}^{s}(-1)^{\sigma}h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha^2(g_\sigma)(h_j^{p-1}x)\\
&\quad+h_1\cdots\widehat{h_j}\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha^2(h_j)(h_j^{p-1}x)\\
&\quad+\sum_{\tau\neq j}h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau^{[p]}\wedge h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha(h_j^{p-1}x)\\
&\quad-\sum_{\tau\neq j}h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau\wedge h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes h_\tau^{p-1}(h_j^{p-1}x)\big)
\end{aligned}
$$

$$(4.10) \quad = -\sum_{j=1}^{t}\Big(\sum_{\sigma=1}^{s}(-1)^{\sigma} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha(g_\sigma h_j^{p-1})\alpha(x)$$

$$(4.11) \quad +h_1\cdots\widehat{h_j}\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes h_j^{p}\alpha(x)$$

$$(4.12) \quad +\sum_{\tau\neq j} h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau^{[p]}\wedge h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha(h_j^{p-1}x)$$

$$(4.13) \quad -\sum_{\tau\neq j} h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau\wedge h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes(h_\tau^{p-1}h_j^{p-1})\alpha(x)\Big).$$

This time the terms in (4.13) cancel in pairs. Moreover, the terms in (4.4) and (4.6) are identical (with $\sigma = i$) except for sign and hence they cancel. The terms in (4.5) and (4.10) cancel in pairs since $\alpha(h_i^{p-1})\alpha(g_j) = \alpha(g_j)\alpha(h_i^{p-1})$. The terms in (4.9) and (4.12) have the same sign but are equal apart from interchanging the first two terms in the alternating part. Finally the terms in (4.7) and (4.11) match except for sign since $h_j^{[p]} = h_j^p$ in $u(g)$ and hence the entire sum is zero as claimed. This completes the proof. □

We next will consider the cohomology of restricted hom-Lie algebras in the case of simpleness. A basis for the space $C_{t,s}$ consists of the monomials

$$e^\mu\otimes e_I\otimes e^r = e_1^{\mu_1}\cdots e_n^{\mu_n}\otimes e_{i_1}\wedge\cdots\wedge e_{i_s}\otimes e_1^{r_1}\cdots e_n^{r_n},$$

where $\mu = (\mu_1,\cdots,\mu_n), I = (i_1,\cdots,i_s), r = (r_1,\cdots,r_n)$ and

$$\mu_j\geq 0, |\mu| = \sum_j \mu_j = t, 1\leq i_1<\cdots<i_s\leq n, 0\leq r_j\leq p-1.$$

For each $i = 1,\cdots,n$ and $e_i\in L_1$, we let

$$c_i = 1\otimes e_i^{[p]}\otimes 1 - 1\otimes e_i\otimes e_i^{p-1}$$

and we easily note that $c_i\in C_{0,1}$ is a cycle for all $i$. Now we define

$$(\partial/\partial e_i\otimes c_i) : C_{t,s}\longrightarrow C_{t-1,s+1}$$

by the formula

$$(\frac{\partial}{\partial e_i}\otimes c_i)(e^\mu\otimes e_I\otimes e^r) = \frac{\partial e^\mu}{\partial e_i}\otimes e_i^{[p]}\wedge\alpha(e_I)\otimes\alpha(e^r) - \frac{\partial e^\mu}{\partial e_i}\otimes e_i\wedge\alpha(e_I)\otimes e_i^{p-1}\alpha(e^r).$$

If $\mu = (\mu_1,\cdots,\mu_n)$ satisfies $|\mu| = t$ and $I = (i_1,\cdots,i_s)$ is increasing, then by the definition we write

$$e^\mu\otimes c_I = \sum_{J\subset\{1,\cdots,s\}}(-1)^{|J|}e^\mu\otimes f_{i_1}\wedge\cdots\wedge f_{i_s}\otimes e_{i_1}^{q_{i_1}}\cdots e_{i_s}^{q_{i_s}}$$

and

$$e^\mu\otimes\alpha(c_I) = \sum_{J\subset\{1,\cdots,s\}}(-1)^{|J|}e^\mu\otimes\alpha(f_{i_1})\wedge\cdots\wedge\alpha(f_{i_s})\otimes\alpha(e_{i_1}^{q_{i_1}})\cdots\alpha(e_{i_s}^{q_{i_s}}),$$

where

$$f_{i_j} = \begin{cases} e_{i_j}, & j\in J\\ e_{i_j}^{[p]}, & j\notin J;\end{cases} \qquad q_{i_j} = \begin{cases} p-1, & j\in J\\ 0, & j\notin J.\end{cases}$$

We then define $\mathfrak{C}_{t,s}$ to be the $\mathbb{F}$-subspace of $C_{t,s}$ spanned by the elements $\{e^\mu\otimes\alpha(c_I) : |\mu| = t$ and $I$ is increasing $\}$ and

$$\mathfrak{C}_k = \bigoplus_{2t+s=k}\mathfrak{C}_{t,s}.$$

The boundary operator $\partial_k = \partial : \mathfrak{C}_k \longrightarrow \mathfrak{C}_{k-1}$ is defined by

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial e_j} \otimes c_j.$$

Then we may show that $\partial^2 = 0$. In fact,

$$\begin{aligned}
&\partial^2(e^\mu \otimes c_I) = \partial(\partial(e^\mu \otimes c_I)) \\
&= \partial(\sum_{j=1}^{n} \frac{\partial}{\partial e_j} \otimes c_j( \sum_{J \subset \{1,\cdots,s\}} (-1)^{|J|} e^\mu \otimes f_{i_1} \wedge \cdots \wedge f_{i_s} \otimes e_{i_1}^{q_{i_1}} \cdots e_{i_s}^{q_{i_s}})) \\
&= \partial(\sum_{j=1}^{n} \sum_{J \subset \{1,\cdots,s\}} (-1)^{|J|} (\frac{\partial e^\mu}{\partial e_j} \otimes e_j^{[p]} \wedge \alpha(f_{i_1}) \wedge \cdots \wedge \alpha(f_{i_s}) \otimes \alpha(e_{i_1}^{q_{i_1}}) \cdots \alpha(e_{i_s}^{q_{i_s}}) \\
&\quad - \frac{\partial e^\mu}{\partial e_j} \otimes e_j \wedge \alpha(f_{i_1}) \wedge \cdots \wedge \alpha(f_{i_s}) \otimes e_j^{p-1} \alpha(e_{i_1}^{q_{i_1}}) \cdots \alpha(e_{i_s}^{q_{i_s}}))) \\
&= \sum_{l=1}^{n} \sum_{j=1}^{n} \sum_{J \subset \{1,\cdots,s\}} (-1)^{|J|} \{ \frac{\partial}{\partial e_l} \otimes c_l (\frac{\partial e^\mu}{\partial e_j} \otimes e_j^{[p]} \wedge \alpha(f_{i_1}) \wedge \cdots \wedge \alpha(f_{i_s}) \otimes \alpha(e_{i_1}^{q_{i_1}}) \cdots \\
&\qquad \alpha(e_{i_s}^{q_{i_s}})) - \frac{\partial}{\partial e_l} \otimes c_l (\frac{\partial e^\mu}{\partial e_j} \otimes e_j \wedge \alpha(f_{i_1}) \wedge \cdots \wedge \alpha(f_{i_s}) \otimes e_j^{p-1} \alpha(e_{i_1}^{q_{i_1}}) \cdots \alpha(e_{i_s}^{q_{i_s}}))\} \\
&\quad = \sum_{l=1}^{n} \sum_{j=1}^{n} \sum_{J \subset \{1,\cdots,s\}} (-1)^{|J|} \{ \frac{\partial(\frac{\partial e^\mu}{\partial e_j})}{\partial e_l} \otimes e_l^{[p]} \wedge \alpha(e_j^{[p]}) \wedge \alpha^2(f_{i_1}) \wedge \cdots \wedge \alpha^2(f_{i_s})
\end{aligned}$$

$$\otimes \alpha^2(e_{i_1}^{q_{i_1}}) \cdots \alpha^2(e_{i_s}^{q_{i_s}}) \tag{4.14}$$

$$- \frac{\partial(\frac{\partial e^\mu}{\partial e_j})}{\partial e_l} \otimes e_l \wedge \alpha(e_j^{[p]}) \wedge \alpha^2(f_{i_1}) \wedge \cdots \wedge \alpha^2(f_{i_s}) \otimes e_l^{p-1} \alpha^2(e_{i_1}^{q_{i_1}}) \cdots \alpha^2(e_{i_s}^{q_{i_s}}) \tag{4.15}$$

$$- \frac{\partial(\frac{\partial e^\mu}{\partial e_j})}{\partial e_l} \otimes e_l^{[p]} \wedge \alpha(e_j) \wedge \alpha^2(f_{i_1}) \wedge \cdots \wedge \alpha^2(f_{i_s}) \otimes \alpha(e_j^{p-1}) \alpha^2(e_{i_1}^{q_{i_1}}) \cdots \alpha^2(e_{i_s}^{q_{i_s}}) \tag{4.16}$$

$$+ \frac{\partial(\frac{\partial e^\mu}{\partial e_j})}{\partial e_l} \otimes e_l \wedge \alpha(e_j) \wedge \alpha^2(f_{i_1}) \wedge \cdots \wedge \alpha^2(f_{i_s}) \otimes e_l^{p-1} \alpha(e_j^{p-1}) \alpha^2(e_{i_1}^{q_{i_1}}) \cdots \alpha^2(e_{i_s}^{q_{i_s}}) \}. \tag{4.17}$$

This time the terms in (4.14) cancel in pairs, and the terms in (4.17) cancel in pairs since $e_l^{p-1} \alpha(e_j^{p-1}) = \alpha(e_l^{p-1}) e_j^{p-1}$. Moreover, the terms in (4.15) and (4.16) are identical except for sign and hence they cancel, so that $\mathfrak{C} = \{\mathfrak{C}_k, \partial_k\}_{k \geq 0}$ is a complex.

**4.6. Theorem.** *If $\mathfrak{C}$ is the complex defined above, we define $H_k(\mathfrak{C}) := \mathrm{Ker}\partial_k / \mathrm{Im}\partial_k$. Then*

$$H_k(\mathfrak{C}) = \begin{cases} U_{res.}(g), & k = 0 \\ 0, & 0 < k < p. \end{cases}$$

*Proof.* Define a map $D : \mathfrak{C}_k \to \mathfrak{C}_{k+1}$ by the formula

$$D(e^\mu \otimes \alpha(c_I)) = \sum_{a=1}^{s} (-1)^{a-1} e^\mu e_{i_a} \otimes c_{i_1} \cdots \widehat{c_{i_a}} \cdots c_{i_s}$$

and compute for any monomial $e^\mu \otimes \alpha(c_I)$ :

$$
\begin{aligned}
D\partial(e^{\mu}\otimes\alpha(c_I)) &= D(\sum_{j=1}^{n}(\tfrac{\partial}{\partial e_j}\otimes c_j)(e^{\mu}\otimes\alpha(c_I)))\\
&= \sum_{j=1,j\neq i_1,\cdots,i_s}^{n} D(\mu_j e_1^{\mu_1}\cdots e_j^{\mu_j-1}\cdots e_n^{\mu_n}\otimes c_j\alpha^2(c_I))\\
&= \sum_{j=1,j\neq i_1,\cdots,i_s}^{n} D(\mu_j e_1^{\mu_1}\cdots e_j^{\mu_j-1}\cdots e_n^{\mu_n}\otimes \alpha(c_j)\alpha^2(c_I))\\
&= \big(\sum_{j=1,j\neq i_1,\cdots,i_s}^{n}\mu_j\big)e^{\mu}\otimes\alpha(c_I)\\
&+ \sum_{j=1,j\neq i_1,\cdots,i_s}^{n}\sum_{a=1}^{s}(-1)^a\mu_j e_1^{\mu_1}\cdots e_j^{\mu_j-1}\cdots e_{i_a}^{\mu_{i_a}+1}\cdots e_n^{\mu_n}\\
&\otimes\alpha(c_j)\alpha(c_{i_1})\cdots\widehat{\alpha(c_{i_a})}\cdots\alpha(c_{i_s})
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
\partial D(e^{\mu}\otimes\alpha(c_I)) &= \partial(\sum_{a=1}^{s}(-1)^{a-1}e^{\mu}e_{i_a}\otimes c_{i_1}\cdots\widehat{c_{i_a}}\cdots c_{i_s})\\
&= \sum_{a=1}^{s}(-1)^{a-1}\partial(e_1^{\mu_1}\cdots e_{i_a}^{\mu_{i_a}+1}\cdots e_n^{\mu_n}\otimes c_{i_1}\cdots\widehat{c_{i_a}}\cdots c_{i_s})\\
&= \big(\sum_{a=1}^{s}\mu_{i_a}+1\big)e^{\mu}\otimes\alpha(c_I)\\
&-\sum_{a=1}^{s}(-1)^a\sum_{j=1,j\neq i_1,\cdots,i_s}^{n}\mu_j e_1^{\mu_1}\cdots e_j^{\mu_j-1}\cdots e_{i_a}^{\mu_{i_a}+1}\cdots e_n^{\mu_n}\\
&\otimes\alpha(c_j)\alpha(c_{i_1})\cdots\widehat{\alpha(c_{i_a})}\cdots\alpha(c_{i_s}).
\end{aligned}
\tag{4.19}
$$

Clearly the terms (4.18) and (4.19) are identical apart from sign so that we have

$$(D\partial+\partial D)(e^{\mu}\otimes\alpha(c_I)) = \big(\sum_{j=1,j\neq i_1,\cdots,i_s}^{n}\mu_j+\sum_{a=1}^{s}\mu_{i_a}+s\big)(e^{\mu}\otimes\alpha(c_I)) = (t+s)(e^{\mu}\otimes\alpha(c_I)).$$

Therefore we see that every cycle in $\mathfrak{C}_k(k=2t+s)$ is a boundary provided that $t+s\neq 0(\mathrm{mod}p)$. In particular, if $0<k<p$, then $0<t+s<p$ so that $H_k(\mathfrak{C})=0$. Moreover, $\mathfrak{C}_1=\mathfrak{C}_{0,1}$ is spanned by the $c_i$ and $\partial c_i=0$ for all $i$. Therefore $H_0(\mathfrak{C})=\mathfrak{C}_0=U_{res.}(g)$, the proof of the theorem is complete. □

**Acknowledgement** The authors would like to thank the referee for carefully reading and checking the paper both for its English as well as its mathematics.

## References

[1] Ammar, F., Ejbehi, Z. and Makhlouf, A., Cohomology and Deformations of Hom-algebras. *J. Lie Theory,* 2011, 21: 813-836.

[2] Bahturin, Y., Mikhalev, A., Petrogradski, V. M. and Zaicev, M., Infinite-dimensional Lie superalgebras, Walter de Gruyter, Berlin, New York, 1992.

[3] Benayadi, S., Makhlouf, A., Hom-Lie Algebras with Symmetric Invariant NonDegenerate Bilinear Forms. *J. Geom. Phys.,* 2014, 76: 38-60.

[4] Dokas, I., Loday, J. L., On restricted Leibniz algebras. *Comm. Algebra,* 2006, 34: 4467-4478.

[5] Evans, T. J., Fuchs, D., A complex for the cohomology of restricted Lie algebras. *J. fixed point theory appl.,* 2008, 3: 159-179.

[6] Farnsteiner, R., Note on Frobenius extensions and restricted Lie superalgebras. *J. Pure Appl. Algebra*, 1996, 108: 241-256.
[7] Hartwig, J. T., Larsson D. and Silvestrov, S. D., Deformations of Lie algebras using $\sigma$-derivations. *J. Algebra,* 2006, 295: 314-361.
[8] Hodge, T. L., Lie triple system, restricted Lie triple system and algebraic groups. *J. Algebra,* 2001, 244: 533-580.
[9] Larsson, D., Silvestrov, S., Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities. *J. Algebra,* 2005, 288: 321-344.
[10] Larsson, D., Silvestrov, S., Quasi-Lie algebras. *Contemp. Math.,* 2005, 391: 241-248.
[11] Makhlouf, A., Silvestrov, S., Notes on formal deformations of hom-associative and hom-Lie algebras. *Forum Math.,* 2010, 22(4): 715-739.
[12] Makhlouf, A., Silvestrov, S., Hom-algebra structures. *J. Gen. Lie Theory Appl.,* 2008, 2(2), 51-64.
[13] Sheng, Y. H., Representations of hom-Lie algebras. *Algebra Representation Theory,* 2012, 15: 1081–1098.
[14] Strade, H., Farnsteiner, R., Modular Lie algebras and their representations. New York: Dekker. 1988.
[15] Strade, H., The classification of the simple modular Lie algebras. *Ann. Math.*, 1991, 133: 577-604.
[16] Yau, D., Hom-Yang-Baxter equation, hom-Lie algebras, and quasi-triangular bialgebras. *J. Phys. A: Math. Theory,* 2009, 42, 165202.
[17] Yau, D., Hom-algebras and homology. *J. Lie Theory,* 2009, 19: 409-421.
[18] Yau, D., Enveloping algebras of Hom-Lie algebras. *Journal of Generalized Lie Theory and Applications,* 2008, 2: 95-108.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 839 – 853

# Generalization of Hermite-Hadamard type inequalities for $n$-times differentiable functions which are $s$-preinvex in the second sense with applications

Muhammad Amer Latif*† and Svere S. Dragomir‡§

**Abstract**

In this paper, Hermite-Hadamard inequality for differentiable preinvex functions is generalized and refined for $n$-times differentiable functions which are $s$-preinvex in the second sense. Some recent results are also improved and applications to special means of positive numbers are given.

*2000 AMS Classification:* 26D07, 26D10, 26D99.

**Keywords:** Hermite-Hadamard's inequality, invex set, preinvex function, Hölder's inequality, $s$-preinvex function.

Received 16/04/2013 : Accepted 18/04/2014 Doi : 10.15672 /HJMS.2015449438

## 1. Introduction

The following definition for convex functions is well known in mathematical literature:

**1.1. Definition.** A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on $I$ if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

---

*School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa, Email: m_amer_latif@hotmail.com

†Corresponding Author.

‡School of Engineering and Science, Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia,

§School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa, Email: sever.dragomir@vu.edu.au

A number of inequalities have been established for convex functions but the following double inequalites, known as Hermite-Hadmard inequalities, are famous in mathematical literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping and $a, b \in I \subseteq \mathbb{R}$ with $a < b$. The inequalities (1.1) hold in reversed direction if $f$ is concave. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements, counterparts and new Hermite-Hadamard-type inequalities and numerous applications, see for instance [7]-[11], [13], [15], [20]-[23], [29], [31], [32]-[34], [36], [37] and the references therein.

In recent years, the classical convexity has been generalized and extended in a diverse manner. One of them is the preinvexity, introduced by Weir et al. [38] as a significant generalization of convex functions. Many researchers have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems, for example see the work of Mohn et al. [24], Noor [26] and Yang et al. [41]. It is well known that the preinvex functions and invex sets may not be convex functions and convex sets respectively.

Let us recall some definitions and known results concerning invexity and preinvexity

**1.2. Definition.** [41] A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to $\eta : K \times K \to \mathbb{R}^n$ if

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

The invex set $K$ is also called an $\eta$-connected set.

**1.3. Definition.** [38] Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to $\eta$, if for all $u, v \in K$ and $t \in [0, 1]$, the following inequality holds

$$f(u + t\eta(v, u)) \leq (1-t)\, f(u) + tf(v).$$

The function $f$ is said to be preincave if and only if $-f$ is preinvex.

It is to be noted that every preinvex function is convex with respect to the map $\eta\,(x, y) = x - y$ but the converse is not true see for instance [38].

Noor [28], proved the following Hermite-Hadamard type inequalities.

**1.4. Theorem.** [28] *Let $f : K = [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers $K^\circ$ (the interior of $K$) and $a$, $b \in K^\circ$ with $\eta(b, a) > 0$. Then the following inequalities holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f\,(x)\,dx \leq \frac{f\,(a)+f\,(b)}{2}. \tag{1.2}$$

Barani et al. in [5], presented the following estimates of the right-side of a Hermite-Hadamard type inequality in which preinvex functions are involved.

**1.5. Theorem.** [5] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function .Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$*

*is preinvex on $K$, for every $a,b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

**1.6. Theorem.** [5]*Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on $K$, for every $a,b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{1.4}$$

Most recently, Li [42] introduced the notion of $s$-preinvexity and established Hermite-Hadamard type inequalities for this class of functions.

**1.7. Definition.** [42] Let $K \subseteq [0,\infty)$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}$. A function $f : K \to \mathbb{R}$ is said to be $s$-preinvex with respect to $\eta$, if for all $u, v \in K$, $t \in [0,1]$ and $s \in (0,1]$, the following inequality holds

$$f(u+t\eta(v,u)) \leq (1-t)^s f(u) + t^s f(v).$$

The function $f$ is said to be $s$-preincave if and only if $-f$ is $s$-preinvex.

**1.8. Theorem.** [42] *Let $f : K = [a, a+\eta(b,a)] \subseteq [0,\infty) \to (0,\infty)$ be a $s$-preinvex function on the interval of the real numbers $K^{\circ}$ (the interior of $K$) and $a$, $b \in K^{\circ}$ with $\eta(b,a) > 0$. Then the following inequalities holds:*

$$2^{s-1} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.5}$$

For more recent results on Hermite-Hadamard type and Simpson's type inequalities for preinvex, log-preinvex functions, $s$-preinvex functions, prequasiinvex functions and $n$-times differentiable preinvex functions, we refer the interested readers to [4, 16, 17, 18, 19, 35, 39, 40, 42].

The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities in Section 2 that are connected with the right-side and left-side of Hermite-Hadamard inequality for $n$-times differentiable $s$-preinvex functions which generalize those results established for $n$-times differentiable preinvex functions and $n$-times differentiable convex functions.

## 2. Main Results

In order to prove our main results, we need the following lemmas:

**2.1. Lemma.** [16] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$*

*is integrable on* $[a, a+\eta(b,a)]$, *then for every* $a, b \in K$ *with* $\eta(b,a) > 0$, *the following equality holds:*

$$
\begin{aligned}
(2.1)\qquad & -\frac{f(a)+f(a+\eta(b,a))}{2}+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \\
& +\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a)) \\
& \qquad = \frac{(-1)^{n-1}(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\,f^{(n)}(a+t\eta(b,a))dt,
\end{aligned}
$$

*where the sum above takes* $0$ *when* $n = 1$ *and* $n = 2$.

**2.2. Lemma.** [16] *Let* $K \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : K \times K \to \mathbb{R}$. *Suppose that* $f : K \to \mathbb{R}$ *is a function such that* $f^{(n)}$ *exists on* $K$ *for* $n \in \mathbb{N}$, $n \geq 1$. *If* $f^{(n)}$ *is integrable on* $[a, a+\eta(b,a)]$, *then for every* $a, b \in K$ *with* $\eta(b,a) > 0$, *the following equality holds:*

$$
\begin{aligned}
(2.2)\qquad & \sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \\
& \qquad = \frac{(-1)^{n+1}(\eta(b,a))^n}{n!}\int_0^1 K_n(t)f^{(n)}(a+t\eta(b,a))dt,
\end{aligned}
$$

*where*

$$
K_n(t) := \begin{cases} t^n, & t \in \left[0, \frac{1}{2}\right] \\ (t-1)^n, & t \in \left(\frac{1}{2}, 1\right] \end{cases}.
$$

We are now ready to present our first result.

**2.3. Theorem.** *Let* $K \subseteq [0, \infty)$ *be an open invex subset with respect to* $\eta : K \times K \to \mathbb{R}$. *Suppose that* $f : K \to \mathbb{R}$ *is a function such that* $f^{(n)}$ *exists on* $K$ *and* $f^{(n)}$ *is integrable on* $[a, a+\eta(b,a)]$ *for* $n \in \mathbb{N}$, $n \geq 2$. *If* $\left|f^{(n)}\right|^q$ *is s-preinvex on* $K$ *for* $q \geq 1$, *then for every* $a, b \in K$ *with* $\eta(b,a) > 0$, *we have the following inequality:*

$$
\begin{aligned}
(2.3)\qquad & \left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right. \\
& \left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right| \\
& \qquad \leq \frac{(\eta(b,a))^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(Q\left|f^{(n)}(a)\right|^q+P\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

*where*

$$
P = \frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)}, Q = nB(n, s+1) - 2B(n+1, s+1)
$$

*and*

$$
B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}
$$

*for* $x > 0$, $y > 0$ *is the Beta function.*

*Proof.* Suppose $n \geq 2$ and $q = 1$. By $s$-preinvexity of $\left|f^{(n)}\right|$ on $K$ and lemma 2.1, we get

$$\begin{aligned}
(2.4)\qquad & \left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \right.\\
& \left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\right|\\
& \leq \frac{(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right| dt\\
& \leq \frac{(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left((1-t)^s\left|f^{(n)}(a)\right| + t^s\left|f^{(n)}(b)\right|\right) dt\\
& = \frac{(\eta(b,a))^n}{2n!}\left(\left|f^{(n)}(b)\right|\int_0^1 t^{n+s-1}(n-2t)\,dt + \left|f^{(n)}(a)\right|\int_0^1 t^{n-1}(n-2t)(1-t)^s\,dt\right).
\end{aligned}$$

Since

$$\int_0^1 t^{n+s-1}(n-2t)\,dt = \frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)} = P$$

and

$$\begin{aligned}
\int_0^1 t^{n-1}(n-2t)(1-t)^s\,dt &= n\int_0^1 t^{n-1}(1-t)^s\,dt - 2\int_0^1 t^n(1-t)^s\,dt\\
&= nB(n,s+1) - 2B(n+1,s+1) = Q.
\end{aligned}$$

Using the above observations in (2.4), we get (2.3). The proof for the case $q = 1$ is complete.

Assume now that $q > 1$, then by the $s$-preinvexity of $\left|f^{(n)}\right|^q$ on $K$, lemma 2.1 and the Hölder's inequality, we have

$$\begin{aligned}
(2.5)\qquad & \left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \right.\\
& \left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\right|\\
& \leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\\
& \leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\\
& \times\left(\left|f^{(n)}(b)\right|^q\int_0^1 t^{n+s-1}(n-2t)\,dt + \left|f^{(n)}(a)\right|^q\int_0^1 t^{n-1}(n-2t)(1-t)^s\right)^{\frac{1}{q}}
\end{aligned}$$

which is the inequality (2.3). Hence the proof of the theorem is completed. □

**2.4. Remark.** If in Theorem 2.3, we take $s = 1$, we get Theorem 8 from [16].

**2.5. Corollary.** *Suppose that the assumptions of Theorem 2.3 are satisfied. Then for $n=2$, we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{(s+3)(s+2)}\right)^{\frac{1}{q}}. \tag{2.6}$$

**2.6. Remark.** If in Corollary 2.5 $s=1$, we get a result proved in Corollary 1 from [16].

**2.7. Remark.** If in Theorem 2.3, we take $\eta(b,a)=b-a$. Then one gets a result proved in Theorem 1.1 from [11].

**2.8. Theorem.** *Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}$. Suppose that $f: K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\geq 2$. If $\left|f^{(n)}\right|^q$ is $s$-preinvex on $K$ for $q\geq 1$, then for every $a,b\in K$ with $\eta(b,a)>0$, we have the following inequality:*

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq\frac{(\eta(b,a))^n(n-1)^{1-1/q}}{2n!}\left[\left\{nB(nq-q+1,s+1)-2B(nq-q+2,s+1)\right\}\left|f^{(n)}(a)\right|^q\right.\\ &\qquad\left.+\left\{\frac{n}{nq-q+s+1}-\frac{2}{nq-q+s+2}\right\}\left|f^{(n)}(b)\right|^q\right]^{1/q},\end{aligned} \tag{2.7}$$

*where*

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}$$

*for $x>0$, $y>0$ is the Beta function.*

*Proof.* The case when $q=1$ is easy to prove so we assume that $q>1$. By making use of Lemma 2.1, the Hölder inequality and the $s$-preinvexity of $\left|f^{(n)}\right|^q$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\leq\frac{(\eta(b,a))^n}{2n!}\left(\int_0^1(n-2t)\,dt\right)^{1-1/q}\\ &\times\left[\int_0^1 t^{q(n-1)}(n-2t)\left((1-t)^s\left|f^{(n)}(a)\right|^q+t^s\left|f^{(n)}(b)\right|^q\right)dt\right]^{1/q}=\frac{(\eta(b,a))^n(n-1)^{1-1/q}}{2n!}\\ &\times\left[\left\{nB(nq-q+1,s+1)-2B(nq-q+2,s+1)\right\}\left|f^{(n)}(a)\right|^q\right.\\ &\qquad\left.+\left\{\frac{n}{nq-q+s+1}-\frac{2}{nq-q+s+2}\right\}\left|f^{(n)}(b)\right|^q\right]^{1/q}.\end{aligned} \tag{2.8}$$

This completes the proof of the theorem. $\square$

**2.9. Corollary.** *Suppose that the conditions of Theorem 2.8 are fulfilled and $n = 2$. Then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.9}$$
$$\leq \frac{(\eta(b,a))^2}{2^{2-1/q}}\times\left[\{B(q+1,s+1)-B(q+2,s+1)\}\left|f''(a)\right|^q\right.$$
$$\left.+\frac{\left|f''(b)\right|^q}{(q+s+1)(q+s+2)}\right]^{1/q},$$

*where $B(x,y)$, $x,y>0$ is the Beta's function.*

**2.10. Corollary.** *If we take $q=1$ and $s=1$ in Corollary 2.10. Then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.10}$$
$$\leq \frac{(\eta(b,a))^2}{24}\left[\left|f''(a)\right|^q+\left|f''(b)\right|\right].$$

**2.11. Remark.** For $\eta(b,a)=b-a$, we obtain new bounds of the difference between the middle and right-side of Hermite-Hadamard inequalities (1.1) in terms of second order derivatives.

Now we give some results related to left-side of Hermite-Hadamard's inequality for $n$-times differentiable $s$-preinvex functions.

**2.12. Theorem.** *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$. Suppose that $f: K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$. If $\left|f^{(n)}\right|^q$ is $s$-preinvex on $K$ for $q>1$, $n\in\mathbb{N}$, $n\geq 1$, $s\in(0,1]$, then for every $a,b\in K$ with $\eta(b,a)>0$, we have the following inequality:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)\right. \tag{2.11}$$
$$\left.-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|\leq\frac{(\eta(b,a))^n}{2^n n!\,(np+1)^{\frac{1}{p}}}\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{s+1}\right]^{\frac{1}{q}},$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* Suppose $n\geq 1$. By using lemma 2.2 and the $s$-preinvexity of $\left|f^{(n)}\right|^q$ on $K$ for $n\in\mathbb{N}$, $n\geq 1$, $q>1$, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.12}$$
$$\leq\frac{(\eta(b,a))^n}{n!}\int_0^1|K_n(t)|\left|f^{(n)}(a+t\eta(b,a))\right|dt$$
$$\leq\frac{(\eta(b,a))^n}{n!}\left(\int_0^1|K_n(t)|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}.$$

Since

$$\int_0^1 |K_n(t)|^p = \int_0^{\frac{1}{2}} t^{np} dt + \int_{\frac{1}{2}}^1 (1-t)^{np}\, dt = \frac{1}{2^{np}\,(np+1)}$$

and

$$\begin{aligned}\int_0^1 \left|f^{(n)}(a+t\eta\,(b,a))\right|^q dt &\leq \int_0^1 (1-t)^s \left|f^{(n)}(a)\right|^q dt + \int_0^1 t^s \left|f^{(n)}(b)\right|^q dt \\ &= \frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{s+1}\end{aligned}$$

An application of the above observations in (2.12), we get the desired inequality (2.11). This completes the proof of the theorem. □

**2.13. Corollary.** *Under the assumptions of Theorem 2.12, if* $n = 2$*, then we obtain the following inequality:*

$$\begin{aligned}\left|f\left(a+\frac{1}{2}\eta\,(b,a)\right) - \frac{1}{\eta\,(b,a)}\int_a^{a+\eta(b,a)} f\,(x)\,dx\right| \\ \leq \frac{(\eta\,(b,a))^2}{8\,(2p+1)^{\frac{1}{p}}}\left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{s+1}\right]^{\frac{1}{q}},\end{aligned} \tag{2.13}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**2.14. Corollary.** *In Corollary 2.13, if we take* $s = 1$*, then one gets the following result:*

$$\begin{aligned}\left|f\left(a+\frac{1}{2}\eta\,(b,a)\right) - \frac{1}{\eta\,(b,a)}\int_a^{a+\eta(b,a)} f\,(x)\,dx\right| \\ \leq \frac{(\eta\,(b,a))^2}{8\,(2p+1)^{\frac{1}{p}}}\left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}},\end{aligned} \tag{2.14}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**2.15. Corollary.** *In Theorem 2.12, if* $\eta\,(b,a) = b-a$*, we have the following inequality:*

$$\begin{aligned}\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}\,(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f\,(x)\,dx\right| \\ \leq \frac{(b-a)^n}{2^n n!\,(np+1)^{\frac{1}{p}}}\left[\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{s+1}\right]^{\frac{1}{q}},\end{aligned} \tag{2.15}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

A different approach leads us to the following result:

**2.16. Theorem.** *Let* $K \subseteq [0,\infty)$ *be an open invex subset with respect to* $\eta : K\times K \to \mathbb{R}$*. Suppose that* $f : K \to \mathbb{R}$ *is a function such that* $f^{(n)}$ *exists on* $K$ *and* $f^{(n)}$ *is integrable* *on* $[a, a+\eta(b,a)]$. *If* $\left|f^{(n)}\right|^q$ *is* $s$*-preinvex on* $K$ *for* $n \in \mathbb{N}$, $n \geq 1$, $q \in \mathbb{R}$, $q > 1$ *and* $s \in (0,1]$, *then for every* $a, b \in K$ *with* $\eta(b,a) > 0$, *we have the following inequality:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.16}$$
$$\leq \frac{(\eta(b,a))^n}{2^{n+\frac{1}{p}}(np+1)^{\frac{1}{p}} n!} \left[\left(\frac{\left(2^{s+1}-1\right)\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}} + \left(\frac{\left|f^{(n)}(a)\right|^q + \left(2^{s+1}-1\right)\left|f^{(n)}(b)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right],$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* From lemma 2.2 and the power-mean integral inequality, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.17}$$
$$\leq \frac{(\eta(b,a))^n}{n!} \left[\left(\int_0^{\frac{1}{2}} t^{np} dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $\left|f^{(n)}\right|^q$ is $s$-preinvex on $K$ in the second sense for $n \in \mathbb{N}$, $n \geq 1$, $q \in \mathbb{R}$, $q > 1$ and $s \in (0,1]$. Hence for every $a, b \in K$ with $\eta(b,a) > 0$, we have

$$\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(a+t\eta(b,a))\right|^q dt \leq \left|f^{(n)}(a)\right|^q \int_0^{\frac{1}{2}} (1-t)^s\,dt + \left|f^{(n)}(b)\right|^q \int_0^{\frac{1}{2}} t^s dt \tag{2.18}$$
$$= \frac{2^{s+1}-1}{2^{s+1}(s+1)} \left|f^{(n)}(a)\right|^q + \frac{1}{2^{s+1}(s+1)} \left|f^{(n)}(b)\right|^q$$

and

$$\int_0^{\frac{1}{2}} \left|f^{(n)}(a+t\eta(b,a))\right|^q dt \leq \left|f^{(n)}(a)\right|^q \int_{\frac{1}{2}}^1 (1-t)^s\,dt + \left|f^{(n)}(b)\right|^q \int_{\frac{1}{2}}^1 t^s dt \tag{2.19}$$
$$= \frac{1}{2^{s+1}(s+1)} \left|f^{(n)}(a)\right|^q + \frac{2^{s+1}-1}{2^{s+1}(s+1)} \left|f^{(n)}(b)\right|^q.$$

Also

$$\int_0^{\frac{1}{2}} t^{np} dt = \int_{\frac{1}{2}}^1 (1-t)^{np}\,dt = \frac{1}{2^{np+1}(np+1)}. \tag{2.20}$$

Using (2.18), (2.19) and (2.20) in (2.17), we get the required inequality (2.16). This completes the proof of the theorem. $\square$

**2.17. Remark.** For $s = 1$, Theorem 2.16 becomes Theorem 11 from [16].

**2.18. Corollary.** *For $s=1$ and $n=2$, we get the following inequality from* [16]*:*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.21}$$
$$\leq \frac{(\eta(b,a))^2}{8\cdot 2^{\frac{1}{p}}(2p+1)^{\frac{1}{p}}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right],$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**2.19. Theorem.** *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$. Suppose that $f: K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$. If $\left|f^{(n)}\right|^q$ is s-preinvex on $K$ for $n\in\mathbb{N}$, $n\geq 1$, $q\in\mathbb{R}$, $q\geq 1$ and $s\in(0,1]$, then for every $a,b\in K$ with $\eta(b,a)>0$, we have the following inequality:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.22}$$
$$\leq \frac{(\eta(b,a))^n(n+1)^{\frac{1}{q}}}{2^{(n+1)\left(1-\frac{1}{q}\right)}(n+1)!}\left[\left(L\left|f^{(n)}(a)\right|^q+M\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}+\left(M\left|f^{(n)}(a)\right|^q+N\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right],$$

*where*

$$L=B\left(\frac{1}{2};n+1,s+1\right),M=\frac{1}{2^{n+s+1}(n+s+1)},$$
$$N=B(s+1,n+1)-B\left(\frac{1}{2};s+1,n+1\right),$$

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}$$

*for $x>0$, $y>0$ is the Beta function and*

$$B(z;x,y)=\int_0^z t^{x-1}(1-t)^{y-1}$$

*is the generalized of the Beta function $B(x,y)$.*

*Proof.* It is not difficult to see that (2.22) holds true for $q = 1$. Suppose that $q > 1$. From lemma 2.2 and the Hölder's integral inequality, we have

$$\begin{aligned}(2.23)\quad &\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\ &\le\frac{(\eta(b,a))^n}{n!}\left[\left(\int_0^{\frac{1}{2}}t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1(1-t)^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is $s$-preinvex on $K$ in the second sense for $n \in \mathbb{N}$, $n \geq 1$, $q \in \mathbb{R}$, $q \geq 1$ and $s \in (0, 1]$. Hence for every $a, b \in K$ with $\eta(b, a) > 0$, we have

(2.24)

$$\begin{aligned}\int_0^{\frac{1}{2}}t^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt&\le\left|f^{(n)}(a)\right|^q\int_0^{\frac{1}{2}}t^n(1-t)^s dt+\left|f^{(n)}(b)\right|^q\int_0^{\frac{1}{2}}t^{n+s}dt\\ &=B\left(\frac{1}{2};n+1,s+1\right)\left|f^{(n)}(a)\right|^q+\frac{\left|f^{(n)}(b)\right|^q}{2^{n+s+1}(n+s+1)}\end{aligned}$$

and

$$\begin{aligned}(2.25)\quad &\int_0^{\frac{1}{2}}(1-t)^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\\ &\le\left|f^{(n)}(a)\right|^q\int_{\frac{1}{2}}^1(1-t)^{n+s}dt+\left|f^{(n)}(b)\right|^q\int_{\frac{1}{2}}^1 t^s(1-t)^n dt\\ &=\frac{\left|f^{(n)}(a)\right|^q}{2^{n+s+1}(n+s+1)}+\left[B(s+1,n+1)-B\left(\frac{1}{2};s+1,n+1\right)\right]\left|f^{(n)}(b)\right|^q.\end{aligned}$$

Using (2.24), (2.25) and

$$(2.26)\quad \int_0^{\frac{1}{2}}t^n dt=\int_{\frac{1}{2}}^1(1-t)^n dt=\frac{1}{2^{n+1}(n+1)}.$$

we get the required inequality (2.22). This completes the proof of the theorem. □

**2.20. Corollary.** *If we choose $n = 2$ and $s = 1$ in the Theorem 2.19, we get the following inequality:*

$$\begin{aligned}(2.27)\quad &\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\ &\le\frac{(\eta(b,a))^2\cdot 3^{\frac{1}{q}-1}}{2^{4-\frac{3}{q}}}\left[\left(\frac{5\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{192}\right)^{\frac{1}{q}}\right.\\ &\qquad\qquad\left.+\left(\frac{3\left|f''(a)\right|^q+5\left|f''(b)\right|^q}{192}\right)^{\frac{1}{q}}\right].\end{aligned}$$

*Proof.* Since for $n = 2$ and $s = 1$, $L = B\left(\frac{1}{2}; 3, 2\right) = \frac{5}{192}$, $M = \frac{1}{64}$ and $N = B(2,3) - B\left(\frac{1}{2}; 2, 3\right) = \frac{5}{192}$ and hence proof follows. $\square$

## 3. Applications to Special Means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

**3.1. Definition.** [3]A function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, is called a Mean function if it has the following properties:

(1) Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
(2) Symmetry : $M(x, y) = M(y, x)$,
(3) Reflexivity : $M(x, x) = x$,
(4) Monotonicity: If $x \le x^{'}$ and $y \le y^{'}$, then $M(x, y) \le M(x^{'}, y^{'})$,
(5) Internality: $\min\{x, y\} \le M(x, y) \le \max\{x, y\}$.

We consider some means for arbitrary positive real numbers $\alpha$, $\beta$ (see for instance [3]).

(1) The arithmetic mean:
$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$
(2) The The geometric mean:
$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$
(3) The harmonic mean:
$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$
(4) The power mean:
$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \; r \ge 1$$
(5) The identric mean:
$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \ne \beta \\ \alpha, & \alpha = \beta \end{cases}$$
(6) The logarithmic mean:
$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \;\; |\alpha| \ne |\beta|$$
(7) The generalized log-mean:
$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right], \; \alpha \ne \beta, \, p \in \mathbb{R} \backslash \{-1, 0\}.$$

It is well known that $L_p$ is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \le G \le L \le I \le A$.

Now, let $a$ and $b$ be positive real numbers such that $a < b$. Consider the function $M := M(a, b) : [a, a+\eta(b, a)] \times [a, a+\eta(b, a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain various inequalities for these means as follows:

Setting $\eta(b,a) = M(b,a)$ in (2.6), (2.13), one can obtain the following interesting inequalities involving means:

$$\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \leq \frac{(M(b,a))^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{(s+3)(s+2)}\right)^{\frac{1}{q}}. \tag{3.1}$$

$$\left|f\left(a+\frac{1}{2}M(b,a)\right) - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \leq \frac{(M(b,a))^2}{8(2p+1)^{\frac{1}{p}}}\left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{s+1}\right]^{\frac{1}{q}}, \tag{3.2}$$

Letting $M = A, G, H, P_r, I, L, L_p$ in (3.1) and in (3.2), we get the inequalities involving means for a particular choice of a twice differentiable $s$-preinvex function $f$, and the details are left to the interested reader.

## References


[1] Antczak, T. *Mean value in invexity analysis*, Nonl. Anal., **60** (2005), 1473-1484.

[2] Avci, M. Kavurmaci, H. and Özdemir, M.E. *New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications*, Applied Mathematics and Computation, **217** (2011) 5171-5176.

[3] Bullen, P.S. Handbook of Means and Their Inequalities, Kluwer Academic Publishers, Dordrecht, 2003.

[4] Barani, A., Ghazanfari, A.G. and Dragomir S.S. *Hermite-Hadamard inequality through prequsiinvex functions*, RGMIA Research Report Collection, **14**(2011), Article 48, 7 pp.

[5] Barani, A., Ghazanfari, A.G. and Dragomir S.S. *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, RGMIA Research Report Collection, **14**(2011), Article 64, 11 pp.

[6] Ben-Israel, A. and Mond, B. *What is invexity*?, J. Austral. Math. Soc., Ser. B, **28**(1986), No. 1, 1-9.

[7] Dragomir, S.S. and Agarwal, R.P. *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., **11**(5) (1998), 91-95.

[8] Dragomir, S.S. and Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[9] Dragomir, S.S. and Fitzpatrick, S. *The Hadamard's inequality for s-convex functions in the second sense*, Demonstratio Math., **32** (4) (1999), 687-696.

[10] Hudzik, H. and Maligranda, L. *Some remraks on s-convex functions*, Aequationes Math. **48** (1994), 100-111.

[11] Jiang, W.-D., Niu, D.-W, Hua, Y. and Qi, F. *Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense*, Analysis (Munich) **32** (2012), 1001–1012; Available online at http://dx.doi.org/10.1524/anly.2012.1161.

[12] Hanson, M.A. *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981) 545-550.

[13] Wang, S.-H., Xi, B.-Y and Qi, F. *Some new inequalities of Hermite-Hadamard type for n-times differentiable functions which are m-convex*, Analysis (Munich) **32** (2012), no. 3, 247-262; Available online at http://dx.doi.org/10.1524/anly.2012.1167.

[14] Hadamard, J. *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann*, J. Math Pures Appl., **58** (1893), 171–215.

[15] Hwang, D.-Y. *Some inequalities for $n$-time differentiable mappings and applications*, Kyungpook Math. J. **43**(2003), 335-343.

[16] Latif, M.A. *On Hermite-Hadamard type integral inequalities for $n$-times differentiable preinvex functions with applications*, Stud. Univ. Babeş-Bolyai Math. **58**(2013), No. 3, 325–343.

[17] Latif, M.A. and Dragomir, S.S. *Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions with applications*, Journal of Inequalities and Applications 2013, 2013:575.

[18] Latif, M.A. *Some inequalities for prequasiinvex functions with applications*, Konurlap Journal of Math.Vol. **1**, no 2, 17-29.

[19] Iscan, I. *Ostrowski type inequalites for functions whose derivatives are preinvex*, arXiv:1204.2010v1.

[20] Kırmacı, U.S. *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., **147** (2004), 137-146.

[21] Kırmacı, U.S. and Özdemir, M.E. *On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., **153** (2004), 361-368.

[22] Kırmacı, U.S. *Improvement and further generalization of inequalities for differentiable mappings and applications,* Computers and Math. with Appl., **55** (2008), 485-493.

[23] Kırmacı, U.S., Bakula, M.K., Özdemir, M.E. and Pečarić, J. *Hadamard-type inequalities for $s$-convex functions*, Appl. Math. and Comput., Volume **193**, Issue 1( 2007), Pages 26-35.

[24] Mohan, S.R. and Neogy, S.K. *On invex sets and preinvex functions*, J. Math. Anal. Appl. **189** (1995), 901–908.

[25] Noor, M.A. *Hermite-Hadamard integral inequalities for log-preinvex functions*, J. Math. Anal. Approx. Theory, **2**(2007), 126-131.

[26] Noor, M.A. *Variational like inequalities, Optimization*, **30**(1994), 323-330.

[27] Noor, M.A. *On Hadamard integral inequalities involving two log-preinvex functions*, J. Inequal. Pure Appl. Math., **8**(2007), No. 3, 1-6, Article 75.

[28] Noor, M.A. *Hadamard integral inequalities for product of two preinvex function*, Nonl. Anal. Forum, **14** (2009), 167-173.

[29] Pearce, C.E.M. and Pečarić, J. *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett., **13**(2) (2000), 51–55.

[30] Pini, R. *Invexity and generalized Convexity*, Optimization **22** (1991) 513-525.

[31] Sarikaya, M.Z., Saglam, A. and Yıldırım, H. *New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex*, International Journal of Open Problems in Computer Science and Mathematics ( IJOPCM), **5**(3), 2012.

[32] Sarikaya, M.Z., Saglam, A. and Yıldırım, H. *On some Hadamard-type inequalities for $h$-convex functions*, Journal of Mathematical Inequalities, Volume 2, Number **3** (2008), 335-341.

[33] Sarikaya, M.Z., Avci, M. and Kavurmaci, H. *On some inequalities of Hermite-Hadamard type for convex functions*, ICMS Iternational Conference on Mathematical Science. AIP Conference Proceedings 1309, **852** (2010).

[34] Sarikaya, M.Z. and Aktan, N. *On the generalization some integral inequalities and their applications,* Mathematical and Computer Modelling, Volume **54**, Issues 9-10, November 2011, Pages 2175-2182.

[35] Sarikaya, M.Z., Bozkurt, H. and Alp, N. *On Haermite-Hadamard type integral inequalities for preinvex and log-preinvex functions*, Contemporary Analysis and Applied Mathematics Vol.**1**, No.2, 237-252, 2013.

[36] Sarikaya, M.Z., Set, E. and Özdemir, M.E. *On new inequalities of Simpson's type for $s$-convex functions*, Computers and Mathematics with Applications, **60**(2010), 2191-2199.

[37] Saglam, A., Sarikaya, M.Z. Yildirim H. *Some new inequalities of Hermite-Hadamard's type*, Kyungpook Mathematical Journal, **50**(2010), 399-410.

[38] Weir, T. and Mond, B. *Preinvex functions in multiobjective optimization*, Journal of Mathematical Analysis and Applications, **136** (1998) 29-38.

[39] Wang, Y., Wang, S. -H. and Qi, F. *Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is $s$-preinvex*, Facta Univ. (NIŠ), Ser. Math. Inform. Vol. **28**, No 2 (2013), 151–159.

[40] Wang, Y., Xi, B.-Y. and Qi, F. *Hermite-Hadamard type integral inequalities when the power of the absolute value of the first derivative of the integrand is preinvex*, Le Matematiche. (to appear)

[41] Yang, X. M. and Li, D. *On properties of preinvex functions*, J. Math. Anal. Appl. **256** (2001) 229-241.

[42] Jue-You, L. *On Hadamard-type inequalities for $s$-preinvex functions*, Journal of Chongqing Normal University (Natural Science) **27**(2010).

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 855 – 866

# Existence and nonexistence results for a fourth-order discrete Dirichlet boundary value problem

Xia Liu * , Yuanbiao Zhang † and Haiping Shi ‡

**Abstract**

In this paper, a fourth-order nonlinear difference equation is considered. By making use of the critical point theory, we establish various sets of sufficient conditions for the existence and nonexistence of solutions for Dirichlet boundary value problem and give some new results. Our results generalize and complement the results in the literature.

*2000 AMS Classification:* 39A10.

**Keywords:** Existence and nonexistence; Dirichlet boundary value problem; Fourth-order; Mountain Pass Lemma; Discrete variational theory.

Received 12/04/2013 : Accepted 27/03/2014 Doi : 10.15672 /HJMS.2015449433

## 1. Introduction

Difference equations have attracted the interest of many researchers in the past twenty years since they provided a natural description of several discrete models. Such discrete models are often investigated in various fields of science and technology such as computer science, economics, *neural networks*, ecology, cybernetics, biological systems, optimal control, and population dynamics. These studies cover many of the branches of difference equations, such as stability, attractivity, periodicity, oscillation, and boundary value *problems*, see [6,12-14,16,18,19,21,26,27] and the references therein.

*Oriental Science and Technology College, Hunan Agricultural University, Changsha 410128, China. Corresponding author.
Science College, Hunan Agricultural University, Changsha 410128, China.
Email: xia991002@163.com
†Packaging Engineering Institute, Jinan University, Zhuhai 519070, China.
‡Modern Business and Management Department, Guangdong Construction Vocational Technology Institute, Guangzhou 510450, China.
This project is supported by the Specialized Research Fund for the Doctoral Program of Higher Eduction of China (No. 20114410110002), National Natural Science Foundation of China (No. 11101098) and Natural Science Foundation of Guangdong Province (No. S2013010014460).

Below **N**, **Z** and **R** denote the sets of all natural numbers, integers and real numbers respectively. $k$ is a positive integer. For any $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a < b$. Besides, * denotes the transpose of a vector.

The present paper considers the fourth-order nonlinear difference equation

$$\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n \Delta u_{n-1}\right) = f(n, u_{n+1}, u_n, u_{n-1}),\ n \in \mathbf{Z}(1,k), \tag{1.1}$$

with boundary value conditions

$$u_{-1} = u_0 = 0,\ u_{k+1} = u_{k+2} = 0, \tag{1.2}$$

where $\Delta$ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $p_n$ is nonzero and real valued for each $n \in \mathbf{Z}(0, k+1)$, $q_n$ is real valued for each $n \in \mathbf{Z}(1, k+1)$, $f \in C(\mathbf{R}^4, \mathbf{R})$.

*In recent years* the study of boundary value problems for differential equations develops at relatively rapid rate. By using various methods and techniques, such as fixed point theory, topological degree theory, coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained in literatures, we refer to [1-3,5,15,30]. And critical point theory is also an important tool to deal with problems on differential equations [9,11,20,25,35]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [12-14] and Shi *et al.* [28] have successfully proved the existence of periodic solutions of second-order nonlinear difference equations. We also refer to [32,33] for the discrete boundary value problems. Compared to *first-order* or second-order difference equations, the study of higher-order equations, and in particular, fourth-order equations, has received considerably less attention (see, for example, [7,8,10,23,24,26,29,31] and the references contained therein). Yan, Liu [31] in 1997 and Thandapani, Arockiasamy [29] in 2001 studied the following fourth-order difference equation of form,

$$\Delta^2\left(p_n \Delta^2 u_n\right) + f(n, u_n) = 0,\ n \in \mathbf{Z}, \tag{1.3}$$

*and obtained* criteria for the oscillation and nonoscillation of solutions for equation (1.3). In 2005, Cai, Yu and Guo [4] have obtained some criteria for the existence of periodic solutions of the fourth-order difference equation

$$\Delta^2\left(p_{n-2}\Delta^2 u_{n-2}\right) + f(n, u_n) = 0,\ n \in \mathbf{Z}. \tag{1.4}$$

In 1995, Peterson and Ridenhour considered the disconjugacy of equation (1.7) when $p_n \equiv 1$ and $f(n, u_n) = q_n u_n$ (see [23]).

The boundary value problem (BVP) for determining the existence of solutions of difference equations has been a very active area of research in the last twenty years, and for surveys of recent results, we refer the reader to the monographs by Agarwal *et al.* [17,21,27]. As far as we know results obtained in the literature for the BVP (1.1) with (1.2) are very scarce. Since $f$ in (1.1) depends on $u_{n+1}$ and $u_{n-1}$, the traditional ways of establishing the functional in [12-14,32-34] are inapplicable to our case. As a result, the goal of this paper is to fill the gap in this area.

Motivated by the above results, *we use* the critical point theory to give some sufficient conditions *for* the existence and nonexistence of solutions for the BVP (1.1) with (1.2). We shall study the *superlinear* and sublinear cases. The main idea in this paper is to transfer the existence of the BVP (1.1) with (1.2) into the existence of the critical points of some functional. The proof is based on the notable Mountain Pass Lemma in combination with variational technique. The purpose of this paper is two-folded. On one hand, we shall further demonstrate the powerfulness of critical point theory in the study of solutions for boundary value problems of difference equations. On the other hand, we shall complement existing results. The motivation for the present work stems from the recent paper in [7].

Let

$$\bar{p} = \max\{p_n : n \in \mathbf{Z}(0, k+1)\},\ \underline{p} = \min\{p_n : n \in \mathbf{Z}(0, k+1)\},$$

$$\bar{q} = \max\{q_n : n \in \mathbf{Z}(1, k+1)\},\ \underline{q} = \min\{q_n : n \in \mathbf{Z}(1, k+1)\}.$$

Our main results are as follows.

**Theorem 1.1.** *Assume that the following hypotheses are satisfied:*
*$(p)$ for any $n \in \boldsymbol{Z}(0, k+1)$, $p_n < 0$;*
*$(q)$ for any $n \in \boldsymbol{Z}(1, k+1)$, $q_n \leq 0$;*
*$(F_1)$ there exists a functional $F(n, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ with $F(0, \cdot) = 0$ such that*

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),\ \forall n \in \boldsymbol{Z}(1, k);$$

*$(F_2)$ there exists a constant $M_0 > 0$ such that for all $(n, v_1, v_2) \in \boldsymbol{Z}(1, k) \times \boldsymbol{R}^2$*

$$\left|\frac{\partial F(n, v_1, v_2)}{\partial v_1}\right| \leq M_0,\ \left|\frac{\partial F(n, v_1, v_2)}{\partial v_2}\right| \leq M_0.$$

*Then the BVP (1.1) with (1.2) possesses at least one solution.*

**Remark 1.1.** Assumption $(F_2)$ implies that there exists a constant $M_1 > 0$ such that
$(F_2')$ $|F(n, v_1, v_2)| \leq M_1 + M_0(|v_1| + |v_2|),\ \forall (n, v_1, v_2) \in \mathbf{Z}(1, k) \times \mathbf{R}^2$.

**Theorem 1.2.** *Suppose that $(F_1)$ and the following hypotheses are satisfied:*
*$(p')$ for any $n \in \boldsymbol{Z}(0, k+1)$, $p_n > 0$;*
*$(q')$ for any $n \in \boldsymbol{Z}(1, k+1)$, $q_n \geq 0$;*
*$(F_3)$ there exists a functional $F(n, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ such that*

$$\lim_{r \to 0} \frac{F(n, v_1, v_2)}{r^2} = 0,\ r = \sqrt{v_1^2 + v_2^2},\ \forall n \in \boldsymbol{Z}(1, k);$$

*$(F_4)$ there exists a constant $\beta > 2$ such that for any $n \in \boldsymbol{Z}(1, k)$,*

$$0 < \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 < \beta F(n, v_1, v_2),\ \forall (v_1, v_2) \neq 0.$$

*Then the BVP (1.1) with (1.2) possesses at least two nontrivial solutions.*

**Remark 1.2.** Assumption $(F_4)$ implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that
$(F_4')$ $F(n, v_1, v_2) > a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^\beta - a_2,\ \forall n \in \mathbf{Z}(1, k)$.

**Theorem 1.3.** *Suppose that $(p')$, $(q')$, $(F_1)$ and the following assumption are satisfied:*
*$(F_5)$ there exist constants $R > 0$ and $1 < \alpha < 2$ such that for $n \in \boldsymbol{Z}(1, k)$ and $\sqrt{v_1^2 + v_2^2} \geq R$,*

$$0 < \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \leq \alpha F(n, v_1, v_2).$$

*Then the BVP (1.1) with (1.2) possesses at least one solution.*

**Remark 1.3.** Assumption $(F_5)$ implies that for each $n \in \mathbf{Z}(1, k)$ there exist constants

$a_3 > 0$ and $a_4 > 0$ such that

$(F_5')$ $F(n, v_1, v_2) \leq a_3 \left(\sqrt{v_1^2 + v_2^2}\right)^{\alpha} + a_4, \ \forall (n, v_1, v_2) \in \mathbf{Z}(1,k) \times \mathbf{R}^2.$

**Theorem 1.4.** *Suppose that* $(p)$, $(q)$, $(F_1)$ *and the following assumption are satisfied:*

$(F_6)$ $v_2 f(n, v_1, v_2, v_3) > 0$, *for* $v_2 \neq 0$, $\forall n \in \boldsymbol{Z}(1,k)$.

*Then the BVP (1.1) with (1.2) has no nontrivial solutions.*

**Remark 1.4.** In the existing literature, results on the nonexistence of solutions of discrete boundary value problems are scarce. Hence, Theorem 1.4 complements existing ones.

The *remainder* of this paper is organized as follows. *First*, in Section 2, we shall establish the variational framework for the BVP (1.1) with (1.2) and transfer the problem of the existence of the BVP (1.1) with (1.2) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give three examples to illustrate the main results.

*For* the basic knowledge *of* variational methods, *the reader is referred to* [20,22,25,35].

## 2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for the BVP (1.1) with (1.2) and give some lemmas which will be of fundamental importance in proving our main results. Firstly, we state some basic notations.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension $k$. Define the inner product on $\mathbf{R}^k$ as follows:

$$(2.1) \qquad \langle u, v \rangle = \sum_{j=1}^{k} u_j v_j, \ \forall u, v \in \mathbf{R}^k,$$

by which the norm $\|\cdot\|$ can be induced by

$$(2.2) \qquad \|u\| = \left(\sum_{j=1}^{k} u_j^2\right)^{\frac{1}{2}}, \ \forall u \in \mathbf{R}^k.$$

On the other hand, we define the norm $\|\cdot\|_r$ on $\mathbf{R}^k$ as follows:

$$(2.3) \qquad \|u\|_r = \left(\sum_{j=1}^{k} |u_j|^r\right)^{\frac{1}{r}},$$

for all $u \in \mathbf{R}^k$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants $c_1$, $c_2$ such that $c_2 \geq c_1 > 0$, and

$$(2.4) \qquad c_1 \|u\|_2 \leq \|u\|_r \leq c_2 \|u\|_2, \ \forall u \in \mathbf{R}^k.$$

Clearly, $\|u\| = \|u\|_2$. For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, for the BVP (1.1) with (1.2), consider the functional $J$ defined on $\mathbf{R}^k$ as follows:

$$(2.5) \qquad J(u) = \frac{1}{2} \sum_{n=-1}^{k} p_{n+1} \left(\Delta^2 u_n\right)^2 + \frac{1}{2} \sum_{n=0}^{k} q_{n+1} \left(\Delta u_n\right)^2 - \sum_{n=1}^{k} F(n, u_{n+1}, u_n),$$

where

$$\frac{\partial F(n-1,v_2,v_3)}{\partial v_2}+\frac{\partial F(n,v_1,v_2)}{\partial v_2}=f(n,v_1,v_2,v_3),$$

$$u_{-1}=u_0=0,\ u_{k+1}=u_{k+2}=0.$$

Clearly, $J\in C^1(\mathbf{R}^k,\mathbf{R})$ and for any $u=\{u_n\}_{n=1}^k=(u_1,u_2,\ldots,u_k)^*$, by using $u_{-1}=u_0=0,\ u_{k+1}=u_{k+2}=0$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n}=\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right)-\Delta\left(q_n\Delta u_{n-1}\right)-f(n,u_{n+1},u_n,u_{n-1}),\ \forall n\in\mathbf{Z}(1,k).$$

Thus, $u$ is a critical point of $J$ on $\mathbf{R}^k$ if and only if

$$\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right)-\Delta\left(q_n\Delta u_{n-1}\right)=f(n,u_{n+1},u_n,u_{n-1}),\ \forall n\in\mathbf{Z}(1,k).$$

We reduce the existence of the BVP (1.1) with (1.2) to the existence of critical points of $J$ on $\mathbf{R}^k$. That is, the functional $J$ is just the variational framework of the BVP (1.1) with (1.2).

Let $P$ and $Q$ be the $k\times k$ matrices defined by

$$P=\begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0\\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0\\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1\\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4\\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 \end{pmatrix},$$

$$Q=\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 2 & -1\\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Clearly, $P$ and $Q$ are positive definite. Let $\lambda_1,\lambda_2,\cdots,\lambda_k$ be the eigenvalues of $P$, $\tilde\lambda_1,\tilde\lambda_2,\cdots,\tilde\lambda_k$ be the eigenvalues of $Q$. Applying matrix theory, we know $\lambda_j>0,\ \tilde\lambda_j>0,\ j=1,2,\cdots,k$. Without loss of generality, we may assume that

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_k, \tag{2.6}$$

$$0<\tilde\lambda_1\le\tilde\lambda_2\le\cdots\le\tilde\lambda_k. \tag{2.7}$$

Let $E$ be a real Banach space, $J\in C^1(E,\mathbf{R})$, i.e., $J$ is a continuously Fréchet-differentiable functional defined on $E$. $J$ is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u^{(l)}\right\}\subset E$ for which $\left\{J\left(u^{(l)}\right)\right\}$ is bounded and $J'\left(u^{(l)}\right)\to 0(l\to\infty)$ possesses a convergent subsequence in $E$.

Let $B_\rho$ denote the open ball in $E$ about 0 of radius $\rho$ and let $\partial B_\rho$ denote its boundary.

**Lemma 2.1** *(Mountain Pass Lemma [25]). Let $E$ be a real Banach space and $J\in C^1(E,\boldsymbol{R})$ satisfy the P.S. condition. If $J(0)=0$ and*
*($J_1$) there exist constants $\rho,\ a>0$ such that $J|_{\partial B_\rho}\ge a$, and*
*($J_2$) there exists $e\in E\setminus B_\rho$ such that $J(e)\le 0$.*
*Then $J$ possesses a critical value $c\ge a$ given by*

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}J(g(s)), \tag{2.8}$$

*where*

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}. \tag{2.9}$$

**Lemma 2.2.** *Suppose that* $(p')$, $(q')$, $(F_1)$, $(F_3)$ *and* $(F_4)$ *are satisfied. Then the functional* $J$ *satisfies the P.S. condition.*

**Proof.** Let $u^{(l)} \in \mathbf{R}^k,\ l \in \mathbf{Z}(1)$ be such that $\left\{J\left(u^{(l)}\right)\right\}$ is bounded. Then there exists a positive constant $M_2$ such that

$$-M_2 \le J\left(u^{(l)}\right) \le M_2,\ \forall l \in \mathbf{N}.$$

By $(F_4')$, we have

$$\begin{aligned}
-M_2 \le J\left(u^{(l)}\right) &= \frac{1}{2}\sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u_n^{(l)}\right)^2 + \frac{1}{2}\sum_{n=0}^{k} q_{n+1}\left(\Delta u_n^{(l)}\right)^2 - \sum_{n=1}^{k} F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right) \\
&\le \frac{1}{2}\bar{p}\sum_{n=-1}^{k}\left(u_{n+2}^{(l)} - 2u_{n+1}^{(l)} + u_n^{(l)}\right)^2 + \frac{1}{2}\bar{q}\sum_{n=0}^{k}\left(u_{n+1}^{(l)} - u_n^{(l)}\right)^2 \\
&\quad - a_1\sum_{n=1}^{k}\left[\sqrt{\left(u_{n+1}^{(l)}\right)^2 + \left(u_n^{(l)}\right)^2}\right]^\beta + a_2 k \\
&\le \frac{1}{2}\bar{p}\left(u^{(l)}\right)^* P u^{(l)} + \frac{1}{2}\bar{q}\left(u^{(l)}\right)^* Q u^{(l)} - a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta + a_2 k \\
&\le \frac{1}{2}\bar{p}\lambda_k \left\|u^{(l)}\right\|^2 + \frac{1}{2}\bar{q}\tilde{\lambda}_k \left\|u^{(l)}\right\|^2 - a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta + a_2 k,
\end{aligned}$$

where $u^{(l)} = \left(u_1^{(l)}, u_2^{(l)}, \cdots, u_k^{(l)}\right)^*,\ u^{(l)} \in \mathbf{R}^k$. That is,

$$a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta - \frac{1}{2}\left(\bar{p}\lambda_k + \bar{q}\tilde{\lambda}_k\right)\left\|u^{(l)}\right\|^2 \le M_2 + a_2 k.$$

Since $\beta > 2$, there exists a constant $M_3 > 0$ such that

$$\left\|u^{(l)}\right\| \le M_3,\ \forall l \in \mathbf{N}.$$

Therefore, $\left\{u^{(l)}\right\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\left\{u^{(l)}\right\}$ possesses a convergence subsequence in $\mathbf{R}^k$. Thus the P.S. condition is verified. □

## 3. Proof of the main results

In this Section, we shall prove our main results by using the critical point theory.

3.1. Proof of Theorem 1.1

**Proof.** By $(F_2')$, for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{2}\sum_{n=0}^{k} q_{n+1}\left(\Delta u_n\right)^2 - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\
&\le \frac{1}{2}\bar{p}\sum_{n=-1}^{k}\left(u_{n+2} - 2u_{n+1} + u_n\right)^2 + \frac{1}{2}\bar{q}\sum_{n=0}^{k}\left(u_{n+1} - u_n\right)^2 + M_0\sum_{n=1}^{k}\left(|u_{n+1}| + |u_n|\right) + M_1 k \\
&\le \frac{1}{2}\bar{p}u^* P u + \frac{1}{2}\bar{q}u^* Q u + 2M_0\sum_{n=1}^{k}|u_n| + M_1 k
\end{aligned}$$

$$\leq \frac{1}{2}\bar{p}\lambda_1\|u\|^2 + \frac{1}{2}\bar{q}\tilde{\lambda}_1\|u\|^2 + 2M_0\sqrt{k}\|u\| + M_1 k$$
$$\to -\infty \text{ as } \|u\| \to +\infty.$$

The above inequality means that $-J(u)$ is coercive. By the continuity of $J(u)$, $J$ attains its maximum at some point, and we denote it $\breve{u}$, that is,

$$J(\breve{u}) = \max\left\{J(u)|u \in \mathbf{R}^k\right\}.$$

Clearly, $\breve{u}$ is a critical point of the functional $J$. This completes the proof of Theorem 1.1. □

3.2. Proof of Theorem 1.2

**Proof.** By $(F_3)$, for any $\epsilon = \frac{1}{8}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)$($\lambda_1$ and $\tilde{\lambda}_1$ can be referred to (2.6) and (2.7)), there exists $\rho > 0$, such that

$$|F(n, v_1, v_2)| \leq \frac{1}{8}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)\left(v_1^2 + v_2^2\right), \forall n \in \mathbf{Z}(1, k),$$

for $\sqrt{v_1^2 + v_2^2} \leq \sqrt{2}\rho$.

For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$ and $\|u\| \leq \rho$, we have $|u_n| \leq \rho,\ n \in \mathbf{Z}(1, k)$.

For any $n \in \mathbf{Z}(1, k)$,

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{2}\sum_{n=0}^{k} q_{n+1}\left(\Delta u_n\right)^2 - \sum_{n=1}^{k} F(n, u_{n+1}, u_n)\\
&\geq \frac{1}{2}\underline{p}\sum_{n=-1}^{k}\left(u_{n+2} - 2u_{n+1} + u_n\right)^2 + \frac{1}{2}\underline{q}\sum_{n=0}^{k}\left(u_{n+1} - u_n\right)^2 - \frac{1}{8}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)\sum_{n=1}^{k}\left(u_{n+1}^2 + u_n^2\right)\\
&\geq \frac{1}{2}\underline{p}u^* P u + \frac{1}{2}\underline{q}u^* Q u - \frac{1}{4}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)\|u\|^2\\
&\geq \frac{1}{2}\underline{p}\lambda_1\|u\|^2 + \frac{1}{2}\underline{q}\tilde{\lambda}_1\|u\|^2 - \frac{1}{4}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)\|u\|^2\\
&= \frac{1}{4}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)\|u\|^2,
\end{aligned}$$

where $u = (u_1, u_2, \cdots, u_k)^*,\ u \in \mathbf{R}^k$.

Take $a = \frac{1}{4}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)\rho^2 > 0$. Therefore,

$$J(u) \geq a > 0,\ \forall u \in \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho} \geq a$. That is to say, $J$ satisfies the condition $(J_1)$ of the Mountain Pass Lemma.

For our setting, clearly $J(0) = 0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify *all* other conditions of the Mountain Pass Lemma. By Lemma 2.2, $J$ satisfies the P.S. condition. So it suffices to verify the condition $(J_2)$.

From the proof of the P.S. condition *in Lemma 2.2*, we know

$$J(u) \leq \frac{1}{2}\left(\bar{p}\lambda_k + \bar{q}\tilde{\lambda}_k\right)\|u\|^2 - a_1 c_1^\beta\|u\|^\beta + a_2 k.$$

Since $\beta > 2$, we can choose $\bar{u}$ large enough to ensure that $J(\bar{u}) < 0$.

By the Mountain Pass Lemma, $J$ possesses a critical value $c \geq a > 0$, where

$$c = \inf_{h\in\Gamma}\sup_{s\in[0,1]} J(h(s)),$$

and

$$\Gamma = \{h \in C([0, 1], \mathbf{R}^k) \mid h(0) = 0,\ h(1) = \bar{u}\}.$$

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value $c$ of $J$, i.e., $J(\tilde{u}) = c$. Similar to the proof of the P.S. condition, we know that there exists $\hat{u} \in \mathbf{R}^k$ such that

$$J(\hat{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s)).$$

Clearly, $\hat{u} \neq 0$. If $\tilde{u} \neq \hat{u}$, then the conclusion of Theorem 1.2 holds. Otherwise, $\tilde{u} = \hat{u}$. Then $c = J(\tilde{u}) = c_{\max} = \max\limits_{s\in[0,1]} J(h(s))$. That is,

$$\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s)).$$

Therefore,

$$c_{\max} = \max_{s\in[0,1]} J(h(s)),\ \forall h \in \Gamma.$$

By the continuity of $J(h(s))$ with respect to $s$, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0,1)$ such that

$$J\left(h\left(s_0\right)\right) = c_{\max}.$$

Choose $h_1,\ h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_2(s) \mid s \in (0,1)\}$ is empty, then there exists $s_1,\ s_2 \in (0,1)$ such that

$$J\left(h_1\left(s_1\right)\right) = J\left(h_2\left(s_2\right)\right) = c_{\max}.$$

Thus, we get two different critical points of $J$ on $\mathbf{R}^k$ denoted by

$$u^1 = h_1\left(s_1\right),\ u^2 = h_2\left(s_2\right).$$

The above argument implies that the BVP (1.1) with (1.2) possesses at least two non-trivial solutions. The proof of Theorem 1.2 is finished. □

3.3. Proof of Theorem 1.3

**Proof.** We only need to find at least one critical point of the functional $J$ defined as in (2.5).

By $(F_5')$, for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{2}\sum_{n=0}^{k} q_{n+1}\left(\Delta u_n\right)^2 - \sum_{n=1}^{k} F(n, u_{n+1}, u_n)\\
&\geq \frac{1}{2}\underline{p}\sum_{n=-1}^{k}\left(u_{n+2} - 2u_{n+1} + u_n\right)^2 + \frac{1}{2}\underline{q}\sum_{n=0}^{k}\left(u_{n+1} - u_n\right)^2 - a_3\sum_{n=1}^{k}\left(\sqrt{u_{n+1}^2 + u_n^2}\right)^{\alpha} - a_4 k\\
&= \frac{1}{2}\underline{p}u^* P u + \frac{1}{2}\underline{q}u^* Q u - a_3\left\{\left[\sum_{n=1}^{k}\left(\sqrt{u_{n+1}^2 + u_n^2}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right\}^{\alpha} - a_4 k\\
&\geq \frac{1}{2}\underline{p}\lambda_1\|u\|^2 + \frac{1}{2}\underline{q}\tilde{\lambda}_1\|u\|^2 - a_3 c_2^{\alpha}\left\{\left[\sum_{n=1}^{k}\left(u_{n+1}^2 + u_n^2\right)\right]^{\frac{1}{2}}\right\}^{\alpha} - a_4 k\\
&\geq \frac{1}{2}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)\|u\|^2 - 2^{\alpha} a_3 c_2^{\alpha}\|u\|^{\alpha} - a_4 k\\
&\to +\infty \text{ as } \|u\| \to +\infty.
\end{aligned}$$

By the continuity of $J$, we know from the above inequality that there exist lower bounds of values of the functional. And this means that $J$ attains its minimal value at some point which is just the critical point of $J$ with the finite norm. $\square$

3.4. Proof of Theorem 1.4

**Proof.** Assume, for the sake of contradiction, that the BVP (1.1) with (1.2) has a nontrivial solution. Then $J$ has a nonzero critical point $u^\star$. Since

$$\frac{\partial J}{\partial u_n} = \Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n \Delta u_{n-1}\right) - f(n, u_{n+1}, u_n, u_{n-1}),$$

we get

$$\sum_{n=1}^{k} f(n, u_{n+1}^\star, u_n^\star, u_{n-1}^\star)u_n^\star = \sum_{n=1}^{k}\left[\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}^\star\right) - \Delta\left(q_n\Delta u_{n-1}^\star\right)\right] u_n^\star$$

$$= \sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u_n^\star\right)^2 + \sum_{n=0}^{k} q_{n+1}\left(\Delta u_n^\star\right)^2 \le 0. \tag{3.1}$$

On the other hand, it follows from $(F_6)$ that

$$\sum_{n=1}^{k} f(n, u_{n+1}^\star, u_n^\star, u_{n-1}^\star)u_n^\star > 0. \tag{3.2}$$

This contradicts (3.1) and hence the proof is complete. $\square$

## 4. Examples

As an application of Theorems 1.2, 1.3 and 1.4, we give three examples to illustrate our main results.

**Example 4.1.** For $n \in \mathbf{Z}(1,k)$, assume that

$$\Delta^4 u_{n-2} - \Delta\left(9^n \Delta u_{n-1}\right) = \beta u_n\left[\varphi(n)\left(u_{n+1}^2 + u_n^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(u_n^2 + u_{n-1}^2\right)^{\frac{\beta}{2}-1}\right], \tag{4.1}$$

with boundary value conditions (1.2), where $\beta > 2$, $\varphi$ is continuously differentiable and $\varphi(n) > 0,\ n \in \mathbf{Z}(1,k)$ with $\varphi(0) = 0$.

We have

$$p_n \equiv 1,\ q_n = 9^n,\ f(n, v_1, v_2, v_3) = \beta v_2\left[\varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(v_2^2 + v_3^2\right)^{\frac{\beta}{2}-1}\right]$$

and

$$F(n, v_1, v_2) = \varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}}.$$

It is easy to verify all the assumptions of Theorem 1.2 are satisfied and then the BVP (4.1) with (1.2) possesses at least two nontrivial solutions.

**Example 4.2.** For $n \in \mathbf{Z}(1,k)$, assume that

$$\Delta^2\left(8^{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(6^n \Delta u_{n-1}\right) = \alpha u_n\left[\psi(n)\left(u_{n+1}^2 + u_n^2\right)^{\frac{\alpha}{2}-1} + \psi(n-1)\left(u_n^2 + u_{n-1}^2\right)^{\frac{\alpha}{2}-1}\right], \tag{4.2}$$

with boundary value conditions (1.2), where $1 < \alpha < 2$, $\psi$ is continuously differentiable and $\psi(n) > 0,\ n \in \mathbf{Z}(1,k)$ with $\psi(0) = 0$.

We have

$$p_n = 8^n,\ q_n = 6^n,\ f(n, v_1, v_2, v_3) = \alpha v_2\left[\psi(n)\left(v_1^2 + v_2^2\right)^{\frac{\alpha}{2}-1} + \psi(n-1)\left(v_2^2 + v_3^2\right)^{\frac{\alpha}{2}-1}\right]$$

and

$$F(n, v_1, v_2) = \psi(n) \left(v_1^2 + v_2^2\right)^{\frac{\alpha}{2}}.$$

It is easy to verify all the assumptions of Theorem 1.3 are satisfied and then the BVP (4.2) with (1.2) possesses at least one solution.

**Example 4.3.** For $n \in \mathbf{Z}(1, k)$, assume that

$$-\Delta^4 u_{n-2} + \Delta\left(7^n \Delta u_{n-1}\right) = \frac{8}{5} u_n \left[\left(u_{n+1}^2 + u_n^2\right)^{-\frac{1}{5}} + \left(u_n^2 + u_{n-1}^2\right)^{-\frac{1}{5}}\right], \tag{4.3}$$

with boundary value conditions (1.2).

We have

$$p_n \equiv -1,\ q_n = -7^n,\ f(n, v_1, v_2, v_3) = \frac{8}{5} v_2 \left[\left(v_1^2 + v_2^2\right)^{-\frac{1}{5}} + \left(v_2^2 + v_3^2\right)^{-\frac{1}{5}}\right]$$

and

$$F(n, v_1, v_2) = \left(v_1^2 + v_2^2\right)^{\frac{4}{5}}.$$

It is easy to verify all the assumptions of Theorem 1.4 are satisfied and then the BVP (4.3) with (1.2) has no nontrivial solutions.

## References


[1] C.D. Ahlbrandt. *Dominant and recessive solutions of symmetric three term recurrences.* J. Differential Equations, **107**(2) (1994) 238-258.

[2] V. Anuradha, C. Maya and R. Shivaji. *Positive solutions for a class of nonlinear boundary value problems with Neumann-Robin boundary conditions.* J. Math. Anal. Appl., **236**(1) (1999) 94-124.

[3] D. Arcoya. *Positive solutions for semilinear Dirichlet problems in an annulus.* J. Differential Equations, **94**(2) (1991) 217-227.

[4] X.C. Cai, J.S.Yu and Z.M. Guo. *Existence of periodic solutions for fourth-order difference equations.* Comput. Math. Appl., **50**(1-2) (2005) 49-55.

[5] M. Cecchi, M. Marini and G. Villari. *On the monotonicity property for a certain class of second order differential equations.* J. Differential Equations, **82**(1) (1989) 15-27.

[6] S.Z. Chen. *Disconjugacy, disfocality, and oscillation of second order difference equations.* J. Differential Equations, **107**(2) (1994) 383-394.

[7] P. Chen and H. Fang. *Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations.* Adv. Difference Equ., **2007** (2007) 1-9.

[8] P. Chen and X.H. Tang. *Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation.* Appl. Math. Comput., **217**(9) (2011) 4408-4415.

[9] P. Chen and X.H. Tang. *New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects.* Math. Comput. Modelling, **55**(3-4) (2012) 723-739.

[10] H. Fang and D.P. Zhao. *Existence of nontrivial homoclinic orbits for fourth-order difference equations.* Appl. Math. Comput., **214**(1) (2009) 163-170.

[11] C.J. Guo, D. O'Regan, Y.T. Xu and R.P. Agarwal. *Existence and multiplicity of homoclinic orbits of a second-order differential difference equation via variational methods.* Appl. Math. Inform. Mech., **4**(1) (2012) 1-15.

[12] Z.M. Guo and J.S. Yu. *Applications of critical point theory to difference equations.* Fields Inst. Commun., **42** (2004) 187-200.

[13] Z.M. Guo and J.S. Yu. *Existence of periodic and subharmonic solutions for second-order superlinear difference equations.* Sci. China Math, **46**(4) (2003) 506-515.

[14] Z.M. Guo and J.S. Yu. *The existence of periodic and subharmonic solutions of subquadratic second order difference equations.* J. London Math. Soc., **68**(2) (2003) 419-430.

[15] J.K. Hale and J. Mawhin. *Coincidence degree and periodic solutions of neutral equations.* J. Differential Equations, **15**(2) (1974) 295-307.

[16] J. Henderson and H.B. Thompson. *Existence of multiple solutions for second-order discrete boundary value problems.* Comput. Math. Appl., **43**(10-11) (2002) 1239-1248.

[17] V.L. Kocic and G. Ladas. *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications.* Kluwer Academic Publishers, Dordrecht (1993).

[18] Y.J. Liu and W.G. Ge. *Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator.* J. Math. Anal. Appl., **278**(2) (2003) 551-561.

[19] H. Matsunaga, T. Hara and S. Sakata. *Global attractivity for a nonlinear difference equation with variable delay.* Computers Math. Appl., **41**(5-6) (2001) 543-551.

[20] J. Mawhin and M. Willem. *Critical Point Theory and Hamiltonian Systems.* Springer, New York (1989).

[21] R.E. Mickens. *Difference Equations: Theory and Application.* Van Nostrand Reinhold, New York (1990).

[22] A. Pankov and N. Zakhrchenko. *On some discrete variational problems.* Acta Appl. Math., **65**(1-3) (2001) 295-303.

[23] A. Peterson and J. Ridenhour. *The (2,2)-disconjugacy of a fourth order difference equation.* J. Difference Equ. Appl., **1**(1) (1995) 87-93.

[24] J. Popenda and E. Schmeidel. *On the solutions of fourth order difference equations.* Rocky Mountain J. Math., **25**(4) (1995) 1485-1499.

[25] P.H. Rabinowitz. *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* Amer. Math. Soc., Providence, RI, New York (1986).

[26] Y. Rodrigues. *On nonlinear discrete boundary value problems.* J. Math. Anal. Appl., **114**(2) (1986) 398-408.

[27] A.N. Sharkovsky, Y.L. Maistrenko and E.Y. Romanenko. *Difference Equations and Their Applications.* Kluwer Academic Publishers, Dordrecht (1993).

[28] H.P. Shi, W.P. Ling, Y.H. Long and H.Q. Zhang. *Periodic and subharmonic solutions for second order nonlinear functional difference equations.* Commun. Math. Anal., **5**(2) (2008) 50-59.

[29] E. Thandapani and I.M. Arockiasamy. *Fourth-order nonlinear oscillations of difference equations.* Comput. Math. Appl., **42**(3-5) (2001) 357-368.

[30] H.Y. Wang. *On the existence of positive solutions for semilinear elliptic equations in the annulus.* J. Differential Equations, **109**(1) (1994) 1-7.

[31] J. Yan and B. Liu. *Oscillatory and asymptotic behavior of fourth order nonlinear difference equations.* Acta. Math. Sinica, **13**(1) (1997) 105-115.

[32] J.S. Yu and Z.M. Guo. *Boundary value problems of discrete generalized Emden-Fowler equation.* Sci. China Math, **49**(10) (2006) 1303-1314.

[33] J.S. Yu and Z.M. Guo. *On boundary value problems for a discrete generalized Emden-Fowler equation.* J. Differential Equations, **231**(1) (2006) 18-31.

[34] Z. Zhou, J.S. Yu and Y.M. Chen. *Periodic solutions of a 2nth-order nonlinear difference equation.* Sci. China Math, **53**(1) (2010) 41-50.

[35] W.M. Zou and M. Schechter. *Critical Point Theory and Its Applications.* Springer, New York (2006).

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 867 – 874

# Boundary value problem for a Sturm-Liouville operator with piecewise continuous coefficient

K. R. Mamedov * and F. A. Cetinkaya †

**Abstract**

In this paper, a self adjoint boundary value problem with a piecewise continuous coefficient on the positive half line $[0, \infty)$ is considered. The resolvent operator is constructed and the expansion formula with respect to eigenfunctions or equivalently Parseval equality is obtained. The spectrum of the operator is discussed.

*2000 AMS Classification:* 34L10, 34L40.

**Keywords:** Sturm-Liouville operator, eigenvalue, eigenfunction, resolvent operator, expansion formula.

Received 20/06/2013 : Accepted 08/04/2014 Doi : 10.15672/HJMS.2015449435

## 1. Introduction

Here, we consider the boundary value problem on the half line $0 < x < \infty$ generated by the differential equation

$$-y'' + q(x)y = \lambda^2 \rho(x)y \tag{1.1}$$

and the boundary condition

$$y'(0) - hy(0) = 0, \tag{1.2}$$

where $\lambda$ is a spectral parameter, $q(x)$ is a real valued function satisfying the condition

$$\int_0^\infty (1 + x)\, |q(x)|\, dx < \infty \tag{1.3}$$

*Mersin University, Faculty of Science and Letters, Department of Mathematics, 33343 Mersin, Turkey.
Email: hanlar@mersin.edu.tr
†Mersin University, Faculty of Science and Letters, Department of Mathematics, 33343 Mersin, Turkey.
Email:faycacetinkaya@mersin.edu.tr

and

$$\rho(x) = \begin{cases} \alpha^2, & 0 \le x < a, \\ 1, & x \ge a, \end{cases}$$

where $0 < \alpha \neq 1$. It is not hard to verify that the function

$$f_0(x,\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^+(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^-(x)}$$

is the solution of equation (1.1) when $q(x) \equiv 0$, where

$$\mu^{\pm}(x) = \pm x\sqrt{\rho(x)} + a(1 \mp \sqrt{\rho(x)}).$$

As it is known from [5, 8] that for $\lambda$ from the closed upper half plane equation (1.1) has a unique solution $f(x,\lambda)$ which can be represented in the form

$$(1.4) \qquad f(x,\lambda) = f_0(x,\lambda) + \int_{\mu^+(x)}^{\infty} K(x,t)e^{i\lambda t}dt,$$

where $K(x,\cdot) \in L_1(\mu^+(x), +\infty)$. The function $f(x,\lambda)$ is called the *Jost solution* of equation (1.1).

Note that, a singular Sturm-Liouville problem in the form of (1.1), (1.2) is encountered when applying separation of variables to mathematical physics problems in nonhomogeneous media, e. g. when $q(x) \equiv 0$ an application of electric prospecting problem, was given in [13, 15]. In this works, expansion formula was obtained by using Titchmarsh's [14] method with the help of integral representation (1.4), for the solution of equation (1.1). When $\rho(x) \equiv 1$ spectral expansion formula, for singular differantial operators on the interval $[0, \infty)$ was investigated with different methods in [14, 10], etc. When $\rho(x) \neq 1$, spectral properties of similar problems were considered in [4, 3, 5, 7, 8, 9]. Also, in this case the direct and inverse problem in a finite interval were examined in [1, 11].

Using (1.4) we have for real $\lambda \neq 0$ that the functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ form the fundamental system of solutions of equation (1.1) and the Wronskian of this system is equal to $2i\lambda$:

$$W\left\{f(x,\lambda), \overline{f(x,\lambda)}\right\} = f'(x,\lambda)\overline{f(x,\lambda)} - f(x,\lambda)\overline{f'(x,\lambda)} = 2i\lambda.$$

By $\omega(x,\lambda)$, we denote the solutions of equation (1.1) satisfying the initial data

$$\omega(0,\lambda) = 1, \ \omega'(0,\lambda) = h.$$

Proof of the following propositions can be done analoguously to [8].

**1.1. Proposition.** *For real $\lambda \neq 0$ the following identity*

$$(1.5) \qquad 2i\lambda\frac{\omega(x,\lambda)}{f'(0,\lambda) - hf(0,\lambda)} = \overline{f(x,\lambda)} - S(\lambda)f(x,\lambda)$$

*holds, here*

$$S(\lambda) = \frac{\overline{f'(0,\lambda) - hf(0,\lambda)}}{f'(0,\lambda) - hf(0,\lambda)} \qquad \text{and} \qquad |S(\lambda)| = 1.$$

$S(\lambda)$ is called the *scattering function* of the boundary value problem (1.1), (1.2).

**1.2. Proposition.** *The function $\varphi(\lambda) \equiv f'(0,\lambda) - hf(0,\lambda) \neq 0$ may have only a finite number of zeros $\lambda_k$, $(k = 1, 2, ..., n)$ in the half plane $Im\lambda > 0$. These zeros are all simple and lie on the imaginary axis.* For $\lambda = i\lambda_j$ $(\lambda_j > 0)$, $j = \overline{1,n}$, we get

$$m_j^{-2} \equiv \int_0^{\infty} \rho(x)\left|f(x,i\lambda_j)\right|^2 dx = -\frac{1}{2i\lambda_j}\dot{\varphi}(i\lambda_j)f(0,i\lambda_j).$$

These values are called the *norming constants* of the boundary value problem (1.1), (1.2).

## 2. Spectrum

This section is devoted to examine the properties of the eigenvalues of the boundary value problem (1.1), (1.2).

**2.1. Theorem.** *The operator $L$ has no eigenvalues on the positive half line.*

*Proof.* Let $\lambda_0^2 > 0$ be an eigenvalue of the operator $L$ and $y_0(x) = y(x, \lambda_0)$ be the corresponding eigenfunction. Since $f(x, \lambda_0)$ and $\overline{f(x, \lambda_0)}$ form the fundamental system of solutions, the general solution of (1.1) can be written in the form

$$y_0(x) = c_1 f(x, \lambda_0) + c_2 \overline{f(x, \lambda_0)}.$$

As $x \to \infty$,

$$f(x, \lambda_0) \to e^{i\lambda_0 x} \quad \textit{and} \quad \overline{f(x, \lambda_0)} \to e^{-i\lambda_0 x},$$

hence

$$y_0(x) = c_1 e^{i\lambda_0 x} + c_2 e^{-i\lambda_0 x} + o(1).$$

Since, its principal part is periodic this function does not belong to $L_2(0, \infty)$ for any values of $c_1$ and $c_2$. □

**2.2. Theorem.** *For $-\lambda_0^2$ $(\lambda_0 \neq 0)$ to be an eigenvalue it is necessary and sufficient that* $\varphi(\lambda_0) = 0$.

*Proof.* Indeed, let $\varphi(\lambda_0) = 0$ $(Im\lambda_0 > 0)$. Thus, $f'(0, \lambda_0) - hf(0, \lambda_0) = 0$. Therefore, $f(x, \lambda_0)$ is a solution of the boundary value problem (1.1), (1.2). While $x \to \infty$ $f(x, \lambda_0)$ decreases exponentially. Hence, $f(x, \lambda_0) \in L_2(0, \infty)$ and for the corresponding eigenvalue $-\lambda_0^2$ $f(x, \lambda_0)$ is the eigenfunction of operator $L$. On the other hand, let $-\lambda_0^2$ $(\lambda_0 \neq 0)$ be an eigenvalue and $y(x, \lambda_0)$ be the suitable eigenfunction of operator $L$. Then $y'(0, \lambda_0) - hy(0, \lambda_0) = 0$. It is clear that, $y(0, \lambda_0) \neq 0$. Without loss of generality assume that $y(0, \lambda_0) = 1$, then $y'(0, \lambda_0) = h$. Since, $f(x, \lambda_0)$ and $\hat{f}(x, \lambda_0)$ form the fundamental system of solutions of equation (1.1) (see [12] p. 297), we can write

$$y(x, \lambda_0) = c_1 f(x, \lambda_0) + c_2 \hat{f}(x, \lambda_0).$$

As $x \to \infty$, we obtain $c_2 = 0$, then $c_1 \neq 0$. Substituting $x = 0$ in the last relation, we get

$$y'(0, \lambda_0) - hy(0, \lambda_0) = c_1$$

i.e.,

$$f'(0, \lambda_0) - hf(0, \lambda_0) = \varphi(\lambda_0) = 0.$$

Thus, for each eigenvalue $-\lambda_0^2$, there is one and only one adequate (up to a multiplicative constant) eigenfunction:

$$y(x, \lambda_0) = cf(x, \lambda_0), \ (c \neq 0).$$

□

The proof of the following theorem can be obtained directly form Theorem 2.1 and Theorem 2.2. **2.3. Theorem.** *The operator $L$ has a finite number of eigenvalues:* $-\lambda_1^2, -\lambda_2^2, ...,$
$-\lambda_n^2$.

Therefore, it is appropriate at this point to note that the spectral problem (1.1), (1.2) has a finite number of negative eigenvalues and it fills positive half line with its continuous spectrum.

## 3. The Resolvent Operator and Expansion Formula for the Eigenfunctions

In the space $L_{2,\rho}(0,\infty)$, we define an inner product by

$$< f, g >:= \int_0^\infty f(x)\overline{g(x)}\rho(x)dx,$$

where $f(x), g(x) \in L_{2,\rho}(0,\infty)$.

Let us define

$$D(L) = \left\{ \begin{array}{c} f(x) \in L_{2,\rho}(0,\infty) : f(x), f'(x) \in AC\,[0,\infty)\,, l(f) \in L_{2,\rho}(0,\infty), \\ f'(0) - hf(0) = 0 \end{array} \right\},$$

as $L : f \to l(f)$ where

$$l(f) = \frac{1}{\rho(x)}\left\{-f''(x) + q(x)f(x)\right\}.$$

The boundary value problem (1.1), (1.2) is equivalent to the equation $Ly = \lambda^2 y$ and the operator $L$ is self-adjoint in the space $L_{2,\rho}(0,\infty)$.

Let us assume that $\lambda^2$ is not a spectrum point of operator $R_{\lambda^2}(L) = (L - \lambda^2 I)^{-1}$ and find the expression of the operator $R_{\lambda^2}(L)$ as all numbers $\lambda^2$ $(Im\lambda \geq 0, \varphi(\lambda) \neq 0)$ belong to the resolvent set of the operator $L$.

**3.1. Theorem.** *The resolvent $R_{\lambda^2}(L)$ is the integral operator*

$$R_{\lambda^2}(L) = \int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt$$

*with the kernel,*

$$G(x,t;\lambda) = -\frac{1}{\varphi(\lambda)}\left\{ \begin{array}{ll} \omega(x,\lambda)\, f(t,\lambda), & t \geq x, \\ f(x,\lambda)\,\omega(t,\lambda), & t \leq x. \end{array} \right. \tag{3.1}$$

*Proof.* Let $g(x) \in D(L)$ and assume that it is a finite function at infinity. To construct the resolvent operator of $L$ we need to solve the boundary value problem

$$-y'' + q(x)y = \lambda^2\rho(x)y + g(x)\rho(x), \tag{3.2}$$

$$y'(0) - hy(0) = 0. \tag{3.3}$$

We know that the functions $w(x,\lambda)$ and $f(x,\lambda)$ are the solutions of homogeneous problem for $Im\lambda > 0$. Now let us find the solutions of the problem (3.2), (3.3) which has the form

$$y(x,\lambda) = c_1(x,\lambda)w(x,\lambda) + c_2(x,\lambda)f(x,\lambda). \tag{3.4}$$

By applying the method of variation of constants, we get the system of equations

$$\left\{ \begin{array}{ll} c_1'(x,\lambda)\,w(x,\lambda) + c_2'(x,\lambda)\,f(x,\lambda) & = 0, \\ c_1'(x,\lambda)\,w'(x,\lambda) + c_2'(x,\lambda)\,f'(x,\lambda) & = -\rho(x)\,g(x). \end{array} \right. \tag{3.5}$$

Since $y(x,\lambda) \in L_{2,\rho}(0,\infty)$, then $c_1(0,\infty) = 0$. By using this relation and the system equations (3.5), we obtain

$$c_1(x,\lambda) = -\frac{1}{\varphi(\lambda)}\int_x^\infty f(t,\lambda)g(t)\rho(t)dt,$$

$$(3.6) \qquad c_2(x,\lambda) = c_2(0,\lambda) - \frac{1}{\varphi(\lambda)} \int_0^x w(t,\lambda) g(t) \rho(t) dt.$$

Substituting (3.6) into (3.4) and taking (3.3) into consideration, the proof of Theorem 3.1 is completed. □

**3.2. Lemma.** *Let $g(x)$ be a twice continuously differential function vanishing outside of some finite interval and $g(x) \in D(L)$. Then, as $|\lambda| \to \infty$, $Im\lambda > 0$ the following holds:*

$$(3.7) \qquad \int_0^\infty G(x,t;\lambda) g(t) \rho(t)\, dt = -\frac{g(x)}{\lambda^2} + \frac{Z(x,\lambda)}{\lambda^2},$$

*where*

$$Z(x,\lambda) = \int_0^\infty G(x,t,\lambda) \tilde{g}(t) \rho(t) dt$$

*as* $\tilde{g}(t) = -g''(t) + q(t) g(t)$.

*Proof.* The proof can be easily seen by using Theorem 3.1 and integrating by parts. □

Bounded solutions of boundary value problem (1.1), (1.2) are given in the following way:

$$u(x,\lambda) = \sqrt{\frac{1}{2\pi}} \left[ \overline{f(x,\lambda)} - S(\lambda) f(x,\lambda) \right], \quad 0 < \lambda^2 < \infty,$$

$$u(x, i\lambda_j) = m_j f(x, i\lambda_j), \qquad j = 1, 2, ..., n.$$

By using the contour integration, it can be shown that they form a complete system.

**3.3. Theorem.** *The expansion formula which is equivalent to Parseval equality*

$$(3.8) \qquad \delta(x-t) = \sum_{j=1}^n u(x, i\lambda_j) u(t, i\lambda_j) \rho(t) + \int_0^\infty u(x,\lambda) \overline{u(t,\lambda)} \rho(t) d\lambda$$

*holds, where $\delta(x)$ is Dirac delta function, also when $x \to \infty$ the following asymptotic formulae are true:*

$$(3.9) \qquad \begin{aligned} u(x,\lambda) &= e^{-i\lambda x} - S(\lambda) e^{i\lambda x} + o(1), \qquad (0 < \lambda^2 < \infty) \\ u(x, i\lambda_j) &= m_j e^{-\lambda_j x} [1 + o(1)], \qquad (j = 1, ..., n). \end{aligned}$$

*Proof.* Let $\Gamma_R$ denote the circle of radius $R$ and center zero which boundary contour is positive oriented. Assume $D = \{z : |z| \le R, |Imz| \ge \epsilon\}$, denote the positive oriented boundary contour of $D$ as $\Gamma_{R,\epsilon}$ and take integration along this contour. By multiplying both sides of (3.7) by $\frac{1}{2\pi i}\lambda$ and integrating it with respect to $\lambda$, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_{R,\epsilon}} \lambda d\lambda \int_0^\infty G(x,t;\lambda) g(t) \rho(t) dt = -\frac{1}{2\pi i} \int_{\Gamma_{R,\epsilon}} \frac{g(x)}{\lambda} d\lambda + Z_{R,\epsilon}(x),$$

where

$$Z_{R,\epsilon}(x) = \frac{1}{2\pi i} \int_{\Gamma_{R,\epsilon}} \frac{Z(x,\lambda)}{\lambda} d\lambda.$$

It can be shown from the properties of the functions $w(x,\lambda)$, $f(x,\lambda)$ that, as $R \to \infty$ and $\epsilon \to 0$, $Z_{R,\epsilon} \to 0$ holds for $\forall x \in [0,T] \subset [0,\infty)$ uniformly. From the last relation, as $R \to \infty$, $\epsilon \to 0$ we can write

$$\frac{1}{2\pi i} \int_{\Gamma_{R,\epsilon}} \lambda d\lambda \int_0^\infty G(x,t;\lambda) g(t) \rho(t) dt \to -g(x) +$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^\infty \lambda d\lambda \int_0^\infty [G(x,t;\lambda + i0) - G(x,t;\lambda - i0)] g(t) \rho(t) dt.$$

On the other hand, using the residue calculus, we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}}\lambda d\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt &= \sum_{j=1}^{n}\underset{\lambda=i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right]+\\ &+\sum_{j=1}^{n}\underset{\lambda=-i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right].\end{aligned}$$

From the last two relations we obtain

$$\begin{aligned}g(x)= &-\sum_{j=1}^{n}\underset{\lambda=i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right]-\\ &-\sum_{j=1}^{n}\underset{\lambda=-i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right]+\\ &+\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\lambda\int_0^\infty[G(x,t;\lambda+i0)-G(x,t;\lambda-i0)]g(t)\rho(t)dt.\end{aligned}$$

Let $\psi(x,\lambda)$ be the solution of (1.1) satisfying the initial conditions

$$\psi(0,\lambda)=0,\qquad \psi'(0,\lambda)=1$$

and $W\{\omega(x,\lambda),f(x,\lambda)\}=1$. From here, we can write

$$f(x,\lambda)=f(0,\lambda)\omega(x,\lambda)-\varphi(\lambda)\psi(x,\lambda).$$

Therefore, from (3.1) we have

$$G(x,t;\lambda)=-\frac{f\left(0,\lambda\right)}{\varphi\left(\lambda\right)}\omega(x,\lambda)\omega(t,\lambda)-\begin{cases}\omega\left(x,\lambda\right)\psi(t,\lambda), & x\le t,\\ \psi(x,\lambda)\omega(t,\lambda), & t\le x.\end{cases}$$

Accordingly for $Im\lambda\ge 0$, we obtain

$$\begin{aligned}\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt= &-\frac{1}{\varphi\left(\lambda\right)}f\left(0,\lambda\right)\omega\left(x,\lambda\right)\int_0^\infty\omega\left(t,\lambda\right)g(t)\rho(t)dt-\\ &-\psi(x,\lambda)\int_0^x\omega\left(t,\lambda\right)g(t)\rho(t)dt-\\ &-\omega(x,\lambda)\int_x^\infty\psi\left(t,\lambda\right)g(t)\rho(t)dt.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\underset{\lambda=i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right]+\underset{\lambda=-i\lambda_j}{Res}\left[\lambda\int_0^\infty\overline{G(x,t;\lambda)}g(t)\rho(t)dt\right]=\\ &=-\frac{2i\lambda_j}{\dot{\varphi}(i\lambda_j)}f(0,i\lambda_j)\omega(x,i\lambda_j)\int_0^\infty\omega(t,i\lambda_j)g(t)\rho(t)dt=\\ &=u(x,i\lambda_j)\int_0^\infty u(t,i\lambda_j)g(t)\rho(t)dt.\end{aligned}$$

We can write

$$\begin{aligned}G(x,t;\lambda+i0)-G(x,t;\lambda-i0) &= \left[-\frac{f(0,\lambda+i0)}{\varphi(\lambda+i0)}+\frac{f(0,\lambda-i0)}{\varphi(\lambda-i0)}\right]\omega(x,\lambda)\omega(t,\lambda)=\\ &=\frac{\varphi(\lambda)\overline{f(0,\lambda)}-\overline{\varphi(\lambda)}f(0,\lambda)}{\left|\varphi(\lambda)\right|^2}\omega(x,\lambda)\omega(t,\lambda)=\\ &=\frac{2i\lambda}{\left|\varphi(\lambda)\right|^2}\omega(x,\lambda)\omega(t,\lambda).\end{aligned}$$

It follows that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda d\lambda\int_0^{\infty}[G(x,t;\lambda+i0)-G(x,t;\lambda-i0)]g(t)\rho(t)dt=$$

$$=\frac{2}{\pi}\int_0^{\infty}\frac{\lambda^2}{|\varphi(\lambda)|^2}\omega(x,\lambda)\int_0^{\infty}\omega(t,\lambda)g(t)\rho(t)dtd\lambda=$$

$$=\int_0^{\infty}u(x,\lambda)\int_0^{\infty}u(t,\lambda)g(t)\rho(t)dtd\lambda.$$

Therefore, from (3.10) we get the expansion formula for the eigenfunctions:

$$g(x)=\sum_{j=1}^{n}u(x,i\lambda_j)\int_0^{\infty}u(t,i\lambda_j)g(t)\rho(t)dt+ \tag{3.11}$$

$$+\int_0^{\infty}u(x,\lambda)\int_0^{\infty}\overline{u(t,\lambda)}g(t)\rho(t)dtd\lambda$$

or we obtain (3.8) that is equivalent to the Parseval equality. Asymptotic expressions (3.9) can be obtained from (1.5) when $x\to\infty$. □

Writing the expansion formula (3.11) in the form of Stieltjes integral we have

$$g(x)=\int_{-\infty}^{\infty}\omega(x,\lambda)\left(\int_0^{\infty}\omega(t,\lambda)g(t)\rho(t)dt\right)d\sigma(\lambda),$$

where

$$d\sigma(\lambda)=\begin{cases}\frac{2}{\pi}\frac{\lambda^2d\lambda}{|\varphi(\lambda)|^2}, & \lambda\geq 0,\\ \sum_{j=1}^{n}\frac{(2i\lambda_j)^2\delta(\lambda-i\lambda_j)}{m_j^2\dot{\varphi}(i\lambda_j)^2}, & \lambda<0\end{cases}$$

is the *spectral function* of operator $L$.

Now taking

$$G(\lambda)=\int_0^{\infty}\omega(x,\lambda)g(x)\rho(x)dx,$$

we get

$$g(x)=\int_{-\infty}^{\infty}G(\lambda)\omega(x,\lambda)d\sigma(\lambda).$$

Multiplying both sides of this equivalence by $g(x)$, we obtain the Parseval equality

$$\int_0^{\infty}g^2(x)dx=\int_{-\infty}^{\infty}G^2(\lambda)d\sigma(\lambda).$$

**Acknowledgment.** This work is supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK).

## References

[1] Akhmedova, E. N. and Huseynov, H. M. *On inverse problem for Sturm-Liouville operator with discontinuous coefficients*, Transactions of Saratov University, (Izv. Sarat. Univ.). **10** (1), 3-9, 2010.

[2] Chadan, K. and Sabatier, P. C. *Inverse problems in quantum scattering theory* (Springer-Verlag, New York, Heidelberg Berlin, 1977).

[3] Darwish, A. A. *The inverse scattering problem for a singular boundary value problem*, New Zea. Jou. Math. **22**, 37-56, 1993.

[4] Gasymov, M G. *The direct and inverse problem of spectral analysis for a class of equations with a discontinuous coefficient*, Non-Classical Methods in Geophysics, Nauka, Novosibirsk, USSR., 37-44, 1977.

[5] Guseinov, I. M. and Pashaev, R. T. *On an inverse problem for a second order differential operator*, In: Usp. Math. Nauk. **57**(3), 597-598, 2002.

[6] Levitan, B. M. and Sargsjan, I. S. *Introduction to spectral theory* (American Mathematical Society, 1975).

[7] Mamedov, Kh. R. *Uniqueness of the solution of the inverse problem of scattering theory for Sturm-Liouville operator with discontinuous coefficient*, Proceedings of IMM of NAS of Azerbaijan, 163-172, 2006.

[8] Mamedov, Kh. R. *On an inverse scattering problem for a discontinuous Sturm-Liouville equation with a spectral parameter in the boundary condition*, Boundary Value Problems, pp. 17, 2010.

[9] Mamedov Kh. R. and Kosar N. P. *Inverse scattering problem for Sturm–Liouville operator with nonlinear dependence on the spectral parameter in the boundary condition*, Mathematical Methods in the Applied Sciences. **34** (2), 231-241, 2011.

[10] Marchenko, V. A. *Sturm-Liouville operators and applications* (AMS Chelsea Publishing, 2011).

[11] Nabiev, A. A. and Amirov, Kh. R. *On the boundary value problem for the Sturm–Liouville equation with the discontinuous coefficient*, Mathematical Methods in the Applied Sciences DOI: 10.1002/mma.2714.

[12] Naimark, M. A. *Linear differential operators, Part II* (Frederick Ungar Publishing, 1967).

[13] Tikhonov, A. N. *On the uniqueness of the solution of the problem of electric prospecting*, Dok. Aka. Nauk SSSR., **69**, 787-80, 1949.

[14] Titchmarsh, E. C. *Eigenfunctions expansions* (Oxford, 1962).

[15] Tikhonov, A. N. and Samarskii, A. A. *Equations of mathematical physics* (Dover Books on Physics, 2011).

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 875 – 886

# A new characterization of $\mathbf{L_2(2^m)}$

Hosein Parvizi Mosaed [*] , Ali Iranmanesh [†] , Mahnaz Foroudi Ghasemabadi [‡] and Abolfazl Tehranian [§]

**Abstract**

Let $G$ be a group and $\pi(G)$ be the set of primes $p$ such that $G$ contains an element of order $p$. Let $nse(G)$ be the set of numbers of elements of $G$ of the same order. In this paper, we prove that the simple group $L_2(2^m)$ is uniquely determined by $nse(L_2(2^m))$, where $|\pi(L_2(2^m))| = 4$.

*2000 AMS Classification:* Primary: 20D60; Secondary: 20D06.

**Keywords:** Element order, set of the numbers of elements of the same order, simple $K_n$-group, projective special linear group.

Received 05/02/2014 : Accepted 28/04/2014 Doi : 10.15672/HJMS.2015449651

## 1. Introduction

Let $G$ be a group. By $\pi(G)$, we denote the set of primes $p$ such that $G$ contains an element of order $p$ and by $\pi_e(G)$ we mean the set of element orders of $G$. If $k \in \pi_e(G)$, then $m_k$ denotes the number of elements of order $k$ in $G$ and we define the set $nse(G) = \{m_k \mid k \in \pi_e(G)\}$.

During the classification of the finite simple groups, it has been observed that some of the known simple groups are characterizable by some of their properties and up to now, different characterizations are investigated for the finite simple groups. For instance, in

*Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
Email: h.parvizi.mosaed@gmail.com

†Department of pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P. O. Box: 14115-137, Tehran, Iran.
Email:iranmana@modares.ac.ir

‡Department of pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran, Iran.
Email:foroudi@modares.ac.ir

§Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
Email:tehranian@srbiau.ac.ir

[16], motivated by one of the Thompson's problem, the authors introduced a new characterization for the finite simple group $G$, by $nse(G)$ and $|G|$. In fact, they proved that if $G$ is a finite simple $K_4$-group, then $G$ is characterizable by $nse(G)$ and $|G|$ (The simple group $G$ is called simple $K_n$-group if $|\pi(G)| = n$). Following this result, in [7] and [17], it is proved that the group $L_2(q)$, where $q \in \{3,4,5,7,8,9,11,13\}$ is determined only by $nse(G)$. Up to the present time, it has been investigated that some other simple groups can be characterized by $nse(G)$ and $|G|$ or only by $nse(G)$ (see for instance [9]-[12]). In this paper, our aim is to show that the simple $K_4$-group $L_2(2^m)$ is characterizable by $nse(L_2(2^m))$. In fact, we improve the results of [16] in the following main theorem:

**Main Theorem.** Let $G$ be a group. If $nse(G) = nse(L_2(2^m))$, where $m$, $2^m - 1$ and $(2^m+1)/3$ are primes greater than 3, then $G \cong L_2(2^m)$.

## 2. Notation and Preliminaries

For a natural number $n$, by $\pi(n)$, we mean the set of all prime divisors of $n$, so it is obvious that if $G$ is a finite group, then $\pi(G) = \pi(|G|)$. A Sylow $p$-subgroup of $G$ is denoted by $G_p$ and by $n_p(G)$, we mean the number of Sylow $p$-subgroups of $G$. Also, the largest element order of $G_p$ is denoted by $exp(G_p)$. Moreover, we denote by $\varphi$, the Euler totient function and by $(a,b)$ the greatest common divisor of integers $a$ and $b$.

In the following, we bring some useful lemmas which will be used in the proof of the main theorem.

**2.1. Lemma.** [2, 6, 15, 20] *Let $G$ be a finite simple $K_n$-group.*

(1) *If $n = 3$, then $G$ is isomorphic to one of the following groups:*
$A_5,\ A_6,\ L_2(7),\ L_2(8),\ L_2(17),\ L_3(3),\ U_3(3),\ U_4(2)$.

(2) *If $n = 4$, then $G$ is isomorphic to one of the following groups:*
   (a) $A_7,\ A_8,\ A_9,\ A_{10},\ M_{11},\ M_{12},\ J_2,\ L_2(16),\ L_2(25),\ L_2(49),$ $L_2(81),\ L_2(97),\ L_2(243),\ L_2(577),\ L_3(4),\ L_3(5),\ L_3(7),$ $L_3(8),\ L_3(17),\ L_4(3),\ S_4(4),\ S_4(5),\ S_4(7),\ S_4(9),\ S_6(2),$ $O_8^+(2),\ G_2(3),\ U_3(4),\ U_3(5),\ U_3(7),\ U_3(8),\ U_3(9),\ U_4(3),$ $U_5(2),\ Sz(8),\ Sz(32),\ {}^3D_4(2),\ {}^2F_4(2)'$;
   (b) $L_2(r)$, *where $r$ is a prime, $r^2 - 1 = 2^a.3^b.v$, $v > 3$ is a prime, $a, b \in \mathbb{N}$;*
   (c) $L_2(2^m)$, *where $m$, $2^m - 1$ and $(2^m+1)/3$ are primes greater than 3;*
   (d) $L_2(3^m)$, *where $m$, $(3^m - 1)/2$ and $(3^m+1)/4$ are odd primes.*

**2.2. Lemma.** [4] *Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

**2.3. Lemma.** [17] *Let $G$ be a group containing more than two elements. Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. If $s = \sup\{m_k \mid k \in \pi_e(G)\}$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.*

**2.4. Lemma.** [13] *Let $G$ be a finite group and $p \in \pi(G) \setminus \{2\}$. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^s m$, where $(p,m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.*

**2.5. Lemma.** [18, Theorem 3] *Let $G$ be a finite group. Then the number of elements whose orders are multiples of $n$ is either zero, or a multiple of the greatest divisor of order $G$ that is prime to $n$.*

**2.6. Lemma.** [14] *Let the finite group $G$ acts on the finite set $X$. If the action is semiregular, then $|G| \mid |X|$.*

**2.7. Lemma.** [5] *Let $G$ be a solvable group and $\pi$ be any set of primes. Then*

(1) *$G$ has a Hall $\pi$-subgroup.*
(2) *If $H$ is a Hall $\pi$-subgroup of $G$ and $V$ is any $\pi$-subgroup of $G$, then $V \leq H^g$ for some $g \in G$. In particular, the Hall $\pi$-subgroups of $G$ form a single conjugacy class of subgroups of $G$.*

**2.8. Lemma.** *Let $G$ be an unsolvable finite group. Then there is a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$, such that $N$ is a solvable normal subgroup of $G$ and $M/N$ is an unsolvable simple group or the direct product of isomorphic unsolvable simple groups.*

*Proof.* Since $G$ is a finite group, it has a chief series $1 = M_0 \trianglelefteq M_1 \trianglelefteq \ldots \trianglelefteq M_{n-1} \trianglelefteq M_n = G$. Also, since $G$ is unsolvable, there is a maximal $i < n$, such that $M_{i-1}$ is solvable. According to the maximality of $i$, we can easily conclude that the chief factor $\frac{M_i}{M_{i-1}}$ is unsolvable. Since each chief factor is a simple group or the direct product of isomorphic simple groups, it is enough to set $N := M_{i-1}$ and $M := M_i$. □

The following number theoretic lemmas play a role in the proof of the main theorem:

**2.9. Lemma.** [19] *Let $q, k, l$ be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$;
(2) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise;} \end{cases}$
(3) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise;} \end{cases}$

*In particular, for every $q \geq 2, k \geq 1$ the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.*

**2.10. Lemma.** *Let $m$ be a natural number. Then*

(1) *3 divides $2^m - 1$ if and only if $m$ is even.*
(2) *3 divides $2^m + 1$ if and only if $m$ is odd.*

*Proof.* On account of Lemma 2.9, the proof is straightforward. □

**2.11. Lemma.** [3, Remark 1] *The only solution of the equation $p^m - q^n = 1$, where $p, q$ are primes and $m, n > 1$, is $3^2 - 2^3 = 1$.*

**2.12. Lemma.** [1] *Let $p$ be a prime number.*

(1) *If $p \neq 3$, then $x^2 \equiv -3 \pmod p$ is solvable if and only if $p \equiv 1 \pmod 3$.*
(2) *The equation $x^2 \equiv -1 \pmod p$ is solvable if and only if $p \equiv 1 \pmod 4$.*

**2.13. Lemma.** [8] *Let $p \neq 3$ be a prime number.*

(1) *If the diophantine equation $3x^2 + 1 = tp^k$ has a solution, then $p \equiv 1 \pmod 3$.*
(2) *If the diophantine equation $x^{2n} + x^n + 1 = tp^k$ or $x^{2n} - x^n + 1 = tp^k$ is solvable, then $p \equiv 1 \pmod 3$.*

**2.14. Lemma.** *Let $m$ be a natural number such that*

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t \end{cases}$$

*with $m \geq 2$, $u$ and $t$ are primes, $t > 3$. Then the following hold:*

(a) $(u-1, t) = 1,\ (u-1, t-1) = t-1,\ (u-1, 2^m) = 2,\ (u+1, t) = 1$;
(b) $(t-1, u) = 1,\ (t-1, 2^m) = 2,\ (t+1, u) = 1$;
(c) $(u, t) = 1,\ (u, 3) = 1,\ (u, 2) = 1,\ (t, 3) = 1,\ (t, 2) = 1$;
(d) $\pi(t-1) \setminus \{2, 3, t, u\} \neq \emptyset$;

(e) $3 \mid (1+2^m u)$ *but* $9 \nmid (1+2^m u)$.

*Proof.* (a) Since $t$ is a prime, $(u-1,t)=1$ or $t$. If $(u-1,t)=t$, then $t \mid (u-1)$. Hence $(2^m+1) \mid 3(2^m-2)=3(2^m+1)-9$. Therefore $(2^m+1) \mid 9$ which implies that $m \in \{1,3\}$ but this contradicts $t>3$. So $(u-1,t)=1$. We have $(u-1,t-1)=(2^m-2,\frac{2^m-2}{3})=\frac{2^m-2}{3}=(t-1)$. Since $(2^{m-1}-1,2^{m-1})=1$, we conclude that $(2^m-2,2^m)=2$ and hence, $(u-1,2^m)=2$. Since $t$ is odd, $(2^m,t)=1$ which implies that $(u+1,t)=1$.
(b) Since $u$ is a prime, $(t-1,u)=1$ or $u$. If $(t-1,u)=u$, then $u \mid (t-1) \mid (u-1)$, which is a contradiction. So $(t-1,u)=1$. Since $(2^{m-1}-1,2^{m-1})=1$, we have $(2^m-2,2^m)=2$ and hence $(t-1,2^m)=2$. According to the hypothesis, $u$ is a prime number and hence, $(t+1,u)=1$ or $u$. If $(t+1,u)=u$, then $(2^m-1) \mid (2^{m-2}+1)$ because $u$ is odd. Thus $(2^m-1) \leq (2^{m-2}+1)$, which is a contradiction. So $(u,t+1)=1$.
(c) It is obvious.
(d) By (b), $(t-1,u)=1$. Thus $u \notin \pi(t-1)$. Also, it is obvious that $t \notin \pi(t-1)$. If $\pi(t-1)=\{2,3\}$, then $2^m-2=2.3^k$. Thus $2^{m-1}-1=3^k$. Therefore $2^{m-1}-3^k=1$, that by Lemma 2.11, is a contradiction. If $\pi(t-1)=\{2\}$, then $\frac{2^m-2}{3}=2$. Hence $2^{m-1}-1=3$. Therefore $m=3$, which is a contradiction. If $\pi(t-1)=\{3\}$, then $t-1$ is odd but we have $2 \mid (t-1)$, which is a contradiction. So there is a prime $p \in \pi(t-1)$ such that $p \neq 2,3,t,u$.
(e) Since $2^m+1=3t$, $3 \mid (2^m+1)$ and hence $3 \mid (2^{2m}-1)$. Thus $3 \mid (2^{2m}-1-2^m-1+3)=(2^{2m}-2^m+1)=(1+2^m u)$. Now, we are going to prove that $9 \nmid (1+2^m u)$. First we claim that $(m,3)=1$. If not, then $(m,3)=3$ and since $3 \mid (2^m+1)$, according to Lemma 2.10(2), we have $m$ is odd and hence, $m=3k$, where $k$ is an odd number. Thus $u=(2^m-1)=(2^{3k}-1)=(8^k-1)=(8-1)(8^{k-1}+8^{k-2}+\ldots+8+1)$ and since $u=2^m-1$ is a prime number, we conclude that $k=1$ and $m=3$, which contradicts $t>3$. Therefore $(m,3)=1$. If $9 \mid (1+2^m u)=(2^{2m}-2^m+1)$, then $27 \mid (2^m+1)(2^{2m}-2^m+1)=(2^{3m}+1)$. Thus $27 \mid (2^{3m}+1,2^{18}-1)$. Since $(m,3)=1$, we have $(18,3m)=3$ and hence Lemma 2.9 (3) implies that $(2^{3m}+1,2^{18}-1)=9$, which is a contradiction. $\square$

**2.15. Lemma.** *Assume that the hypotheses of Lemma 2.14 are fulfilled. Further let* $x=2^m$ *and let* $p$ *be a prime number such that* $p \notin \{2,3,t,u\}$ *and* $(p,u-1)=1$.

(1) *Let* $p \mid x^3-3x^2+2x+3$.
  (a) *If* $p \mid x+4$, *then* $p=13$;
  (b) *If* $p \mid x^2+x-4$, *then* $p=101$;
  (c) *If* $p \mid x^2+x+3$, *then* $p=23$;
  (d) *If* $p \mid x^2+4x+6$, *then* $p=43$;
  (e) *If* $p \mid x^2-2$, *then* $p=23$;
  (f) $p \nmid 2x+1$.
(2) *Let* $p \mid x^2-4x+6$.
  (a) *If* $p \mid 2x+1$, *then* $p=11$;
  (b) *If* $p \mid x+4$, *then* $p=19$;
  (c) *If* $p \mid x^2+x-4$, *then* $p=5$;
  (d) *If* $p \mid x^2+x+3$, *then* $p=11$;
  (e) $p \nmid x^2+4x+6$ *and* $p \nmid x^2-2$.
(3) *Let* $p \mid x^2-2$.
  (a) *If* $p \mid 2x+1$, *then* $p=7$;
  (b) *If* $p \mid x+4$, *then* $p=7$ *and* $p \mid 2x+1$;
  (c) $p \nmid x^2+x-4$.

*Proof.*
• Let $p \mid x^3-3x^2+2x+3$.
If $p \mid x+4$, then $p \mid (x^3-3x^2+2x+3)-(x^2-7x)(x+4)=3(10x+1)$ and since $(p,3)=1$,

we conclude that $p \mid 10x+1$. Therefore, $p \mid (10x+1)-10(x+4) = -3(13)$ which implies that $p = 13$. If $p \mid x^2+x-4$, then $p \mid (x^3-3x^2+2x+3)-(x-4)(x^2+x-4) = 10x-13$. Thus $p \mid -13(x^2+x-4)+4(10x-13) = -x(13x-27)$ and since $(p,x) = 1$, we conclude that $p \mid 13x-27$. Therefore, $p \mid 10(13x-27)-13(10x-13) = -101$ which implies that $p = 101$. If $p \mid x^2+x+3$, then $p \mid (x^3-3x^2+2x+3)-(x-4)(x^2+x+3) = 3(x+5)$. Thus $p \mid (x^2+x+3)-(x-4)(x+5) = 23$ and hence, $p = 23$. If $p \mid x^2+4x+6$, then $p \mid -2(x^3-3x^2+2x+3)+(x^2+4x+6) = x^2(-2x+7)$. Thus $p \mid -2x+7$. On the other hand, $p \mid (x^3-3x^2+2x+3)-(x-7)(x^2+4x+6) = 24x+45$. Therefore, $p \mid (24x+45)+12(-2x+7) = 3(43)$ which implies that $p = 43$. If $p \mid x^2-2$, then $p \mid (x^3-3x^2+2x+3)-(x-3)(x^2-2) = 4x-3$. On the other hand, $p \mid (x^2-2)+(4x-3) = (x-1)(x+5)$ and since $(p,x-1) = 1$, we conclude that $p \mid x+5$. Thus $p \mid -4(x+5)+(4x-3) = -23$ which implies that $p = 23$. If $p \mid 2x+1$, then $p \mid (x^3-3x^2+2x+3)-3(2x+1) = x(x+1)(x-4)$ and since $(p,x) = (p,x+1) = 1$, we conclude that $p \mid x-4$. Thus $p \mid (2x+1)-2(x-4) = 9$, which is a contradiction to the fact that $(p,3) = 1$.

• Let $p \mid x^2-4x+6$.

If $p \mid 2x+1$, then $p \mid -2(x^2-4x+6)+x(2x+1) = 3(3x-4)$ and since $(p,3) = 1$, we conclude that $p \mid 3x-4$. Therefore, $p \mid 3(2x+1)-2(3x-4) = 11$ which implies that $p = 11$. If $p \mid x+4$, then $p \mid (x^2-4x+6)-x(x+4) = -2(4x-3)$ and since $(p,2) = 1$, we conclude that $p \mid 4x-3$. Thus $p \mid (4x-3)-4(x+4) = -19$ and hence, $p = 19$. If $p \mid x^2+x-4$, then $p \mid 4(x^2-4x+6)+6(x^2+x-4) = 10x(x-1)$ and since $(p,x-1) = (p,2) = 1$, we conclude that $p = 5$. If $p \mid x^2+x+3$, then $p \mid -(x^2-4x+6)+(x^2+x+3) = (5x-3)$. Thus $p \mid (x^2+x+3)+(5x-3) = x(x+6)$ and since $(p,2) = 1$, we conclude that $p \mid x+6$. Therefore, $p \mid 5(x+6)-(5x-3) = 3(11)$ which implies that $p = 11$. If $p \mid x^2+4x+6$, then $p \mid -(x^2-4x+6)+(x^2+4x+6) = 8x$. Thus $p \mid 2$ which is a contradiction to the fact that $(2,p) = 1$. If $p \mid x^2-2$, then $p \mid (x^2-4x+6)-(x^2-2) = -4(x-2)$. Since $(p,2) = (p,x-2) = 1$, we get a contradiction.

• Let $p \mid x^2-2$.

If $p \mid 2x+1$, then $p \mid -2(x^2-2)+x(2x+1) = (x+4)$. Therefore, $p \mid (2x+1)-2(x+4) = -7$ which implies that $p = 7$. If $p \mid x+4$, then $p \mid -(x^2-2)+x(x+4) = 2(2x+1)$ and since $(p,2) = 1$, we conclude that $p \mid 2x+1$. Thus $p \mid (2x+1)-2(x+4) = -7$ and hence, $p = 7$. If $p \mid x^2+x-4$, then $p \mid -(x^2-2)+(x^2+x-4) = (x-2)$. Since $(p,x-2) = 1$, we get a contradiction. □

## 3. Proof of the Main Theorem

We know that $nse(G) = nse(L_2(2^m))$, where $m$ satisfies

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t \end{cases}$$

$m \geq 2$, $u$ and $t$ are primes, $t > 3$. Denote $x = 2^m$. According to [16], we know that $\pi(L_2(2^m)) = \{2, 3, t, u\}$ and

$$nse(L_2(2^m)) = \{1, 3tu, 2^m u, (t-1)2^m u, 1/2(t-1)2^m u, 1/2(u-1)2^m 3t\}.$$

We have divided the proof into a sequence of lemmas.

**3.1. Lemma.** *The group $G$ is finite. If $i \in \pi_e(G)$, then*

$$\begin{cases} \varphi(i) \mid m_i \\ i \mid \sum_{d|i} m_d \end{cases} \tag{3.1}$$

*and if $i > 2$, then $m_i$ is even.*

*Proof.* Since $nse(G) = nse(L_2(2^m))$, according to Lemma 2.3, $G$ is a finite group. Now, if $i \in \pi_e(G)$, then Lemma 2.2 implies that $i \mid \sum_{d|i} m_d$. We know that the number of elements of order $i$ in a cyclic group of order $i$ is equal to $\varphi(i)$. Thus $m_i = \varphi(i)k$, where $k$ is the number of cyclic subgroups of order $i$ in $G$ and hence, $\varphi(i) \mid m_i$. Also, it is known that if $i > 2$, then $\varphi(i)$ is even and since $\varphi(i) \mid m_i$, we conclude that $m_i$ is even as well. $\square$

**3.2. Lemma.** $|\pi(G)| \geq 2$.

*Proof.* Since $3tu \in nse(G)$, Lemma 3.1 yields $2 \in \pi(G)$ and $m_2 = 3tu$. Let $\pi(G) = \{2\}$. Then $|G| = 2^k$. If $exp(G_2) > 2^{m+2}$, then $2^{m+3} \in \pi_e(G)$ and hence $2^{m+2} = \varphi(2^{m+3}) \mid m_{2^{m+3}}$, which is a contradiction. Thus $exp(G_2) \leq 2^{m+2}$ and we have

$$|G| = 1 + 3tu + k_1 2^m u + k_2(t-1)2^m u + \tag{3.2}$$

$$k_3 1/2(t-1)2^m u + k_4 1/2(u-1)2^m 3t$$

where $k_1, k_2, k_3$ and $k_4$ are natural numbers and $k_1+k_2+k_3+k_4 \leq m+1$. Since $u = x-1$ and $t = (x+1)/3$, we can conclude that $|G|$ divides

$$(2k_2 + k_3 + 3k_4)x^3 + (6 + 6k_1 - 6k_2 - 3k_3 - 3k_4)x^2 + (-6k_1 + 4k_2 + 2k_3 - 6k_4)x.$$

Moreover, since $1 + m_2 = 2^{2m}$, we conclude that $2^{2m} < 2^k$ and hence $x^2 \mid |G|$. Thus $x^2$ divides

$$(2k_2 + k_3 + 3k_4)x^3 + (6 + 6k_1 - 6k_2 - 3k_3 - 3k_4)x^2 + (-6k_1 + 4k_2 + 2k_3 - 6k_4)x$$

which implies that $x \mid 6k_1 - 4k_2 - 2k_3 + 6k_4$. Since

$$6k_1 - 4k_2 - 2k_3 + 6k_4 < 6(k_1 + k_2 + k_3 + k_4) \leq 6(m+1),$$

we conclude that $2^m \leq (6m+6)$. Thus $m = 5$ which implies that $u = 31$ and $t = 11$. From (3.2) we have

$$2^k = 1 + 1023 + 992k_1 + 9920k_2 + 4960k_3 + 15840k_4,$$

where $k_1 + k_2 + k_3 + k_4 \leq 6$ and it is easy to check that this equation has no solution. $\square$

**3.3. Lemma.** $\pi(G) \neq \{2, 3\}$.

*Proof.* Let $\pi(G) = \{2, 3\}$. If $G_3$ is a cyclic group of order $3^k$, then $n_3(G) = \frac{m_{3^k}}{\varphi(3^k)} = \frac{m_{3^k}}{2(3^{k-1})}$ and hence, according to $nse(G)$ and Lemma 2.14(c), we can conclude that $t$ or $u$ divides $n_3(G)$. On the other hand, since $n_3(G)$ divides $|G|$, we can get a contradiction. Thus $G_3$ is not cyclic and according to Lemmas 2.2 and 2.4, we have $9 \mid 1 + m_3$. If $m_3 = 2^m u$, then since by Lemma 2.14, $9 \nmid 1 + 2^m u$, we can get a contradiction. Also, since $(3, m_3) = 1$, we conclude that $m_3 \neq 1/2(u-1)2^m 3t$. Thus $m_3 \in \{(t-1)2^m u, 1/2(t-1)2^m u\}$ which implies that

$$(3, t-1) = 1. \tag{3.3}$$

If $6 \notin \pi_e(G)$, then by Lemma 2.6, $|G_3| \mid m_2$. According to Lemma 3.2, $m_2 = 3tu$ and hence Lemma 2.14 implies that $G_3$ is cyclic, which is a contradiction. Thus $6 \in \pi_e(G)$. Since $6 \mid 1 + m_2 + m_3 + m_6$ and $3 \mid 1 + m_2 + m_3$, we conclude that $3 \mid m_6$. Now according to $nse(G)$ and (3.3), we have $m_6 = 1/2(u-1)2^m 3t$ and hence, $9 \mid m_6$.

Now we have the following two cases:

**Case 1.** Let $exp(G_3) = 3$. Then by Lemma 2.5, $9 \mid \sum_{i \geq 2} m_{2^i} + \sum_{i \geq 2} m_{2^i 3}$ and $9 \mid \sum_{i \geq 1} m_{2^i} + \sum_{i \geq 1} m_{2^i 3}$. Thus $9 \mid m_2 + m_6$ and since $9 \mid m_6$, we conclude that $9 \mid m_2$, which is a contradiction.

**Case 2.** Let $exp(G_3) > 3$. If $18 \notin \pi_e(G)$, then similar to Case 1, we can get a contradiction. If $18 \in \pi_e(G)$, then according to Lemma 2.4, $9 \mid m_{2^i 3^j}$, where $i \geq 0,\ j \geq 2$. Since $18 \in \pi_e(G)$, we have $18 \mid 1 + m_2 + m_3 + m_6 + m_9 + m_{18}$. On the other hand, $9 \mid m_6$ and according to Lemma 3.1, $9 \mid 1 + m_3 + m_9$ and hence, $9 \mid m_2$, which is a contradiction. □

**3.4. Lemma.** $\pi(G) \subseteq \{2, 3, t, u\}$.

*Proof.* Suppose, contrary to our claim, that $p \in \pi(G) \setminus \{2, 3, t, u\}$. To obtain a contradiction, in the following six steps we will prove that there is no choice for $m_p$ in $nse(G)$.
**Step 1.** $m_p \neq 2^m u$ and $(p, t-1) = 1$.
If $m_p = 2^m u$, then according to (3.1), $p \mid (1 + m_p) = (2^{2m} - 2^m + 1)$. Thus Lemma 2.13 implies that $3 \mid (p-1)$. On the other hand, by (3.1), we have $p - 1 \mid m_p$ and hence, $3 \mid m_p$, which is impossible according to Lemma 2.14. Therefore, $m_p \in \{(t-1)2^m u, 1/2(t-1)2^m u, 1/2(u-1)2^m 3t\}$. Since $(p, m_p) = 1$, we conclude that $(p, t-1) = 1$.
**Step 2.** $exp(G_p) = p$.
If $exp(G_p) > p$, then $p^2 \in \pi_e(G)$. Since $p(p-1) = \varphi(p^2) \mid m_{p^2}$, we conclude that $p$ divides one of the numbers $2, 3, t, u, (t-1)$, which is a contradiction. So $exp(G_p) = p$.
**Step 3.** If $q \in \pi_e(G) \setminus \{1\}$ and $(q, p) = 1$, then $qp \in \pi_e(G)$ and $p \mid m_q + m_{qp}$.
If $qp \notin \pi_e(G)$, then Lemma 2.6 implies that $|G_p| \mid m_q$. Now according to $nse(G)$, we conclude that $p$ divides one of the numbers $2, 3, t, u, (t-1)$, which is a contradiction. Thus $qp \in \pi_e(G)$. Let $q = q_1^{s_1} \ldots q_k^{s_k}$, where $q_1, \ldots, q_k$ are distinct prime numbers and $k, s_1, \ldots, s_k$ are natural numbers. We prove $p \mid m_q + m_{qp}$ by induction on $s = s_1 + \ldots + s_k$. Let $s = 1$. Then $q$ is a prime number and according to (3.1), we have $p \mid 1 + m_p + m_q + m_{qp}$ and since $p \mid 1 + m_p$, we can easily conclude that $p \mid m_q + m_{qp}$. Let $s = 2$. Then there exist $1 \leq i < j \leq k$ such that $q = q_i q_j$ or $q = q_i^2$. If $q = q_i q_j$, then we have $p \mid 1 + m_p + m_{q_i} + m_{q_j} + m_{q_i p} + m_{q_j p} + m_{q_i q_j} + m_{q_i q_j p}$ and since $p \mid 1 + m_p,\ m_{q_i} + m_{q_i p},\ m_{q_j} + m_{q_j p}$, we conclude that $p \mid m_{q_i q_j} + m_{q_i q_j p}$, as desired. The case $q = q_i^2$ is similar and we omit the details for the sake of convenience. Now, assume the statement is true for the values less than $s$. We have

$$p \mid \sum_{d \mid qp} m_d = \sum_{\substack{d \mid qp \\ d \neq q, qp}} m_d + m_q + m_{qp}.$$

Moreover, according to induction hypothesis, $p \mid \sum_{\substack{d \mid qp \\ d \neq q, qp}} m_d$. Therefore, $p \mid m_q + m_{qp}$.
**Step 4.** There is $q \in \pi_e(G)$ such that $(q, p) = 1$, $m_q = 2^m u$ or $m_{qp} = 2^m u$. Moreover, we have $p \mid m_q + m_{pq}$.
According to $nse(G)$, there exists $i \in \pi_e(G)$ such that $m_i = 2^m u$. If $(i, p) = 1$, then according to Step 3, we have $p \mid m_i + m_{ip}$. So it is enough to assume $q := i$. If $(i, p) \neq 1$, then since according to Step 2, $exp(G_p) = p$, we have $i = qp$, where $(q, p) = 1$ and $q \in \pi_e(G) \setminus \{1\}$. According to Step 3, we have $p \mid m_i + m_{ip}$.
**Step 5.** $m_p \neq (t-1)2^m u$.
If $m_p = (t-1)2^m u$, then since $p \mid 1 + m_p$, we have $p \mid x^3 - 3x^2 + 2x + 3$. By using Step 4, we have the following five cases:
**Case 1.** If $\{m_q, m_{qp}\} = \{2^m u, 3tu\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid 2x + 1$, which is impossible according to Lemma 2.15(1).
**Case 2.** If $\{m_q, m_{qp}\} = \{2^m u, 2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p = 2$ or $u$, which is contradiction.
**Case 3.** If $\{m_q, m_{qp}\} = \{2^m u, (t-1)2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p = 2$ or $t$ or $u$, which is contradiction.
**Case 4.** If $\{m_q, m_{qp}\} = \{2^m u, 1/2(t-1)2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid x + 4$.

Thus Lemma 2.15(1) implies that $p = 13$. On the other hand, in this case $q \neq 2$ and hence Step 3 implies that $p \mid m_2 + m_{2p}$. Thus $p$ divides one of the numbers $(2x+1)$, $(x^2+x+3)$, $(x^2+4x+6)$ or $(x^2-2)$. Lemma 2.15 now yields $p \in \{23, 43\}$, a contradiction.
**Case 5.** If $\{m_q, m_{qp}\} = \{2^m u, 1/2(u-1)2^m 3t\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid x^2+x-4$. Thus Lemma 2.15(1) implies that $p = 101$. On the other hand, similar to Case 4, $p \mid m_2 + m_{2p}$ and hence $p = 23$ or $43$, which is a contradiction.
**Step 6.** $m_p \notin \{1/2(t-1)2^m u, 1/2(u-1)2^m 3t\}$.
If $m_p = 1/2(t-1)2^m u$ or $m_p = 1/2(u-1)2^m 3t$, then since $p \mid 1+m_p$, we have $p \mid x^2-4x+6$ or $p \mid x^2-2$, respectively. In the former case, similar argument as stated in Step 5 leads us to a contradiction. So, it is enough to consider the case $p \mid x^2-2$ for $m_p = 1/2(u-1)2^m 3t$. According to Step 4, we have the following five cases:
**Case 1.** If $\{m_q, m_{qp}\} = \{2^m u, 3tu\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid 2x+1$. Thus Lemma 2.15(3) implies that $p = 7$. On the other hand, $p \mid 2x+1$, hence Lemma 2.12 implies that $4 \mid (p-1) = 6$, which is contradiction.
**Case 2.** If $\{m_q, m_{qp}\} = \{2^m u, 2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p = 2$ or $u$, which is contradiction.
**Case 3.** If $\{m_q, m_{qp}\} = \{2^m u, (t-1)2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p = 2$ or $t$ or $u$, which is contradiction.
**Case 4.** If $\{m_q, m_{qp}\} = \{2^m u, 1/2(t-1)2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid x+4$. Thus Lemma 2.15(3) implies that $p = 7$. On the other hand, $p \mid 2x+1$, hence Lemma 2.12 implies that $4 \mid (p-1) = 6$, which is contradiction.
**Case 5.** If $\{m_q, m_{qp}\} = \{2^m u, 1/2(u-1)2^m 3t\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid x^2+x-4$. Thus Lemma 2.15(3) implies a contradiction. □

**3.5. Lemma.** *If $t \in \pi(G)$, then $u \in \pi(G)$.*

*Proof.* The proof will be divided into the following four steps.
**Step 1.** $m_t = 1/2(t-1)2^m u$.
According to Lemma 3.1, we have $m_t \neq 1$ and $(m_t, t) = 1$ and hence $m_t \neq 3tu, 1/2(u-1)2^m 3t$. If $m_t = 2^m u$, then Lemma 3.1 implies that $t \mid 1 + m_t$ and hence $x+1 \mid 3x^2 - 3x + 3 = (x+1)(3x-6) + 9$. Thus $x+1 \mid 9$. So $m = 3$, which is a contradiction. If $m_t = (t-1)2^m u$, then $t \mid 1+m_t$ and hence $x+1 \mid x^3-3x^2+2x+3 = (x+1)(x^2-4x+6)-3$. Thus $x+1 \mid 3$. So $m = 1$, which is a contradiction. Therefore, $m_t = 1/2(t-1)2^m u$.
**Step 2.** $t^2 \notin \pi_e(G)$.
If $t^2 \in \pi_e(G)$, then by (3.1), we have $t(t-1) = \varphi(t^2) \mid m_{t^2}$. Hence Lemma 2.14 implies that $m_{t^2} = 1/2(u-1)2^m 3t$. Since $t^2 \mid 1 + m_t + m_{t^2}$, we conclude that $(x+1)^2 \mid (x+1)^2(6x-21) + 30(x+1)$. So $(x+1) \mid 30$, which is a contradiction.
**Step 3.** $|G_t| = t$ and $n_t(G) = \frac{m_t}{\varphi(t)} = 1/2(2^m u)$.
Since $t^2 \notin \pi_e(G)$, Lemma 2.2 implies that $|G_t| \mid 1 + m_t$. If $t^2 \mid |G_t|$, then $2(x+1)^2 \mid (x+1)^2(3x-15) + 33(x+1)$. Thus $(x+1) \mid 33$ which implies that $m = 5$, $t = 11$ and $nse(G) = \{1, 992, 1023, 4960, 9920, 15840\}$. Since $2 \in \pi(G)$, there is the largest element $2 \leq i$ of $\pi_e(G)$ such that $(i, 11) = 1$. By Step 2, $11^2 \notin \pi_e(G)$. Thus $\sum_{i|d} m_d = m_i + m_{11i}$ or $m_i$ and hence Lemma 2.5 implies that $11^2 \mid |G_{11}| \mid m_i + m_{11i}$ or $m_i$. But according to $nse(G)$, we can get a contradiction. Therefore, $|G_t| = t$ which implies that $n_t(G) = \frac{m_t}{\varphi(t)} = 1/2(2^m u)$.
**Step 4.** $u \in \pi(G)$.
According to Step 3, since $n_t(G) = 1/2(2^m u)$ and $n_t(G) \mid |G|$, we conclude that $u \in \pi(G)$.
□

**3.6. Lemma.** $\pi(G) = \{2, 3, t, u\}$.

*Proof.* According to Lemmas 3.2-3.5, we can conclude that $\{2, u\} \subseteq \pi(G) \subseteq \{2, 3, t, u\}$. In the following three steps, we show $n_u(G) = 2^{m-1}3t$ which completes the proof.
**Step 1.** $m_u = 1/2(u-1)2^m 3t$.
According to Lemma 3.1, we have $m_u \neq 1$ and $(m_u, u) = 1$ and hence, according to $nse(G)$, it is obvious that $m_u = 1/2(u-1)2^m 3t$.
**Step 2.** $u^2 \notin \pi_e(G)$.
If $u^2 \in \pi_e(G)$, then by (3.1), $u(u-1) = \varphi(u^2) \mid m_{u^2}$. But according to Lemma 2.14 and $nse(G)$ we can easily see that there is no choice for $m_{u^2}$. Therefore, $u^2 \notin \pi_e(G)$.
**Step 3.** $|G_u| = u$.
Since $u^2 \notin \pi_e(G)$, Lemma 2.2 implies that $|G_u| \mid 1 + m_u$. If $u^2 \mid 1 + m_u$, then $(x-1)^2 \mid (x-1)^2(x+1) - (x-1)$ which implies that $(x-1) \mid 1$, a contradiction. So $|G_u| = u$ and $n_u(G) = \frac{m_u}{\varphi(u)} = 2^{m-1}3t$. □

**3.7. Lemma.** $m_3 = 2^m u$.

*Proof.* According to Lemma 3.1, we have $m_3 \neq 1$ and $(m_3, 3) = 1$ and hence, $m_3 \neq 3tu,\ 1/2(u-1)2^m 3t$. If $m_3 = 1/2(t-1)2^m u$, then by (3.1), we have $3 \mid 1 + m_3$. Thus $18 \mid (x+1)(x^2 - 4x + 6)$. Lemma 2.14 now yields $3 \mid (x^2 - 4x + 6)$ and hence, $3 \mid (x-4)$ which implies that $3 \mid (2^{m-2} - 1)$. Thus according to Lemma 2.10, $3 \mid (2^m - 1) = u$, which contradicts Lemma 2.14(c). Also, if $m_3 = (t-1)2^m u$, then by (3.1), we have $3 \mid 1 + m_3$ and hence, $9 \mid 3 + (x-2)x(x-1)$. This implies that $3 \mid (x-2)x(x-1)$ and $9 \nmid (x-2)x(x-1)$. Since according to Lemma 2.14(c), we have $(2, 3) = (u, 3) = 1$, so $3 \mid (x-2)$ and $9 \nmid (x-2)$. Now we claim that $3t \notin \pi_e(G)$. Indeed, if $3t \in \pi_e(G)$, then $m_{3t} = \varphi(3t)n_t(G)k$, where $k$ is the number of cyclic subgroups of order 3 in $C_G(G_t)$. Actually, this follows from the fact that all centralizers of Sylow $t$-subgroups of $G$ in $G$ are conjugate in $G$. So we have $(t-1)2^m u = \varphi(3t)n_t(G) \mid m_{3t}$ which implies that $m_{3t} = (t-1)2^m u$. Since by (3.1), $3t \mid 1 + m_3 + m_t + m_{3t}$ and $t \mid 1 + m_t$ and $m_3 = m_{3t}$, we conclude that $t \mid (2m_3) = (t-1)2^{m+1}u$, which is a contradiction according to Lemma 2.14(c). Therefore, $3t \notin \pi_e(G)$ which implies that $G_3$ acts fixed point freely on the set of elements of order $t$ by conjugation. Lemma 2.6 now leads to $|G_3| \mid m_t$. Now, according to Lemma 2.14(c), we conclude that $|G_3| \mid 1/3(x-2)$. Since $3 \mid (x-2)$ but $9 \nmid (x-2)$, we conclude that $|G_3| = 1$, which is a contradiction. □

**3.8. Lemma.** $9 \notin \pi_e(G)$.

*Proof.* If $9 \in \pi_e(G)$, then according to (3.1), we have $6 = \varphi(9) \mid m_9$ and by Lemma 2.14 and $nse(G)$, we conclude that $m_9 \in \{(t-1)2^m u, 1/2(t-1)2^m u, 1/2(u-1)2^m 3t\}$. So we have the following two cases:
**Case 1.** If $m_9 = 1/2(u-1)2^m 3t = 1/2(t-1)2^m 9t$, then $9 \mid m_9$. On the other hand, (3.1) implies that $9 \mid 1 + m_3 + m_9$ and hence, $9 \mid 1 + m_3$, which contradicts Lemma 2.14(e).
**Case 2.** If $m_9 = (t-1)2^m u$ or $1/2(t-1)2^m u$, then by (3.1), $9 \mid 1 + m_3 + m_9$. Since by Lemma 2.14(e), $3 \mid 1 + m_3$ and $9 \nmid 1 + m_3$, we conclude that $3 \mid m_9$ and $9 \nmid m_9$ and hence $3 \mid (t-1)$ and $9 \nmid (t-1)$. Lemma 2.4 yields $G_3$ is a cyclic group of order $3^k$, where $k \geq 2$. Thus by (3.1), $n_3(G) = \frac{m_{3^k}}{\varphi(3^k)} = \frac{m_{3^k}}{2(3^{k-1})}$ and also, from (3.1) and Lemma 2.14, we conclude that $m_{3^k} \in \{(t-1)2^{m-1}9t, (t-1)2^{m-1}u, (t-1)2^m u\}$. Therefore, $n_3(G) \in \{\frac{(t-1)2^{m-2}9t}{3^{k-1}}, \frac{(t-1)2^{m-2}u}{3^{k-1}}, \frac{(t-1)2^{m-1}u}{3^{k-1}}\}$. Moreover, according to Lemma 2.14(d), there is a prime $p \in \pi(t-1) \setminus \{2, 3, t, u\}$ which implies that $p \mid n_3(G)$. But since $n_3(G) \mid |G|$, we conclude that $p \in \pi(G)$, a contradiction. □

**3.9. Lemma.** $|G_u| = u,\ |G_t| = t,\ |G_2| \mid 2^m,\ |G_3| = 3$ *and hence,* $|G| = 2^k 3tu$, *where* $k \leq m$.

*Proof.* According to Lemmas 3.5 and 3.6, we have $|G_u| = u$ and $|G_t| = t$. Since $9 \notin \pi_e(G)$, Lemma 2.2 implies $|G_3| \mid 1 + m_3$ and hence, Lemma 2.14(e) leads to $|G_3| = 3$. We know that $2u \notin \pi_e(G)$. Actually, this follows by the same method as in Lemma 3.7. Therefore, $G_2$ acts fixed point freely on the set of elements of order $u$ by conjugation and Lemma 2.6 implies that $|G_2| \mid m_u$ and hence, according to Lemma 2.14, we have $|G_2| \mid 2^m$. □

**3.10. Lemma.** *$G$ is unsolvable.*

*Proof.* If $G$ is solvable, then by Lemma 2.7, $G$ has a Hall $\pi$-subgroup $H$, where $\pi = \{3, t, u\}$ and all the Hall $\pi$-subgroups of $G$ are conjugate and hence, $|G : N_G(H)| \mid 2^m$. Since $|H| = 3tu$, we conclude that $n_u(H) \in \{1, 3, t, 3t\}$ and according to Sylow theorem, we have $n_u(H) \equiv 1 \pmod u$ and hence Lemma 2.14 implies that $n_u(H) = 1$. On the other hand, we can easily see that

$$n_u(G) \mid n_u(H).|G : N_G(H)|.|N_G(H) : H| \mid 2^{m+k}.$$

Also, since the Sylow $u$-subgroups of $G$ are cyclic, we have $m_u = (u-1).n_u(G)$ and hence, $m_u \mid 2^{m+k}(u-1)$, but according to Lemma 3.6, Step 1, we have $m_u = 1/2(u-1)2^m 3t$, which is a contradiction. □

**3.11. Lemma.** $G \cong L_2(2^m)$.

*Proof.* Since $G$ is a finite unsolvable group, according to Lemma 2.8, there is a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$, such that $N$ is a normal solvable subgroup of $G$ and $M/N$ is an unsolvable simple group or the direct product of isomorphic unsolvable simple groups. Let $M/N \cong S_1 \times \ldots \times S_r$, where $S_1$ is an unsolvable simple group and $S_1 \cong \ldots \cong S_r$. According to $|G| = 2^k.3.t.u$, where $k \leq m$ and the structure of $M/N$, we can easily conclude that $r = 1$ and $M/N$ is a simple $K_3$-group or a simple $K_4$-group.

**Case 1.** If $M/N$ is a simple $K_3$-group, then according to Lemma 2.1, we have $\pi(M/N) \cap \{5, 7, 13, 17\} \neq \emptyset$. But since $\pi(M/N) \subseteq \pi(G)$ and $|G| = 2^k.3.t.u$, where $k \leq m$, we can get a contradiction.

**Case 2.** If $M/N$ is a simple $K_4$-group, then by Lemma 2.1, $M/N$ is isomorphic to one of the following groups:

- If $M/N \cong A_7,\ A_8,\ A_9,\ A_{10},\ M_{11},\ M_{12},\ J_2,\ L_2(81),\ L_2(243),\ L_2(577),$ $L_3(4),\ L_3(7),\ L_3(8),\ L_3(17),\ L_4(3),\ S_4(4),\ S_4(5),\ S_4(7),\ S_4(9),\ S_6(2),$ $O_8^+(2),\ G_2(3),\ U_3(5),\ U_3(8),\ U_3(9),\ U_4(3),\ U_5(2),\ {}^3D_4(2),\ {}^2F_4(2)'$ or $L_2(3^m)$, where $m$, $(3^m - 1)/2$ and $(3^m + 1)/4$ are odd primes, then $3^2 \mid |M/N|$, a contradiction.
- If $M/N \cong L_2(25),\ L_2(49),\ L_3(5),\ U_3(4),\ Sz(32)$, then $5^2 \mid |M/N|$, a contradiction.
- If $M/N \cong L_2(97),\ U_3(7)$, then $7^2 \mid |M/N|$, a contradiction.
- If $M/N \cong Sz(8)$, then $3 \nmid |M/N|$, a contradiction.
- If $M/N \cong L_2(16)$, then $t = 5$, a contradiction.
- If $M/N \cong L_2(r)$, where $r$ is a prime, $r^2 - 1 = 2^a.3^b.v$, $v > 3$ is a prime, $a, b \in \mathbb{N}$, then $|M/N| = |L_2(r)| = \frac{1}{(r-1,2)} r(r^2 - 1) = \frac{1}{(r-1,2)} r.2^a.3^b.v$ and hence, $\pi(M/N) = \{2, 3, r, v\}$. Since $\pi(M/N) \subseteq \pi(G)$, we have $v = t,\ r = u$ or $v = u,\ r = t$. But since $v$ is a prime number which divides $r^2 - 1$, according to Lemma 2.14(a-b) we can get a contradiction.
- If $M/N \cong L_2(2^{m'})$, where $m'$ satisfies

$$\begin{cases} 2^{m'} - 1 = u' \\ 2^{m'} + 1 = 3t' \end{cases}$$

with $m' \geq 2,\ u', t'$ are primes, $t' > 3$, then $|M/N| = 2^{m'}.3.t'.u'$. Since $|M/N| \mid |G|$ and $|G| = 2^k.3.t.u$, where $k \leq m$, we conclude that $m' \leq m$ and $t' = t$ or $u$. If $t' = u$, then

$\frac{2^{m'}+1}{3} = 2^m - 1$. Thus $2^{m'}(3.2^{m-m'} - 1) = 4$, which is a contradiction . So we conclude $t' = t$ and this implies that $m = m'$ and $u' = u$. Therefore, $M/N \cong L_2(2^m)$, where $m$ satisfies

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t \end{cases}$$

with $m \geq 2,\ u, t$ are primes, $t > 3$.

Since $2^m.3tu = |M/N| \mid |G| = 2^k.3.t.u$, where $k \leq m$, we conclude that $N = 1$ and $M = G = L_2(2^m)$. $\square$

According to the main theorem, we pose the following problem:

**Problem:** Is a group $G$ isomorphic to $L_2(2^m)(m \geq 2)$ if and only if $nse(G) = nse(L_2(2^m))$?

## Acknowledgments

The authors would like to express their deep gratitude to the referee for invaluable comments and suggestions. Partial support by the Center of Excellence of Algebraic Hyperstructures and its Applications of Tarbiat Modares University (CEAHA) is gratefully acknowledged by the second author (AI).

## References

[1] Adams, W.W., Goldstein, L.J.: Introduction to number theory, Prentice-Hall. Inc (1976)

[2] Bugeaud, Y., Cao, Z., Mignotte, M.: On simple $K_4$-groups, J. Algebra 241(2), 658-668 (2001)

[3] Crescenzo, P.: A diophantine equation which arises in the theory of finite groups, Advances in Math. 17, 25-29 (1975)

[4] Frobenius, G.: Verallgemeinerung des sylowschen satze, Berliner sitz, pp. 981-993 (1895)

[5] Hall, P.: A note on soluble groups, J. London Math. Soc. 3, 98-105 (1928)

[6] Herzog, M.: On finite simple groups of order divisble by three primes only, J. Algebra 10, 383-388 (1968)

[7] Khatami, M., Khosravi, B., Akhlaghi, Z.: A new characterization for some linear groups, Monatsh. Math. 163, 39-50 (2011)

[8] Khosravi, B., Moghanjoghi, A.Z.: Quasirecognition by prime graph of some alternating groups, Int. J. Contemp. Math. Sci. 2, no. 25-28, 1351-1358 (2007)

[9] Khalili Asboei, A., Salehi Amiri, S.S., Iranmanesh, A., Tehranian, A.: A new characterization of symmetric groups for some $n$.(submitted)

[10] Khalili Asboei, A., Salehi Amiri, S.S., Iranmanesh A., Tehranian, A.: A characterization of Matheiu groups by NSE. ( submitted)

[11] Khalili Asboei, A., Salehi Amiri, S.S., Iranmanesh, A., Tehranian, A.: A new characterization of $A_7$, $A_8$, Anale Stintifice ale Universitatii Ovidius constanta 21, 43-50 (2013)

[12] Khalili Asboei, A., Salehi Amiri, S.S., Iranmanesh, A., Tehranian, A.: A new characterization of $PSL(2, p)$ where $p$ is a prime number with its NSE (submitted)

[13] Miller, G.: Addition to a theorem due to Frobenius, Bull. Am. Math. Soc. 11, 6-7 (1904)

[14] Passman, D.: Permutation Groups, W. A. Benjamin New York (1968)

[15] Shi, W. J.: On simple $K_4$-groups, Chinese Science Bull. 36(17), 1281-1283 (in Chinese) (1991)

[16] Shao, C.G., Shi, W.J., Jiang, Q.H.: Characterization of simple $K_4$-groups, Front. Math. China 3, 355-370 (2008)

[17] Shen, R., Shao, C., Jiang, Q., Shi, W., Mazurov, V.: A new characterization of $A_5$, Monatsh. Math. 160, 337-341 (2010)

[18] Weisner, L.: On the number of elements of a group which have a power in a given conjugate set, Bull. Amer. Math. Soc. 31, 492-496 (1925)

[19] Zavarnitsine, A.V.: Recognition of the simple groups $L_3(q)$ by element orders, J. Group Theory 7, 81-97 (2004)
[20] Zhang, S., Shi, W.: Revisiting the number of simple $K_4$-groups, arXiv: 1307. 8079v1 [math.NT] (2013)

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 887 – 892

# Skew-commuting mappings on semiprime and prime rings

Abbas Najati * and M. Mohammadi Saem †

**Abstract**

In this paper we study some maps which are skew-commuting on rings. Also we present some results concerning derivations in generalized case.

*2000 AMS Classification:* 16N60, 16W25.

**Keywords:** skew-commuting map, additive map, generalized derivation, semiprime ring, prime ring.

Received 09/02/2014 : Accepted 30/04/2014 Doi : 10.15672/HJMS.2015449652

## 1. Introductions and Preliminaries

Throughout this paper, $R$ be a ring with center $Z(R)$. For an integer $n > 1$, a ring $R$ is called *n-torsion free* if $nx = 0$, $(x \in R)$ implies $x = 0$. As usual we write $[x, y]$ for $xy - yx$ and use the identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for $x, y, z \in R$. Recall that a ring $R$ is *prime* if $xRy = \{0\}$ implies $x = 0$ or $y = 0$ and is *semiprime* if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d$ from $R$ into itself is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $f : R \to R$ is said to be a *generalized derivation* if there exists a derivation $d : R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. A mapping $f : R \to R$ is called *skew-commuting* on $R$ if $f(x)x + xf(x) = 0$ and is called *commuting* on $R$ if $[f(x), x] = 0$ for all $x \in R$.

*Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran.
Email: a.najati@uma.ac.ir, a.nejati@yahoo.com
†Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran.
Email:m.mohammadisaem@yahoo.com

## 2. Main Results

We shall make use of the following results.

**2.1. Theorem.** [2] *Let $R$ be a 2-torsion free semiprime ring. If an additive mapping $f : R \to R$ is skew-commuting on $R$, then $f = 0$.*

**2.2. Theorem.** [3] *Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $F$ a nonzero generalized derivation of $R$ associated with a derivation $D$. If $F(xy) = F(x)F(y)$ for all $x, y \in R$, then $D(I) = 0$.*

**2.3. Lemma.** [5] *Let $R$ be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case, $(a + c)xb = 0$ is satisfied for all $x \in R$.*

**2.4. Lemma.** [6] *Let $R$ be a semiprime ring and $D$ be a derivation on $R$. If $D(x)D(y) = 0$ for all $x, y \in R$, then $D = 0$.*

**2.5. Theorem.** *Let $R$ be a 2-torsion free semiprime ring, $D$ be a derivation and $\alpha$ be a homomorphism on $R$. Suppose that the mapping $x \mapsto (D(x) + (\alpha(x) - x))$ is skew-commuting on $R$. In this case $\alpha = I$ and $D = 0$.*

*Proof.* Put $G(x) = \alpha(x) - x$. By Theorem 2.1, we have $D = -G$. Therefore $D(x)D(y) = 0$ for all $x, y \in R$. Hence $D = 0$ by Lemma 2.4. So we get $\alpha = I$. □

**2.6. Theorem.** *Let $R$ be a nonzero 2-torsion free semiprime ring, $D$ be a derivation and $\alpha$ be a homomorphism on $R$. Suppose that the mapping $x \mapsto D(x) + \alpha(x)$ is skew-commuting on $R$. In this case $\alpha = D = 0$.*

*Proof.* The result follows by Theorems 2.1 and 2.2. □

In [4] Vukman proved that on a 2-torsion free semiprime ring $R$, if the mapping $x \mapsto D(x)x + x\alpha(x)$ is commuting on $R$, then $D$ and $\alpha - I$ map $R$ into $Z(R)$. The next theorem is a version of this result in case of skew-commuting map. We will use the following lemmas in the proofs of next theorems.

**2.7. Lemma.** [4] *Let $R$ be a semiprime ring and let $f : R \to R$ be an additive mapping. If either $f(x)x = 0$ or $xf(x) = 0$ holds for all $x \in R$, then $f = 0$.*

**2.8. Lemma.** [4] *Let $R$ be a 2-torsion free semiprime ring and $\alpha : R \to R$ be an automorphism such that $x[\alpha(x), x] = 0$ or $[\alpha(x), x]x = 0$ for all $x \in R$. Then $\alpha - I$ maps $R$ into $Z(R)$.*

**2.9. Lemma.** [1] *Let $R$ be a prime ring and let $F : R \to R$ be an additive map. If there exists a positive integer $n$ such that $F(x)x^n = 0$ for all $x \in R$, then $F = 0$.*

**2.10. Theorem.** *Let $R$ ba a 2 and 3-torsion free semiprime ring. Suppose that $D$ is a derivation and $\alpha : R \to R$ is an onto homomorphism such that the mapping $x \mapsto D(x)x + x\alpha(x)$ is skew-commuting on $R$. In this case $\alpha - I$ maps $R$ into $Z(R)$.*

*Proof.* The assumption of the theorem can be written in the form

$$(D(x)x + x\alpha(x))x + x(D(x)x + x\alpha(x)) = 0, \quad x \in R. \tag{2.1}$$

Using the linearization of (2.1), a routine calculation gives

$$\begin{aligned} &A(x)y + D(x)yx + D(y)x^2 + x\alpha(y)x + y\alpha(x)x + x\alpha(x)y + xD(y)x \\ &\quad + x^2\alpha(y) + xy\alpha(x) + yD(x)x + yx\alpha(x) = 0, \quad x, y \in R, \end{aligned}$$

where $A(x) = D(x)x + xD(x)$. Replacing $y$ by $yx$ in the above relation, we get

$$A(x)yx + D(x)yx^2 + D(y)x^3 + yD(x)x^2 + x\alpha(y)\alpha(x)x + yx\alpha(x)x + x\alpha(x)yx$$
$$+ xD(y)x^2 + xyD(x)x + x^2\alpha(y)\alpha(x) + xyx\alpha(x) + yxD(x)x + yx^2\alpha(x) = 0.$$

It follows from the above relations that

$$(xy + yx)D(x)x + x^2\alpha(y)G(x) + x\alpha(y)G(x)x + xy[x, \alpha(x)] + yx[x, \alpha(x)] + y[x, \alpha(x)]x = 0, \quad x, y \in R, \tag{2.2}$$

where $G(x) = \alpha(x) - x$. Replacing $y$ by $xy$ in (2.2), we get

$$\begin{aligned} &x(xy + yx)D(x)x + x^2\alpha(x)\alpha(y)G(x) + x\alpha(x)\alpha(y)G(x)x \\ &+ x^2y[x, \alpha(x)] + xyx[x, \alpha(x)] + xy[x, \alpha(x)]x = 0. \end{aligned} \tag{2.3}$$

Multiplying the left side of (2.2) by $x$ and then subtracting the obtained relation from (2.3), we obtain

$$x^2G(x)\alpha(y)G(x) + xG(x)\alpha(y)G(x)x = 0, \quad x, y \in R.$$

Since $\alpha$ is onto, therefore

$$x^2G(x)yG(x) + xG(x)yG(x)x = 0, \quad x, y \in R.$$

Hence

$$x^2G(x)yxG(x) + xG(x)yxG(x)x = 0, \quad x, y \in R.$$

By Lemma 2.3, we have

$$(x^2G(x) + xG(x)x)yxG(x) = 0, \quad x, y \in R. \tag{2.4}$$

Replacing $y$ by $yx$ in (2.4), we get

$$(x^2G(x) + xG(x)x)yx^2G(x) = 0, \quad x, y \in R. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$(x^2G(x) + xG(x)x)y(x^2G(x) + xG(x)x) = 0, \quad x, y \in R.$$

Since $R$ is semiprime, this implies

$$x(xG(x) + G(x)x) = 0, \quad x \in R. \tag{2.6}$$

Using the linearization of (2.6), a routine calculation gives

$$x(xG(y) + yG(x) + G(x)y + G(y)x) + y(xG(x) + G(x)x) = 0. \tag{2.7}$$

Replacing $y$ by $xy$ in (2.7), we obtain

$$\begin{aligned} &x(xG(xy) + xyG(x) + G(x)xy + G(xy)x) \\ &+ xy(xG(x) + G(x)x) = 0, \quad x, y \in R. \end{aligned} \tag{2.8}$$

Multiplying the left side of (2.7) by $x$ and then subtracting the obtained relation from (2.8), we get

$$x[\alpha(x), x]y + x^2G(x)\alpha(y) + xG(x)\alpha(y)x = 0, \quad x, y \in R.$$

Using (2.6), we get

$$x[\alpha(x), x]y + xG(x)[\alpha(y), x] = 0, \quad x, y \in R. \tag{2.9}$$

Replacing $y$ by $yz$ in (2.9), we obtain

$$\begin{aligned} 0 &= x[\alpha(x), x]yz + xG(x)\alpha(y)[\alpha(z), x] + xG(x)[\alpha(y), x]\alpha(z) \\ &= -xG(x)[\alpha(y), x]z + xG(x)\alpha(y)[\alpha(z), x] + xG(x)[\alpha(y), x]\alpha(z) \\ &= xG(x)[\alpha(y), x]G(z) + xG(x)\alpha(y)[\alpha(z), x], \quad x, y, z \in R. \end{aligned}$$

Since $\alpha$ is onto, we have

$$xG(x)[y,x]G(z) + xG(x)y[\alpha(z),x] = 0, \quad x,y,z \in R.$$

Putting $y = x$ in the above relation, we infer $xG(x)x[z,x] = 0$ for all $x, z \in R$. If we replace $z$ by $zy$, then $xG(x)xz[y,x] = 0$ for all $x, y, z \in R$. Putting $y = G(x)$, we have

$$xG(x)xz[G(x),x] = 0, \quad x,z \in R. \tag{2.10}$$

Replacing $z$ by $xz$ in (2.10), we get $xG(x)x^2z[G(x),x] = 0$ for all $x, z \in R$. On the other hand (2.10) gives $x^2G(x)xz[G(x),x] = 0$. Subtracting these two recent relations, we obtain

$$x[G(x),x]xz[G(x),x] = 0, \quad x,z \in R.$$

Hence $x[G(x),x]x = 0$ by semiprimeness of $R$. According to (2.6), we get $x^2G(x)x = 0$ for all $x \in R$. Therefore $x^2[G(x),x] = 0$ for all $x \in R$. The linearization with a simple calculation leads to

$$x^2[G(y),y] + (xy+yx)([G(x),y] + [G(y),x]) + y^2[G(x),x] = 0, \quad x,y \in R.$$

Replacing $y$ by $x + y$ in the above relation, we get

$$x^2([G(x),y] + [G(y),x]) + (xy+yx)[G(x),x] = 0, \quad x,y \in R. \tag{2.11}$$

Left multiplication (2.11) by $x[G(x),x]$ and using $x[G(x),x]x = 0$, we get

$$x[G(x),x]yx[G(x),x] = 0, \quad x,y \in R.$$

Since $R$ is semiprime, $x[G(x),x] = 0$ for all $x \in R$. So $x[\alpha(x),x] = 0$ for all $x \in R$. Therefore $\alpha - I$ maps $R$ into $Z(R)$ by Lemma 2.8. □

**2.11. Theorem.** *Let $R$ ba a 2 and 3-torsion free prime ring. Suppose that $D$ is a derivation and $\alpha : R \to R$ is an onto homomorphism such that the mapping $x \mapsto D(x)x + x\alpha(x)$ is skew-commuting on $R$. The only case for $R$ is $R = \{0\}$.*

*Proof.* By Theorem 2.10 we obtain that $\alpha - I$ maps $R$ into $Z(R)$. So relation (2.6) gives us $G(x)x^2 = 0$. Hence $\alpha = I$ by Lemma 2.9. Therefore (2.1) and (2.2) give

$$D(x)x^2 + xD(x)x = -2x^3, \quad x^2D(x)x = 0, \quad x \in R. \tag{2.12}$$

Hence we have

$$D(x^3) - x^2D(x) = -2x^3, \quad x \in R. \tag{2.13}$$

It follows from (2.12) that

$$xD(x)x^2 = -2x^4, \quad x \in R. \tag{2.14}$$

Right multiplication of (2.12) by $x$ and then using (2.14), we get $D(x)x^3 = 0$. Hence $D = 0$ by Lemma 2.9. So (2.14) implies $x^4 = 0$ for all $x \in R$. So we get $R = \{0\}$ by Lemma 2.9. □

Vukman [4] proved the result below.

**2.12. Theorem.** [4] *Let $R$ be a 2-torsion free semiprime ring and $D : R \to R$ be a derivation such that $x[D(x),x] = 0$ or $[D(x),x]x = 0$ for all $x \in R$. Then $D$ maps $R$ into $Z(R)$.*

In the following theorem we generalize this result.

**2.13. Theorem.** *Let $R$ be a 2-torsion free semiprime ring and $F$ be a generalized derivation associated with a derivation $D$ on $R$. Also let $[F(x),x]x = 0$ for all $x \in R$. In this case $D$ maps $R$ into $Z(R)$.*

*Proof.* The linearization of $[F(x), x]x = 0$ gives

(2.15) $\quad [F(x), y]x + [F(y), x]x + [F(x), x]y = 0, \quad x, y \in R.$

Replacing $y$ by $yx$ in (2.15), we get

$$[F(x), y]x^2 + [F(y), x]x^2 + y[D(x), x]x + [y, x]D(x)x + [F(x), x]yx = 0.$$

Right multiplication of (2.15) by $x$ and subtracting the obtained relation from the above relation, gives

(2.16) $\quad y[D(x), x]x + [y, x]D(x)x = 0, \quad x, y \in R.$

Replacing $y$ by $D(x)y$ in (2.16), we have

$$D(x)y[D(x), x]x + D(x)[y, x]D(x)x + [D(x), x]yD(x)x = 0, \quad x, y \in R.$$

Using (2.16), we infer $[D(x), x]yD(x)x = 0$. Hence (2.16) implies that $[D(x), x]xy[D(x), x]x = 0$ for all $x, y \in R$. Since $R$ is semiprime, $[D(x), x]x = 0$ for all $x \in R$. Therefore $D$ maps $R$ into $Z(R)$ by Theorem 2.12. □

**2.14. Theorem.** *Let $R$ be a 2-torsion free semiprime ring and let $D$ and $G$ be two derivations on $R$. Suppose that $(D(x)x + xG(x))x = 0$ for all $x \in R$. In this case $D$ and $G$ map $R$ into $Z(R)$.*

*Proof.* A routine calculation shows that

(2.17) $\quad D(x)yx + D(y)x^2 + xG(y)x + yG(x)x + D(x)xy + xG(x)y = 0.$

Let $y$ be $yx$ in (2.17). Then

$$\begin{aligned} &D(x)yx^2 + D(y)x^3 + yD(x)x^2 + xG(y)x^2 \\ &+ xyG(x)x + yxG(x)x + D(x)xyx + xG(x)yx = 0, \quad x, y \in R. \end{aligned}$$

Multiplying (2.17) from the right by $x$ and then subtracting the obtained relation from the above relation, we get

$$y(D(x)x^2 + xG(x)x) + xyG(x)x - yG(x)x^2 = 0, \quad x, y \in R.$$

Hence by the assumption, we get

$$xyG(x)x - yG(x)x^2 = 0, \quad x, y \in R.$$

Replacing $y$ by $G(x)xy$, we get

$$xG(x)xyG(x)x + G(x)xy(-G(x)x^2) = 0, \quad x, y \in R.$$

By Lemma 2.3 we get

(2.18) $\quad [G(x), x]xyG(x)x = 0, \quad x, y \in R.$

If we replace $y$ by $yx$ in (2.18), then

$$[G(x), x]xyxG(x)x = 0, \quad x, y \in R.$$

Multiplying (2.18) from the right by $x$ and subtracting the obtained relation from the above relation, we obtain

$$[G(x), x]xy[G(x), x]x = 0, \quad x, y \in R.$$

Since $R$ is semiprime, $[G(x), x]x = 0$ for all $x \in R$. Hence $G$ maps $R$ into $Z(R)$ by Theorem 2.12. Also using same argument shows that $D$ maps $R$ into $Z(R)$. □

**2.15. Theorem.** *Let $R$ be a 2-torsion free prime ring and let $D$ and $G$ be two derivations on $R$. Suppose that $(D(x)x + xG(x))x = 0$ for all $x \in R$. In this case $D = -G$ and $R$ is commutative, unless $D = G = 0$.*

*Proof.* By Theorem 2.14 we get that $G$ maps $R$ into $Z(R)$. So by the assumption, we obtain $(D+G)(x)x^2 = 0$ for all $x \in R$. Therefore $D+G=0$ by Lemma 2.9, and we conclude that $D$ maps $R$ into $Z(R)$. $\square$

Our last result generalizes a result of [4].

**2.16. Theorem.** *Let $R$ be a semiprime ring, $F$ be a generalized derivation associated with a derivation $D$ on $R$ and $\alpha : R \to R$ be an onto homomorphism. If $F(x)x+x(\alpha(x)-x)=0$ holds for all $x \in R$, then $\alpha = I$ and $F=D=0$.*

*Proof.* The linearization of $F(x)x + x(\alpha(x) - x) = 0$ gives

(2.19) $\quad F(x)y + F(y)x + xG(y) + yG(x) = 0,$

where $G(x) = \alpha(x) - x$. Substituting $yx$ for $y$ in (2.19), we have

$$\begin{aligned} 0 &= F(x)yx + F(y)x^2 + yD(x)x + x\alpha(y)\alpha(x) - xyx + yxG(x) \\ &= (F(x)y + F(y)x - xy)x + yD(x)x + x\alpha(y)\alpha(x) + yxG(x) \\ &= -x\alpha(y)x - yG(x)x + yD(x)x + x\alpha(y)\alpha(x) + yxG(x). \end{aligned}$$

Therefore

(2.20) $\quad x\alpha(y)G(x) + y[x, G(x)] + yD(x)x = 0, \quad x, y \in R.$

Replacing $y$ by $xy$ in (2.20), we get

$$x\alpha(x)\alpha(y)G(x) + xy[x, G(x)] + xyD(x)x = 0, \quad x, y \in R.$$

Multiplying (2.20) from the left by $x$ and subtracting the obtained relation from the above relation, we get

$$xG(x)\alpha(y)G(x) = 0, \quad x, y \in R.$$

Since $\alpha$ is onto, $xG(x)yG(x) = 0$ for all $x, y \in R$. Hence $xG(x)yxG(x) = 0$ for all $x, y \in R$. Since $R$ is semiprime, $xG(x) = 0$ for all $x \in R$. By Lemma 2.7 we get $G = 0$, which implies that $\alpha = I$. Now by (2.19) we have $F(x)x = 0$ for all $x \in R$. Hence $F = 0$ by Lemma 2.7. On the other hand, we have $F(xy) = F(x)y + xD(y)$ for all $x, y \in R$. So $xD(y) = 0$ for all $x, y \in R$. Since $R$ is semiprime, we infer $D = 0$. $\square$

## Acknowledgments

The authors would like to thank the referee for his (her) useful comments.

## References

[1] D. Benkovič and D. Eremita, *Characterizing left centralizers by their action on a polynomial,* Publ. Math. Debercen **64** (2004), 343–351.
[2] M. Brešar, *On Skew-Commuting mappings of rings,* Bull. Austral. Math. Soc. **47** (1993), 291–296.
[3] B. Dhara, *Generalized derivations acting as a homomorphism or anti-homomorphism in semiprime ring,* Beitr. Algebra Geom. **53** (2012), 203–209.
[4] J. Vukman, *Identities with derivations and automorphisms on semiprime rings,* Int. J. Math. Math. Sci. **7** (2005), 1031–1038.
[5] J. Vukman, *Centralizers on semiprime rings,* Comment. Math. Univ. Carolin. **42** (2001), 237–245.
[6] B. Zalar, *On centralizers of semiprime rings,* Comment. Math. Univ. Carolin. **32** (1991), 609–614.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 893 – 899

# Autoisoclinism classes and autocommutativity degrees of finite groups

Mohammad Reza Rismanchian * and Zohreh Sepehrizadeh †

**Abstract**

The notion of autoisoclinism was first introduced by Moghaddam et. al., in 2013. In this article we derive more properties of autoisoclinism and define autocommutativity degrees of finite groups. This work also generalizes some results of Lescot in 1995. Among the other results, we determine an upper bound for autocommutativity degree of finite groups.

*2000 AMS Classification:* 20D15, 20F28, 20P05.

**Keywords:** Autocommutativity degree, Autoabelian, Autoisoclinism, Absolute centre.

Received 03/02/2014 : Accepted 28/04/2014 Doi : 10.15672 /HJMS.2015449442

## 1. Introduction

If $G$ is a finite group then the commutativity degree $d(G)$ of $G$ is defined as

$$d(G) = \frac{1}{|G|^2} \left|\{(x, y) \in G \times G | [x, y] = 1\}\right|,$$

which is the probability that two randomly chosen elements of $G$ commute, where $[x, y] = x^{-1}y^{-1}xy$. The commutative degree first studied by Gustafson in 1973, where he showed that $d(G) \leq \frac{5}{8}$ for every non-abelian finite group $G$. The equality holds when $G/Z(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. In 1995, Lescot investigated this concept by considering the notion of isoclinism of groups. Whence he obtained certain results in this regard. In 2007, Erfanian et. al., introduced the concept of relative commutativity degree $d(H, G)$ of a subgroup $H$ in a given group $G$ as

$$d(H, G) = \frac{1}{|H||G|} \left|\{(x, y) \in H \times G | [x, y] = 1\}\right|,$$

*Department of Pure Mathematics, Shahrekord University, Shahrekord, Iran. Corresponding author.
Email: `rismanchian@sci.sku.ac.ir`

†Department of Pure Mathematics, Shahrekord University, Shahrekord, Iran.
Email: `zohreh.sepehri@gmail.com`

which is the probability that an arbitrary element of $H$ commutes with an element of $G$. In this article, we introduce the *autocommutativity degree* and the *relative autocommutativity degree of a subgroup $H$ of $G$*, denoted by $d_{aut}(G)$ and $d_{aut}(H,G)$, respectively, which are defined as follows:

$$d_{aut}(G) = \frac{1}{|G||Aut(G)|}\left|\{(x,\alpha) \in G \times Aut(G) | [x,\alpha] = 1\}\right|,$$

and

$$d_{aut}(H,G) = \frac{1}{|H||Aut(G)|}\left|\{(x,\alpha) \in H \times Aut(G) | [x,\alpha] = 1\}\right|,$$

where $[x,\alpha] = x^{-1}x^{\alpha}$. Clearly, $d_{aut}(G) = 1$ if and only if $Aut(G) = \{1\}$, that is, if and only if $|G| \leq 2$.
In Hegarty [5], the characteristic subgroups $K(G)$ and $L(G)$ of $G$ are defined as follows:

$$K(G) = \langle\ [x,\alpha] \mid x \in G, \alpha \in Aut(G)\ \rangle,$$

and

$$L(G) = \{\ x \mid [x,\alpha] = 1,\ \forall \alpha \in Aut(G)\ \},$$

which are called *autocommutator subgroup* and *absolute centre* of $G$, respectively. One can easily check that $K(G)$ contains the derived subgruop $G'$ of $G$ and $L(G)$ is contained in the centre, $Z(G)$, of $G$.

## 2. Results on the relative autocommutativity degree

Let $G$ be a group, and $\alpha$ be an automorphism of $G$. The subgroup $C_G(\alpha)$ of $G$ is defined by

$$C_G(\alpha) = \{\ x \in G \mid [x,\alpha] = 1\ \ \}.$$

The following lemma gives an upper bound for $d_{aut}(G)$, which is similar to Lemma 1.3 of [6].

**2.1. Lemma.** *Let $G$ be a finite nontrivial group. If $p$ is the smallest prime divisor of $|G|$, then $d_{aut}(G) \leq \frac{p-1}{p|Aut(G)|} + \frac{1}{p}$.*

*Proof.* Let $p$ be the smallest prime divisor of $|G|$. Then $|C_G(\alpha)| \leq |G|/p$ for $\alpha \neq 1$ which $\alpha \in Aut(G)$ and hence

$$\begin{aligned}
|G||Aut(G)|d_{aut}(G) &= \sum_{\alpha\in Aut(G)} |C_G(\alpha)| \\
&= \sum_{\alpha=1} |C_G(\alpha)| + \sum_{\alpha\in Aut(G)\setminus\{1\}} |C_G(\alpha)| \\
&= |G| + |C_G(\alpha)|(|Aut(G)|-1) \\
&\leq |G| + \frac{|G|}{p}(|Aut(G)|-1) \\
&= \frac{|G|}{p}\left((p-1)+|Aut(G)|\right).
\end{aligned}$$

Therefore

$$d_{aut}(G) \leq \frac{p-1}{p|Aut(G)|} + \frac{1}{p}.$$

□

**2.2. Example.** Consider the Klein four-group,

$$V_4 = \langle\ a, b \mid a^2 = b^2 = (ab)^2 = 1\ \rangle.$$

It is easy to see that $|Aut(V_4)| = 6$. Thus

$$\begin{aligned} d_{aut}(V_4) &= \frac{|\{(1,\alpha)|\alpha(1)=1\}| + 3|\{(b,\alpha)|\alpha(b)=b\}|}{|V_4||Aut(V_4)|} \\ &= \frac{6+6}{24} = \frac{1}{2} < \frac{7}{12}, \end{aligned}$$

$(\alpha \in Aut(V_4))$ which satisfies Lemma 2.1.

The following theorem is a generalization of Lemma 2.1 of [1]. Moreover its first part implies Lemma 2.2, Proposition 2.1 and Theorem 2.2 of [8].

**2.3. Theorem.** *Let $G$ be a finite group and $H \leq K \leq G$. Then $d_{aut}(K,G) \leq d_{aut}(H,G)$, and equality holds if and only if $K = HC_K(\alpha)$ for all $\alpha \in Aut(G)$.*

*Proof.* Put $A = \{h \in H | h^\alpha = h\}$ and $B = \{k \in K | k^\alpha = k\}$. Clearly the map $\{hA | h \in H\} \to \{kB | k \in K\}$, with $hA \mapsto hB$ is one-to-one. So we have $\frac{|H|}{|\{\ h\ |\ \ h^\alpha = h\ \}|} \leq \frac{|K|}{|\{\ k\ |\ \ k^\alpha = k\ \}|}$, that is $\frac{|C_K(\alpha)|}{|K|} \leq \frac{|C_H(\alpha)|}{|H|}$ for each $\alpha \in Aut(G)$. Hence

$$\begin{aligned} d_{aut}(K,G) &= \frac{1}{|Aut(G)|} \sum_{\alpha \in Aut(G)} \frac{|C_K(\alpha)|}{|K|} \\ &\leq \frac{1}{|Aut(G)|} \sum_{\alpha \in Aut(G)} \frac{|C_H(\alpha)|}{|H|} \\ &= d_{aut}(H,G). \end{aligned}$$

Also $d_{aut}(K,G) = d_{aut}(H,G)$ if and only if $\frac{|C_K(\alpha)|}{|K|} = \frac{|C_H(\alpha)|}{|H|}$ for all $\alpha \in Aut(G)$, which is equivalent to $K = HC_K(\alpha)$ for all $\alpha \in Aut(G)$. □

Clearly for any subgroup $H \leq G$, we have $d_{aut}(G) \leq d_{aut}(H,G)$. For example, in the Klein four-group we have $d_{aut}(V_4) = \frac{1}{2} \leq \frac{2}{3} = d_{aut}(< a >, V_4)$.

**2.4. Definition.** Two groups $G$ and $H$ are autoisoclinic, (written $G \sim_{aut} H$), if there exists isomorphisms $\alpha$, $\beta$ and $\gamma$, as follows:

$$\alpha : \frac{G}{L(G)} \longrightarrow \frac{H}{L(H)}$$

$$\beta : K(G) \longrightarrow K(H)$$

$$\gamma : Aut(G) \longrightarrow Aut(H),$$

where $\alpha$ induces $\beta$ in following sense: if $g \in G$, $h \in \alpha(gL(G))$ and if $\varphi_H = \gamma(\varphi_G)$ , then $\beta([g, \varphi_G]) = [h, \varphi_H]$.
The pair $(\alpha \times \gamma, \beta)$ is called an autoisoclinism between $G$ and $H$, see also [7] (and [6] for isoclinism).

Let $(\alpha \times \gamma, \beta)$ be an autoisoclinism between the groups $G$ and $H$, then the following diagram is commutative:

$$\begin{tikzcd}
\frac{G}{L(G)} \times Aut(G) \arrow[r, "\alpha \times \gamma"] \arrow[d] & \frac{H}{L(H)} \times Aut(H) \arrow[d] \\
K(G) \arrow[r, "\beta"] & K(H)
\end{tikzcd}$$

Next, we present the following lemma which is similar to Lemma 2.4 of [6].

**2.5. Lemma.** *Let $G$ and $H$ be two autoisoclinic finite groups, then* $d_{aut}(G) = d_{aut}(H)$.

*Proof.* Let pair $(\alpha \times \gamma, \beta)$ be an autoisoclinism from $G$ to $H$, then one has

$$\begin{aligned}
|\frac{G}{L(G)}||Aut(G)|d_{aut}(G) &= \frac{1}{|L(G)|}|G||Aut(G)|d_{aut}(G) \\
&= \frac{1}{|L(G)|}|\{(g, \varphi_G) \in G \times Aut(G)|\ [g, \varphi_G] = 1\ \}| \\
&= |\{(gL(G), \varphi_G) \in \frac{G}{L(G)} \times Aut(G)|\ \beta([g, \varphi_G]) = 1\ \}| \\
&= |\{(hL(H), \varphi_H) \in \frac{H}{L(H)} \times Aut(H)|\ [h, \varphi_H] = 1\ \}| \\
&= \frac{1}{|L(H)|}|\{(h, \varphi_H) \in H \times Aut(H)|\ [h, \varphi_H] = 1\ \}| \\
&= \frac{1}{|L(H)|}|H||Aut(H)|d_{aut}(H) \\
&= |\frac{H}{L(H)}||Aut(H)|d_{aut}(H).
\end{aligned}$$

But $\frac{G}{L(G)}$ and $\frac{H}{L(H)}$ are isomorphic, so $|\frac{G}{L(G)}| = |\frac{H}{L(H)}|$. From the fact that $Aut(G)$ and $Aut(H)$ are isomorphic, we conclude $|Aut(G)| = |Aut(H)|$, from which the equality $d_{aut}(G) = d_{aut}(H)$ follows. □

## 3. Results on autoisoclinism

We begin this section by establishing some elementary lemmas which will be used to derive the results on autoisoclinism (see also [2, 6]). Now, we present the following lemma which is similar to Lemma 2.6 of [6].

**3.1. Lemma.** *Let $S$ be a characteristically simple group of order more than two, then any group $G$ autoisoclinic to $S$ is isomorphic to $S \times L(G)$.*

*Proof.* By order of $S$ we have $K(S) \neq < 1 >$ and $L(S) \neq S$. Suppose $G \sim_{aut} S$. Hence $K(G) \simeq K(S)$. Since $S$ is a characteristically simple group, then $K(S) = S \simeq K(G)$. Thus $K(G) \cap L(G) \subseteq L(K(G)) = \{1\}$. On the other hand $\frac{G}{L(G)} \simeq \frac{S}{L(S)} \simeq S$ because $G \sim_{aut} S$ and $S$ is a characteristically simple group. Hence

$$\begin{aligned}
&K(\frac{G}{L(G)}) \simeq K(S) = S \simeq \frac{G}{L(G)} \\
&\Rightarrow \frac{L(G)K(G)}{L(G)} = K(\frac{G}{L(G)}) = \frac{G}{L(G)} \\
&\Rightarrow G = K(G)L(G) = K(G) \times L(G) \simeq S \times L(G).
\end{aligned}$$

□

**3.2. Lemma.** *Let $H$ be a finite subgroup of $G$, where $H$ and $G$ are autoisoclinic, then $G = HL(G)$.*

*Proof.* If $H$ is finite group autoisoclinic to $G$, then $\frac{G}{L(G)} \simeq \frac{H}{L(H)}$ is also finite. But

$$\begin{aligned} |\frac{G}{L(G)}| &\geq |\frac{HL(G)}{L(G)}| \\ &= |\frac{H}{H \cap L(G)}| \\ &= |\frac{H}{L(H)}||\frac{L(H)}{H \cap L(H)}| \\ &\geq |\frac{H}{L(H)}| \\ &= |\frac{G}{L(G)}|. \end{aligned}$$

This implies that $G = HL(G)$. □

**3.3. Lemma.** *Let $H$ be a characteristic subgroup of finite group $G$, and $G = HL(G)$ such that $Aut(H) \simeq Aut(G)$. Then $G$ and $H$ are autoisoclinic.*

*Proof.* If $H$ is a characteristic subgroup of $G$, then $H \cap L(G) \subseteq L(H)$. Also $L(H) \simeq H \cap L(G)$ because $Aut(H) \simeq Aut(G)$, and

$$\begin{aligned} \frac{H}{L(H)} &= \frac{H}{L(G) \cap H} \\ &\simeq \frac{HL(G)}{L(G)} \\ &= \frac{G}{L(G)}, \end{aligned}$$

the isomorphism $i_1 : H/L(H) \longrightarrow G/L(G)$ being induced by the inclusion $i : H \longrightarrow G$. Furthermore, let $g \in G$, $\alpha \in Aut(G)$, then $g = lh$ for some $l \in L(G)$ and $h \in H$. Hence $[g, \alpha] = [lh, \alpha] = (lh)^{-1}(lh)^{\alpha} = h^{-1}l^{-1}l^{\alpha}h^{\alpha} = h^{-1}h^{\alpha} = [h, \alpha] \in K(H)$, On the other hand $K(H) \simeq \langle [h, \alpha] \mid h \in H, \alpha \in Aut(G) \rangle \subseteq K(G)$, and so $K(G) = K(H)$. This argument shows that $(i_1 \times 1_{Aut(G)}, 1_{K(G)})$ is an autoisoclinism from $H$ to $G$. □

**3.4. Theorem.** *Let $G$ be a finite group such that $G = HL(G)$ where $H$ is a characteristic subgroup of $G$ with $Aut(H) \simeq Aut(G)$. Then $d_{aut}(G) = d_{aut}(H)$.*

*Proof.* It follows from Lemma 2.5 and Lemma 3.3 . □

The following lemma is similar to Lemma 1.3 of [2], will be used in the next theorem.

**3.5. Lemma.** *Let $G$ be a group with characteristic subgroup $N$. Then $G/N \sim_{aut} G/(N \cap K(G))$. In particular, if $N \cap K(G) = \{1\}$ , then $G \sim_{aut} G/N$. Conversely, if $|K(G)| < \infty$ and $G \sim_{aut} G/N$, then $N \cap K(G) = \{1\}$.*

*Proof.* We set $\bar{G} = G/N$ and $\hat{G} = G/(N \cap K(G))$. For any $k_1, k_2 \in K(G)$, $\bar{k}_1 = \bar{k}_2 \Leftrightarrow \hat{k}_1 = \hat{k}_2$. For $g \in G$ and $\varphi \in Aut(G)$, we have therefore, $[\bar{g}, \bar{\varphi}] = \bar{1} \Leftrightarrow [\hat{g}, \hat{\varphi}] = \hat{1}$(because $N$ is characteristic subgroup of $G$), where $\bar{\varphi} : gN \to \varphi(g)N$ and $\hat{\varphi} : g(N \cap K(G)) \to \varphi(g)(N \cap K(G))$. This implies that $\bar{g} \in L(\bar{G})$ if and only if $\hat{g} \in L(\hat{G})$. Let $\alpha(\bar{g}L(\bar{G})) = \hat{g}L(\hat{G})$. Then $\alpha$ is an isomorphism of $\bar{G}/L(\bar{G})$ onto $\hat{G}/L(\hat{G})$. If $\gamma(\bar{\varphi}_{\bar{G}}) = \hat{\varphi}_{\hat{G}}$, then $\gamma$ is an isomorphism of $Aut(\bar{G})$ onto $Aut(\hat{G})$. Let $k \in K(G)$ and denote $\beta(\bar{k}) = \hat{k}$. Then $\beta$ defines an isomorphism of $K(\bar{G})$ onto $K(\hat{G})$ and $\beta$ is induced by $\alpha$ in Definition 2.4 .

Conversely, if $N \trianglelefteq G$ and $G \sim_{aut} G/N$, then

$$K(G) \simeq K(G/N) = K(G)N/N \simeq K(G)/(N \cap K(G)).$$

Thus, if $|K(G)| < \infty$, then $N \cap K(G) = \{1\}$. □

The following theorem follows from the above lemma and is similar to Theorem 1.4 of [2].

**3.6. Theorem.** *Let $G$ and $H$ be finite groups. Then $G$ and $H$ are autoisoclinic if and only if there exists finite groups $C$, $L_G$, $L_H$, $C_G$ and $C_H$ such that $G \simeq C/L_H$ and $H \simeq C/L_G$ and the following two (equivalent) properties hold:*

(i) $G \simeq C/L_H \sim_{aut} C \sim_{aut} C/L_G \simeq H$,

(ii) $C/L_H \times C/K(C) \sim_{aut} C_H \simeq C \simeq C_G \sim_{aut} C/L_G \times C/K(C)$, *where $C_H$ and $C_G$ are subgroups of $C/L_H \times C/K(C)$ and $C/L_G \times C/K(C)$ respectively.*

*Proof.* One part of the theorem is trivial. Assume that $G \sim_{aut} H$, and let $\beta$ be the isomorphism between $K(G)$ and $K(H)$ given in Definition 2.4. Finally, let $C$ be the direct product of $G$ and $H$ with identified factor groups $G/L(G)$ and $H/L(H)$. If

$$L_H = \{(1,l) \mid l \in L(H)\ \} \text{ and } L_G = \{(l,1) \mid l \in L(G)\ \},$$

then we have $C/L_H \simeq G$ and $C/L_G \simeq H$, where $L_H \simeq L(H)$ and $L_G \simeq L(G)$.

(i) It follows from Definition 2.4 that $K(C)$ is generated by elements of the form

$$([g,\varphi_G],\beta([g,\varphi_G]))$$

where $g \in G, \varphi_G \in Aut(G)$. We claim that

$$K(C) \cap L_G = K(C) \cap L_H = 1.$$

For, if $(1,l) = (g,h) \in K(C)$, then $h = \beta(g) = 1$. Similarly for $K(C) \cap L_G$. By Lemma 3.5 we therefore have

$$C/L_H \sim_{aut} C \sim_{aut} C/L_G.$$

(ii) Let

$$C_G = \{(cL_G, cK(C) \mid c \in C\}.$$

$C_G$ is a group, isomorphic to $C$, since $K(C) \cap L_G = \{1\}$. Moreover, it follows from Lemma 3.3 that $C_G \sim_{aut} C/L_G \times C/K(C)$. Now we have the equality

$$C_G L(C/L_G \times C/K(C)) = C/L_G \times C/K(C).$$

To see this, let $x = (c_1 L_G, c_2 K(C))$ be an element of the direct product of the groups $C/L_G$ and $C/K(C)$. Then $x = yl$, where $y = (c_1 L_G, c_1 K(C)) \in C_G$, and $l = (L_G, {c_1}^{-1} c_2 K(C))$. Since $C/K(C)$ is an autoabelian group, then $L(C/K(C)) = C/K(C)$ and $L_G$ is identity of $C/L_G$. It follows that $l \in L(C/L_G \times C/K(C))$.
Similarly

$$C \simeq C_H \sim_{aut} C/L_H \times C/K(C).$$

□

**Acknowledgments**. The authors are grateful to referee for the useful comments, which improved the presentation of the paper. The first author was also partially supported by the Center of Excellence for Mathematics, Shahrekord University, Iran.

## References


[1] R. Barzegar, A. Erfanian and M. Farrokhi D.G, Finite groups with three relative commutativity degrees, *Bull. Iranian Math. Soc.*, **39**(2) (2013) 271-280.

[2] J. C. Bioch, On n-isoclinic groups, *Indag. Math.*, **79** (1976) 400-407.

[3] A. Erfanian, R. Rezaei and P. Lescot, On the relative commutativity degree of a subgroup of a finite group, *Comm. Algebra*, **35**(12) (2007) 4183-4197.

[4] W. H. Gustafson, What is the probability that two group elements commute ?, *Amer. Math. Monthly*, **80** (1973) 1031-1034.

[5] P. V. Hegarty, The absolute centre of a group, *J. Algebra*, **169**(3) (1994) 929-935.

[6] P. Lescot, Isoclinism classes and commutativity degrees of finite groups, *J. Algebra*, **177** (1995) 847-869.

[7] M. R. R. Moghaddam, M. J. Sadeghifard and M. Eshrati, Some properties of autoisoclinism of groups, *Fifth International group theory conference, Islamic Azad university, Mashhad-Iran*, 13-15 March 2013.

[8] M. R. R. Moghaddam, F. Saeedi and E. Khamseh, The probability of an automorphism fixing a subgroup element of a finite group, *Asian-Eur. J. Math.*, **4**(2) (2011) 301-308.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 901 – 908

# The frobenius problem for some numerical semigroups with embedding dimension equal to three

Aureliano M. Robles-Pérez* † and José Carlos Rosales* ‡

**Abstract**

If $S$ is a numerical semigroup with embedding dimension equal to three whose minimal generators are pairwise relatively prime numbers, then $S = \langle a, b, cb - da \rangle$ with $a, b, c, d$ positive integers such that $\gcd(a, b) = \gcd(a, c) = \gcd(b, d) = 1$, $c \in \{2, \ldots, a-1\}$, and $a < b < cb - da$. In this paper we give formulas, in terms of $a, b, c, d$, for the genus, the Frobenius number, and the set of pseudo-Frobenius numbers of $\langle a, b, cb - da \rangle$ in the case in which the interval $\left[\frac{a}{c}, \frac{b}{d}\right]$ contains some integer.

*2000 AMS Classification:* 11D07, 20M14.

**Keywords:** Frobenius problem, numerical semigroups, Apéry set.

Received 03/11/2012 : Accepted 27/03/2014 Doi : 10.15672 /HJMS.2015449432

## 1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of nonnegative integers, respectively. A *numerical semigroup* is a subset $S$ of $\mathbb{N}$ such that it is closed under addition, $0 \in S$, and $\mathbb{N} \setminus S$ is finite. The elements of $\mathbb{N} \setminus S$ are the *gaps* of $S$, and the cardinality of such set is called the *genus* of $S$, denoted by $\mathrm{g}(S)$. The *Frobenius number* of $S$ is the largest integer that does not belong to $S$ and it is denoted by $\mathrm{F}(S)$.

If $A \subseteq \mathbb{N}$ is a nonempty set, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by $A$, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\},\ a_1, \ldots, a_n \in A, \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

*Both of the authors are supported by FQM-343 (Junta de Andalucía), MTM2010-15595 (MICINN, Spain), and FEDER funds. The second author is also partially supported by Junta de Andalucía/Feder grant number FQM-5849.

†Departamento de Matemática Aplicada, Universidad de Granada, 18071-Granada, Spain. Email: `arobles@ugr.es`

‡Departamento de Álgebra, Universidad de Granada, 18071-Granada, Spain. Email: `jrosales@ugr.es`

In [16] it is proved that $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A)=1$, where gcd means *greatest common divisor*.

It is well known (see [16]) that every numerical semigroup $S$ is finitely generated, that is, there exists a finite subset $G\subseteq S$ such that $S=\langle G\rangle$. In addition, if no proper subset of $G$ generates $S$, then we say that $G$ is a *minimal system of generators* of $S$. In [16] it is proved that every numerical semigroup admits a unique minimal system of generators. The cardinality of such a set is known as the *embedding dimension* of $S$, denoted by $\mathrm{e}(S)$.

The Frobenius problem (see [8]) consists of finding formulas that allow us to compute, in terms of the minimal system of generators of a numerical semigroup, the Frobenius number and the genus of such a numerical semigroup. This problem was solved by Sylvester and Curran Sharp (see [18, 19, 20]) when the embedding dimension is equal to two. In fact, if $S$ is a numerical semigroup with minimal system of generators $\{n_1,n_2\}$, then $\mathrm{F}(S)=n_1n_2-n_1-n_2$ and $\mathrm{g}(S)=\frac{(n_1-1)(n_2-1)}{2}$.

At present, the Frobenius problem is open for the case of embedding dimension equal to three. To be precise, Curtis proved in [2] that it is impossible to find a polynomial formula that solves the problem of Frobenius number. On the other hand, algorithms that compute Frobenius number, quasi-formulas, and upper bounds for such number are the topic of several contributions (see [4, 8, 9, 11, 12]). In addition, the authors showed in [10] that, if the multiplicity of $S$ is fixed, then it is possible to give explicit formulas for the Frobenius number. In this paper, our purpose is to give simple formulas in a particular but extensive case (in the line of [7] and some results collected in [8]).

If $\{n_1,n_2,n_3\}$ is the minimal system of generators of a numerical semigroup $S$ and $d=\gcd(n_1,n_2)$, then we have (see [6, 11]) that $\mathrm{F}(S)=d\,\mathrm{F}\left(\langle\frac{n_1}{d},\frac{n_2}{d},n_3\rangle\right)+(d-1)n_3$ and $\mathrm{g}(S)=d\,\mathrm{g}\left(\langle\frac{n_1}{d},\frac{n_2}{d},n_3\rangle\right)+\frac{(d-1)(n_3-1)}{2}$. Therefore, in order to solve the Frobenius problem for numerical semigroups with embedding dimension equal to three, we focus our attention on numerical semigroups whose three minimal generators are pairwise relatively prime numbers.

If $S$ is a numerical semigroup and $m\in S\setminus\{0\}$, then the *Apéry set* of $m$ in $S$ (see [1]) is $\mathrm{Ap}(S,m)=\{s\in S\mid s-m\notin S\}$. Obviously, $\mathrm{Ap}(S,m)=\{w(0)=0,w(1),\ldots,w(m-1)\}$, where $w(i)$ is the least element of $S$ congruent with $i$ modulo $m$, for all $i\in\{0,\ldots,m-1\}$. It is clear that $\mathrm{F}(S)=\max\{\mathrm{Ap}(S,m)\}-m$, and a formula for $\mathrm{g}(S)$ in terms of $\mathrm{Ap}(S,m)$ is given in [17] (see Lemma 3.1).

Following the notation introduced in [14], we say that $x\in\mathbb{Z}\setminus S$ is a *pseudo-Frobenius number* of $S$ if $x+s\in S$ for all $s\in S\setminus\{0\}$. We will denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of $S$, and its cardinality is the *type* of $S$, denoted by $\mathrm{t}(S)$. From the definition it follows that $\mathrm{F}(S)=\max\{\mathrm{PF}(S)\}$.

In [15] it is shown that, if $S$ is a numerical semigroup with embedding dimension equal to three whose three minimal generators are pairwise relatively prime numbers, it is possible to describe $S$ in function of six positive integers $r_{12},r_{13},r_{21},r_{23},r_{31},r_{32}$. Then, it is possible to give formulas for $\mathrm{F}(S)$, $\mathrm{g}(S)$, and $\mathrm{PF}(S)$ in terms of such parameters. In this paper we will show that we can reduce the number of parameters to four $(a,b,c,d)$ and we will give the formulas for $\mathrm{F}(S)$, $\mathrm{g}(S)$, and $\mathrm{PF}(S)$ in the case in which $\left[\frac{a}{c},\frac{b}{d}\right]\cap\mathbb{N}\neq\emptyset$. We left as an open problem the case $\left[\frac{a}{c},\frac{b}{d}\right]\cap\mathbb{N}=\emptyset$.

We summarize the content of this paper. Let us denote

$$\begin{aligned}\mathcal{F}=\{\langle a,b,cb-da\rangle\mid\ & a,b,c,d\in\mathbb{N}\setminus\{0\},\ \gcd(a,b)=\gcd(a,c)=\gcd(b,d)=1,\\ & 2\le c\le a-1,\ \text{and}\ a<b<cb-da\}\,.\end{aligned}$$

First of all, we observe that $\mathcal{F}$ is the set of all numerical semigroups with embedding dimension equal to three whose minimal generators are pairwise relatively prime numbers. Let us consider $S=\langle a,b,cb-da\rangle\in\mathcal{F}$. The main result in Section 2 is Theorem 2.8, where we give $\mathrm{Ap}(S,a)$ in an explicit way when $\left[\frac{a}{c},\frac{b}{d}\right]\cap\mathbb{N}\neq\emptyset$. As a consequence of this result, in Section 3, we give the formulas for $\mathrm{F}(S)$, $\mathrm{g}(S)$, and $\mathrm{PF}(S)$ in the above mentioned case.

## 2. The Apéry set

Our purpose in this section is to prove Theorem 2.8, where we will show explicitly an Apéry set for a particular family of numerical semigroups with embedding dimension three. It is well known that the Apéry set allows us to solve easily the Frobenius problem, as well as simplify many questions about numerical semigroups, such as the membership problem (that is, determine if a positive integer belongs to a numerical semigroup).

First we need to introduce some results. The following lemma has an immediate proof (see [16, Lemma 2.6]).

**2.1. Lemma.** *Let $S$ be a numerical semigroup and $m\in S\setminus\{0\}$. Then, for every $x\in\mathbb{Z}$, there exist a unique $(\lambda,w)\in\mathbb{Z}\times\mathrm{Ap}(S,m)$ such that $x=\lambda m+w$. Moreover, $x\in S$ if and only if $\lambda\in\mathbb{N}$.*

Let $p,q$ be two integers such that $q\neq 0$. We denote by $\left\lfloor\frac{p}{q}\right\rfloor$ and $p \bmod q$ the quotient and the remainder of the integer division of $p$ by $q$, respectively. The next result follows from [13, Lemma 3.3].

**2.2. Lemma.** *Let $m$ be a positive integer. Let $\{X(0)=0,X(1),\dots,X(m-1)\}$ be a subset of $\mathbb{N}$ such that $X(i) \bmod m = i$, for all $i\in\{0,1,\dots,m-1\}$. Let $S=\langle m,X(1),\dots,X(m-1)\rangle$. Then $\mathrm{Ap}(S,m)=\{X(0),X(1),\dots,X(m-1)\}$ if and only if $X(i)+X(j)\geq X((i+j) \bmod m)$, for all $i,j\in\{1,\dots,m-1\}$.*

The following lemma is well known (see, for instance, [17]).

**2.3. Lemma.** *Let $n_1,n_2$ be two positive integers such that $\gcd(n_1,n_2)=1$. Let $S=\langle n_1,n_2\rangle$. Then $\mathrm{Ap}(S,n_1)=\{0,n_2,2n_2,\dots,(n_1-1)n_2\}$.*

**2.4. Lemma.** *Let $S$ be a numerical semigroup with minimal system of generators given by $\{n_1,n_2,n_3\}$. If $\gcd(n_1,n_2)=1$, then there exist two unique numbers $k\in\{2,\dots,n_1-1\}$ and $t\in\{1,\dots,n_2-1\}$ such that $n_3=kn_2-tn_1$.*

*Proof.* Since $n_3\notin\langle n_1,n_2\rangle$, by Lemmas 2.1 and 2.3, we deduce that there exist unique numbers $k\in\{0,1,\dots,n_1-1\}$ and $t\in\mathbb{N}\setminus\{0\}$ such that $n_3=kn_2-tn_1$. Since $n_3>0$, it is obvious that $t\in\{1,\dots,n_2-1\}$. In order to finish the proof we have to see that $k\notin\{0,1\}$. If $k=0$, then $n_3=-tn_1$, which is a contradiction to the positiveness of $n_3$. If $k=1$, then $n_2=n_3+tn_1\in\langle n_1,n_3\rangle$, which is a contradiction because $\{n_1,n_2,n_3\}$ is a minimal system of generators. □

**2.5. Remark.** By using Euclidean algorithm, we have that, if $\gcd\{n_1,n_2\}=1$, then there exist two positive integers $u,v$ such that $un_2-vn_1=1$. Thus, $n_3=(un_3)n_2-(vn_3)n_1=\left(\left\lfloor\frac{un_3}{n_1}\right\rfloor n_1+(un_3) \bmod n_1\right)n_2-(vn_3)n_1$. Therefore $n_3=((un_3) \bmod n_1)n_2-\left(vn_3-\left\lfloor\frac{un_3}{n_1}\right\rfloor n_2\right)n_1$. We conclude that, in Lemma 2.4, $k=(un_3) \bmod n_1$ and $t=vn_3-\left\lfloor\frac{un_3}{n_1}\right\rfloor n_2$.

**2.6. Remark.** Let $S$ be a numerical semigroup with minimal system of generators $\{n_1,n_2,n_3\}$. If $n_1,n_2,n_3$ are pairwise relatively prime numbers and $n_1<n_2<n_3$, taking $a=n_1$, $b=n_2$, $c=k$, $d=t$, and $cb-da=kn_2-tn_1=n_3$, we deduce from Lemma 2.4 that $S\in\mathcal{F}$. On the other hand, it is easy to see that any element of $\mathcal{F}$ is a numerical semigroup with embedding dimension equal to three whose minimal generators are pairwise relatively prime numbers.

**2.7. Lemma.** *Let $a, b, c, d$ be four positive integers such that* $\gcd(a, b) = 1$ *and* $cb - da \geq d((-a) \bmod c)$. *Let* $S = \left\langle \left\{ \alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \mid \alpha \in \{0, \ldots, a-1\} \right\} \cup \{a\} \right\rangle$. *Then*

$$\mathrm{Ap}(S, a) = \left\{ \alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \mid \alpha \in \{0, \ldots, a-1\} \right\}.$$

*Proof.* Since $cb - da \geq 0$, it follows that $\alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \in \mathbb{N}$, for all $\alpha \in \{0, \ldots, a-1\}$. Moreover, $\left\{ \alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \mid \alpha \in \{0, \ldots, a-1\} \right\}$ is a subset of $\mathbb{N}$ such that it has the form $\{X(0) = 0, X(1), \ldots, X(a-1)\}$ with $X(i) \bmod a = i$, for all $i \in \{0, 1, \ldots, a-1\}$. Then, by using Lemma 2.2, we conclude the proof if we show both of the next two statements.

(1) If $\alpha, \beta \in \{1, \ldots, a-1\}$ and $\alpha + \beta \leq a - 1$, then

$$\alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da + \beta b - \left\lfloor \frac{\beta}{c} \right\rfloor da \geq (\alpha + \beta) b - \left\lfloor \frac{\alpha + \beta}{c} \right\rfloor da.$$

(2) If $\alpha, \beta \in \{1, \ldots, a-1\}$ and $\alpha + \beta \geq a$, then

$$\alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da + \beta b - \left\lfloor \frac{\beta}{c} \right\rfloor da \geq (\alpha + \beta - a) b - \left\lfloor \frac{\alpha + \beta - a}{c} \right\rfloor da.$$

The first one follows from the inequality $\left\lfloor \frac{\alpha+\beta}{c} \right\rfloor \geq \left\lfloor \frac{\alpha}{c} \right\rfloor + \left\lfloor \frac{\beta}{c} \right\rfloor$. The second one is equivalent to $\frac{b}{d} \geq \left\lfloor \frac{\alpha}{c} \right\rfloor + \left\lfloor \frac{\beta}{c} \right\rfloor - \left\lfloor \frac{\alpha+\beta-a}{c} \right\rfloor$. Since $\left\lfloor \frac{x}{c} \right\rfloor c = x - x \bmod c$, for all $x \in \mathbb{N}$, if we multiply both sides of the inequality by $c$, then it suffices to prove that

$$\frac{cb}{d} \geq \alpha - \alpha \bmod c + \beta - \beta \bmod c - (\alpha + \beta - a) + (\alpha + \beta - a) \bmod c,$$

or equivalently, that

$$\frac{cb}{d} \geq a + (\alpha + \beta - a) \bmod c - \alpha \bmod c - \beta \bmod c.$$

Now, let us observe that $(\alpha + \beta - a) \bmod c \leq \alpha \bmod c + \beta \bmod c + (-a) \bmod c$. Since $cb - da \geq d((-a) \bmod c)$, we conclude that

$$\begin{aligned}\frac{cb}{d} &\geq a + (-a) \bmod c \\ &= a + (-a) \bmod c + \alpha \bmod c + \beta \bmod c - \alpha \bmod c - \beta \bmod c \\ &\geq a + (\alpha + \beta - a) \bmod c - \alpha \bmod c - \beta \bmod c. \qquad \square\end{aligned}$$

Let $q$ be a rational number. As usual, we denote by $\lceil q \rceil$ the minimum of the set $\{z \in \mathbb{Z} \mid q \leq z\}$. At this point, we are in a position to prove the main result of this section.

**2.8. Theorem.** *Let* $S = \langle a, b, cb - da \rangle \in \mathcal{F}$. *If* $\left[ \frac{a}{c}, \frac{b}{d} \right] \cap \mathbb{N} \neq \emptyset$, *then*

$$\mathrm{Ap}(S, a) = \left\{ \alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \mid \alpha \in \{0, \ldots, a-1\} \right\}.$$

*Proof.* First of all, let us observe that, since $\left\lceil \frac{x}{c} \right\rceil c = x + (-x) \bmod c$, for all $x \in \mathbb{N}$, then

$$cb - da \geq d((-a) \bmod c) \Leftrightarrow cb - da \geq d\left( \left\lceil \frac{a}{c} \right\rceil c - a \right) \Leftrightarrow \frac{b}{d} \geq \left\lceil \frac{a}{c} \right\rceil.$$

Obviously, the last inequality is precisely the condition $\left[ \frac{a}{c}, \frac{b}{d} \right] \cap \mathbb{N} \neq \emptyset$.

Now, from Lemma 2.7, if $\bar{S} = \left\langle \left\{ \alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \mid \alpha \in \{0, \ldots, a-1\} \right\} \cup \{a\} \right\rangle$, then we have that $\mathrm{Ap}(\bar{S}, a) = \left\{ \alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \mid \alpha \in \{0, \ldots, a-1\} \right\}$. Therefore, to finish the proof, it is enough to show that $S = \bar{S}$.

Since $c \geq 2$, then $\left\lfloor \frac{1}{c} \right\rfloor = 0$ and $b = b - \left\lfloor \frac{1}{c} \right\rfloor da \in \bar{S}$. Moreover, it is obvious that $a \in \bar{S}$ and $cb - da = cb - \left\lfloor \frac{c}{c} \right\rfloor da \in \bar{S}$. Therefore, $S = \langle a, b, cb - da \rangle \subseteq \bar{S}$. For the other inclusion, let us take $x \in \bar{S}$. From Lemma 2.1, we deduce that there exist $\lambda \in \mathbb{N}$ and

$\alpha \in \{0,\dots,a-1\}$ such that $x = \lambda a + \left(\alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da\right)$. Since $x = \lambda a + \left(\left\lfloor \frac{\alpha}{c} \right\rfloor c + \alpha \bmod c\right) b - \left\lfloor \frac{\alpha}{c} \right\rfloor da = \lambda a + (\alpha \bmod c) b + \left\lfloor \frac{\alpha}{c} \right\rfloor (cb - da)$, then $x \in \langle a, b, cb - da \rangle = S$. $\square$

**2.9. Remark.** Let us observe that $\mathcal{F}$ contains infinitely many numerical semigroups (in fact, as it is pointed out in Remark 2.6, all numerical semigroups with embedding dimension equal to three whose minimal generators are pairwise relatively prime numbers) but not all of them satisfy the condition of Theorem 2.8, that is, $\left[\frac{a}{c}, \frac{b}{d}\right]$ contains some integer. For example, the numerical semigroup $S = \langle 16, 19, 7 \times 19 - 7 \times 16 \rangle = \langle 16, 19, 21 \rangle$ does not satisfy that condition. Even more (see Remark 3.8 and Example 3.13), no possible combination of $a, b, c, d$ for this numerical semigroup satisfies the condition of Theorem 2.8.

**2.10. Example.** Let $S = \langle 5, 7, 3 \times 7 - 2 \times 5 \rangle = \langle 5, 7, 11 \rangle$. Since $\frac{7}{2} \geq \left\lceil \frac{5}{3} \right\rceil$, by Theorem 2.8, we have that

$$\operatorname{Ap}(S, 5) = \left\{ \alpha \times 7 - \left\lfloor \frac{\alpha}{3} \right\rfloor \times 2 \times 5 \mid \alpha \in \{0, 1, 2, 3, 4\} \right\} = \{0, 7, 14, 11, 18\}.$$

## 3. The genus and the pseudo-Frobenius numbers

Along this section $S$ is a numerical semigroup which belongs to $\mathcal{F}$. Therefore, $S = \langle a, b, cb - da \rangle$ with $a, b, c, d$ positive integers such that $\gcd(a, b) = \gcd(a, c) = \gcd(b, d) = 1$, $2 \leq c \leq a - 1$, and $a < b < cb - da$. Moreover, we suppose that $\frac{b}{d} \geq \left\lceil \frac{a}{c} \right\rceil$. Our purpose is to give formulas for $\mathrm{g}(S)$, $\mathrm{PF}(S)$, and $\mathrm{F}(S)$.

The following result appears in [17].

**3.1. Lemma.** *If $T$ is a numerical semigroup and $m \in T \setminus \{0\}$, then*

$$\mathrm{g}(T) = \left( \frac{1}{m} \sum_{w \in \operatorname{Ap}(T, m)} w \right) - \frac{m-1}{2}.$$

Let us show a formula to compute $\mathrm{g}(S)$.

**3.2. Proposition.** *Let $S$ be a numerical semigroup which satisfies the conditions stated at the beginning of this section. Then*

$$\mathrm{g}(S) = \frac{(b-1)(a-1)}{2} - d \left\lfloor \frac{a-1}{c} \right\rfloor \left( a - \frac{c}{2} \left( \left\lfloor \frac{a-1}{c} \right\rfloor + 1 \right) \right).$$

*Proof.* From Theorem 2.8 and Lemma 3.1, we have that

$$\begin{aligned}
\mathrm{g}(S) &= \frac{1}{a} \sum_{\alpha=1}^{a-1} \left( \alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \right) - \frac{a-1}{2} = \frac{1}{a} \left( b \sum_{\alpha=1}^{a-1} \alpha - da \sum_{\alpha=1}^{a-1} \left\lfloor \frac{\alpha}{c} \right\rfloor \right) - \frac{a-1}{2} \\
&= \frac{1}{a} \left( b \frac{a(a-1)}{2} - da \sum_{\alpha=1}^{a-1} \left\lfloor \frac{\alpha}{c} \right\rfloor \right) - \frac{a-1}{2} = \frac{(b-1)(a-1)}{2} - d \sum_{\alpha=1}^{a-1} \left\lfloor \frac{\alpha}{c} \right\rfloor.
\end{aligned}$$

In order to finish the proof we need to show that

$$\sum_{\alpha=1}^{a-1} \left\lfloor \frac{\alpha}{c} \right\rfloor = \left\lfloor \frac{a-1}{c} \right\rfloor \left( a - \frac{c}{2} \left( \left\lfloor \frac{a-1}{c} \right\rfloor + 1 \right) \right).$$

Let us observe that $\alpha \in \{ic, ic+1, \dots, ic+(c-1)\}$ if and only if $\left\lfloor \frac{\alpha}{c} \right\rfloor = i$. Therefore,

$$\begin{aligned}\sum_{\alpha=1}^{a-1} \left\lfloor \frac{\alpha}{c} \right\rfloor &= \sum_{j=1}^{\left\lfloor \frac{a-1}{c} \right\rfloor -1} \left( \sum_{\alpha=0}^{c-1} j \right) + \sum_{\alpha=0}^{a-1-\left\lfloor \frac{a-1}{c} \right\rfloor c} \left\lfloor \frac{a-1}{c} \right\rfloor \\ &= c \sum_{j=1}^{\left\lfloor \frac{a-1}{c} \right\rfloor -1} j + \left\lfloor \frac{a-1}{c} \right\rfloor \left( a - \left\lfloor \frac{a-1}{c} \right\rfloor c \right) \\ &= \left\lfloor \frac{a-1}{c} \right\rfloor \left( \frac{c}{2} \left( \left\lfloor \frac{a-1}{c} \right\rfloor - 1 \right) + a - c \left\lfloor \frac{a-1}{c} \right\rfloor \right). \end{aligned}$$ □

**3.3. Example.** Let $S = \langle 5, 7, 3 \times 7 - 2 \times 5 \rangle = \langle 5, 7, 11 \rangle$. Since $\frac{7}{2} \geq \left\lceil \frac{5}{3} \right\rceil$, applying Proposition 3.2, we have that

$$\mathrm{g}(S) = \frac{6 \times 4}{2} - 2 \left\lfloor \frac{4}{3} \right\rfloor \left( 5 - \frac{3}{2} \left( \left\lfloor \frac{4}{3} \right\rfloor + 1 \right) \right) = 12 - 4 = 8.$$

In [3] it is shown that, if $T$ is a numerical semigroup with embedding dimension equal to three, then $\mathrm{t}(T) \in \{1, 2\}$. Moreover, $t(T) = 1$ if and only if $T$ is a symmetric numerical semigroup. In [5] it is proved that a numerical semigroup $T$ with embedding dimension equal to three is symmetric if and only if is a complete intersection numerical semigroup, and then the minimal generators of $T$ can not be pairwise relatively prime numbers. Therefore, if $S$ is a numerical semigroup such as at the beginning of this section, then $\mathrm{t}(S) = 2$. We can give explicitly the elements of $\mathrm{PF}(S)$. But first we need a lemma.

Let $T$ be a numerical semigroup. We define in $T$ the partial order

$$x \leq_T y \text{ if } y - x \in T.$$

If $A \subseteq T$, we denote by $\max_{\leq_T}\{A\}$ the set of maximals elements of $A$ with respect to the previous partial order. From [3, Proposition 7] we deduce the following result.

**3.4. Lemma.** *Let $T$ be a numerical semigroup and $m \in T \backslash \{0\}$. If we set $\{w_{i1}, \dots, w_{it}\} = \max_{\leq_T}\{\mathrm{Ap}(T, m)\}$, then $\mathrm{PF}(T) = \{w_{i1} - m, \dots, w_{it} - m\}$.*

**3.5. Proposition.** *Let $S$ be a numerical semigroup such as at the beginning of this section. Then*

$$\mathrm{PF}(S) = \left\{ \left\lfloor \frac{a-1}{c} \right\rfloor (cb - da) + da - b - a, (a-1)b - \left\lfloor \frac{a-1}{c} \right\rfloor da - a \right\}.$$

*Proof.* From Theorem 2.8 we know that

$$\mathrm{Ap}(S, a) = \left\{ \alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \mid \alpha \in \{0, \dots, a-1\} \right\}.$$

Since $\alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da = \left( \left\lfloor \frac{\alpha}{c} \right\rfloor c + \alpha \bmod c \right) b - \left\lfloor \frac{\alpha}{c} \right\rfloor da = \left\lfloor \frac{\alpha}{c} \right\rfloor (cb - da) + (\alpha \bmod c)\, b$ and $(a-1) \bmod c \neq c - 1$ (in case of equality, we have $a \bmod c = 0$, and then $c | a$, which is a contradiction with $\gcd(a, c) = 1$), we have that $\max_{\leq_S}\{\mathrm{Ap}(S, a)\} =$

$$\left\{ \left( \left\lfloor \frac{a-1}{c} \right\rfloor - 1 \right) (cb - da) + (c-1)b, \left\lfloor \frac{a-1}{c} \right\rfloor (cb - da) + ((a-1) \bmod c)\, b \right\}.$$

Having in mind that $(a-1) \bmod c = a - 1 - \left\lfloor \frac{a-1}{c} \right\rfloor c$, then

$$\left\lfloor \frac{a-1}{c} \right\rfloor (cb - da) + ((a-1) \bmod c)\, b = - \left\lfloor \frac{a-1}{c} \right\rfloor da + (a-1)b.$$

Therefore,

$$\max_{\leq_S}\{\mathrm{Ap}(S, a)\} = \left\{ \left\lfloor \frac{a-1}{c} \right\rfloor (cb - da) + da - b, (a-1)b - \left\lfloor \frac{a-1}{c} \right\rfloor da \right\}.$$

From Lemma 3.4 we conclude the proof. $\square$

**3.6. Example.** Let $S = \langle 5, 7, 3\times 7 - 2\times 5\rangle = \langle 5, 7, 11\rangle$. Since $\frac{7}{2} \geq \left\lceil \frac{5}{3} \right\rceil$, applying Proposition 3.5, we have that $\mathrm{PF}(S) = \{9, 13\}$.

**3.7. Corollary.** *Let $S$ be a numerical semigroup such as at the beginning of this section. Then*

$$\mathrm{F}(S) = \begin{cases} (a-1)b - \left\lfloor \frac{a-1}{c} \right\rfloor da - a, & \textit{if } 1 > \left\lfloor \frac{a-1}{c} \right\rfloor \frac{c}{a} + \frac{d}{b}, \\ \left\lfloor \frac{a-1}{c} \right\rfloor (cb - da) + da - b - a, & \textit{in other case.} \end{cases}$$

*Proof.* Since $\mathrm{F}(S) = \max\{\mathrm{PF}(S)\}$, then it is enough to apply Proposition 3.5 and note that $(a-1)b - \left\lfloor \frac{a-1}{c} \right\rfloor da - a > \left\lfloor \frac{a-1}{c} \right\rfloor (cb-da) + da - b - a$ if and only if $1 > \left\lfloor \frac{a-1}{c} \right\rfloor \frac{c}{a} + \frac{d}{b}$. $\square$

**3.8. Remark.** It is possible to improve the results of this paper. Indeed, we only need to impose the conditions $0 < cb - da$ and $\mathrm{e}(S) = 3$ instead of the condition $a < b < cb - da$. This way, if we consider the set

$$\begin{aligned} \mathcal{F}^* = \{ &\langle a, b, cb - da\rangle \mid a, b, c, d \in \mathbb{N} \setminus \{0\},\ \gcd(a, b) = \gcd(a, c) = \gcd(b, d) = 1, \\ &2 \leq c \leq a - 1,\ cb - da > 0,\ \text{and}\ \mathrm{e}(\langle a, b, cb - da\rangle) = 3\}, \end{aligned}$$

it is easy to see that $S = \langle a, b, cb - da\rangle \in \mathcal{F}^*$ is a numerical semigroup with minimal system of generators whose elements are pairwise relatively prime numbers. Reciprocally, following the ideas of Remarks 2.5 and 2.6, if $S$ is a numerical semigroup with minimal system of generators $\{n_1, n_2, n_3\}$ and $n_1, n_2, n_3$ are pairwise relatively prime numbers, then $S \in \mathcal{F}^*$. Moreover, if we make some minor changes at the exposed reasonings in this section and the previous one, we get that Theorem 2.8, Propositions 3.2 and 3.5, and Corollary 3.7 remain true.

**3.9. Remark.** In fact, as sets, $\mathcal{F} = \mathcal{F}^*$. The difference between both of them is that, if $S$ is a numerical semigroup with embedding dimension three and minimal system of generators formed by pairwise relatively prime numbers, then $S$ has a unique representation in $\mathcal{F}$ and six representations in $\mathcal{F}^*$. On the other hand, we have that all the numerical semigroups in $\mathcal{F}$ has dimension three automatically, but in $\mathcal{F}^*$ we have to impose explicitly such a condition.

**3.10. Example.** Let $S = \langle 6, 7, 11\rangle$. We have that $(6, 7, 5, 4)$ is the unique combination associated to $S$ in $\mathcal{F}$. The six combinations for $S$ in $\mathcal{F}^*$ are $(6, 7, 5, 4)$, $(6, 11, 5, 8)$, $(7, 6, 3, 1)$, $(7, 11, 5, 7)$, $(11, 6, 3, 1)$, and $(11, 7, 4, 2)$.

**3.11. Example.** If we take the combination $(a, b, c, d) = (3, 7, 2, 4)$, we have the numerical semigroup $S = \langle 3, 7, 2\times 7 - 4\times 3\rangle = \langle 3, 7, 2\rangle = \langle 2, 3\rangle$. If we try to apply our results, we will have wrong answers.

**3.12. Example.** Let $S = \langle 6, 7, 11\rangle$. If we take $a = 6$, $b = 7$, $c = 5$, and $d = 4$, then $\frac{7}{4} < \left\lceil \frac{6}{5} \right\rceil$. Therefore, we can not apply the results. However, if we take $a = 7$, $b = 6$, $c = 3$, and $d = 1$, then $\frac{6}{1} \geq \left\lceil \frac{7}{3} \right\rceil$. In this case we have that,

- by Theorem 2.8,

$$\begin{aligned} \mathrm{Ap}(S, 7) = &\left\{ \alpha \times 6 - \left\lfloor \frac{\alpha}{3} \right\rfloor \times 1 \times 7 \mid \alpha \in \{0, 1, \ldots, 6\} \right\} = \\ &\{0, 6, 12, 11, 17, 23, 22\}; \end{aligned}$$

- by Proposition 3.2,

$$\mathrm{g}(S) = \frac{5\times 6}{2} - 1 \left\lfloor \frac{6}{3} \right\rfloor \left( 7 - \frac{3}{2} \left( \left\lfloor \frac{6}{3} \right\rfloor + 1 \right) \right) = 15 - 5 = 10;$$

- by Proposition 3.5, $\mathrm{PF}(S) = \{16, 15\}$.

**3.13. Example.** Let $S$ be the numerical semigroup generated by $\{16, 19, 21\}$. It is easy to check that there not exists a combination of $a, b, c, d$ (associated to $S$) such that the condition $\left[\frac{a}{c}, \frac{b}{d}\right] \cap \mathbb{N} \neq \emptyset$ is satisfied.

Let $S$ be a numerical semigroup. We denote by $\mathrm{m}(S) = \min(S \setminus \{0\})$, which it is called the *multiplicity* of $S$. We finish this paper with the following conjecture.

**3.14. Conjecture.** *Let $S$ be a numerical semigroup such that $\mathrm{m}(S) \leq 15$. Then it is possible to find a combination of $a, b, c, d$ (associated to $S$) in such a way the condition $\left[\frac{a}{c}, \frac{b}{d}\right] \cap \mathbb{N} \neq \emptyset$ is satisfied.*

**Acknowledgements.** The authors would like to thank the referees for their useful comments and suggestions that helped to improve this work.

## References

[1] Apéry, R. *Sur les branches superlinéaires des courbes algébriques*, C. R. Acad. Sci. Paris **222**, 1198–1200, 1946.

[2] Curtis, F. *On formulas for the Frobenius number of a numerical semigroup*, Math. Scand. **67**, 190–192, 1990.

[3] Fröberg, R., Gottlieb, G. and Häggkvist, R. *On numerical semigroups*, Semigroup Forum **35**, 63–83, 1987.

[4] Greenberg, H. *Solution to a Diophantine equation for nonnegative integers*, J. Algorithms **9**, 343–353, 1988.

[5] Herzog, J. *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3**, 175–193, 1970.

[6] Johnson, S. M. *A linear Diophantine problem*, Canad. J. Math. **12**, 390–398, 1960.

[7] Marín, J. M., Ramírez Alfonsín, J. L. and Revuelta, M. P. *On the Frobenius number of Fibonacci numerical semigroups*, Integers **7**, 7pp., 2007.

[8] Ramírez Alfonsín, J. L. *The Diophantine Frobenius problem* (Oxford Lecture Series in Mathematics and its Applications, **30**, Oxford University Press, London, 2005).

[9] Ramírez Alfonsín, J. L. and Rødseth, Ø. J. *Numerical semigroups: Apéry sets and Hilbert series*, Semigroup Forum **79**, 323–340, 2009.

[10] Robles-Pérez, A. M. and Rosales, J. C. *The Frobenius problem for numerical semigroups with embedding dimension equal to three*, Math. Comput. **81**, 1609–1617, 2012.

[11] Rödseth, Ö. J. *On a linear Diophantine problem of Frobenius*, J. Reine Angew. Math. **301**, 171–178, 1978.

[12] Rödseth, Ö. J. *O*n a linear Diophantine problem of Frobenius II, J. Reine Angew. Math. **307/308**, 431–440, 1979.

[13] Rosales, J. C. *On numericals semigroups*, Semigroup Forum **52**, 307–318, 1996.

[14] Rosales, J. C. and Branco, M. B. *Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups*, J. Pure Appl. Algebra **171**, 303–314, 2002.

[15] Rosales, J. C. and García-Sánchez, P. A. *Numerical semigroups with embedding dimension three*, Arch. Math. **83**, 488–496, 2004.

[16] Rosales, J. C. and García-Sánchez, P. A. *Numerical semigroups* (Developments in Mathematics, vol. **20**, Springer, New York, 2009).

[17] Selmer, E. S. *On the linear Diophantine problem of Frobenius*, J. Reine Angew. Math. **293/294**, 1–17, 1977.

[18] Sylvester, J. J. *On subvariants, i.e. semi-invariants to binary quantics of an unlimited order*, Amer. J. Math. **5**, 79–136, 1882.

[19] Sylvester, J. J. *Problem 7382*, The Educational Times, and Journal of the College of Preceptors, New Series, **36 (266)**, 177, 1883. Solution by W. J. Curran Sharp, ibid., **36 (271)**, 315, 1883. Republished as [20].

[20] Sylvester, J. J. *Problem 7382*, in W. J. C. Miller (Ed.), Mathematical questions, with their solutions, from "The Educational Times", vol. **41**, page 21, Francis Hodgson, London, 1884.

# STATISTICS

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 909 – 921

# Analysis of ruin measures for two classes of risk processes with stochastic income

Wuyuan Jiang* †‡ and Chaoqun Ma§

**Abstract**

In this paper, we consider the ruin measures for two classes of risk processes. We assume that the claim number processes are independent Poisson and generalized Erlang(n) processes, respectively. Historically, it has been assumed that the premium size is a constant. In this contribution, the premium income arrival process is a Poisson process. In this framework, both the integro-differential equation and the Laplace transform for the expected discounted penalty function are established. Explicit expressions for the expected discounted penalty function are derived when the claim amount distributions belong to the rational family. Finally, Numerical examples are considered.

*2000 AMS Classification:* 62P05, 91B30.

**Keywords:** Two classes of risk processes, Expected discounted penalty function, Integro-differential equation, Laplace transform, Stochastic income.

Received 12/05/2014 : Accepted 08/09/2014 Doi : 10.15672/HJMS.2014307481

## 1. Introduction

In the actuarial literature, many researchers studied the ruin measures for a risk model involving two independent classes of risks. Among them, [9] considered the expected discounted penalty functions for two classes of risk processes by assuming that the two claim number processes are independent Poisson and generalized Erlang(2) processes, respectively. A system of integro-differential equations for the expected discounted penalty functions were derived and explicit results when the claim sizes are exponentially distributed were obtained. [13] extended the model of [9], by considering the claim number process of the second class to be a renewal process with generalized Erlang(n) inter-arrival

*School of Business Administration, Hunan University, Changsha 410082, China,
†Department of Mathematics, Hunan Institute of Science and Technology, Yueyang 414006, China Email: csujw@163.com
‡Corresponding Author.
§School of Business Administration, Hunan University, Changsha 410082, China, Email: cqma1998@hnu.edu.cn

times. The authors derived an integro-differential equation system for the expected discounted penalty functions, and obtained their Laplace transforms when the corresponding Lundberg equation has distinct roots. [5] investigated the risk model with two classes of renewal risk processes by assuming that both of the two claim number processes have phase-type inter-claim times. A system of integro-differential equations for the expected discounted penalty function was derived and solved. For more related references on two classes of risk processes problem, the reader may consult the following publications and references therein, [12], [8], [3], etc.

Under the above risk models, premiums are assumed to be received by insurance companies at a constant rate over time. In fact, the insurance company may have lump sums of income. For example, insurances of traveling art collections or ship and plane insurances might be expected to have a significant impact on the premium income. [2] first considered the risk model with stochastic premium income by adding a compound Poisson process with positive jumps to the classical risk model. Subsequently, [1] and [10] studied the ruin probabilities for the risk models with stochastic premiums. Recently, [6] considered a risk model with stochastic premium income, where both premiums and claims follow compound Poisson processes. Both a defective renewal equation and an integral equation satisfied by the expected discounted penalty function are established. [14] extended the model in [6] by assuming that there exists a dependence structure among the claim sizes, inter-claim times and premium sizes. [11] studied a risk model with a dependence setting where there exists a specific structure among the time between two claim occurrences, premium sizes and claim sizes. Given that the premium size is exponentially distributed, both the Laplace transforms and defective renewal equations for the expected discounted penalty functions are obtained.

To the best of our knowledge, there is less work in the literature on two classes of risk models with stochastic premiums. Henceforth, the purpose of this paper is to investigate the expected discounted penalty functions in a risk model involving two independent classes of risks and the premium income arrival process is a Poisson process, in which the claim number processes are independent Poisson and generalized Erlang(n) processes, respectively. The structure of the paper is as follows. Section 2 describes two classes of risk processes with stochastic income. In Section 3, we derive the system of integro-differential equations for the expected discounted penalty functions. Then Section 4 presents the Laplace solutions of the expected discounted penalty functions and provides closed forms for rational family claim-size distribution. Numerical examples are considered in Section 5. Last, Section 6 concludes.

## 2. Model and assumptions

The surplus process $R(t)$ is given by

$$R(t) = u + \sum_{i=1}^{M(t)} X_i - S(t), \quad t \geq 0, \tag{2.1}$$

where $u \geq 0$ is the initial surplus, $M(t)$ denotes the number of insurer's premium income up to time $t$ and follows a Poisson process with intensity $\mu > 0$. $\{X_1, X_2, \cdots\}$ are independent and identically distributed (i.i.d.) positive random variables (r.v.'s) representing the individual premium amounts with common distribution $P$, probability density function (p.d.f.) $p$ and Laplace transform (LT) $\tilde{p}(s) = \int_0^\infty e^{-sx} p(x)dx$. The aggregate-claim process $\{S(t) : t \geq 0\}$ is defined by

$$S(t) = \sum_{i=1}^{N_1(t)} Y_i + \sum_{i=1}^{N_2(t)} Z_i, \quad t \geq 0,$$

where $\{Y_1, Y_2, \cdots\}$ are i.i.d. positive r.v.'s representing the successive individual claim amounts from the first class. These r.v.'s are assumed to have common cumulative distribution function $F(x), x \geq 0$, with p.d.f. $f(x) = F'(x)$, of which the LT is $\tilde{f}(s) = \int_0^\infty e^{-sx} f(x)dx$, while $\{Z_1, Z_2, \cdots\}$ are i.i.d. positive r.v.'s representing the claim amounts from the second class with common cumulative distribution function $G(x), x \geq 0$ and p.d.f. $g(x) = G'(x)$, of which the LT is $\tilde{g}(s) = \int_0^\infty e^{-sx} g(x)dx$.

The counting process $\{N_1(t); t \geq 0\}$ is assumed to be a Poisson process with parameter $\lambda$, representing the number of claims from the first class up to time $t$. While the counting process $\{N_2(t); t \geq 0\}$, representing the number of claims from the second class up to time $t$, is defined as follows. $N_2(t) = \sup\{n : W_1 + W_2 + \cdots + W_n \leq t\}$, where $\{W_1, W_2, \cdots\}$ are the i.i.d. positive r.v.'s representing the second class inter-claim times. In this paper, we suppose that $W_i's$ are generalized Erlang(n) distributed with $n$ possibly different parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $W_i$ can be expressed as $W_i = W_{i1} + W_{i2} + \cdots + W_{in}$, where $W_{ij}$ is exponentially distributed with parameter $\frac{1}{\lambda_i}$.

In addition, we assume that $\{X_1, X_2, \cdots\}$, $\{Y_1, Y_2, \cdots\}$, $\{Z_1, Z_2, \cdots\}$, $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are mutually independent, and $\mu E(X_1) > \lambda E(Y_1) + \frac{E(Z_1)}{\sum_{i=1}^{n} \frac{1}{\lambda_i}}$, providing a positive safety loading factor.

The time of (ultimate) ruin is $T = \inf\{t | R(t) < 0\}$, where $T = \infty$ if $R(t) \geq 0$ for all $t \geq 0$. The probability of ruin is $\psi(u) = Pr(T < \infty)$.

For $x_1, x_2 \geq 0, k = 1, 2$, let $w_k(x_1, x_2)$ be two possibly distinct non-negative value functions. For $\delta \geq 0$, the expected discounted penalty function at ruin if the ruin is caused by a claim from class $k$ is defined by

$$m_k(u) = E[e^{-\delta T} w_k(R(T-), |R(T)|) I(T < \infty, J = k) | R(0) = u], \quad u \geq 0,$$

where $J$ is defined to be the cause-of-ruin random variable, and $J = k$ if the ruin is caused by a claim of class $k, k = 1, 2$. $R(T-)$ is the surplus immediately before ruin, $|R(T)|$ is the deficit at ruin, $I(\cdot)$ is an indicator function.

When $\delta = 0$ and $w_k(R(T-), |R(T)|) = 1$, let

$$\psi_k(u) = E[I(T < \infty, J = k) | R(0) = u], \quad u \geq 0, k = 1, 2,$$

is the ruin probability due to a claim from class $k$. The probability of ruin $\psi(u)$ can be decomposed as $\psi(u) = \psi_1(u) + \psi_2(u)$.

## 3. System of integro-differential equations

In this section, we derive the integro-differential equations for the expected discounted penalty function. Since every inter-claim time with generalized Erlang(n) distribution can be decomposed into the independent sum of $n$ exponential r.v.'s with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, each causing a sub-claim of size 0 and at the time of the $n$th sub-claim an actual claim with distribution function $G$ occurs. This can be realized by considering $n$ states of the risk process (2.1) for the second class claim. Starting at time 0 in state 1, every sub-claim causes a transition to the next state and at the time of the occurrence of the $n$th sub-claim, an actual claim with distribution function $G$ occurs and the risk process jumps into state 1 again. We define the corresponding expected discounted penalty function by $m_{kj}, j = 1, 2, \ldots, n$, when ruin is caused by a claim from class $k, k = 1, 2$ and the risk process is in state $j$. Obviously, $m_{k1}(u) = m_k(u)$.

Considering an infinitesimal time interval $(0, dt)$, there are five possible events regarding to the occurrence of the premium and claim and change of the state: (1) no premium and claim arrival and no change of state; (2) a premium arrival but no claim arrival and no change of state; (3) a claim arrival but no premium arrival and no change of state; (4) a change of state but no claim and premium arrival; (5) two or more events occur.

By conditioning on the above five events in $(0, dt)$ when $j = 1, 2, \ldots, n-1$, we have

$$
\begin{array}{rcl}
m_{1j}(u) & = & (1-\mu dt)(1-\lambda dt)(1-\lambda_j dt)e^{-\delta dt} m_{1j}(u) \\
 & + & \mu dt(1-\lambda dt)(1-\lambda_j dt)e^{-\delta dt} \int_0^\infty m_{1j}(u+x)p(x)dx \\
 & + & (1-\mu dt)\lambda dt(1-\lambda_j dt)e^{-\delta dt} \times \\
 & & \left[\int_0^u m_{1j}(u-x)f(x)dx + \int_u^\infty w_1(u, x-u)f(x)dx\right] \\
 & + & (1-\mu dt)(1-\lambda dt)\lambda_j dt e^{-\delta dt} m_{1,j+1}(u) + o(dt).
\end{array} \tag{3.1}
$$

From (3.1) it follows that

$$
\begin{array}{rcl}
m_{1j}(u) & = & \frac{\mu}{\lambda_j^*+\delta} \int_0^\infty m_{1j}(u+x)p(x)dx \\
 & + & \frac{\lambda}{\lambda_j^*+\delta} \left[\int_0^u m_{1j}(u-x)f(x)dx + \zeta_1(u)\right] \\
 & + & \frac{\lambda_j}{\lambda_j^*+\delta} m_{1,j+1}(u),
\end{array} \tag{3.2}
$$

where $\lambda_j^* = \mu + \lambda + \lambda_j$, $\zeta_1(u) = \int_u^\infty w_1(u, x-u)f(x)dx$.

When $j = n$, we obtain

$$
\begin{array}{rcl}
m_{1n}(u) & = & (1-\mu dt)(1-\lambda dt)(1-\lambda_n dt)e^{-\delta dt} m_{1n}(u) \\
 & + & \mu dt(1-\lambda dt)(1-\lambda_n dt)e^{-\delta dt} \int_0^\infty m_{1n}(u+x)p(x)dx \\
 & + & (1-\mu dt)\lambda dt(1-\lambda_n dt)e^{-\delta dt} \times \\
 & & \left[\int_0^u m_{1n}(u-x)f(x)dx + \int_u^\infty w_1(u, x-u)f(x)dx\right] \\
 & + & (1-\mu dt)(1-\lambda dt)\lambda_n dt e^{-\delta dt} \int_0^u m_1(u-x)g(x)dx + o(dt).
\end{array} \tag{3.3}
$$

Which results in

$$
\begin{array}{rcl}
m_{1n}(u) & = & \frac{\mu}{\lambda_n^*+\delta} \int_0^\infty m_{1n}(u+x)p(x)dx \\
 & + & \frac{\lambda}{\lambda_n^*+\delta} \left[\int_0^u m_{1n}(u-x)f(x)dx + \zeta_1(u)\right] \\
 & + & \frac{\lambda_n}{\lambda_n^*+\delta} \int_0^u m_1(u-x)g(x)dx,
\end{array} \tag{3.4}
$$

where $\lambda_n^* = \mu + \lambda + \lambda_n$.

By similar arguments, we get

$$
\begin{array}{rcl}
m_{2j}(u) & = & \frac{\mu}{\lambda_j^*+\delta} \int_0^\infty m_{2j}(u+x)p(x)dx \\
 & + & \frac{\lambda}{\lambda_j^*+\delta} \int_0^u m_{2j}(u-x)f(x)dx \\
 & + & \frac{\lambda_j}{\lambda_j^*+\delta} m_{2,j+1}(u), \quad j = 1, 2, \ldots, n-1.
\end{array} \tag{3.5}
$$

and

$$
\begin{array}{rcl}
m_{2n}(u) & = & \frac{\mu}{\lambda_n^*+\delta} \int_0^\infty m_{2n}(u+x)p(x)dx \\
 & + & \frac{\lambda}{\lambda_n^*+\delta} \int_0^u m_{2n}(u-x)f(x)dx \\
 & + & \frac{\lambda_n}{\lambda_n^*+\delta} \left[\int_0^u m_2(u-x)g(x)dx + \zeta_2(u)\right],
\end{array} \tag{3.6}
$$

where $\zeta_2(u) = \int_u^\infty w_2(u, x-u)g(x)dx$.

## 4. Analysis of the integro-differential equations with exponential premiums

In this section, we assume that the premium sizes are exponentially distributed with p.d.f. $p(x) = \beta e^{-\beta x}, \beta > 0, x \geq 0$. Throughout this paper, we will use a hat $\sim$ to designate the Laplace transform of a function $f$, namely, $\tilde{f}(s) = \int_0^\infty e^{-sx} f(x)dx$. Now, we introduce a complex operator $T_r$ of an integrable real-valued function $f$ which will be necessary in order to obtain the main results. $T_r$ is defined as

$$
T_r f(x) = \int_x^\infty e^{-r(u-x)} f(u)du, \quad r \in \mathbb{C}, x \geq 0,
$$

where $r$ has a non-negative real part, $\Re(r) \geq 0$. [7] provide a list of properties of the operator $T_r$ and we recall two of them that will be used in the following:

(1) $T_r f(0) = \int_0^\infty e^{-ru} f(u)du = \tilde{f}(r), r \in \mathbb{C}$, is the Laplace transform of $f$.

(2) $T_r T_s f(x) = T_s T_r f(x) = \frac{T_s f(x) - T_r f(x)}{r-s}, s \neq r \in \mathbb{C}, x \geq 0$.

**4.1. Laplace transform.** In the following, for notational convenience, Let $H_{kj}(u) = \int_0^\infty m_{kj}(u+x)p(x)dx, k = 1, 2, j = 1, 2, \ldots, n$. Taking Laplace transforms on both sides of (3.2) and (3.4) yields

$$\tilde{m}_{1j}(s) = \frac{\mu}{\lambda_j^* + \delta}\tilde{H}_{1j}(s) + \frac{\lambda}{\lambda_j^* + \delta}\tilde{m}_{1j}(s)\tilde{f}(s) + \frac{\lambda}{\lambda_j^* + \delta}\tilde{\zeta}_1(s) + \frac{\lambda_j}{\lambda_j^* + \delta}\tilde{m}_{1,j+1}(s), \tag{4.1}$$

and

$$\tilde{m}_{1n}(s) = \frac{\mu}{\lambda_n^* + \delta}\tilde{H}_{1n}(s) + \frac{\lambda}{\lambda_n^* + \delta}\tilde{m}_{1n}(s)\tilde{f}(s) + \frac{\lambda}{\lambda_n^* + \delta}\tilde{\zeta}_1(s) + \frac{\lambda_n}{\lambda_n^* + \delta}\tilde{m}_1(s)\tilde{g}(s). \tag{4.2}$$

Since, for $j = 1, 2, \ldots, n$, $s \neq \beta$,

$$\begin{aligned} \tilde{H}_{1j}(s) &= \int_0^\infty e^{-su} \int_0^\infty m_{1j}(u+x)\beta e^{-\beta x} dx du \\ &= \int_0^\infty \{\int_0^\infty e^{-su} m_{1j}(u+x)du\}\beta e^{-\beta x} dx \\ &= \int_0^\infty T_s m_{1j}(x)\beta e^{-\beta x} dx = \beta T_\beta T_s m_{1j}(0) \\ &= \beta\frac{\tilde{m}_{1j}(s) - \tilde{m}_{1j}(\beta)}{\beta - s}. \end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.1) and (4.2), respectively, we have

$$\left[\frac{\mu\beta}{\beta - s} - \lambda_j^* - \delta + \lambda\tilde{f}(s)\right]\tilde{m}_{1j}(s) + \lambda_j\tilde{m}_{1,j+1}(s) = \frac{\mu\beta}{\beta - s}\tilde{m}_{1j}(\beta) - \lambda\tilde{\zeta}_1(s), \tag{4.4}$$

and

$$\left[\frac{\mu\beta}{\beta - s} - \lambda_n^* - \delta + \lambda\tilde{f}(s)\right]\tilde{m}_{1n}(s) + \lambda_n\tilde{g}(s)\tilde{m}_1(s) = \frac{\mu\beta}{\beta - s}\tilde{m}_{1n}(\beta) - \lambda\tilde{\zeta}_1(s). \tag{4.5}$$

Let $\tilde{\mathbf{m}}_k(s) = (\tilde{m}_{k1}(s), \tilde{m}_{k2}(s), \ldots, \tilde{m}_{kn}(s))^\top$, $\tilde{\mathbf{m}}_k(\beta) = (\tilde{m}_{k1}(\beta), \tilde{m}_{k2}(\beta), \ldots, \tilde{m}_{kn}(\beta))^\top$, $k = 1, 2$, $\mathbf{m}^\top$ denotes the transpose of $\mathbf{m}$, and

$$\mathbf{A}_\delta(s) = \begin{pmatrix} \frac{\mu\beta}{\beta-s} - \lambda_1^* - \delta + \lambda\tilde{f}(s) & \lambda_1 & 0 & \cdots & 0 \\ 0 & \frac{\mu\beta}{\beta-s} - \lambda_2^* - \delta + \lambda\tilde{f}(s) & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} \\ \lambda_n\tilde{g}(s) & 0 & 0 & \cdots & \frac{\mu\beta}{\beta-s} - \lambda_n^* - \delta + \lambda\tilde{f}(s) \end{pmatrix}.$$

Then (4.4) and (4.5) can be rewritten as the following matrix form

$$\mathbf{A}_\delta(s)\tilde{\mathbf{m}}_1(s) = \frac{\mu\beta}{\beta - s}\tilde{\mathbf{m}}_1(\beta) - \lambda\tilde{\zeta}_1(s)\mathbf{e}_1, \tag{4.6}$$

where $\mathbf{e}_1$ denotes a column vector of length n with all elements being one.

Similarly, from (3.5) and (3.6) we can obtain the following matrix form for $\tilde{\mathbf{m}}_2(s)$

$$\mathbf{A}_\delta(s)\tilde{\mathbf{m}}_2(s) = \frac{\mu\beta}{\beta - s}\tilde{\mathbf{m}}_2(\beta) - \lambda_n\tilde{\zeta}_2(s)\mathbf{e}_2, \tag{4.7}$$

where $\mathbf{e}_2 = (0, 0, \ldots, 0, 1)^\top$ denotes a $n \times 1$ column vector.

When $det[\mathbf{A}_\delta(s)] \neq 0$, solving the linear systems (4.6) and (4.7), we obtain

$$\tilde{\mathbf{m}}_1(s) = \frac{\mathbf{A}_\delta^\star(s)\left[\frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta) - \lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]}{det[\mathbf{A}_\delta(s)]}, \tag{4.8}$$

and

$$\tilde{\mathbf{m}}_2(s) = \frac{\mathbf{A}_\delta^\star(s)\left[\frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_2(\beta) - \lambda_n\tilde{\zeta}_2(s)\mathbf{e}_2\right]}{det[\mathbf{A}_\delta(s)]}, \tag{4.9}$$

where $\mathbf{A}_\delta^\star(s)$ is the adjoint matrix of $\mathbf{A}_\delta(s)$.

**4.1. Theorem.** *For $\delta > 0$, the generalized Lundberg's fundamental equation $det[\mathbf{A}_\delta(s)] = 0$ has exactly $n$ roots, say $\rho_1, \rho_2, \ldots, \rho_n$ with $\Re(\rho_i) > 0$.*

*Proof.* $det[\mathbf{A}_\delta(s)] = 0$ can be rewritten as

$$\frac{1}{(\beta-s)^n}\left\{\prod_{i=1}^n\{\mu\beta - [\lambda_i^* + \delta - \lambda\tilde{f}(s)](\beta-s)\} - (\prod_{i=1}^n\lambda_i)\tilde{g}(s)(\beta-s)^n\right\} = 0.$$

Thus, it is only needed to prove

$$\prod_{i=1}^n\{\mu\beta - [\lambda_i^* + \delta - \lambda\tilde{f}(s)](\beta-s)\} - (\prod_{i=1}^n\lambda_i)\tilde{g}(s)(\beta-s)^n = 0 \tag{4.10}$$

has exactly $n$ roots in the right half complex plane. Let $z = (\beta - s)/\beta$, then (4.10) may be expressed as

$$\prod_{i=1}^n\{\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1-z))]z\} - (\prod_{i=1}^n\lambda_i)\tilde{g}(\beta(1-z))z^n = 0. \tag{4.11}$$

When $\delta > 0$, choose $r \in (0,1)$ such that $(\mu+\delta)r > \mu$, and denote $C_z = \{z \in \mathbb{C}||z| = r\}$. Obviously, $\prod_{i=1}^n\{\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1-z))]z\}$ and $(\prod_{i=1}^n\lambda_i)\tilde{g}(\beta(1-z))z^n$ are analytic on and inside the contour $C_z$.

We first prove that each of equations $\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1-z))]z = 0$, $i = 1, \cdots, n$ has exactly one root in the interior of $C_z$. For any $z \in C_z$, we have

$$|(\lambda_i^* + \delta)z - \mu| \geq |(\lambda_i^* + \delta)z| - \mu > (\lambda_i + \lambda)|z| > \lambda|z| \geq |\lambda\tilde{f}(\beta(1-z))]z|.$$

By virtue of Rouché's theorem, $(\lambda_i^* + \delta)z - \mu = 0$ and $\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1-z))]z = 0$ have the same number of roots inside $C_z$. Thus $\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1-z))]z = 0$ has exactly one root inside $C_z$. It implies that

$$\prod_{i=1}^n\{\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1-z))]z\} = 0 \tag{4.12}$$

has exactly $n$ roots inside $C_z$.

Furthermore, for any $z \in C_z$,

$$\begin{aligned}
&|\prod_{i=1}^n\{\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1-z))]z\}| \\
= \;& \prod_{i=1}^n|(\lambda_i^* + \delta)z - \mu - \lambda\tilde{f}(\beta(1-z))z| \\
\geq \;& \prod_{i=1}^n\{|(\lambda_i^* + \delta)z - \mu| - |\lambda\tilde{f}(\beta(1-z))z|\} \\
\geq \;& \prod_{i=1}^n\{|(\lambda_i^* + \delta)z - \mu| - |\lambda z|\} \\
= \;& \prod_{i=1}^n\{|(\lambda_i + \lambda)z + (\mu+\delta)z - \mu| - |\lambda z|\} \\
> \;& \prod_{i=1}^n|\lambda_i z| \geq |(\prod_{i=1}^n\lambda_i)\tilde{g}(\beta(1-z))z^n|.
\end{aligned}$$

In the last second step, we use $z \in C_z = \{z \in \mathbb{C}||z| = r\}$ and $r \in (\mu/(\mu+\delta), 1)$.

By Rouché's theorem, both Eq. (4.12) and Eq. (4.11) have the same number of roots inside $C_z$. Then, we conclude that the equation Eq. (4.11) has exactly $n$ roots inside $C_z$. That is to say, Lundberg's equation $det[\mathbf{A}_\delta(s)] = 0$ has exactly $n$ roots in $C_s = \{s \in \mathbb{C} \| \beta - s| = r\beta\}$. From $r \in (\mu/(\mu+\delta), 1)$, the interior of $C_s$ is entirely contained in the right half complex plane. This completes the proof. □

**4.2. Remark.** If $\delta \to 0+$ then $\rho_i(\delta) \to \rho_i(0)$ for $i = 1, \cdots, n$, and we have that $s = 0$ is one of the roots from Lundberg's equation $det[\mathbf{A}_\delta(s)] = 0$.

In what follows, we assume that $\rho_1, \rho_2, \ldots, \rho_n$ are distinct.

Divided difference plays an important role in the present paper. Now we recall divided differences of a matrix $\mathbf{L}(s)$ with respect to distinct numbers $r_1, r_2, \cdots$, which are defined recursively as follows:

$$\mathbf{L}[r_1, s] = \frac{\mathbf{L}(s) - \mathbf{L}(r_1)}{s - r_1}, \quad \mathbf{L}[r_1, r_2, s] = \frac{\mathbf{L}[r_1, s] - \mathbf{L}[r_1, r_2]}{s - r_2},$$

and so on.

**4.3. Theorem.** *$\tilde{\mathbf{m}}_1(\beta)$ and $\tilde{\mathbf{m}}_2(\beta)$ are given by*

(4.13)

$$\tilde{\mathbf{m}}_1(\beta) = \frac{\lambda}{\mu\beta}\left(\sum_{i=1}^{n} \mathbf{A}^\star_\delta[\rho_1, \cdots, \rho_i]\frac{1}{\prod_{l=i}^{n}(\beta - \rho_l)}\right)^{-1}\left(\sum_{i=1}^{n} \mathbf{A}^\star_\delta[\rho_1, \cdots, \rho_i]\tilde{\zeta}_1[\rho_i, \cdots, \rho_n]\right)\mathbf{e}_1,$$

(4.14)

$$\tilde{\mathbf{m}}_2(\beta) = \frac{\lambda_n}{\mu\beta}\left(\sum_{i=1}^{n} \mathbf{A}^\star_\delta[\rho_1, \cdots, \rho_i]\frac{1}{\prod_{l=i}^{n}(\beta - \rho_l)}\right)^{-1}\left(\sum_{i=1}^{n} \mathbf{A}^\star_\delta[\rho_1, \cdots, \rho_i]\tilde{\zeta}_2[\rho_i, \cdots, \rho_n]\right)\mathbf{e}_2.$$

*Proof.* Since $\tilde{m}_{kj}(s)$ is finite for $k = 1, 2, j = 1, 2, \ldots, n$, from (4.8), we have, for distinct numbers $\rho_1, \rho_2, \ldots, \rho_n$,

$$\mathbf{A}^\star_\delta(\rho_i)\frac{\mu\beta}{\beta - \rho_i}\tilde{\mathbf{m}}_1(\beta) = \mathbf{A}^\star_\delta(\rho_i)\tilde{\zeta}_1(\rho_i)\lambda\mathbf{e}_1.$$

Hence

$$\left[\mathbf{A}^\star_\delta(\rho_1)\frac{\mu\beta}{\beta - \rho_1} - \mathbf{A}^\star_\delta(\rho_2)\frac{\mu\beta}{\beta - \rho_2}\right]\tilde{\mathbf{m}}_1(\beta) = [\mathbf{A}^\star_\delta(\rho_1)\tilde{\zeta}_1(\rho_1) - \mathbf{A}^\star_\delta(\rho_2)\tilde{\zeta}_1(\rho_2)]\lambda\mathbf{e}_1.$$

Namely

$$\begin{aligned} &\left[\mathbf{A}^\star_\delta(\rho_1)\tfrac{1}{\beta-\rho_1} - \mathbf{A}^\star_\delta(\rho_1)\tfrac{1}{\beta-\rho_2} + \mathbf{A}^\star_\delta(\rho_1)\tfrac{1}{\beta-\rho_2} - \mathbf{A}^\star_\delta(\rho_2)\tfrac{1}{\beta-\rho_2}\right]\mu\beta\tilde{\mathbf{m}}_1(\beta) \\ = \;&[\mathbf{A}^\star_\delta(\rho_1)\tilde{\zeta}_1(\rho_1) - \mathbf{A}^\star_\delta(\rho_1)\tilde{\zeta}_1(\rho_2) + \mathbf{A}^\star_\delta(\rho_1)\tilde{\zeta}_1(\rho_2) - \mathbf{A}^\star_\delta(\rho_2)\tilde{\zeta}_1(\rho_2)]\lambda\mathbf{e}_1. \end{aligned}$$

Using the divided difference, we derive

$$\begin{aligned} &\left[\mathbf{A}^\star_\delta(\rho_1)\tfrac{1}{(\beta-\rho_1)(\beta-\rho_2)} + \mathbf{A}^\star_\delta[\rho_1, \rho_2]\tfrac{1}{\beta-\rho_2}\right]\mu\beta\tilde{\mathbf{m}}_1(\beta) \\ = \;&\{\mathbf{A}^\star_\delta(\rho_1)\tilde{\zeta}_1[\rho_1, \rho_2] + \mathbf{A}^\star_\delta[\rho_1, \rho_2]\tilde{\zeta}_1(\rho_2)\}\lambda\mathbf{e}_1. \end{aligned}$$

We finally have by recursively deriving

(4.15)

$$\left(\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{1}{\prod\limits_{l=i}^{n}(\beta-\rho_l)}\right)\mu\beta\tilde{\mathbf{m}}_1(\beta)=\lambda\left(\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_1[\rho_i,\cdots,\rho_n]\right)\mathbf{e}_1,$$

which leads to (4.13).

Similarly, we can obtain (4.14) from (4.9). □

Applying the divided difference repeatedly to the numerators of (4.8) and (4.9), respectively, we obtain the following theorem.

**4.4. Theorem.** *The Laplace transforms of the expected discounted penalty function are given by*

$$(4.16)\quad \begin{aligned}\tilde{\mathbf{m}}_1(s) \;=\;& \frac{\prod_{i=1}^{n}(s-\rho_i)}{det[\mathbf{A}_\delta(s)]}\Bigg\{\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_n,s]\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]\\ &+\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{1}{\prod_{l=i}^{n}(\beta-\rho_l)}\left(\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)\right)\\ &-\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_1[\rho_i,\cdots,\rho_n,s](\lambda\mathbf{e}_1)\Bigg\},\end{aligned}$$

*and*

$$(4.17)\quad \begin{aligned}\tilde{\mathbf{m}}_2(s) \;=\;& \frac{\prod_{i=1}^{n}(s-\rho_i)}{det[\mathbf{A}_\delta(s)]}\Bigg\{\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_n,s]\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_2(\beta)-\lambda_n\tilde{\zeta}_2(s)\mathbf{e}_2\right]\\ &+\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{1}{\prod_{l=i}^{n}(\beta-\rho_l)}\left(\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_2(\beta)\right)\\ &-\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_2[\rho_i,\cdots,\rho_n,s](\lambda_n\mathbf{e}_2)\Bigg\}.\end{aligned}$$

*Proof.* By the fact that $s=\rho_1$ is a root of the numerator in (4.8), we have

$$(4.18)\quad \begin{aligned}&\mathbf{A}_\delta^\star(s)\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]\\ =\;&\mathbf{A}_\delta^\star(s)\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]-\mathbf{A}_\delta^\star(\rho_1)\left[\tfrac{\mu\beta}{\beta-\rho_1}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(\rho_1)\mathbf{e}_1\right]\\ =\;&(s-\rho_1)\left\{\begin{array}{c}\mathbf{A}_\delta^\star[\rho_1,s]\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]+\\ \mathbf{A}_\delta^\star(\rho_1)\tfrac{1}{(\beta-\rho_1)}\left(\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)\right)-\mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1[\rho_1,s](\lambda\mathbf{e}_1)\end{array}\right\}.\end{aligned}$$

Since $s=\rho_2$ is also a root of numerator in (4.8), it shows that $s=\rho_2$ is a zero of the expression within the brace in (4.18), namely

$$(4.19)\quad \begin{aligned}&\left(\mathbf{A}_\delta^\star[\rho_1,s]+\mathbf{A}_\delta^\star(\rho_1)\tfrac{1}{\beta-\rho_1}\right)\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\left(\mathbf{A}_\delta^\star[\rho_1,s]\tilde{\zeta}_1(s)+\mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1[\rho_1,s]\right)\lambda\mathbf{e}_1\\ =\;&\left(\mathbf{A}_\delta^\star[\rho_1,s]+\mathbf{A}_\delta^\star(\rho_1)\tfrac{1}{\beta-\rho_1}\right)\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\left(\mathbf{A}_\delta^\star[\rho_1,s]\tilde{\zeta}_1(s)+\mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1[\rho_1,s]\right)\lambda\mathbf{e}_1\\ &-\left(\mathbf{A}_\delta^\star[\rho_1,\rho_2]+\mathbf{A}_\delta^\star(\rho_1)\tfrac{1}{\beta-\rho_1}\right)\tfrac{\mu\beta}{\beta-\rho_2}\tilde{\mathbf{m}}_1(\beta)+\left(\mathbf{A}_\delta^\star[\rho_1,\rho_2]\tilde{\zeta}_1(\rho_2)+\mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1[\rho_1,\rho_2]\right)\lambda\mathbf{e}_1\\ =\;&(s-\rho_2)\left\{\begin{array}{c}\mathbf{A}_\delta^\star[\rho_1,\rho_2,s]\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]+\\ \sum_{i=1}^{2}\mathbf{A}_\delta^\star[\rho_1,\rho_i]\frac{1}{\prod_{l=i}^{2}(\beta-\rho_l)}\left(\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)\right)-\sum_{i=1}^{2}\mathbf{A}_\delta^\star[\rho_1,\rho_i]\tilde{\zeta}_1[\rho_i,\rho_2,s](\lambda\mathbf{e}_1)\end{array}\right\},\end{aligned}$$

where we denote $\mathbf{A}_\delta^\star[\rho_1,\rho_i]=\mathbf{A}_\delta^\star(\rho_1)$, when $i=1$.

Substituting (4.19) into (4.18), recursively from the fact $s=\rho_3,\ldots,\rho_n$ are roots of the numerator in (4.8), (4.16) is derived.

By similar arguments, we obtain (4.17) from (4.9). □

**4.2. Closed forms for rational family claim-size distribution.** Now, we restrict the further analysis to the case of the claim amount distributions $F(x)$ and $G(x)$ both with rational Laplace transforms, viz,

$$\tilde{f}(s) = \frac{f_{r_1-1}(s)}{f_{r_1}(s)}, \quad \tilde{g}(s) = \frac{g_{r_2-1}(s)}{g_{r_2}(s)}, \quad r_1, r_2 \in \mathbb{N}^+,$$

where $f_{r_1-1}(s)$, $g_{r_2-1}(s)$ are polynomials of degree $r_1 - 1$ and $r_2 - 1$ or less, respectively, while $f_{r_1}(s)$ and $g_{r_2}(s)$ are polynomials of degree $r_1$ and $r_2$ with only negative roots, and satisfy $f_{r_1-1}(0) = f_{r_1}(0)$, $g_{r_2-1}(0) = g_{r_2}(0)$. Without loss of generality, we assume that $f_{r_1}(s)$ and $g_{r_2}(s)$ have leading coefficient 1. This wide class of distributions includes the phase-type distributions, and in particular, it includes the Erlang, Coxian and exponential distribution and all the mixtures of them.

In what follows, let $h(s) = (s - \beta)^n [f_{r_1}(s)]^n g_{r_2}(s)$. Multiplying both numerator and denominator of (4.16) by $h(s)$, we get

$$\begin{aligned} \tilde{\mathbf{m}}_1(s) \quad = \quad & \frac{\prod_{i=1}^{n}(s-\rho_i)}{h(s)det[\mathbf{A}_\delta(s)]} \Big\{ \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s] h(s) \left[ \tfrac{\mu\beta}{\beta - s} \tilde{\mathbf{m}}_1(\beta) - \lambda \tilde{\zeta}_1(s) \mathbf{e}_1 \right] \\ & + h(s) \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \frac{1}{\prod_{l=i}^{n}(\beta - \rho_l)} \left( \tfrac{\mu\beta}{\beta - s} \tilde{\mathbf{m}}_1(\beta) \right) \\ & - h(s) \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \tilde{\zeta}_1[\rho_i, \cdots, \rho_n, s](\lambda \mathbf{e}_1) \Big\}. \end{aligned} \tag{4.20}$$

It is obvious that the factor $h(s)det[\mathbf{A}_\delta(s)]$ of the denominator is a polynomial of degree $n(r_1+1)+r_2$ with leading coefficient $\prod_{i=1}^{n}(\lambda_i^* + \delta)$. Therefore, the equation $h(s)det[\mathbf{A}_\delta(s)] = 0$ has $n(r_1+1)+r_2$ roots on the complex plane. We can factorize $h(s)det[\mathbf{A}_\delta(s)]$ as follows

$$h(s)det[\mathbf{A}_\delta(s)] = \prod_{i=1}^{n}(\lambda_i^* + \delta) \prod_{j=1}^{n}(s - \rho_j) \prod_{j=1}^{nr_1+r_2}(s + R_j), \tag{4.21}$$

where $R_j$ for each $j$ has positive real part and we assume that all of them are distinct from each other.

Substituting (4.21) into (4.20) then canceling the same factor $\prod_{j=1}^{n}(s - \rho_{lj})$, we derive from (4.20) that

$$\begin{aligned} \tilde{\mathbf{m}}_1(s) \quad = \quad & \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta) \prod_{j=1}^{nr_1+r_2}(s+R_j)} \Big\{ \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s] h(s) \left[ \tfrac{\mu\beta}{\beta - s} \tilde{\mathbf{m}}_1(\beta) - \lambda \tilde{\zeta}_1(s) \mathbf{e}_1 \right] \\ & + h(s) \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \frac{1}{\prod_{l=i}^{n}(\beta - \rho_l)} \left( \tfrac{\mu\beta}{\beta - s} \tilde{\mathbf{m}}_1(\beta) \right) \\ & - h(s) \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \tilde{\zeta}_1[\rho_i, \cdots, \rho_n, s](\lambda \mathbf{e}_1) \Big\}. \end{aligned} \tag{4.22}$$

It is easy to find that the elements in matrix $h(s)\mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s]$ are polynomials of degree less than $nr_1 + r_2$, of course, the elements in matrix $h(s)\mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s]\frac{1}{\beta - s}$ are polynomials of degree less than $nr_1 + r_2 - 1$, and each $\mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i]$ for $i = 1, 2, \cdots, n$ is constant. Therefore, we have the following partial fractions:

$$\frac{h(s)\mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s]}{\prod_{j=1}^{nr_1+r_2}(s + R_j)} = \sum_{j=1}^{nr_1+r_2} \frac{\mathbf{Q}_j}{s + R_j},$$

$$\frac{h(s)\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_n,s]\frac{1}{\beta-s}}{\prod_{j=1}^{nr_1+r_2}(s+R_j)}=\sum_{j=1}^{nr_1+r_2}\frac{\mathbf{D}_j}{s+R_j},$$

and

$$\frac{h(s)\frac{1}{\beta-s}}{\prod_{j=1}^{nr_1+r_2}(s+R_j)}=\sum_{j=1}^{nr_1+r_2}\frac{\varsigma_j}{s+R_j},\quad \frac{h(s)}{\prod_{j=1}^{nr_1+r_2}(s+R_j)}=1+\sum_{j=1}^{nr_1+r_2}\frac{\tau_j}{s+R_j},$$

where $\mathbf{Q}_j$, $\mathbf{D}_j$, $\tau_j$ and $\varsigma_j$ are given respectively by

$$\mathbf{Q}_j=\frac{h(-R_j)\mathbf{A}_\delta{}^\star[\rho_1,\cdots,\rho_n,-R_j]}{\prod_{i=1,i\neq j}^{nr_1+r_2}(R_i-R_j)}, \tag{4.23}$$

$$\mathbf{D}_j=\frac{h(-R_j)\mathbf{A}_\delta{}^\star[\rho_1,\cdots,\rho_n,-R_j]\frac{1}{\beta+R_j}}{\prod_{i=1,i\neq j}^{nr_1+r_2}(R_i-R_j)}, \tag{4.24}$$

and

$$\varsigma_j=\frac{h(-R_j)\frac{1}{\beta+R_j}}{\prod_{i=1,i\neq j}^{nr_1+r_2}(R_i-R_j)},\quad \tau_j=\frac{h(-R_j)}{\prod_{i=1,i\neq j}^{nr_1+r_2}(R_i-R_j)}. \tag{4.25}$$

In view of the above partial fractions, (4.22) can be rewritten as

$$\begin{aligned}\tilde{\mathbf{m}}_1(s)\quad=\quad&\frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{j=1}^{nr_1+r_2}\frac{1}{s+R_j}\Big\{\mathbf{D}_j\mu\beta\tilde{\mathbf{m}}_1(\beta)-\mathbf{Q}_j\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\\&+\varsigma_j\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{\mu\beta}{\prod_{l=i}^{n}(\beta-\rho_l)}\tilde{\mathbf{m}}_1(\beta)\\&-\tau_j\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_1[\rho_i,\cdots,\rho_n,s](\lambda\mathbf{e}_1)\Big\}\\&-\frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_1[\rho_i,\cdots,\rho_n,s](\lambda\mathbf{e}_1).\end{aligned} \tag{4.26}$$

By the same arguments, we have

$$\begin{aligned}\tilde{\mathbf{m}}_2(s)\quad=\quad&\frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{j=1}^{nr_1+r_2}\frac{1}{s+R_j}\Big\{\mathbf{D}_j\mu\beta\tilde{\mathbf{m}}_2(\beta)-\mathbf{Q}_j\lambda_n\tilde{\zeta}_2(s)\mathbf{e}_2\\&+\varsigma_j\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{\mu\beta}{\prod_{l=i}^{n}(\beta-\rho_l)}\tilde{\mathbf{m}}_2(\beta)\\&-\tau_j\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_2[\rho_i,\cdots,\rho_n,s](\lambda_n\mathbf{e}_2)\Big\}\\&-\frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_2[\rho_i,\cdots,\rho_n,s](\lambda_n\mathbf{e}_2).\end{aligned} \tag{4.27}$$

From [4], we have the Laplace inverse of $\tilde{\zeta}[\rho_1,\rho_2,\cdots,\rho_n,s]$ as follows

$$\mathcal{L}^{-1}\left(\tilde{\zeta}[\rho_1,\rho_2,\cdots,\rho_n,s]\right)=(-1)^n\left(\prod_{i=1}^{n}T_{\rho_i}\right)\zeta(x). \tag{4.28}$$

Thus, by inverting (4.26) and (4.27) results in the following theorem

**4.5. Theorem.** *If the claim-size distributions $F(x)$ and $G(x)$ both belong to the rational family, the expected discounted penalty function are given by*

$$
\begin{aligned}
\mathbf{m}_1(u) &= \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)} \sum_{j=1}^{nr_1+r_2} \Big\{ e^{-R_j u}\mathbf{D}_j \mu\beta \tilde{\mathbf{m}}_1(\beta) - \mathbf{Q}_j \lambda [e^{-R_j u} \circledast \zeta_1(u)]\mathbf{e}_1 \\
&\quad + \varsigma_j e^{-R_j u} \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \frac{\mu\beta}{\prod_{l=i}^{n}(\beta-\rho_l)} \tilde{\mathbf{m}}_1(\beta) \\
&\quad + \tau_j e^{-R_j u} \circledast \left( \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i](-1)^{n-i} \left( \prod_{l=i}^{n} T_{\rho_l} \right) \zeta_1(u) \right)(\lambda \mathbf{e}_1) \Big\} \\
&\quad + \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)} \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i](-1)^{n-i} \left( \prod_{l=i}^{n} T_{\rho_l} \right) \zeta_1(u)(\lambda \mathbf{e}_1), u \geq 0,
\end{aligned}
\tag{4.29}
$$

*and*

$$
\begin{aligned}
\mathbf{m}_2(u) &= \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)} \sum_{j=1}^{nr_1+r_2} \Big\{ e^{-R_j u}\mathbf{D}_j \mu\beta \tilde{\mathbf{m}}_2(\beta) - \mathbf{Q}_j \lambda_n [e^{-R_j u} \circledast \zeta_2(u)]\mathbf{e}_2 \\
&\quad + \varsigma_j e^{-R_j u} \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \frac{\mu\beta}{\prod_{l=i}^{n}(\beta-\rho_l)} \tilde{\mathbf{m}}_2(\beta) \\
&\quad + \tau_j e^{-R_j u} \circledast \left( \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i](-1)^{n-i} \left( \prod_{l=i}^{n} T_{\rho_l} \right) \zeta_2(u) \right)(\lambda_n \mathbf{e}_2) \Big\} \\
&\quad + \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)} \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i](-1)^{n-i} \left( \prod_{l=i}^{n} T_{\rho_l} \right) \zeta_2(u)(\lambda_n \mathbf{e}_2), u \geq 0,
\end{aligned}
\tag{4.30}
$$

*where $\circledast$ represents the convolution operator. $\mathbf{Q}_j$, $\mathbf{D}_j$, $\tau_j$ and $\varsigma_j$ are given respectively by (4.23)-(4.25).*

## 5. Numerical illustrations

In this section, we present a numerical example to illustrate an application of the main results in this paper. We suppose that the claim amounts from class 1 and class 2 have density functions, respectively,

$$f(x) = \mu_1 e^{-\mu_1 x}, \quad \mu_1 > 0, x > 0, \quad g(y) = \mu_2 e^{-\mu_2 y}, \quad \mu_2 > 0, y > 0.$$

Hence, LTs $\tilde{f}(s) = \frac{\mu_1}{s+\mu_1}$, $\tilde{g}(s) = \frac{\mu_2}{s+\mu_2}$. The inter-claim times from class 1 occur following a Poisson process with parameter $\lambda$, and inter-claim times from class 2 occur following a generalized Erlang(2) distribution with parameters $\lambda_1, \lambda_2$. In addition, the number of insurer's premium income $M(t)$ follows a Poisson process with parameter $\mu > 0$ and the premium sizes are exponentially distributed with parameter $\beta > 0$.

In order to obtain the probability of ultimate ruin, we assume $\delta = 0$ and $w_1(x_1, x_2) = w_2(x_1, x_2) = 1$. Thus

$$\mathbf{A}_0(s) = \begin{pmatrix} \frac{\mu\beta}{\beta-s} - \lambda_1^* + \lambda\tilde{f}(s) & \lambda_1 \\ \lambda_2\tilde{g}(s) & \frac{\mu\beta}{\beta-s} - \lambda_2^* + \lambda\tilde{f}(s) \end{pmatrix}.$$

Now, $m_{kj}(u), k = 1, 2, j = 1, 2, \ldots, n$ simplify to the probability of ultimate ruin $\psi_{kj}(u), k = 1, 2, j = 1, 2, \ldots, n$. Eventually, we are only interested in $\psi_k(u) = \psi_{k1}(u), k = 1, 2$.

For illustration purpose, we set $\mu_1 = 1, \mu_2 = 2, \lambda = 2, \lambda_1 = 1, \lambda_2 = 3, \mu = 3, \beta = 1$. It is easy to check that the positive security loading conditions are satisfied. Under this hypothesis, the solutions of $h(s)det[\mathbf{A}_0(s)] = 0$ are $-R_1 = -1.9087, -R_2 = -0.7394, -R_3 = -0.1222, \rho_1 = 0, \rho_2 = 0.6037$. From Theorem 4.2, we have $\tilde{\mathbf{m}}_1(\beta) =$

$\tilde{\mathbf{m}}_1(1) = \begin{pmatrix} 0.6906 \\ 0.5948 \end{pmatrix}$ and $\tilde{\mathbf{m}}_2(\beta) = \tilde{\mathbf{m}}_2(1) = \begin{pmatrix} 0.0911 \\ 0.2267 \end{pmatrix}$. Substituting $\tilde{\mathbf{m}}_1(1)$, $\tilde{\mathbf{m}}_2(1)$ into (4.29) and (4.30), respectively, we obtain the probability of ruin due to a claim from class $k$,

$$\psi_1(u) = -0.0214e^{-1.9087u} - 0.2504e^{-0.7394u} + 0.8202e^{-0.1222u} + 0.3333e^{-u}, u \geq 0, \tag{5.1}$$

$$\psi_2(u) = 0.0040e^{-1.9087u} - 0.0303e^{-0.7394u} + 0.0958e^{-0.1222u}, u \geq 0. \tag{5.2}$$

Thus, in view of $\psi(u) = \psi_1(u) + \psi_2(u)$, we can obtain the probability of ruin $\psi(u)$. Figure 1 shows the probabilities of ruin $\psi_1(u)$, $\psi_2(u)$ and $\psi(u)$ for different values of $u \in [0, 10]$.

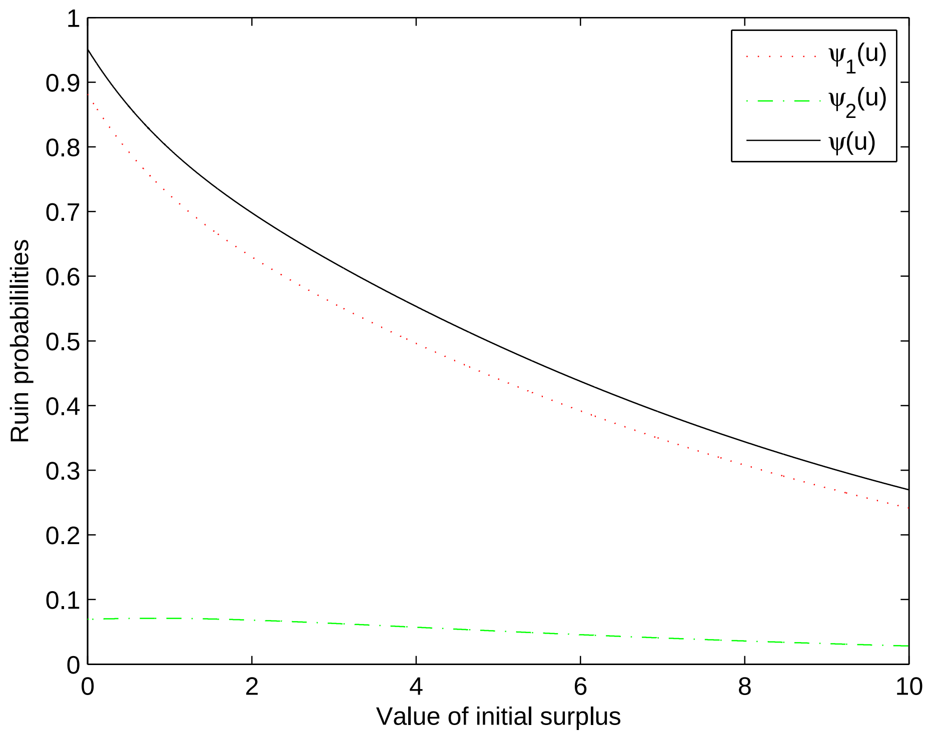


**Figure 1.** Ruin probabilities for different values of $u \in [0, 10]$.

## 6. Concluding remarks

In present paper, we investigate the expected discounted penalty functions in a risk model involving two independent classes of risks with stochastic income, in which the claim number processes are independent Poisson and generalized Erlang(n) processes, respectively. Namely, we extend the model in [13] by assuming that the premium income arrival process is a Poisson process. The integro-differential equations for the expected discounted penalty functions are established. By aid of Dickson-Hipp operator and divided difference, the Laplace transforms for the expected discounted penalty functions are obtained, and explicit expressions are derived when the claim amount distributions belong to the rational family.

The results in our paper can be extended. For example, the premium income arrival process may be a renewal process, the model can also be perturbed by diffusion. We remark that it is very challenging to obtain closed form solutions for the expected discounted penalty functions if we move away from the exponential assumption for the premium sizes. Of course, we can find the solutions numerically for some complicated premium size distributions.

## Acknowledgements

The authors would like to thank the referees for their helpful comments and suggestions which led to the improvement of the paper. This work was supported by the National Natural Science Foundation of China (71221001, 71371068, 71431008), the Hunan Provincial Academy of Social Sciences Fund (14YBA189), China Postdoctoral Science Foundation (2014M562105), Scientific Research Fund of Hunan Provincial Education Department of China (13B034) and the aid program for Science and Technology Research Team in Higher Educational Institutions of Hunan Province of China.

## References

[1] Boikov,A.V. *The Cramér-Lundberg model with stochastic premium process*, Theory of Probabilistic Applications **47**, 489-493, 2002.

[2] Boucherie, R.J., Boxma,O.J. and Sigman, K. *A note on negative customers, GI/G/I workload, and risk processes*, Probability in Engineering and Informational System **11**, 305-311, 1997.

[3] Chadjiconstantinidis, S. and Papaioannou, A.D. *Analysis of the Gerber-Shiu function and dividend barrier problems for a risk process with two classes of claims*, Insurance: Mathematics and Economics **45**, 470-484, 2009.

[4] Gerber, H.U. and Shiu, E.S.W. *The time value of ruin in a Sparre Andersen model*, North American Actuarial Journal **9**, 49-69, 2005.

[5] Ji, L. and Zhang, C. *The Gerber-Shiu penalty functions for two classes of renewal risk processes*, Journal of Computational and Applied Mathematics **233**, 2575-2589, 2010.

[6] Labbé, C. and Sendova, K.P. *The expected discounted penalty function under a risk model with stochastic income*, Applied Mathematics and Computation **215**, 1852-1867, 2009.

[7] Li, S. and Garrido, J. *On ruin for the Erlang(n) risk process*, Insurance: Mathematics and Economics **34**, 391-408, 2004.

[8] Li, S. and Garrido, J. *Ruin probabilities for two classes of risk processes*, ASTIN Bulletin **35**, 61-77, 2005.

[9] Li, S. and Lu, Y. *On the expected discounted penalty functions for two classes of risk processes*, Insurance: Mathematics and Economics **36**, 179-193, 2005.

[10] Temnov,G. *Risk processes with random income*, Journal of Mathematical Sciences **123**, 3780-3794, 2004.

[11] Xie, J.H. and Zou, W. *On a risk model with random incomes and dependence between claim sizes and claim intervals*, Indagationes Mathematicae **24**, 557-580, 2013.

[12] Yuen, K.C., Guo, J. and Wu, X. *On a correlated aggregate claims model with Poisson and Erlang risk processes*, Insurance: Mathematics and Economics **31**, 205-214, 2002.

[13] Zhang, Z.M., Li, S. and Yang, H. *The Gerber-Shiu discounted penalty functions for a risk model with two classes of claims*, Journal of Computational and Applied Mathematics **230**, 643–655, 2009.

[14] Zhang, Z.M. and Yang,H. *On a risk model with stochastic premiums income and dependence between income and loss*, Journal of Computational and Applied Mathematics **234**, 44-57, 2010.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 923 – 931

# Improved ratio-type estimators using maximum and minimum values under simple random sampling scheme

Mursala Khan*†, Saif Ullah‡, Abdullah Y. Al-Hossain§ and Neelam Bashir¶

**Abstract**

This paper presents a class of ratio-type estimators for the evaluation of finite population mean under maximum and minimum values by using knowledge of the auxiliary variable. The properties of the proposed estimators in terms of biases and mean square errors are derived up to first order of approximation. Also, the performance of the proposed class of estimators is shown theoretically and these theoretical conditions are, then, verified numerically by taking three natural populations under which the proposed class of estimators performed better than other competing estimators.

*2000 AMS Classification:* 62D05.

**Keywords:** Study variable, Auxiliary variable, Ratio estimators, Maximum and Minimum values, Simple random sampling, Mean squared error, Efficiency.

Received 18/04/2014 : Accepted 06/09/2014 Doi : 10.15672/HJMS.2014297480

*Department of Mathematics, COMSATS Institute of Information Technology Abbottabad, 22060, Pakistan. Email: mursala.khan@yahoo.com

†Corresponding Author.

‡Department of Mathematics, Government College University Lahore, 54000, Pakistan. Email: dr.saifullah@gcu.edu.pk

§Department of Mathematics, Faculty of Science, Jazan University, Jazan 2097, Saudi Arabia.

¶Department of Mathematics, Government College University Lahore, 54000, Pakistan.

## 1. Introduction

A lot of work has been done for the estimation of finite population mean using auxiliary information for improving the efficiency of the estimators. Das and Tripathi (1980, 1981), Uphadhyaya and Singh (1999), Singh (2004), Sisodia and Dwivedi (1981) proposed ratio estimator using coefficient of variation of an auxiliary variable. Kadilar and Cingi (2005) suggested ratio estimators in stratified random sampling. In the same way, Kadilar and Cingi (2006) proposed an improvement in estimating the population mean by using the correlation coefficient. Khan and Shabbir (2013) proposed a ratio-type estimator for the estimation of population variance using the knowledge of quartiles and their functions as auxiliary information. They proposed different modified estimators for the estimation of finite population mean using maximum and minimum values. Recently Hossain and Khan (2014) worked on the estimation of population mean using maximum and minimum values under simple random sampling by incorporating the knowledge of two auxiliary variables.

Let us consider a finite population of size $N$ of different units $U = \{U_1, U_2, U_3, ....., U_N\}$. Let $y$ and $x$ be the study and the auxiliary variable with corresponding values $y_i$ and $x_i$ respectively for the $i$-th unit $i = \{1, 2, 3, ...., N\}$ defined on a finite population $U$. Let $\overline{Y} = (1/N) \sum_{i=1}^{N} y_i$ and $\overline{X} = (1/N) \sum_{i=1}^{N} x_i$ be the population means of the study and the auxiliary variable, respectively. Also $S_y^2 = (1/N-1) \sum_{i=1}^{N}(y_i - \overline{Y})^2$ and $S_x^2 = (1/N-1) \sum_{i=1}^{N}(x_i - \overline{X})^2$ be the corresponding population mean square error of the study and the auxiliary variable respectively, and let $C_y = S_y/\overline{Y}$ and $C_x = S_x/\overline{X}$ be the coefficients of variation of the study and the auxiliary variable respectively, and $\rho_{yx} = S_{yx}/S_y S_x$ be the population correlation coefficient between $x$ and $y$.

In order to estimate the unknown population parameters we take a random sample of size $n$ units from the finite population $U$ by using simple random sample without replacement. Let $\overline{y} = (1/n) \sum_{i=1}^{n} y_i$ and $\overline{x} = (1/n) \sum_{i=1}^{n} x_i$ be the corresponding sample means of the study and the auxiliary variable respectively, and their corresponding sample variances are $\hat{S}_y^2 = (1/n-1) \sum_{i=1}^{n}(y_i - \overline{y})^2$ and $\hat{S}_x^2 = (1/n-1) \sum_{i=1}^{n}(x_i - \overline{x})^2$ respectively.

When there is no auxiliary information the usual unbiased estimator for the population mean of the study variable is:

$$\overline{y} = \frac{\sum_{i=1}^{n} y_i}{n} \tag{1.1}$$

The variance of the estimator $\overline{y}$ is given by:

$$\text{var}(\overline{y}) = \theta\, S_y^2\,, \quad \text{where } \theta = \frac{1}{n} - \frac{1}{N}. \tag{1.2}$$

In many populations there exist some large $(y_{max})$ or small $(y_{min})$ values and to estimate the population parameters without considering this information is very sensitive. In either case the result will be overestimated or underestimated. In order to handle this situation Sarndal (1972) suggested the following unbiased estimator for the assessment of finite population mean:

$$\overline{y}_s = \begin{cases} \overline{y} + c & \text{if sample contains } y_{min} \text{ but not } y_{max} \\ \overline{y} - c & \text{if sample contains } y_{max} \text{ but not } y_{min} \\ \overline{y} & \text{for all other samples,} \end{cases} \tag{1.3}$$

were $c$ is a constant whose value is to be found for minimum variance.

The minimum variance of the estimator $\overline{y}_s$ up to first order of approximation is given as under:

$$\text{(1.4)}\qquad \text{var}\,(\overline{y}_s)_{min} = \text{var}\,(\overline{y}) - \frac{\theta(y_{max} - y_{min})^2}{2(N-1)}\,,$$

where the optimum value of $c_{opt}$ is

$$c_{opt} = \frac{(y_{max} - y_{min})}{2n}.$$

The classical ratio estimator for finding the population mean of the study variable is given by:

$$\text{(1.5)}\qquad \hat{\overline{Y}}_R = \overline{y}\ \frac{\overline{X}}{\overline{x}}$$

The bias and mean square errors of the estimator $\hat{\overline{Y}}_R$ up to first order of approximation are given by:

$$\text{(1.6)}\qquad \text{Bias}\left(\hat{\overline{Y}}_R\right) = \frac{\theta}{\overline{X}}\ (RS_x^2 - S_{yx})$$

$$\text{(1.7)}\qquad \text{MSE}\left(\hat{\overline{Y}}_R\right) = \theta(S_y^2 + R^2 S_x^2 - 2RS_{yx})$$

Similarly, Sisodid and Dwivedi (1981) suggested the following ratio estimator using the knowledge of coefficient of variation of the auxiliary variable:

$$\text{(1.8)}\qquad \hat{\overline{Y}}_{SD} = \overline{y}\left(\frac{\overline{X} + C_x}{\overline{x} + C_x}\right)$$

The bias and mean square errors of the estimator $\hat{\overline{Y}}_{SD}$ up to first order of approximation are as follows:

$$\text{(1.9)}\qquad \text{Bias}\left(\hat{\overline{Y}}_{SD}\right) = \frac{\theta\,\alpha_1}{\overline{Y}}\ (R\alpha_1 S_x^2 - S_{yx}),$$

$$\text{(1.10)}\qquad \text{MSE}\left(\hat{\overline{Y}}_{SD}\right) = \theta(S_y^2 + \alpha_1^2 S_x^2 - 2\alpha_1 S_{yx}),\quad \text{where}\ \ \alpha_1 = \frac{\overline{Y}}{\overline{X} + C_x}\,.$$

## 2. The proposed class of estimators

On the lines of Sarndal (1972), we propose a class of ratio-type estimators for the estimation of finite population mean using knowledge of the coefficient of variation and coefficient of correlation of an auxiliary variable. Usually when the correlation between the study variable ($y$) and the auxiliary variable ($x$) is positive, then the selection of the larger value of the auxiliary variable ($x$), the larger value of study variable ($y$) is to be expected, and the smaller the value of auxiliary variable ($x$), the smaller the value of study variable ($y$). Using such type of information, we propose the following class of estimators given by:

$$\text{(2.1)}\qquad \hat{\overline{Y}}_{P_1} = \overline{y}_{c_1}\left(\frac{\overline{X} + C_x}{\overline{x}_{c_2} + C_x}\right),$$

$$\text{(2.2)}\qquad \hat{\overline{Y}}_{P_2} = \overline{y}_{c_1}\left(\frac{\overline{X} + \rho_{yx}}{\overline{x}_{c_2} + \rho_{yx}}\right),$$

$$\text{(2.3)}\qquad \hat{\overline{Y}}_{P_3} = \overline{y}_{c_1}\left(\frac{\overline{X}C_x + \rho_{yx}}{\overline{x}_{c_2}C_x + \rho_{yx}}\right),$$

$$\hat{\overline{Y}}_{P_4} = \overline{y}_{c_1}\left(\frac{\overline{X}\rho_{yx} + C_x}{\overline{x}_{c_2}\rho_{yx} + C_x}\right), \tag{2.4}$$

where ($\overline{y}_{c_1} = \overline{y} + c_1$, $\overline{x}_{c_2} = \overline{x} + c_2$), also $c_1$ and $c_2$ are unknown constants.

To obtain the properties of $\hat{\overline{Y}}_{P_i}$ in terms of Bias and Mean square error, we define the following relative error terms and their expectations:

$$\zeta_0 = \frac{\overline{y}_{c_1} - \overline{Y}}{\overline{Y}}\ , \quad \zeta_1 = \frac{\overline{x}_{c_2} - \overline{X}}{\overline{X}}\ , \text{ such that } \mathrm{E}(\zeta_0) = \mathrm{E}(\zeta_1) = 0,$$

Also,

$$\mathrm{E}\Big(\zeta_0^2\Big) = \frac{\theta}{\overline{Y}^2}\Big(S_y^2 - \frac{2nc_1}{N-1}(y_{max} - y_{min} - nc_1)\Big),\ \ \mathrm{E}\Big(\zeta_1^2\Big) = \frac{\theta}{\overline{X}^2}\Big(S_x^2 - \frac{2nc_2}{N-1}(x_{max} - x_{min} - nc_2)\Big)$$

$$\text{and} \quad \mathrm{E}(\zeta_0\zeta_1) = \frac{\theta}{\overline{Y}\,\overline{X}}\Big(S_{yx} - \frac{n}{N-1}(c_2(y_{max} - y_{min}) + c_1(x_{max} - x_{min}) - 2nc_1c_2)\Big)\,.$$

$$\begin{aligned}\text{Where } \theta \quad = \quad & \frac{1}{n} - \frac{1}{N}\ ,\ R = \frac{\overline{Y}}{\overline{X}}\ ,\ \alpha_{P_1} = \frac{\overline{X}}{\overline{X} + C_x}\ ,\ \alpha_{P_2} = \frac{\overline{X}}{\overline{X} + \rho_{yx}}\ ,\ \alpha_{P_3} = \frac{\overline{X}C_x}{\overline{X}C_x + \rho_{yx}}\ , \\ & \alpha_{P_4} = \frac{\overline{X}\rho_{yx}}{\overline{X}\rho_{yx} + C_x}\ ,\ k_{P1} = \frac{\overline{Y}}{\overline{X} + C_x}\ ,\ k_{P2} = \frac{\overline{Y}}{\overline{X} + \rho_{yx}}\ ,\ k_{P3} = \frac{\overline{Y}C_x}{\overline{X}C_x + \rho_{yx}} \\ & \text{and} \quad k_{P4} = \frac{\overline{Y}\rho_{yx}}{\overline{X}\rho_{yx} + C_x}.\end{aligned}$$

Rewriting $\hat{\overline{Y}}_{P_i}$ in terms of $\zeta_i$'s, we have

$$\hat{\overline{Y}}_{P_i} = \overline{Y}\ \Big(1 + \zeta_0\Big)\Big(1 + \alpha_{P_i}\zeta_1\Big)^{-1},$$

where $\hat{\overline{Y}}_{P_i}$ represent the proposed class of estimators for $i$=1, 2, 3, 4.

Expanding the right hand side of the equation given above and including terms up to second powers of $\zeta_i$'s i.e., up to first order of approximation, we have:

$$\hat{\overline{Y}}_{P_i} - \overline{Y} = \overline{Y}\Big(\zeta_0 - \alpha_{P_i}\zeta_1 + \alpha_{Pi}^2\zeta_1^2 - \alpha_{P_i}\zeta_0\zeta_1\Big) \tag{2.5}$$

Taking expectation on both sides of (2.5), we get bias up to first order of approximation which is given as:

$$\begin{aligned}\mathrm{Bias}(\hat{\overline{Y}}_{P_i}) \quad = \quad & \frac{\theta\, k_{P_i}}{\overline{Y}}\Big[k_{P_i}\Big(S_x^2 - \frac{2nc_2}{N-1}(x_{max} - x_{min} - nc_2)\Big) - S_{yx} \\ & + \frac{n}{N-1}(c_2(y_{max} - y_{min}) + c_1(x_{max} - x_{min}) - 2nc_1c_2)\Big]\end{aligned} \tag{2.6}$$

On squaring both sides of (2.5), and keeping $\zeta_i$'s powers up to first order of approximation, we get:

$$\Big(\hat{\overline{Y}}_{P_i} - \overline{Y}\Big)^2 = \overline{Y}^2\ \Big(\zeta_0^2 + \alpha_{P_i}^2\zeta_1^2 - 2\alpha_{P_i}\zeta_0\zeta_1\Big) \tag{2.7}$$

Taking expectation on both sides of (2.7), we get mean square error up to first order of approximation, given as under:

$$\begin{aligned}\mathrm{MSE}\Big(\hat{\overline{Y}}_{P_i}\Big) \quad = \quad & \theta\Big[\Big(S_y^2 + k_{P_i}^2 S_x^2 - 2k_{P_i}S_{yx}\Big) - \frac{2n}{N-1}\Big\{(c_1 - c_2k_{P_i})((y_{max} - y_{min}) \\ & - n(c_1 - c_2k_{P_i}) - k_{P_i}(x_{max} - x_{min}))\Big\}\Big]\end{aligned} \tag{2.8}$$

The optimum values of $c_1$ and $c_2$ are given in the following lines:

$$\left\{\begin{array}{l} c_1 = \frac{(y_{max}-y_{min})}{2n} \\ \\ c_2 = \frac{(x_{max}-x_{min})}{2n} \end{array}\right. \tag{2.9}$$

On substituting the optimum value of $c_1$ and $c_2$ in (2.8), we get the minimum mean square error of the proposed estimators as follows:

$$\begin{aligned} \text{MSE}\left(\hat{\bar{Y}}_{P_i}\right)_{min} &= \theta\Big[\Big(S_y^2 + k_{P_i}^2 S_x^2 - 2k_{P_i} S_{yx}\Big) \\ &\quad -\frac{1}{2(N-1)}\Big((y_{max}-y_{min}) - k_{P_i}(x_{max}-x_{min})\Big)^2\Big] \end{aligned} \tag{2.10}$$

## 3. Comparison of estimators

In this section, we compare the proposed class of estimators with other existing estimators and some of their efficiency comparison conditions have been carried out under which the proposed class of estimators perform better than the other existing estimators discussed in the literature above.

(i) By (1.2) and (2.10),

$\Big[\text{MSE }(\overline{y}) - \text{MSE }(\hat{\bar{Y}}_{P_i})_{min}\Big] \geq 0,$ if

$$\Big[\frac{1}{2(N-1)}\Big\{(y_{max}-y_{min}) - k_{P_i}(x_{max}-x_{min})\Big\}^2 - k_{P_i}^2 S_x^2 + 2k_{P_i}S_{yx}\Big] \geq 0.$$

(ii) By (1.4) and (2.10),

$\Big[\text{MSE }(\overline{y}_s) - \text{MSE }(\hat{\bar{Y}}_{P_i})_{min}\Big] \geq 0,$ if

$$\Big[k_{P_i}\Big(\frac{(x_{max}-x_{min})^2}{2(N-1)} - S_x^2\Big) - \Big(\frac{(y_{max}-y_{min})(x_{max}-x_{min})}{N-1} - 2S_{yx}\Big)\Big] \geq 0.$$

(iii) By (1.7) and (2.10),

$\Big[\text{MSE }(\hat{\bar{Y}}_R) - \text{MSE }(\hat{\bar{Y}}_{P_i})_{min}\Big] \geq 0,$ if

$$\Big[\frac{1}{2(N-1)}\Big((y_{max}-y_{min}) - k_{P_i}(x_{max}-x_{min})\Big)^2 + S_x^2(R-k_{P_i})(R+k_{P_i}-2\,\delta)\Big] \geq 0.$$

(iv) By (1.10) and (2.10),

$\Big[\text{MSE }(\hat{\bar{Y}}_{SD}) - \text{MSE }(\hat{\bar{Y}}_{P_i})_{min}\Big] \geq 0,$ if

$$\Big[\frac{1}{2(N-1)}\Big((y_{max}-y_{min}) - k_{P_i}(x_{max}-x_{min})\Big)^2 + S_x^2(R\alpha_1-k_{P_i})(R\alpha_1+k_{P_i}-2\delta)\Big] \geq 0.$$

Where

$$\delta = \frac{\rho_{yx}S_y}{S_x}.$$

## 4. Numerical illustration

In this section, we illustrate the performance of the proposed class of estimators in comparison with various other existing estimators through three natural populations. The description and the necessary data statistics are given by:

**Population-1:** [Source: Singh and Mangat (1996), p.193]

$Y$: be the milk yield in kg after new food, and

$X$: be the yield in kg before new yield.

$$\begin{aligned} N &= 27,\ n = 12,\ \overline{X} = 10.4111,\ \overline{Y} = 11.2519,\ y_{max} = 14.8,\ y_{min} = 7.9,\ x_{max} = 14.5, \\ & x_{min} = 6.5,\ S_y^2 = 4.103\,,\ S_x^2 = 4.931,\ S_{yx} = 4.454,\ \rho_{yx} = 0.990. \end{aligned}$$

**Population-2:** [Source: Murthy (1967), p.399]

$Y$: be the area under wheat crop in 1964, and

$X$: be the area under wheat crop in 1963.

$$\begin{aligned} N &= 34,\ n = 12,\ \overline{X} = 208.882,\ \overline{Y} = 199.441,\ y_{max} = 634,\ y_{min} = 6,\ x_{max} = 564, \\ & x_{min} = 5,\ S_y^2 = 22564.56,\ S_x^2 = 22652.05,\ S_{yx} = 22158.05,\ \rho_{yx} = 0.980. \end{aligned}$$

**Population-3:** [Source: Cochran (1977), p.152]

$Y$: be the population size in 1930 (in 1000), and

$X$: be the population size in 1920 (in 1000).

$$\begin{aligned} N &= 49,\ n = 12,\ \overline{X} = 103.1429,\ \overline{Y} = 127.7959,\ y_{max} = 634,\ y_{min} = 46,\ x_{max} = 507, \\ & x_{min} = 2,\ S_y^2 = 15158.83,\ S_x^2 = 10900.42,\ S_{yx} = 12619.78,\ \rho_{yx} = 0.98. \end{aligned}$$

The mean squared error of the proposed class and the existing estimators are shown in Table-1.

Table-1: *MSE* of the Competing and the Proposed Class of Estimators

| Estimator | | Population 1 | Population 2 | Population 3 |
|---|---|---|---|---|
| | | *MSE* (.) | *MSE* (.) | *MSE* (.) |
| Existing | $\bar{y}$ | 0.1900 | 1220.9455 | 953.4904 |
| | $\bar{y}_S$ | 0.1476 | 898.8652 | 726.9560 |
| | $\hat{\bar{Y}}_R$ | 0.0109 | 48.5723 | 39.0823 |
| | $\hat{\bar{Y}}_{SD}$ | 0.0092 | 48.6879 | 37.8472 |
| Proposed | $\hat{\bar{Y}}_{P1}$ | 0.0070 | 46.8878 | 37.1675 |
| | $\hat{\bar{Y}}_{P2}$ | 0.0044 | 41.2479 | 37.2200 |
| | $\hat{\bar{Y}}_{P3}$ | 0.0085 | 41.2162 | 37.2378 |
| | $\hat{\bar{Y}}_{P4}$ | 0.0070 | 41.1896 | 37.1704 |

For the percent relative efficiencies (PREs) of the proposed class and the existing estimators, we use the following expression for efficiency comparison. The results are, then, shown in Table-2.

$$\mathrm{PRE}\left(\hat{\bar{Y}}_g, \bar{y}\right) = \frac{\mathrm{MSE}\left(\bar{y}\right)}{\mathrm{MSE}\left(\hat{\bar{Y}}_g\right)} \times 100, \quad \text{where} \quad g = S,\ R,\ SD,\ P_1,\ P_2,\ P_3 \text{ and } P_4.$$

Table-2: $PRE$ of Different Estimators with Respect to $\bar{y}$

| Estimator | | Population 1 | Population 2 | Population 3 |
|---|---|---|---|---|
| | | $PRE\,(.,\hat{\bar{Y}})$ | $PRE\,(.,\hat{\bar{Y}})$ | $PRE\,(.,\hat{\bar{Y}})$ |
| Existing | $\bar{y}$ | 100.00 | 100.00 | 100.00 |
| | $\bar{y}_S$ | 128.7263 | 135.8319 | 131.1621 |
| | $\hat{\bar{Y}}_R$ | 1743.1193 | 2513.6662 | 2439.6988 |
| | $\hat{\bar{Y}}_{SD}$ | 2065.2174 | 2507.6414 | 2519.3156 |
| Proposed | $\hat{\bar{Y}}_{P1}$ | 2714.2857 | 2603.9727 | 2565.3875 |
| | $\hat{\bar{Y}}_{P2}$ | 4318.1818 | 2960.0186 | 2561.7689 |
| | $\hat{\bar{Y}}_{P3}$ | 2235.2941 | 2962.2952 | 2560.5444 |
| | $\hat{\bar{Y}}_{P4}$ | 2714.2857 | 2964.2082 | 2565.1874 |

## 5. Conclusion

In this study, we have developed some ratio-type estimators under maximum and minimum values using knowledge of the coefficient of variation and coefficient of correlation of the auxiliary variable. We have found some theoretical possibilities under which the proposed class of estimators have smaller mean squared errors than the usual unbiased estimator; the classical ratio estimator; and the other competing estimators suggested by statisticians. Theoretical results are also verified with the help of three natural populations and their statistics are shown in table 1 and table 2, which clearly indicates that the proposed estimators have smaller mean squared errors and larger percent relative efficiency than the other estimators discussed in the literature. Thus the proposed estimators under maximum and minimum values may be preferred over the existing estimators for the use of practical applications.

## 6. Acknowledgements

The authors wish to thank the reviewers for their careful assessment, constructive comments and suggestions, which significantly improved this paper.

## References


[1] Al-Hossain, Y. and Khan, M. *Efficiency of ratio, product and regression estimators under maximum and minimum values using two auxiliary variables*, Journal of Applied Mathematics **2014**, Atricle ID 693782, 2014.

[2] Cochran, W. G. *Sampling Techniques*, (John Wiley and Sons, New York, 1977).

[3] Das, A. K. Tripathi, T. P. *A class of sampling strategies for population mean using information on mean and variance of an auxiliary character*, Proceeding of the Indian Statistical Institute; Golden Jubilee International Conference on Statistic: Applications and New Directions, Calcutta, 16-19 December, 1981, 174-181, 1981.

[4] Das, A. K. and Tripathi, T. P. *Sampling strategies for population mean when the coefficient of variation of an auxiliary character is known*, Sankhya-C **42**, 76-86, 1980.

[5] Kadilar, C. and Cingi, H. *A new ratio estimator in stratified random sampling*, Communication in Statistics: Theory and Methods **34**, 597-602, 2005.

[6] Kadilar, C. and Cingi, H. *An improvement in estimating the population mean by using the correlation cofficient*, Hacettepe Journal of Mathematics and Statistics **35** (1), 103-109, 2006.

[7] Khan, M. and Shabbir, J. *A ratio type estimator for the estimation of population variance using quartiles and its functions of an auxiliary variables*, Journal of Statistics Applications & Probability **2** (3), 157-162, 2013.

[8] Khan, M. and Shabbir, J. *Some improved ratio, product and regression estimators of finite population mean when using minimum and maximum values*, The Scientific World Journal, 1-7, 2013.

[9] Murthy, N. M. *Product method of estimation*, Sankhya-A **16**, 69-74, 1964.

[10] Sarndal, C. E. *Sampling survey theory vs general statistical theory: Estimation of the population mean*, International Statistical Institue **40**, 1-12, 1972.

[11] Singh, R. and Mangat, N. S. *Elements of Survey Sampling*, (Kluwer Academic Publishers, Dordrecht, Boston, London, 1996).

[12] Singh, S. *Golden and Silver Jubilee year-2003 of the linear regression estimators*, Proceeding of the American Statistical Association, Survey Method Section [CD-ROM], Toronto, Canada: American Statistical Association, 4382–4389, 2004.

[13] Sisodia, B. V. S. and Dwivedi, V. K. *A modified ratio estimator using coefficient of variation of auxiliary variable*, Journal of the Indian Society of Agricultural Statistics **33** (1), 13-18, 1981.

[14] Upadhyaya, L. N. and singh, H. P. *Use of transformed auxiliary variable in estimating the finite population mean*, Biometrical Journal **41** (5), 627–636, 1999.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 933 – 948

# Multivariate generalization of the Gauss hypergeometric distribution

Daya K. Nagar* , Danilo Bedoya-Valencia† and Saralees Nadarajah‡

**Abstract**

The Gauss hypergeometric distribution with the density proportional to $x^{\alpha-1}(1-x)^{\beta-1}(1+\xi x)^{-\gamma}$, $0<x<1$ arises in connection with the prior distribution of the parameter $\rho$ $(0<\rho<1)$ representing traffic intensity in a $M/M/1$ queue system. In this article, we define and study a multivariate generalization of this distribution and derive some of its properties like marginal densities, joint moments, and factorizations. A data application is given.

*2000 AMS Classification:* Primary 33E99; Secondary 62H99.

**Keywords:** Beta function; Dirichlet function; Gamma function; Gauss hypergeometric function; Liouville integral; Multivariate; Transformation.

Received 15/11/2013 : Accepted 01/09/2014 Doi : 10.15672/HJMS.2014277478

## 1. Introduction

A random variable $X$ is said to have a Gauss hypergeometric distribution with parameters $\alpha>0$, $\beta>0$, $-\infty<\gamma<\infty$ and $\xi>-1$, denoted by $X\sim \mathrm{GH}(\alpha,\beta,\gamma,\xi)$, if its probability density function (p.d.f.) is given by

$$(1.1)\qquad f_{\mathrm{GH}}(x;\alpha,\beta,\gamma,\xi)=C(\alpha,\beta,\gamma,\xi)\,\frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1+\xi x)^{\gamma}},\quad 0<x<1,$$

where the normalizing constant $C(\alpha,\beta,\gamma,\xi)$ is given by

$$(1.2)\qquad C(\alpha,\beta,\gamma,\xi)=\left[B(\alpha,\beta)\ {}_2F_1(\gamma,\alpha;\alpha+\beta;-\xi)\right]^{-1},$$

*Instituto de Matemáticas, Universidad de Antioquia, Calle 67, No. 53-108, Medellín, Colombia.
Email: dayaknagar@yahoo.com

†Instituto de Matemáticas, Universidad de Antioquia, Calle 67, No. 53-108, Medellín, Colombia.
Email:danilo.bv@gmail.com

‡School of Mathematics, University of Manchester, Manchester, M13 9PL, UK.
Email:Saralees.Nadarajah@manchester.ac.uk

with $B(\alpha, \beta)$ being the beta function is defined by

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)},$$

and ${}_2F_1$ is the Gauss hypergeometric function (Luke [13]). Note that the Gauss hypergeometric function ${}_2F_1$ in (1.2) can be expanded in series form if $-1 < \xi < 1$. If $\xi > 1$, then the function can be suitably transformed such that the absolute value of its argument is less than one, see (2.5).

The above distribution was suggested by Armero and Bayarri [1] in connection with the prior distribution of the parameter $\rho$, $0 < \rho < 1$ representing the traffic intensity in a $M/M/1$ queueing system. A brief introduction of this distribution is given in the encyclopedic work of Johnson, Kotz and Balakrishnan [10, p. 253]. In the context of Bayesian analysis of unreported Poisson count data, while deriving the marginal posterior distribution of the reporting probability $p$, Fader and Hardie [5] have shown that $q = 1-p$ has a Gauss hypergeometric distribution. The Gauss hypergeometric distribution has also been used by Dauxois [4] to introduce conjugate priors in the Bayesian inference for linear growth birth and death processes. Sarabia and Castillo [21] have pointed out that this distribution is conjugate prior for the binomial distribution.

When either $\gamma$ or $\xi$ equals to zero, the Gauss hypergeometric p.d.f. reduces to a beta type 1 p.d.f. given by (Johnson, Kotz and Balakrishnan [10]),

$$f_{\mathrm{B1}}(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1. \tag{1.3}$$

Further, for $\gamma = \alpha + \beta$ and $\xi = 1$ the Gauss hypergeometric distribution simplifies to a beta type 3 distribution given by the p.d.f. (Cardeño, Nagar and Sánchez [3], Sánchez and Nagar [20]),

$$f_{\mathrm{B3}}(x; \alpha, \beta) = \frac{2^{\alpha} x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)(1+x)^{\alpha+\beta}}, \quad 0 < x < 1. \tag{1.4}$$

The matrix variate generalizations of beta type 1 and beta type 3 distributions have been defined and studied extensively. For example, see Gupta and Nagar [6, 7]. For $\gamma = \alpha + \beta$ and $\xi = -(1-\lambda)$ the GH distribution slides to a three parameter generalized beta type 1 distribution (Libby and Novic [12], Pham-Gia and Duong [19], Nadarajah [15], Nagar and Rada-Mora [17]) defined by the p.d.f.

$$f_{\mathrm{GB1}}(x; \alpha, \beta; \lambda) = \frac{\lambda^{\alpha} x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)[1-(1-\lambda)x]^{\alpha+\beta}}, \quad 0 < x < 1, \tag{1.5}$$

where $\alpha > 0$ and $\beta > 0$.

In this article, we propose a multivariate generalization of the Gauss hypergeometric distribution which is a new members of the Liouville family of distributions. We define the multivariate generalization of (1.1) and study some of its properties such as marginal p.d.f.s, joint moments, variance and covariances. We also derive the distribution of partial sums of random variables jointly distributed as multivariate Gauss hypergeometric and several results on factorizations in terms of known distributions. Finally, a data application of the multivariate Gauss hypergeometric p.d.f. is illustrated.

Multivariate Liouville family of distributions was proposed by Marshall and Olkin [14]. Sivazlian [22] introduced Liouville distributions as generalizations of gamma and Dirichlet distributions. The Dirichlet and Liouville distributions arise in a variety of context including Bayesian analysis, modeling of multivariate data, order statistics, limit laws, multivariate analysis, reliability theory and stochastic processes. These distributions have been widely used in geology, biology, chemistry, forensic science, and statistical genetics. A comprehensive account of some applications and other aspects of these distributions can be found in Gupta and Song [9], Gupta and Richards [8], Marshall and Olkin [14], Nagar, Bran-Cardona and Gupta [16], Nagar and Sepúlveda-Murillo [18], and Song and Gupta [23].

## 2. Preliminaries

In this section we give definitions and results that will be used in subsequent sections. Throughout this work we will use the Pochhammer symbol $(a)_n$ defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots,$ and $(a)_0 = 1$.

The generalized hypergeometric function of scalar argument is defined by

$$ {}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}, \tag{2.1} $$

where $a_i$, $i = 1, \ldots, p$; $b_j$, $j = 1, \ldots, q$ are complex numbers with suitable restrictions and $x$ is a complex variable.

Conditions for the convergence of the series in (2.1) are available in the literature, see Luke [13]. From (2.1) it is easy to see that

$$ {}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \quad |x| < 1. \tag{2.2} $$

The integral representation of the Gauss hypergeometric function is

$$ {}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b} \, \mathrm{d}t, \tag{2.3} $$

where $\operatorname{Re}(c) > \operatorname{Re}(a) > 0$ and $|\arg(1-x)| < \pi$. Note that, the series expansion for ${}_2F_1$ given in (2.2) can be obtained by expanding $(1-xt)^{-b}$, $|xt| < 1$, in (2.3) and integrating $t$. Substituting $x = 1$ in (2.3) and integrating, we obtain

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \operatorname{Re}(c-a-b) > 0, \tag{2.4} $$

$c \neq 0, -1, -2, \ldots$ . The Gauss' hypergeometric function ${}_2F_1$ satisfies the following relations

$$ \begin{aligned} {}_2F_1(a, b; c; x) &= (1-x)^{-b} \, {}_2F_1(c-a, b; c; -x(1-x)^{-1}) \\ &= (1-x)^{c-a-b} \, {}_2F_1(c-a, c-b; c; x). \end{aligned} \tag{2.5} $$

Let $f$ be a continuous function and $\operatorname{Re}(\alpha_i) > 0$, $i = 1, \ldots, n$, the integral

$$ D_n(\alpha_1, \ldots, \alpha_n; f) = \underset{\substack{x_1>0,\ldots,x_n>0 \\ \sum_{i=1}^n x_i<1}}{\int \cdots \int} \prod_{i=1}^{n} x_i^{\alpha_i - 1} f\left(\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} \mathrm{d}x_i, $$

is known as the Liouville-Dirichlet integral. Making the substitution $y_i = x_i/x$, $i = 1, \ldots, n-1$ and $x = \sum_{i=1}^n x_i$ with the Jacobian $J(x_1, \ldots, x_n \to y_1, \ldots, y_{n-1}, x) = x^{n-1}$ and integrating, we obtain

$$ D_n(\alpha_1, \ldots, \alpha_n; f) = \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^n \alpha_i\right)} \int_0^1 x^{\sum_{i=1}^n \alpha_i - 1} f(x) \, \mathrm{d}x. \tag{2.6} $$

In particular, for $f(x) = (1-x)^{\beta-1}/(1+\xi x)^{\gamma}$, we obtain

$$ D_n\left(\alpha_1, \ldots, \alpha_n; f\right) = \frac{\prod_{i=1}^n \Gamma(\alpha_i)\Gamma(\beta)}{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta\right)} {}_2F_1\left(\gamma, \sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta; -\xi\right). \tag{2.7} $$

## 3. The Density Function

We propose a multivariate generalization of the Gauss hypergeometric distribution as follows.

**3.1. Definition.** The random variables $X_1, \ldots, X_n$ are said to have a multivariate Gauss hypergeometric distribution with parameters $\alpha_i > 0$, $i = 1, \ldots, n$, $\beta > 0$, $-\infty < \gamma < \infty$ and $\xi > -1$, denoted as $(X_1, \ldots, X_n) \sim \mathrm{GH}(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)$, if their joint p.d.f. is

$$C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi) \, \frac{\prod_{i=1}^n x_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^n x_i\right)^{\beta - 1}}{\left(1 + \xi \sum_{i=1}^n x_i\right)^{\gamma}}, \tag{3.1}$$

$$x_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^n x_i < 1,$$

where $C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)$ is the normalizing constant.

Since, the integration of the p.d.f. (3.1) over its support set is one, we have

$$C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi) \underset{\substack{x_1 > 0, \ldots, x_n > 0 \\ \sum_{i=1}^n x_i < 1}}{\int \cdots \int} \frac{\prod_{i=1}^n x_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^n x_i\right)^{\beta - 1}}{\left(1 + \xi \sum_{i=1}^n x_i\right)^{\gamma}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n = 1,$$

and, using (2.7), we obtain the expression for $C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)$ as

$$[C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)]^{-1} = \frac{\prod_{i=1}^n \Gamma(\alpha_i) \Gamma(\beta)}{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta\right)} \, {}_2F_1\left(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi\right). \tag{3.2}$$

For specific values of the parameters, we obtain several known multivariate distributions. For $\xi = 0$ or $\gamma = 0$, the p.d.f. (3.1), takes the form of a Dirichlet type 1 p.d.f., $(X_1, \ldots, X_n) \sim \mathrm{D1}(\alpha_1, \ldots, \alpha_n; \beta)$, given by

$$\frac{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta\right)}{\prod_{i=1}^n \Gamma(\alpha_i) \Gamma(\beta)} \prod_{i=1}^n x_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^n x_i\right)^{\beta - 1}, \quad x_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^n x_i < 1.$$

For $\gamma = \sum_{i=1}^n \alpha_i + \beta$ and $\xi = 1$, the p.d.f. (3.1) reduces to a Dirichlet type 3 p.d.f., $(X_1, \ldots, X_n) \sim \mathrm{D3}(\alpha_1, \ldots, \alpha_n; \beta)$, stated as

$$\frac{2^{\sum_{i=1}^n \alpha_i} \Gamma\left(\sum_{i=1}^n \alpha_i + \beta\right)}{\prod_{i=1}^n \Gamma(\alpha_i) \Gamma(\beta)} \frac{\prod_{i=1}^n x_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^n x_i\right)^{\beta - 1}}{\left(1 + \sum_{i=1}^n x_i\right)^{\sum_{i=1}^n \alpha_i + \beta}},$$

$$x_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^n x_i < 1.$$

The Dirichlet type 1 and Dirichlet type 3 distributions have been studied extensively in the literature. For example, see, Kotz, Balakrishnana and Johnson [11] and Cardeño, Nagar and Sánchez [3].

In Bayesian probability theory, if the posterior distribution belongs to the same family as the prior distribution, then the prior and posterior are called conjugate distributions, and the prior is called a conjugate prior. In case of multinomial distribution, the usual conjugate prior is the Dirichlet distribution. In the present case, if

$$p(s_1, \ldots, s_n, f | x_1, \ldots, x_n) = \binom{s_1 + \cdots + s_n + f}{s_1, \ldots, s_n, f} x_1^{s_1} \cdots x_n^{s_n} (1 - x_1 - \cdots - x_n)^f$$

and

$$p(x_1, \ldots, x_n) = C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi) \frac{x_1^{\alpha_1 - 1} \cdots x_n^{\alpha_n - 1} (1 - x_1 - \cdots - x_n)^{\beta - 1}}{[1 + \xi(x_1 + \cdots + x_n)]^{\gamma}},$$

where $x_1 > 0, \ldots, x_n > 0$, and $x_1 + \cdots + x_n < 1$, then

$$\begin{aligned} p(x_1, \ldots, x_n | s_1, \ldots, s_n, f) &= C(\alpha_1 + s_1, \ldots, \alpha_n + s_n, \beta + f, \gamma, \xi) \\ &\quad \times \frac{x_1^{\alpha_1 + s_1 - 1} \cdots x_n^{\alpha_n + s_n - 1} (1 - x_1 - \cdots - x_n)^{\beta + f - 1}}{[1 + \xi(x_1 + \cdots + x_n)]^\gamma}. \end{aligned}$$

Thus, the multivariate family of distributions considered in this article is conjugate prior for the multinomial distribution.

Figure 1 gives some graphs of the p.d.f. define by (3.1) for different values of the parameters. A wide range of shapes arise out of the multivariate Gauss hypergeometric p.d.f.

In the next theorem, by applying a linear transformation to the multivariate Gauss hypergeometric variables, we define a generalization of the Dirichlet type 2 distribution.

**3.2. Theorem.** *Let $(X_1, \ldots, X_n) \sim \mathrm{GH}(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)$. Define $Y_1, \ldots, Y_n$ as $Y_i = X_i/(1 - \sum_{i=1}^n X_i)$, $i = 1, \ldots, n$. Then, the p.d.f. of $(Y_1, \ldots, Y_n)$ is*

$$C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi) \frac{\prod_{i=1}^n y_i^{\alpha_i - 1} \left(1 + \sum_{i=1}^n y_i\right)^{\gamma - \sum_{i=1}^n \alpha_i - \beta}}{\left[1 + (1 + \xi) \sum_{i=1}^n y_i\right]^\gamma},$$

*where $y_i > 0$, $i = 1, \ldots, n$ and $C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)$ is the normalizing constant given in (3.2).*

*Proof.* Transforming $X_i = Y_i/(1+\sum_{i=1}^n Y_i)$ with the Jacobian $J(x_1, \ldots, x_n \to y_1, \ldots, y_n) = (1 + \sum_{i=1}^n y_i)^{-(n+1)}$ in the p.d.f. (3.1), we obtain the desired result. □

Note that, if $\xi = 0$ or $\gamma = 0$, then the p.d.f. given in the above theorem slides to a Dirichlet type 2 p.d.f. given by

$$\frac{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta\right)}{\prod_{i=1}^n \Gamma(\alpha_i)\Gamma(\beta)} \frac{\prod_{i=1}^n y_i^{\alpha_i - 1}}{\left(1 + \sum_{i=1}^n y_i\right)^{\sum_{i=1}^n \alpha_i + \beta}}, \quad y_i > 0, \quad i = 1, \ldots, n,$$

and in this case we write $(Y_1, \ldots, Y_n) \sim \mathrm{D2}(\alpha_1, \ldots, \alpha_n; \beta)$.

## 4. Marginal Distribution

It is well known that if $(X_1, \ldots, X_n) \sim \mathrm{D1}(\alpha_1, \ldots, \alpha_n; \beta)$, then for $1 \le s \le n$, $(X_1, \ldots, X_s) \sim \mathrm{D1}(\alpha_1, \ldots, \alpha_s; \beta + \sum_{i=s+1}^n \alpha_i)$. In this section, we derive similar result for multivariate generalization of the Gauss hypergeometric distribution defined by the p.d.f. (3.1).

**4.1. Theorem.** *Let $(X_1, \ldots, X_n) \sim \mathrm{GH}(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)$. Then, for $1 \le s \le n$, the joint p.d.f. of $X_1, \ldots, X_s$ is*

$$\begin{aligned} &K_1(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi) \frac{\prod_{i=1}^s x_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^s x_i\right)^{\beta + \sum_{i=s+1}^n \alpha_i - 1}}{\left(1 + \xi \sum_{i=1}^s x_i\right)^\gamma} \\ &\quad \times {}_2F_1\left(\sum_{i=s+1}^n \alpha_i, \gamma; \sum_{i=s+1}^n \alpha_i + \beta; -\frac{\xi\left(1 - \sum_{i=1}^s x_i\right)}{1 + \xi \sum_{i=1}^s x_i}\right), \end{aligned} \tag{4.1}$$

*for $x_1 > 0, \ldots, x_s > 0$ and $\sum_{i=1}^s x_i < 1$, where*

$$\begin{aligned} &[K_1(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)]^{-1} \\ &\quad = \frac{\prod_{i=1}^s \Gamma(\alpha_i) \Gamma\left(\sum_{i=s+1}^n \alpha_i + \beta\right)}{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta\right)} {}_2F_1\left(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi\right). \end{aligned}$$

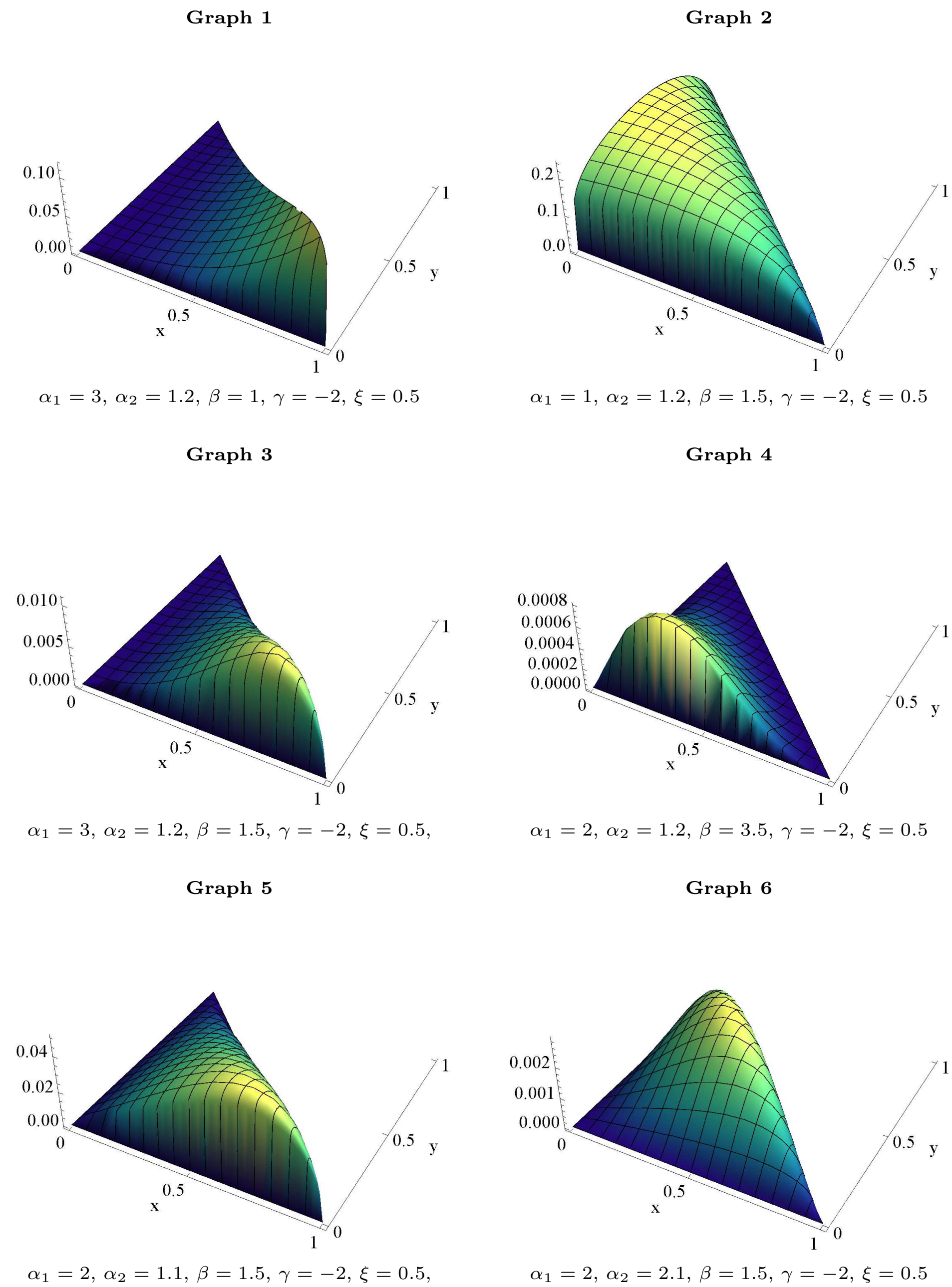


**Figure 1.** p.d.f. of the multivariate Gauss hypergeometric distribution.

*Proof.* To calculate the marginal p.d.f. of $X_1, \dots, X_s$, we integrate (3.1) with respect to $x_{s+1}, \dots, x_n$, to obtain

$$
\begin{aligned}
&C(\alpha_1, \dots, \alpha_n, \beta, \gamma, \xi) \prod_{i=1}^{s} x_i^{\alpha_i - 1} \\
&\times \underset{\substack{x_{s+1}>0,\dots,x_n>0 \\ \sum_{i=s+1}^n x_i < 1 - \sum_{i=1}^s x_i}}{\int \cdots \int} \frac{\prod_{i=s+1}^n x_i^{\alpha_i - 1} (1 - \sum_{i=1}^s x_i - \sum_{i=s+1}^n x_i)^{\beta - 1}}{(1 + \xi \sum_{i=1}^s x_i + \xi \sum_{i=s+1}^n x_i)^{\gamma}} \mathrm{d}x_{s+1} \cdots \mathrm{d}x_n,
\end{aligned}
$$

where $0 < x_i$, $i = 1, \dots, s$. Now, substituting $z_j = x_j/(1 - \sum_{i=1}^s x_i)$ for $j = s+1, \dots, n$ with the Jacobian $J(x_{s+1}, \dots, x_n \to z_{s+1}, \dots, z_n) = \left(1 - \sum_{i=1}^s x_i\right)^{n-s}$, the marginal p.d.f. of $X_1, \dots, X_s$ is obtained as

$$
\begin{aligned}
(4.2) \qquad &C(\alpha_1, \dots, \alpha_n, \beta, \gamma, \xi) \frac{\prod_{i=1}^s x_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^s x_i\right)^{\beta + \sum_{i=s+1}^n \alpha_i - 1}}{\left(1 + \xi \sum_{i=1}^s x_i\right)^{\gamma}} \\
&\times \underset{\substack{z_{s+1}>0,\dots,z_n>0 \\ \sum_{i=s+1}^n z_i < 1}}{\int \cdots \int} \frac{\prod_{i=s+1}^n z_i^{\alpha_i - 1} (1 - \sum_{i=s+1}^n z_i)^{\beta - 1}}{\left[1 + \xi (1 - \sum_{i=1}^s x_i) \sum_{i=s+1}^n z_i / (1 + \xi \sum_{i=1}^s x_i)\right]^{\gamma}} \, \mathrm{d}z_{s+1} \cdots \mathrm{d}z_n.
\end{aligned}
$$

Further, using the Liouville-Dirichlet integral (2.7), we can evaluate the above integral as

$$
\begin{aligned}
&\frac{\prod_{i=s+1}^n \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=s+1}^n \alpha_i\right)} \int_0^1 \frac{z^{\sum_{i=s+1}^n \alpha_i - 1} (1-z)^{\beta - 1}}{\left[1 + \xi \left(1 - \sum_{i=1}^s x_i\right) z / (1 + \xi \sum_{i=1}^s x_i)\right]^{\gamma}} \, \mathrm{d}z \\
&= \frac{\prod_{i=s+1}^n \Gamma(\alpha_i) \Gamma(\beta)}{\Gamma\left(\sum_{i=s+1}^n \alpha_i + \beta\right)} \, {}_2F_1 \left( \sum_{i=s+1}^n \alpha_i, \gamma; \sum_{i=s+1}^n \alpha_i + \beta; -\frac{\xi \left(1 - \sum_{i=1}^s x_i\right)}{1 + \xi \sum_{i=1}^s x_i} \right).
\end{aligned}
$$

Finally, substituting this last expression, as well as the value of $C(\alpha_1, \dots, \alpha_n, \beta, \gamma, \xi)$, in (4.2) and simplifying, we obtain the desired result. □

Note that the marginal p.d.f. of $X_1, \dots, X_s$, obtained in the previous theorem, differs from the multivariate Gauss hypergeometric p.d.f. by a factor that involves the ${}_2F_1$ function.

**4.2. Corollary.** *Let $(X_1, \dots, X_n) \sim \mathrm{GH}(\alpha_1, \dots, \alpha_n, \beta, \gamma, \xi)$. Then, for $k$=1, …, $n$, the p.d.f. of $X_k$ is*

$$
\begin{aligned}
&K_2(\alpha_1, \dots, \alpha_n, \beta, \gamma, \xi) \; \frac{x_k^{\alpha_k - 1} (1 - x_k)^{\beta + \sum_{i(\neq k)=1}^n \alpha_i - 1}}{(1 + \xi x_k)^{\gamma}} \\
&\times {}_2F_1 \left( \sum_{i(\neq k)=1}^n \alpha_i, \gamma; \sum_{i(\neq k)=1}^n \alpha_i + \beta; -\frac{\xi (1 - x_k)}{1 + \xi x_k} \right),
\end{aligned}
$$

*for $0 < x_k < 1$, where*

$$
\begin{aligned}
&\left[K_2(\alpha_1, \dots, \alpha_n, \beta, \gamma, \xi)\right]^{-1} \\
&\quad = \frac{\Gamma(\alpha_k) \Gamma(\sum_{i(\neq k)=1}^n \alpha_i + \beta)}{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta\right)} \, {}_2F_1 \left( \sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi \right).
\end{aligned}
$$

**4.3. Theorem.** *Let $(X_1, \dots, X_n) \sim \mathrm{GH}(\alpha_1, \dots, \alpha_n, \beta, \gamma, \xi)$ and for $s = 1$, …, $n-1$, define the random variables $Y_i = X_i/(1 - \sum_{i=1}^s X_i)$, $i = s+1, \dots, n$. Then, the p.d.f. of*

$(Y_{s+1},\dots,Y_n)$ *is*

$$K_3(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)\frac{\prod_{i=s+1}^n y_i^{\alpha_i-1}(1-\sum_{i=s+1}^n y_i)^{\beta-1}}{\left(1+\xi\sum_{i=s+1}^n y_i\right)^{\gamma}}$$
$$\times\,{}_2F_1\left(\sum_{i=1}^s\alpha_i,\gamma;\sum_{i=1}^n\alpha_i+\beta;-\frac{\xi(1-\sum_{i=s+1}^n y_i)}{1+\xi\sum_{i=s+1}^n y_i}\right),\quad y_i>0,\quad i=1,\dots,n,$$

*where*

$$[K_3(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)]^{-1}=\frac{\prod_{i=s+1}^n\Gamma(\alpha_i)\Gamma(\beta)}{\Gamma(\sum_{i=s+1}^n\alpha_i+\beta)}\,{}_2F_1\left(\sum_{i=1}^n\alpha_i,\gamma;\sum_{i=1}^n\alpha_i+\beta;-\xi\right).$$

*Proof.* Applying the transformation $Y_i=X_i/(1-\sum_{i=1}^s X_i)$, $i=s+1,\dots,n$ with the Jacobian $J(x_{s+1},\dots,x_n\to y_{s+1},\dots,y_n)=(1-\sum_{i=1}^s x_i)^{n-s}$ in (3.1) and integrating with respect to $x_1,\dots,x_s$, we obtain the p.d.f. of $(Y_{s+1},\dots,Y_n)$ as

$$C(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)\ \frac{\prod_{i=s+1}^n y_i^{\alpha_i-1}(1-\sum_{i=s+1}^n y_i)^{\beta-1}}{(1+\xi\sum_{i=s+1}^n y_i)^{\gamma}}$$
$$\times\underset{\substack{x_1>0,\dots,x_s>0,\\ \sum_{i=1}^s x_i<1}}{\int\cdots\int}\frac{\prod_{i=1}^s x_i^{\alpha_i-1}(1-\sum_{i=1}^s x_i)^{\sum_{i=s+1}^n\alpha_i+\beta-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_s}{\left[1+\xi(1-\sum_{i=s+1}^n y_i)\sum_{i=1}^s x_i/(1+\xi\sum_{i=s+1}^n y_i)\right]^{\gamma}},$$

where $0<y_i, i=s+1,\dots,n$ and $\sum_{i=s+1}^n y_i<1$. Now, evaluating the above integral using the Liouville-Dirichlet integral (2.7), we get

$$C(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)\ \frac{\prod_{i=s+1}^n y_i^{\alpha_i-1}(1-\sum_{i=s+1}^n y_i)^{\beta-1}}{(1+\xi\sum_{i=s+1}^n y_i)^{\gamma}}$$
$$\times\frac{\prod_{i=1}^s\Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^s\alpha_i)}\frac{\Gamma(\sum_{i=1}^s\alpha_i)\Gamma(\sum_{i=s+1}^n\alpha_i+\beta)}{\Gamma(\sum_{i=1}^n\alpha_i+\beta)}$$
$$\times\,{}_2F_1\left(\sum_{i=1}^s\alpha_i,\gamma;\sum_{i=1}^n\alpha_i+\beta;-\frac{\xi(1-\sum_{i=s+1}^n y_i)}{1+\xi\sum_{i=s+1}^n y_i}\right).$$

Finally, substituting for $C(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)$ in the above expression and simplifying, we obtain the desired result. □

The following theorem gives the distribution of partial sums of random variables whose joint distribution is multivariate Gauss hypergeometric.

**4.4. Theorem.** *Let* $(X_1,\dots,X_n)\sim\mathrm{GH}(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)$ *and* $n_1,\dots,n_\ell$ *be non-negative integers such as* $\sum_{i=1}^\ell n_i=n$. *Define,* $\alpha_{(i)}=\sum_{j=n_{i-1}^*+1}^{n_i^*}\alpha_i$, $n_0^*=0$, $n_i^*=\sum_{j=1}^i n_j$, $i=1,\dots,\ell$, $Z_j=X_j/X_{(i)}$, $j=n_{i-1}^*+1,\dots,n_i^*-1$ *and* $X_{(i)}=\sum_{j=n_{i-1}^*+1}^{n_i^*}X_j$, $i=1,\dots,\ell$. *Then*

*(i)* $(X_{(1)},\dots,X_{(\ell)})$ *and* $(Z_{n_{i-1}^*+1},\dots,Z_{n_i^*-1})$, $i=1,\dots,\ell$, *are independently distributed,*
*(ii)* $(X_{(1)},\dots,X_{(\ell)})\sim\mathrm{GH}(\alpha_{(1)},\dots,\alpha_{(\ell)},\beta,\gamma,\xi)$ *and*
*(iii)* $(Z_{n_{i-1}^*+1},\dots,Z_{n_i^*-1})\sim\mathrm{D1}(\alpha_{n_{i-1}^*+1},\dots,\alpha_{n_i^*-1};\alpha_{n_i^*})$, $i=1,\dots,\ell$.

*Proof.* Transforming $Z_j=X_j/X_{(i)}$ and $X_{(i)}=\sum_{j=n_{i-1}^*+1}^{n_i^*}X_j$, $j=n_{i-1}^*+1,\dots,n_i^*-1$, $i=1,\dots,\ell$, with the Jacobian

$$J(x_1,\dots,x_n\to z_1,\dots,z_{n_1-1},x_{(1)},\dots,z_{n_{\ell-1}^*+1},\dots,z_{n_\ell^*-1},x_{(\ell)})=\prod_{i=1}^\ell x_{(i)}^{n_i-1},$$

in the p.d.f. (3.1), we obtain the joint p.d.f. of $Z_{n^*_{i-1}+1},\ldots,Z_{n^*_i-1}$, $X_{(i)}$, $i=1,\ldots,\ell$, as being proportional to

$$\frac{\prod_{i=1}^{\ell} x_{(i)}^{\alpha_{(i)}-1}(1-\sum_{i=1}^{\ell} x_{(i)})^{\beta-1}}{(1+\xi\sum_{i=1}^{\ell} x_{(i)})^{\gamma}}\prod_{i=1}^{\ell}\left[\prod_{j=n^*_{i-1}+1}^{n^*_i-1} z_j^{\alpha_j-1}\left(1-\sum_{j=n^*_{i-1}+1}^{n^*_i-1} z_j\right)^{\alpha_{n^*_i}-1}\right], \tag{4.3}$$

where $x_{(i)}>0$, $i=1,\ldots,\ell$, $\sum_{i=1}^{\ell} x_{(i)}<1$, $z_j>0$, $j=n^*_{i-1}+1,\ldots,n^*_i-1$, $\sum_{j=n^*_{i-1}+1}^{n^*_i-1} z_j<1, i=1,\ldots,\ell$. From the factorization (4.3), it is clear that $(X_{(1)},\ldots,X_{(\ell)})$ and $(Z_{n^*_{i-1}+1},\ldots,Z_{n^*_i-1})$, $i=1,\ldots,\ell$, are independently distributed. $(X_{(1)},\ldots,X_{(\ell)})\sim \mathrm{GH}(\alpha_{(1)},\ldots,\alpha_{(\ell)},\beta,\gamma,\xi)$ and $(Z_{n^*_{i-1}+1},\ldots,Z_{n^*_i-1})\sim \mathrm{D1}(\alpha_{n^*_{i-1}+1},\ldots,\alpha_{n^*_i-1};\alpha_{n^*_i})$, $i=1,\ldots,\ell$. □

**4.5. Corollary.** *Let* $(X_1,\ldots,X_n)\sim \mathrm{GH}(\alpha_1,\ldots,\alpha_n,\beta,\gamma,\xi)$ *and define* $Z_i=X_i/Z$ *for* $i=1,\ldots,n-1$ *and* $Z=\sum_{j=1}^{n} X_j$. *Then,* $(Z_1,\ldots,Z_{n-1})$ *and* $Z$ *are independent,* $(Z_1,\ldots,Z_{n-1})\sim \mathrm{D1}(\alpha_1,\ldots,\alpha_{n-1};\alpha_n)$ *and* $Z\sim \mathrm{GH}(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi)$.

**4.6. Corollary.** *If* $(X_1,\ldots,X_n)\sim \mathrm{GH}(\alpha_1,\ldots,\alpha_n,\beta,\gamma,\xi)$, *then*

$$\frac{\sum_{i=1}^{s} X_i}{\sum_{j=1}^{n} X_j}\sim \mathrm{B1}\left(\sum_{i=1}^{s}\alpha_i,\sum_{i=s+1}^{n}\alpha_i\right),\quad s<n.$$

**4.7. Corollary.** *If* $(X_1,\ldots,X_n)\sim \mathrm{GH}(\alpha_1,\ldots,\alpha_n,\beta,\gamma,\xi)$, *then* $(X_1,\ldots,X_i+X_j,\ldots,X_n)\sim \mathrm{GH}(\alpha_1,\ldots,\alpha_i+\alpha_j,\ldots,\alpha_n,\beta,\gamma,\xi)$.

**4.8. Theorem.** *Let* $(X_1,\ldots,X_n)\sim \mathrm{GH}(\alpha_1,\ldots,\alpha_n,\beta,\gamma,\xi)$ *and* $n_1,\ldots,n_\ell$ *be non-negative integers such as* $\sum_{i=1}^{\ell} n_i=n$. *Define,* $\alpha_{(i)}=\sum_{j=n^*_{i-1}+1}^{n^*_i}\alpha_i$, $n^*_0=0$, $n^*_i=\sum_{j=1}^{i} n_j$, $i=1,\ldots,\ell$, $W_j=X_j/X_{n^*_i}$, $j=n^*_{i-1}+1,\ldots,n^*_i-1$ *and* $X_{(i)}=\sum_{j=n^*_{i-1}+1}^{n^*_i} X_j$, $i=1,\ldots,\ell$. *Then*

*(i)* $(X_{(1)},\ldots,X_{(\ell)})$ *and* $(W_{n^*_{i-1}+1},\ldots,W_{n^*_i-1})$, $i=1,\ldots,\ell$, *are independently distributed,*
*(ii)* $(X_{(1)},\ldots,X_{(\ell)})\sim \mathrm{GH}(\alpha_{(1)},\ldots,\alpha_{(\ell)},\beta,\gamma,\xi)$ *and*
*(iii)* $(W_{n^*_{i-1}+1},\ldots,W_{n^*_i-1})\sim \mathrm{D2}(\alpha_{n^*_{i-1}+1},\ldots,\alpha_{n^*_i-1};\alpha_{n^*_i})$, $i=1,\ldots,\ell$.

**4.9. Corollary.** *Let* $(X_1,\ldots,X_n)\sim \mathrm{GH}(\alpha_1,\ldots,\alpha_n,\beta,\gamma,\xi)$ *and define* $W_i=X_i/X_n$ *for* $i=1,\ldots,n-1$ *and* $Z=\sum_{j=1}^{n} X_j$. *Then,* $(W_1,\ldots,W_{n-1})$ *and* $Z$ *are independent,* $(Z_1,\ldots,Z_{n-1})\sim \mathrm{D2}(\alpha_1,\ldots,\alpha_{n-1};\alpha_n)$ *and* $Z\sim \mathrm{GH}(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi)$.

**4.10. Corollary.** *If* $(X_1,\ldots,X_n)\sim \mathrm{GH}(\alpha_1,\ldots,\alpha_n,\beta,\gamma,\xi)$, *then*

$$\frac{\sum_{i=1}^{s} X_i}{\sum_{j=s+1}^{n} X_j}\sim \mathrm{B2}\left(\sum_{i=1}^{s}\alpha_i,\sum_{i=s+1}^{n}\alpha_i\right),\quad s<n.$$

## 5. Joint Moments

We derive the joint moments of random variables jointly distributed as multivariate Gauss hypergeometric. These moments will facilitate us to compute several expected values such as mean and variance.

Using (3.1) and (3.2), the the joint moments of $X_1, \ldots, X_n$ are obtained as

$$\begin{aligned} E(X_1^{r_1} \cdots X_n^{r_n}) &= \frac{C(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)}{C(\alpha_1 + r_1, \ldots, \alpha_n + r_n, \beta, \gamma, \xi)} \\ &= \frac{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta\right) \prod_{i=1}^n \Gamma(\alpha_i + r_i)}{\Gamma\left[\sum_{i=1}^n (\alpha_i + r_i) + \beta\right] \prod_{i=1}^n \Gamma(\alpha_i)} \\ &\quad \times \frac{{}_2F_1(\sum_{i=1}^n (\alpha_i + r_i), \gamma; \sum_{i=1}^n (\alpha_i + r_i) + \beta; -\xi)}{{}_2F_1(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi)}. \end{aligned}$$

By substituting appropriately in the above expression, the following expected values can easily be obtained:

$$E(X_i) = \frac{\alpha_i}{\sum_{i=1}^n \alpha_i + \beta} \frac{{}_2F_1(\sum_{i=1}^n \alpha_i + 1, \gamma; \sum_{i=1}^n \alpha_i + \beta + 1; -\xi)}{{}_2F_1(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi)},$$

$$\begin{aligned} E(X_i^2) &= \frac{\alpha_i(\alpha_i + 1)}{(\sum_{i=1}^n \alpha_i + \beta)(\sum_{i=1}^n \alpha_i + \beta + 1)} \\ &\quad \times \frac{{}_2F_1(\sum_{i=1}^n \alpha_i + 2, \gamma; \sum_{i=1}^n \alpha_i + \beta + 2; -\xi)}{{}_2F_1(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi)}, \end{aligned}$$

$$\begin{aligned} \operatorname{Var}(X_i) &= \frac{\alpha_i}{\sum_{i=1}^n \alpha_i + \beta} \left[ \frac{(\alpha_i + 1)\ {}_2F_1(\sum_{i=1}^n \alpha_i + 2, \gamma; \sum_{i=1}^n \alpha_i + \beta + 2; -\xi)}{(\sum_{i=1}^n \alpha_i + \beta + 1)\ {}_2F_1(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi)} \right. \\ &\quad \left. - \frac{\alpha_i}{\sum_{i=1}^n \alpha_i + \beta} \left\{ \frac{{}_2F_1(\sum_{i=1}^n \alpha_i + 1, \gamma; \sum_{i=1}^n \alpha_i + \beta + 1; -\xi)}{{}_2F_1(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi)} \right\}^2 \right], \end{aligned}$$

$$\begin{aligned} E(X_i X_j) &= \frac{\alpha_i \alpha_j}{(\sum_{i=1}^n \alpha_i + \beta)(\sum_{i=1}^n \alpha_i + \beta + 1)} \\ &\quad \times \frac{{}_2F_1(\sum_{i=1}^n \alpha_i + 2, \gamma; \sum_{i=1}^n \alpha_i + \beta + 2; -\xi)}{{}_2F_1(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi)}, \quad i \neq j, \end{aligned}$$

and finally for $i \neq j$,

$$\begin{aligned} \operatorname{Cov}(X_i, X_j) &= \frac{\alpha_i \alpha_j}{\sum_{i=1}^n \alpha_i + \beta} \left[ \frac{{}_2F_1(\sum_{i=1}^n \alpha_i + 2, \gamma; \sum_{i=1}^n \alpha_i + \beta + 2; -\xi)}{(\sum_{i=1}^n \alpha_i + \beta + 1) {}_2F_1(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi)} \right. \\ &\quad \left. - \frac{1}{\sum_{i=1}^n \alpha_i + \beta} \left\{ \frac{{}_2F_1(\sum_{i=1}^n \alpha_i + 1, \gamma; \sum_{i=1}^n \alpha_i + \beta + 1; -\xi)}{{}_2F_1(\sum_{i=1}^n \alpha_i, \gamma; \sum_{i=1}^n \alpha_i + \beta; -\xi)} \right\}^2 \right]. \end{aligned}$$

Using the definition and the above expressions, one can calculate the correlation between $X_i$ and $X_j$.

## 6. Factorizations

In this section we give several factorizations of the multivariate Gauss hypergeometric p.d.f..

**6.1. Theorem.** *Let* $(X_1, \ldots, X_n) \sim \mathrm{GH}(\alpha_1, \ldots, \alpha_n, \beta, \gamma, \xi)$. *For* $i = 1, \ldots, n-1$ *define* $Y_i = \sum_{j=1}^i X_j / \sum_{j=1}^{i+1} X_j$ *and* $Y_n = \sum_{j=1}^n X_j$. *Then, the random variables* $Y_1, \ldots, Y_n$ *are independent,* $Y_i \sim \mathrm{B1}(\sum_{j=1}^i \alpha_j, \alpha_{i+1})$, $i = 1, \ldots, n-1$ *and* $Y_n \sim \mathrm{GH}(\sum_{i=1}^n \alpha_i, \beta, \gamma, \xi)$.

*Proof.* From the transformation given in the theorem, we obtain $x_1 = y_n \prod_{i=1}^{n-1} y_i$, $x_2 = y_n(1 - y_1) \prod_{i=2}^{n-1} y_i, \ldots, x_{n-1} = y_n(1 - y_{n-2}) y_{n-1}$, and $x_n = y_n(1 - y_{n-1})$. with the Jacobian $J(x_1,\dots,x_n \to y_1,\dots,y_n) = \prod_{i=2}^{n} y_i^{i-1}$. Now, making appropriate substitutions in the joint p.d.f. of $X_1,\dots,X_n$, we obtain

$$C(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)\ \left(y_n\prod_{i=1}^{n-1} y_i\right)^{\alpha_1-1}\prod_{j=2}^{n}\left[y_n(1-y_{j-1})\prod_{i=j}^{n-1} y_i\right]^{\alpha_j-1}$$
$$\times\frac{(1-y_n)^{\beta-1}}{(1+\xi y_n)^{\gamma}}\prod_{i=2}^{n} y_i^{i-1}.$$

Further, writing

$$C(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi) = \prod_{j=1}^{n-1}\left[B\left(\sum_{i=1}^{j}\alpha_i,\alpha_{j+1}\right)\right]^{-1}$$
$$\times\left[B\left(\sum_{i=1}^{n}\alpha_i,\beta\right){}_2F_1\left(\sum_{i=1}^{n}\alpha_i,\gamma;\sum_{i=1}^{n}\alpha_i+\beta;-\xi\right)\right]^{-1},$$

the above expression is simplified as

$$\left[\prod_{j=1}^{n-1}\frac{y_j^{\sum_{i=1}^{j}\alpha_i-1}(1-y_j)^{\alpha_{j+1}-1}}{B(\sum_{i=1}^{j}\alpha_i,\alpha_{j+1})}\right] C\left(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi\right)\frac{y_n^{\sum_{i=1}^{n}\alpha_i-1}(1-y_n)^{\beta-1}}{(1+\xi y_n)^{\gamma}},$$

where $0 < y_1,\dots,y_n < 1$. Now, from the above factorization, we get the result. □

**6.2. Theorem.** *Let* $(X_1,\dots,X_n) \sim \mathrm{GH}(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)$. *Define* $Z_n = \sum_{j=1}^{n} X_j$ *and* $Z_i = X_{i+1}/\sum_{j=1}^{i} X_j$, *for* $i = 1,\dots,n-1$. *Then,* $Z_1, \dots, Z_n$ *are independent,* $Z_i \sim \mathrm{B2}(\alpha_{i+1},\sum_{j=1}^{i}\alpha_j)$, $i = 1,\dots,n-1$ *and* $Z_n \sim \mathrm{GH}(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi)$.

*Proof.* This result is obtain from Theorem 6.1, by observing that $Z_i = (1-Y_i)/Y_i$, for $i = 1,\dots,n-1$, $Z_n = Y_n$ and $(1-Y_i)/Y_i \sim \mathrm{B2}(\alpha_{i+1},\sum_{j=1}^{i}\alpha_j)$, where $Y_i \sim \mathrm{B1}(\sum_{j=1}^{i}\alpha_j,\alpha_{i+1})$. □

**6.3. Theorem.** *Let* $(X_1,\dots,X_n) \sim \mathrm{GH}(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)$. *Define* $W_n = \sum_{j=1}^{n} X_j$ *and* $W_i = \sum_{j=1}^{i} X_j/X_{i+1}$, *for* $i = 1,\dots,n-1$. *Then,* $W_1, \dots, W_n$, *are independent,* $W_i \sim \mathrm{B2}(\sum_{j=1}^{i}\alpha_j,\alpha_{i+1})$ *for* $i = 1,\dots,n-1$ *and* $W_n \sim \mathrm{GH}(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi)$.

*Proof.* The result is obtained from Theorem 6.2 by taking into account that $W_i = 1/Z_i$ for $i = 1,\dots,n-1$, $W_n = Z_n$ and $1/Z_i \sim \mathrm{B2}(\sum_{j=1}^{i}\alpha_j,\alpha_{i+1})$, where $Z_i \sim \mathrm{B2}(\alpha_{i+1},\sum_{j=1}^{i}\alpha_j)$. □

**6.4. Theorem.** *Let* $(X_1,\dots,X_n) \sim \mathrm{GH}(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)$. *Define* $Y_n = \sum_{j=1}^{n} X_j$ *and for* $i = 1,\dots,n-1$ $Y_i = X_i/\sum_{j=i}^{n} X_j$. *Then,* $Y_1,\dots,Y_n$ *are independent,* $Y_i \sim \mathrm{B1}(\alpha_i,\sum_{j=i+1}^{n}\alpha_j)$, *for* $i = 1,\dots,n-1$ *and* $Y_n \sim \mathrm{GH}(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi)$.

*Proof.* Making the substitution $x_1 = y_n y_1$, $x_2 = y_n y_2(1-y_1)$, $\dots$, $x_{n-1} = y_n y_{n-1}(1-y_1)\cdots(1-y_{n-2})$ and $x_n = y_n(1-y_1)\cdots(1-y_{n-1})$ with the Jacobian $J(x_1,\dots,x_n \to y_1,\dots,y_n) = y_n^{n-1}\prod_{i=1}^{n-2}(1-y_i)^{n-i-1}$ in (3.1), we obtain the joint p.d.f. of $Y_1,\dots,Y_n$ as

$$C(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)\prod_{i=1}^{n-1}\left[y_n y_i\prod_{j=1}^{i-1}(1-y_j)\right]^{\alpha_i-1}\left[y_n\prod_{j=1}^{n-1}(1-y_j)\right]^{\alpha_n-1}$$
$$\times\frac{(1-y_n)^{\beta-1}}{(1+\xi y_n)^{\gamma}}y_n^{n-1}\prod_{i=1}^{n-2}(1-y_i)^{n-i-1},$$

which can be re-written as

$$\left[\prod_{i=1}^{n-1} \frac{y_i^{\alpha_i-1}(1-y_i)^{\sum_{j=i+1}^{n}\alpha_j-1}}{B(\alpha_i,\sum_{j=i+1}^{n}\alpha_j)}\right] C\left(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi\right) \frac{y_n^{\sum_{i=1}^{n}\alpha_i-1}(1-y_n)^{\beta-1}}{(1+\xi y_n)^{\gamma}}.$$

Now, the desired result follows from the above factorization. □

**6.5. Theorem.** *Let* $(X_1,\dots,X_n) \sim \mathrm{GH}(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)$. *Define* $Z_n = \sum_{j=1}^{n} X_j$ *and* $Z_i = X_i/\sum_{j=i+1}^{n} X_j$, *for* $i = 1,\dots,n-1$. *Then,* $Z_1, \dots, Z_n$, *are independent,* $Z_i \sim \mathrm{B2}(\alpha_i,\sum_{j=i+1}^{n}\alpha_j)$, *for* $i = 1,\dots,n-1$ *and* $Z_n \sim \mathrm{GH}(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi)$.

*Proof.* The result is obtained from Theorem 6.4, by noting that $Z_i = Y_i/(1-Y_i)$ for $i = 1,\dots,n-1$, $Z_n = Y_n$ and $Y_i/(1-Y_i) \sim \mathrm{B2}(\alpha_i,\sum_{j=i+1}^{n}\alpha_j)$ for $Y_i \sim \mathrm{B1}(\alpha_i,\sum_{j=i+1}^{n}\alpha_j)$. □

**6.6. Theorem.** *Let* $(X_1,\dots,X_n) \sim \mathrm{GH}(\alpha_1,\dots,\alpha_n,\beta,\gamma,\xi)$. *Define* $W_n = \sum_{j=1}^{n} X_j$ *and* $W_i = \sum_{j=i+1}^{n} X_j/X_i$, *for* $i = 1,\dots,n-1$. *Then,* $W_1, \dots, W_n$ *are independent,* $W_i \sim \mathrm{B2}(\sum_{j=i+1}^{n}\alpha_j,\alpha_i)$ *for* $i = 1,\dots,n-1$ *and* $W_n \sim \mathrm{GH}(\sum_{i=1}^{n}\alpha_i,\beta,\gamma,\xi)$.

*Proof.* This result follows from Theorem 6.5, by observing that $W_i = 1/Z_i$ for $i = 1,\dots,n-1$, $W_n = Z_n$ and $1/Z_i \sim \mathrm{B2}(\sum_{j=i+1}^{n}\alpha_j,\alpha_i)$, where $Z_i \sim \mathrm{B2}(\alpha_i,\sum_{j=i+1}^{n}\alpha_j)$. □

## 7. Data application

Here, we illustrate the use of the multivariate Gauss hypergeometric p.d.f. We use the following data taken from Aitchison [2]:

| | sand | silt | clay |
|---|---|---|---|
| 1 | 0.775 | 0.195 | 0.030 |
| 2 | 0.719 | 0.249 | 0.032 |
| 3 | 0.507 | 0.361 | 0.132 |
| 4 | 0.522 | 0.409 | 0.066 |
| 5 | 0.700 | 0.265 | 0.035 |
| 6 | 0.665 | 0.322 | 0.013 |
| 7 | 0.431 | 0.553 | 0.016 |
| 8 | 0.534 | 0.368 | 0.098 |
| 9 | 0.155 | 0.544 | 0.301 |
| 10 | 0.317 | 0.415 | 0.268 |
| 11 | 0.657 | 0.278 | 0.065 |
| 12 | 0.704 | 0.290 | 0.006 |
| 13 | 0.174 | 0.536 | 0.290 |
| 14 | 0.106 | 0.698 | 0.196 |
| 15 | 0.382 | 0.431 | 0.187 |
| 16 | 0.108 | 0.527 | 0.365 |
| 17 | 0.184 | 0.507 | 0.309 |
| 18 | 0.046 | 0.474 | 0.480 |
| 19 | 0.156 | 0.504 | 0.340 |
| 20 | 0.319 | 0.451 | 0.230 |
| 21 | 0.095 | 0.535 | 0.370 |
| 22 | 0.171 | 0.480 | 0.349 |
| 23 | 0.105 | 0.554 | 0.341 |
| 24 | 0.048 | 0.547 | 0.410 |
| 25 | 0.026 | 0.452 | 0.522 |
| 26 | 0.114 | 0.527 | 0.359 |

| | | | |
|---|---|---|---|
| 27 | 0.067 | 0.469 | 0.464 |
| 28 | 0.069 | 0.497 | 0.434 |
| 29 | 0.040 | 0.449 | 0.511 |
| 30 | 0.074 | 0.516 | 0.409 |
| 31 | 0.048 | 0.495 | 0.457 |
| 32 | 0.045 | 0.485 | 0.470 |
| 33 | 0.066 | 0.521 | 0.413 |
| 34 | 0.067 | 0.473 | 0.459 |
| 35 | 0.074 | 0.456 | 0.469 |
| 36 | 0.060 | 0.489 | 0.451 |
| 37 | 0.063 | 0.538 | 0.399 |
| 38 | 0.025 | 0.480 | 0.495 |
| 39 | 0.020 | 0.478 | 0.502 |

The data are on the sediment composition in an Arctic lake. The second column gives relative frequencies of sand. The third column gives relative frequencies of silt. The fourth column gives relative frequencies of clay.

We fitted the multivariate Gauss hypergeometric p.d.f. in (3.1) to the pairwise data on (sand, clay) and (silt, clay). We also fitted the Dirichlet p.d.f., the particular case of (3.1) for $\gamma = 0$ and $\xi = 0$. The method of maximum likelihood was used for the fitting.

For the first pair, we obtained the estimates:

- $\widehat{\alpha}_1 = 6.574 \times 10^{-1}(1.330 \times 10^{-1})$, $\widehat{\alpha}_2 = 7.957 \times 10^{-1}(1.686 \times 10^{-1})$, $\widehat{\beta} = 10.452(2.339)$, $\widehat{\gamma} = -11.033(2.818)$, $\widehat{\xi} = 24540.9(1288584)$ with $\log L = 60.7$ for the bivariate Gauss hypergeometric p.d.f.
- $\widehat{\alpha}_1 = 1.021(1.698 \times 10^{-1})$, $\widehat{\alpha}_2 = 1.299(2.186 \times 10^{-1})$, $\widehat{\beta} = 2.319(4.004 \times 10^{-1})$ with $\log L = 39.5$ for the Dirichlet p.d.f.

For the second pair, we obtained the estimates:

- $\widehat{\alpha}_1 = 4.182(9.407 \times 10^{-1})$, $\widehat{\alpha}_2 = 2.168(4.617 \times 10^{-1})$, $\widehat{\beta} = 7.802 \times 10^{-1}(1.638 \times 10^{-1})$, $\widehat{\gamma} = 4.072(1.635)$, $\widehat{\xi} = 302.0(5031.1)$ with $\log L = 44.6$ for the bivariate Gauss hypergeometric p.d.f.
- $\widehat{\alpha_1} = 2.316(3.999 \times 10^{-1})$, $\widehat{\alpha_2} = 1.297(2.183 \times 10^{-1})$, $\widehat{\beta} = 1.019 \times 10^{-1}(1.695 \times 10^{-1})$ with $\log L = 39.5$ for the Dirichlet p.d.f.

Here, $\log L$ denotes the maximized log-likelihood value and the numbers within brackets are the standard errors computed by inverting the observation information matrices.

It follows by the standard likelihood ratio test that the bivariate Gauss hypergeometric distribution provides a significantly better fit for both data sets. Contours of the fitted bivariate Gauss hypergeometric p.d.f.s are shown in Figures 2 and 3. Also shown in the figures are the actual observed data. The fitted p.d.f.s do appear to capture the pattern in the data.

## Acknowledgement

The research work of DKN and DB-V was supported by the Comité para el Desarrollo de la Investigación, Universidad de Antioquia research grant No. IN560CE.

## References

[1] C. Armero and M. Bayarri, Prior assessments for predictions in queues, *The Statistician*, **43** (1994), no. 1, 139–153.

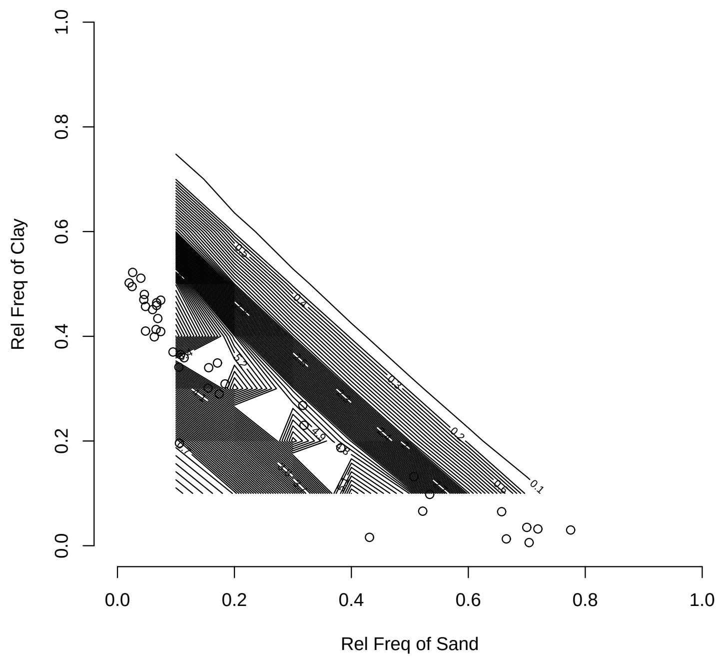


**Figure 2.** Relative frequencies of sand and clay and contours of the fitted bivariate Gauss hypergeometric p.d.f.


[2] J. Aitchison, *The Statistical Analysis of Compositional Data*, The Blackburn Press, Caldwell, NJ, 2003.

[3] Liliam Cardeño, Daya K. Nagar and Luz Estela Sánchez, Beta type 3 distribution and its multivariate generalization, *Tamsui Oxford Journal of Mathematical Sciences*, **21**(2005), no. 2, 225–241.

[4] J.-Y. Dauxois, Bayesian inference for linear growth birth and death processes, *Journal of Statistical Planning and Inference*, **121** (2004), no. 1, 1–19.

[5] Peter S. Fader and Bruce G. S. Hardie, A note on modelling underreported Poisson counts, *Journal of Applied Statistics*, **27** (2000), no. 8, 953–964 .

[6] A. K. Gupta and D. K. Nagar, *Matrix Variate Distributions*, Chapman & Hall/CRC, Boca Raton, 2000.

[7] A. K. Gupta and D. K. Nagar, Properties of matrix variate beta type 3 distribution, *International Journal of Mathematics and Mathematical Sciences*, **2009**(2009), Art. ID 308518, 18 pp.

[8] Rameshwar D. Gupta and Donald St. P. Richards, The history of Dirichlet and Liouville distributions, *Internationals Statistical Review*, **69**(2001), no. 3, 433-446.

[9] A. K. Gupta and D. Song, Generalized Lioville distribution, *Computters & Mathematics with Applicattions*, **32**(1996), no. 2, 103-109.

[10] N. L. Johnson, S. Kotz and N. Balakrishnan, *Continuous Univariate Distributions-2*, Second Edition, John Wiley & Sons, New York, 1994.

[11] S. Kotz, N. Balakrishnana and N. L. Johnson, *Continuous Multivariate Distributions-1*, Second Edition, John Wiley & Sons, New York (2000).

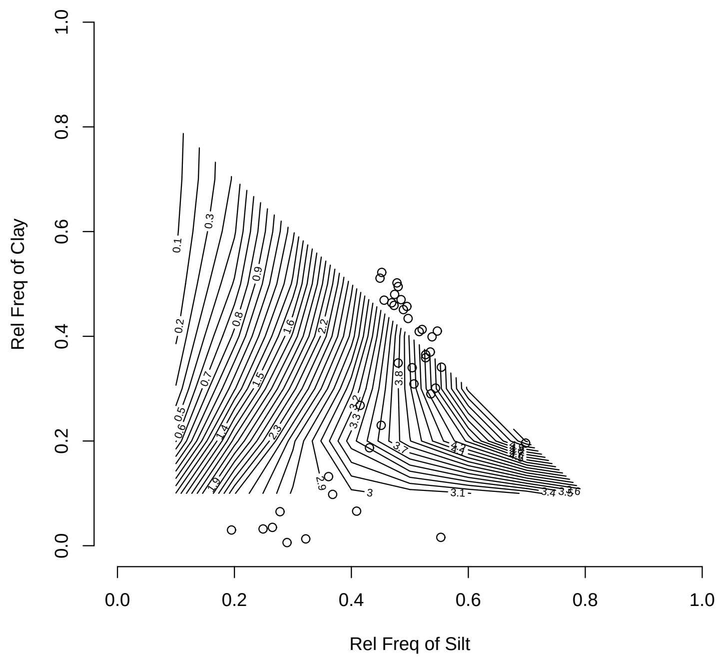


**Figure 3.** Relative frequencies of silt and clay and contours of the fitted bivariate Gauss hypergeometric p.d.f.


[12] D. L. Libby and M. R. Novic, Multivariate generalized beta distributions with applications to utility assessment, *J. Educ. Statist*, **7** (1982), no. 4, 271–294.

[13] Y. L. Luke, *The Special Functions and Their Approximations, Vol. 1*, Academic Press, New York, 1969.

[14] Albert W. Marshall and Ingram Olkin, Inequalities: theory of majorization and its applications. *Mathematics in Science and Engineering*, 143, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York - London, 1979.

[15] S. Nadarajah, Sums, products and ratios of generalized beta variables, *Statistical Papers*, **47**(2006), no. 1, 69–90.

[16] Daya K. Nagar, Paula Andrea Bran-Cardona and A. K. Gupta, Multivariate generalization of the hypergeometric function type 1 distribution, *Acta Applicandae Mathematicae*, **105**(2009), no. 1, 111-122.

[17] Daya K. Nagar and Erika Alejandra Rada-Mora, Properties of multivariate beta distributions, *Far East Journal of Theoretical Statistics*, **24**(2008), no. 1, 73–94.

[18] Daya K. Nagar and Fabio Humberto Sepúlveda-Murillo, Multivariate generalization of the confluent hypergeometric function kind 1 distribution, *International Journal of Mathematics and Mathematical Sciences*, **2008**, Art. ID 152808, 13 pp.

[19] T. Pham-Gia and Q. P. Duong, The generalized beta- and $F$-distributions in statistical modelling, *Mathematics and Computer Modelling*, **12** (1989), no.12, 1613-1625.

[20] Luz Estela Sánchez and Daya K. Nagar, Distribution of the product and the quotient of independent beta type 3 variables, *Far East Journal of Theoretical Statistics*, **17**(2005), no. 2, 239–251.

[21] José María Sarabia and Enrique Castillo, Bivariate distributions based on the generalized three-parameter beta distribution, *Advances in Distribution Theory, Order Statistics, and Inference*, Stat. Ind. Technol., Birkhäuser Boston, Boston, MA, 2006.

[22] B. D. Sivazlian, On multivariate extensión of the gamma and beta distributions, *Siam Journal of Applied Mathematics*, **41**(1981), no. 2, 205-209.

[23] D. Song and A. K. Gupta, Properties of generalized Liouville distribution, *Random Operators and Stochastic Equations*, **5**(1997), no. 5, 337-348.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 949 – 964

# Estimation of $P\{X \leq Y\}$ for geometric-Poisson model

Marko Obradović *, Milan Jovanović †, Bojana Milošević ‡ and Vesna Jevremović §

**Abstract**

In this paper we estimate $R = P\{X \leq Y\}$ when $X$ and $Y$ are independent random variables from geometric and Poisson distribution respectively. We find maximum likelihood estimator of $R$ and its asymptotic distribution. This asymptotic distribution is used to construct asymptotic confidence intervals. A procedure for deriving bootstrap confidence intervals is presented. UMVUE of $R$ and UMVUE of its variance are derived and also the Bayes estimator of $R$ for conjugate prior distributions is obtained. Finally, we perform a simulation study in order to compare these estimators.

*2000 AMS Classification:* 62F10, 62F12, 62F15, 62F25, 62F40.

**Keywords:** stress-strength, geometric distribution, Poisson distribution, maximum likelihood estimator, Bayes estimator, UMVUE, bootstrap confidence intervals.

Received 12/03/2014 : Accepted 19/08/2014 Doi : 10.15672/HJMS.2014267477

---

*Faculty of Mathematics, University of Belgrade, Studenski trg 16, Belgrade, Serbia. Email: marcone@matf.bg.ac.rs

†Faculty of Mathematics, University of Belgrade, Studenski trg 16, Belgrade, Serbia. Email:mjovanovic@matf.bg.ac.rs

‡Faculty of Mathematics, University of Belgrade, Studenski trg 16, Belgrade, Serbia. Email:bojana@matf.bg.ac.rs

§Faculty of Mathematics, University of Belgrade, Studenski trg 16, Belgrade, Serbia. Email:v_jevremovic@matf.bg.ac.rs

## 1. Introduction

In reliability theory the main parameter is the reliability of a system. The system fails if the applied stress $X$ is greater than strength $Y$, so $R = P\{X \leq Y\}$ is a measure of system performance. Its estimation is one of the main goals and it has been widely studied in statistical literature.

The problem was first introduced by Birnbaum [4]. The estimation of $R$ when $X$ and $Y$ are normally distributed has been considered by Downtown [7], Govidarajulu [9], Woodward and Kelley [26] and Owen [20]. Tong [24],[25], studied the case when $X$ and $Y$ were exponentially distributed. Exponential case with common location parameter was examined by Baklizi and Quader El-Masri [2]. The gamma case was studied by Constantine and Karson [5], Ismail et al. [12] and Constantine et al. [6]. Kundu and Gupta considered generalized exponetial case [16]. Kakade et al. [14] studied the exponentiated Gumbel case. Gompertz distribution was examined by Saraçoglu et al. [22], and the generalized Pareto case was considered by Rezaei et al. [21]. Kundu and Gupta [17] examined the case of Weibull distribution. Recently, the Topp-Leone distribution was studied by Genç [8]. Most of results are collected in Kotz et al. [15].

The majority of papers in this area deal with continuous probability distributions. However, there are some applications where stress and strength can have discrete distributions. For example, this is the case when the stress is the number of shocks the product undergoes and the strength is the number of shocks the product can withstand. Maiti [19] and Ahmad et al. [1] studied the geometric case. The negative binomial distribution was considered by Ivshin and Lumelskii [13] and Sathe and Dixit [23]. Belyaev and Lumelskii [3] examined the Poisson case.

In all mentioned papers both stress and strength come from the same type of distribution. In this paper we focus on the case when $X$ and $Y$ follow different types of distribution, namely geometric and Poisson distribution.

If we consider the stress to be the demand for some product, and the strength its supply, which are discrete in nature, then it might be convenient to model them with geometric and Poisson distributions.

Another motivating example can be the following. An employer is interviewing potential candidates for a vacant position. The number of interviews he needs to conduct until he finds suitable candidate follows geometric distribution, while the number of persons that apply for that job during a certain period of time follows Poisson distribution. Therefore $R$ is the probability that the employer will find the right candidate.

Let $X$ and $Y$ be independent random variables with geometric $\mathcal{G}(p)$ and Poisson $\mathcal{P}(\lambda)$ distribution, respectively, where probability $p$ and positive value $\lambda$ are unknown parameters. Their probability mass functions are

$$P\{X = x\} = (1-p)^{x-1}p, \quad x = 1, 2, \ldots,$$

and

$$P\{Y = y\} = \frac{e^{-\lambda}\lambda^y}{y!}, \quad y = 0, 1, \ldots.$$

Then the reliability of the system is

$$\begin{aligned}
R &= P\{X \le Y\} = \sum_{y=1}^{\infty}\sum_{x=1}^{y} P\{X = x, Y = y\} \\
&= \sum_{y=1}^{\infty}\sum_{x=1}^{y}(1-p)^{x-1}p\frac{e^{-\lambda}\lambda^y}{y!} = \sum_{y=1}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!}(1-(1-p)^y) \\
&= 1 - e^{-\lambda} - \sum_{y=1}^{\infty}\frac{e^{-\lambda}(\lambda(1-p))^y}{y!} \\
&= 1 - e^{-\lambda} - e^{-\lambda p}\sum_{y=1}^{\infty}\frac{e^{-\lambda(1-p)}(\lambda(1-p))^y}{y!} \\
&= 1 - e^{-\lambda p}. \qquad (1.1)
\end{aligned}$$

In the following sections we study various estimators of $R$. In section 2 the maximum likelihood estimator (MLE) of $R$ and its asymptotic distribution are derived. We use that to construct asymptotic and bootstrap confidence intervals. The uniformly minimum variance unbiased estimator (UMVUE) of $R$ and UMVUE of its variance are obtained in section 3. Bayes estimator of $R$ with respect to mean square error is found in section 4. In section 5 we perform a simulation study and compare the obtained estimators.

## 2. MLE of $R$ and its Asymptotics

Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ be the samples from the distributions of random variables $X$ and $Y$. Therefore, the log-likelihood function of combined sample is

$$\ln L(p,\lambda) = \big(\sum_{k=1}^{n} x_k - n\big)\ln(1-p) + n\ln p - m\lambda + \ln\lambda\sum_{k=1}^{m} y_k - \ln\prod_{k=1}^{m} y_k!\ .$$

Solving the likelihood equations with respect to $p$ and $\lambda$ we get that the MLEs for $p$ and $\lambda$ are

$$\widetilde{p} = \frac{1}{\overline{X}}, \quad \widetilde{\lambda} = \overline{Y}.$$

Using the invariance property of MLE, from (1.1) we get the MLE of $R$

$$\widetilde{R} = 1 - e^{-\frac{\overline{Y}}{\overline{X}}}. \qquad (2.1)$$

**2.1. Asymptotic Distribution.** In the following two theorems we shall find the asymptotic distributions of $(\tilde{p}, \tilde{\lambda})$ and $\tilde{R}$.

**2.1. Theorem.** *Let the ratio $\frac{n}{m}$ converge to a positive number $s$ when both $n$ and $m$ tend to infinity. Then*

$$(\sqrt{n}(\widetilde{p} - p), \sqrt{n}(\widetilde{\lambda} - \lambda)) \xrightarrow[n\to\infty]{D} \mathcal{N}_2(\mathbf{0}, J(p,\lambda)),$$

*where*

$$J(p,\lambda) = \begin{bmatrix} p^2(1-p) & 0 \\ 0 & s\lambda \end{bmatrix}.$$

**Proof.** Since

$$-E\Big(\frac{\partial^2 \ln L}{\partial p^2}\Big) = \frac{n}{p^2(1-p)}$$

and

$$-E\Big(\frac{\partial^2 \ln L}{\partial \lambda^2}\Big) = \frac{m}{\lambda},$$

from the asymptotic normality of maximum likelihood estimator (see [11]) it follows that

$$\sqrt{n}(\widetilde{p} - p) \xrightarrow[n\to\infty]{D} \mathcal{N}(0, p^2(1-p))$$

and

$$\sqrt{m}(\widetilde{\lambda} - \lambda) \xrightarrow[m\to\infty]{D} \mathcal{N}(0, \lambda).$$

Then

$$\sqrt{n}(\widetilde{\lambda} - \lambda) \xrightarrow[n\to\infty]{D} \mathcal{N}(0, s\lambda).$$

From the independence of $\widetilde{p}$ and $\widetilde{\lambda}$ we get the statement of the theorem. □

**2.2. Theorem.** *Let the ratio $\frac{n}{m}$ converge to a positive number $s$ when both $n$ and $m$ tend to infinity. Then*

$$\sqrt{n}(\widetilde{R} - R) \xrightarrow{D} \mathcal{N}(0, e^{-2\lambda p} p^2 \lambda(\lambda(1-p) + s)).$$

**Proof**. In order to prove this theorem we shall use the method from [11]. Since $R = R(p, \lambda)$ is the transformation such that the matrix of partial derivatives

$$B = \begin{bmatrix} \frac{\partial R}{\partial p} & \frac{\partial R}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \lambda e^{-\lambda p} & p e^{-\lambda p} \end{bmatrix}$$

has continuous elements and does not vanish in the neighbourhood of $(p, \lambda)$, then we have

$$\sqrt{n}(\widetilde{R} - R) \xrightarrow{D} \mathcal{N}(0, BJB').$$

Inserting the values of $B$ and $J$ we get the statement of the theorem. □

Using this theorem we can construct the asymptotic confidence interval for $R$. Denote $\widetilde{\sigma}^2 = e^{-2\widetilde{\lambda}\widetilde{p}} \widetilde{p}^2 \widetilde{\lambda}(\widetilde{\lambda}(1-\widetilde{p}) + s)$. Then the estimator of the variance of $\widetilde{R}$ is

$$\widetilde{Var}(\widetilde{R}) = \frac{\widetilde{\sigma}^2}{n}. \tag{2.2}$$

The interval of confidence level $1 - \alpha$ is given by

$$I_R = \left(\widetilde{R} - \frac{z_{1-\frac{\alpha}{2}} \widetilde{\sigma}}{\sqrt{n}}, \widetilde{R} + \frac{z_{1-\frac{\alpha}{2}} \widetilde{\sigma}}{\sqrt{n}}\right), \tag{2.3}$$

where $z_\gamma$ is the $\gamma$th quantile from standard normal distribution.

**2.2. Bootstrap-t Confidence Interval.** The confidence intervals based on the asymptotic distribution do not perform very well for small sample sizes. Therefore, we propose a construction of the confidence interval based on bootstrap-t method (see [10]). The algorithm is illustrated below.

Step 1: From initial samples $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_m)$ calculate MLEs $\widetilde{p}$ and $\widetilde{\lambda}$.

Step 2: Use those estimates to generate bootstrap samples $\mathbf{x}^*$ and $\mathbf{y}^*$ and compute bootstrap sample estimates $R^*$ of $R$ using (2.1).

Step 3: Repeat step 2, $N$ boot times.

Step 4: For each $R_i^*$, $1 \le i \le N$, calculate the following statistic

$$T_i^* = \frac{R_i^* - \widetilde{R}}{\sqrt{Var(R^*)}},$$

where $Var(R^*) = \frac{\sum_{i=1}^{N}(R_i^* - \overline{R^*})^2}{N-1}$ and $\overline{R^*} = \frac{\sum_{i=1}^{N} R_i^*}{N}$.

Step 5: For sample of $T_i^*$ obtained in step 4, calculate sample quantiles of order $\frac{\alpha}{2}$ $(t_{\frac{\alpha}{2}})$ and $1 - \frac{\alpha}{2}$ $(t_{1-\frac{\alpha}{2}})$. Then, the bootstrap-t confidence interval is given by

$$\left(\widetilde{R} - t_{1-\frac{\alpha}{2}}\sqrt{Var(R^*)}, \widetilde{R} - t_{\frac{\alpha}{2}}\sqrt{Var(R^*)}\right). \tag{2.4}$$

## 3. UMVUE of $R$

In this section we find the UMVUE of $R$, denoted by $\widehat{R}$, and UMVUE of the variance of $\widehat{R}$.

The complete sufficient statistics for $p$ and $\lambda$ are $T_X = \sum_{j=1}^{n} X_j$ and $T_Y = \sum_{j=1}^{m} Y_j$. The statistic $T_X$, as a sum of $n$ independent identically distributed random variables with geometric distribution, has negative binomial distribution with parameters $n$ and $p$, and the statistic $T_Y$, as a sum of $m$ independent identically distributed random variables with Poisson distribution, has Poisson distribution with parameter $m\lambda$.

An unbiased estimator for $R$ is $I\{X_1 \le Y_1\}$. Then

$$E(I\{X_1 \le Y_1\}|T_X = t_X, T_Y = t_Y) = P\{X_1 \le Y_1|T_X = t_X, T_Y = t_Y\}$$

$$= \frac{P\{X_1 \le Y_1, \sum_{j=1}^{n} X_j = t_X, \sum_{j=1}^{m} Y_j = t_Y\}}{P\{\sum_{j=1}^{n} X_j = t_X, \sum_{j=1}^{m} Y_j = t_Y\}}$$

$$= \frac{\sum_{y=1}^{t_Y}\sum_{x=1}^{M} P\{X_1 = x\}P\{Y_1 = y\}P\{\sum_{j=2}^{n} X_j = t_X - x\}P\{\sum_{j=2}^{m} Y_j = t_Y - y\}}{P\{\sum_{j=1}^{n} X_j = t_X\}P\{\sum_{j=1}^{m} Y_j = t_Y\}}$$

$$= \frac{\sum_{y=1}^{t_Y}\sum_{x=1}^{M}(1-p)^{x-1}p\frac{e^{-\lambda}\lambda^y}{y!}\binom{t_X-x-1}{n-2}p^{n-1}(1-p)^{t_X-x-n+1}\frac{e^{-(m-1)\lambda}((m-1)\lambda)^{t_Y-y}}{(t_Y-y)!}}{\binom{t_X-1}{n-1}p^n(1-p)^{t_X-n}\frac{e^{-m\lambda}(m\lambda)^{t_Y}}{t_Y!}}$$

$$= \frac{\sum_{y=1}^{t_Y}\binom{t_Y}{y}(m-1)^{t_Y-y}\sum_{x=1}^{M}\binom{t_X-x-1}{n-2}}{\binom{t_X-1}{n-1}m^{t_Y}},$$

where $M = \min\{t_X - n + 1, y\}$.

Using the identity

$$\sum_{s=0}^{n}\binom{s}{c} = \binom{n+1}{c+1},$$

we get that

$$
\begin{aligned}
E(I\{X_1 \le Y_1\}|T_X{=}t_X, T_Y{=}t_Y) &= \frac{\sum_{y=1}^{t_Y} \binom{t_Y}{y}(m-1)^{t_Y-y} \sum_{x=1}^{M} \binom{t_X-x-1}{n-2}}{\binom{t_X-1}{n-1} m^{t_Y}} \\
&= \frac{\sum_{y=1}^{t_Y} \binom{t_Y}{y}(m-1)^{t_Y-y} \sum_{s=t_X-M-1}^{t_X-2} \binom{s}{n-2}}{\binom{t_X-1}{n-1} m^{t_Y}} \\
&= \frac{\sum_{y=1}^{t_Y} \binom{t_Y}{y}(m-1)^{t_Y-y} \left( \sum_{s=0}^{t_X-2} \binom{s}{n-2} - \sum_{s=0}^{t_X-M-2} \binom{s}{n-2} \right)}{\binom{t_X-1}{n-1} m^{t_Y}} \\
&= \frac{\sum_{y=1}^{t_Y} \binom{t_Y}{y}(m-1)^{t_Y-y} \left( \binom{t_X-1}{n-1} - \binom{t_X-M-1}{n-1} \right)}{\binom{t_X-1}{n-1} m^{t_Y}}.
\end{aligned}
$$

Using Rao-Blackwell and Lehmann-Sheffé theorems we get that the UMVUE of $R$ is

$$
\widehat{R} = 1 - \left(1 - \frac{1}{m}\right)^{T_Y} - \sum_{y=1}^{T_Y} \frac{\binom{T_Y}{y}\binom{T_X-M-1}{n-1}}{\binom{T_X-1}{n-1}} \left(1 - \frac{1}{m}\right)^{T_Y-y} \left(\frac{1}{m}\right)^{y}. \tag{3.1}
$$

This formula is valid for $T_Y > 0$. If $T_Y = 0$, then $\widehat{R} = 0$.

Now, in order to find the UMVUE of variance of $\widehat{R}$, we calculate the UMVUE of $R^2$. An unbiased estimator for $R^2$ is $I\{X_1 \leq Y_1, X_2 \leq Y_2\}$. Then

$$
\begin{aligned}
& E(I\{X_1 \leq Y_1, X_2 \leq Y_2\}|T_X = t_X, T_Y = t_Y) \\
= & \frac{P\{X_1 \leq Y_1, X_2 \leq Y_2, \sum_{j=1}^{n} X_j = t_X, \sum_{j=1}^{m} Y_j = t_Y\}}{P\{\sum_{j=1}^{n} X_j = t_X, \sum_{j=1}^{m} Y_j = t_Y\}} \\
= & \frac{1}{P\{\sum_{j=1}^{n} X_j = t_X\} P\{\sum_{j=1}^{m} Y_j = t_Y\}} \\
\times & \sum_{y_1=1}^{t_Y-1} \sum_{y_2=1}^{t_Y-y_1} \sum_{x_1=1}^{M_1} \sum_{x_2=1}^{M_2} P\{X_1 = x_1\} P\{X_2 = x_2\} P\{Y_1 = y_1\} P\{Y_2 = y_2\} \\
\times & P\Big\{\sum_{j=3}^{n} X_j = t_X - x_1 - x_2\Big\} P\Big\{\sum_{j=3}^{m} Y_j = t_Y - y_1 - y_2\Big\} \\
= & \frac{1}{\binom{t_X-1}{n-1} p^n (1-p)^{t_X-n} \frac{e^{-m\lambda}(m\lambda)^{t_Y}}{t_Y!}} \\
\times & \sum_{y_1=1}^{t_Y-1} \sum_{y_2=1}^{t_Y-y_1} \sum_{x_1=1}^{M_1} \sum_{x_2=1}^{M_2} p(1-p)^{x_1-1} p(1-p)^{x_2-1} \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \\
\times & \binom{t_X - x_1 - x_2 - 1}{n-3} p^{n-2} (1-p)^{t_x - x_1 - x_2 - n + 2} \frac{e^{-(m-2)\lambda}((m-2)\lambda)^{t_Y - y_1 - y_2}}{(t_Y - y_1 - y_2)!} \\
(3.2) = & \frac{\sum_{y_1=1}^{t_Y-1} \sum_{y_2=1}^{t_Y-y_1} \binom{t_Y}{y_1+y_2} \binom{y_1+y_2}{y_1} (m-2)^{t_Y-y_1-y_2} \sum_{x_1=1}^{M_1} \sum_{x_2=1}^{M_2} \binom{t_X-x_1-x_2-1}{n-3}}{\binom{t_X-1}{n-1} m^{t_Y}},
\end{aligned}
$$

where $M_1 = \min\{y_1, t_X - n + 1\}$ and $M_2 = \min\{y_2, t_X - n + 2 - x_1\}$. Using similar technique as when finding $\widehat{R}$, we get that

$$
\begin{aligned}
& \sum_{x_1=1}^{M_1} \sum_{x_2=1}^{M_2} \binom{t_X - x_1 - x_2 - 1}{n-3} = \sum_{x_1=1}^{M_1} \sum_{s=t_X-x_1-M_2-1}^{t_X-x_1-2} \binom{s}{n-3} \\
= & \sum_{x_1=1}^{M_1} \left( \binom{t_X - x_1 - 1}{n-2} - \binom{t_X - x_1 - M_2 - 1}{n-2} \right) \\
= & \sum_{s=0}^{t_X-2} \binom{s}{n-2} - \sum_{s=0}^{t_X-M_1-2} \binom{s}{n-2} - \sum_{x_1=1}^{M_1} \binom{t_X - x_1 - M_2 - 1}{n-2} \\
= & \binom{t_X-1}{n-1} - \binom{t_X - M_1 - 1}{n-1} - \sum_{x_1=1}^{M_1} \binom{t_X - x_1 - M_2 - 1}{n-2}.
\end{aligned}
$$

Inserting this into (3.2) and using Rao-Blackwell and Lehmann-Sheffé theorems we get that the UMVUE of $R^2$ is

$$\begin{aligned}
\widehat{R^2} &= \frac{1}{\binom{T_X-1}{n-1} m^{T_Y}} \sum_{y_1=1}^{T_Y-1} \sum_{y_2=1}^{T_Y-y_1} \binom{T_Y}{y_1+y_2} \binom{y_1+y_2}{y_1} (m-2)^{T_Y-y_1-y_2} \\
&\quad \times \left( \binom{T_X-1}{n-1} - \binom{T_X-M_1-1}{n-1} - \sum_{x_1=1}^{M_1} \binom{T_X-x_1-M_2-1}{n-2} \right). \qquad (3.3)
\end{aligned}$$

This formula is valid for $T_Y > 1$. If $T_Y \leq 1$, then $\widehat{R^2} = 0$.

Finally, we obtain the UMVUE of variance of $\widehat{R}$ using the following theorem.

**3.1. Theorem.** *The UMVUE of $Var(\widehat{R})$ is given by*

$$\widehat{Var}(\widehat{R}) = (\widehat{R})^2 - \widehat{R^2}, \qquad (3.4)$$

*where $\widehat{R}$ and $\widehat{R^2}$ are given by* (3.1) *and* (3.3).

The proof follows from general result obtained in [18] and [13].

## 4. Bayes Estimator of $R$

In this section we shall find the Bayes estimator of $R$ with respect to mean square error. Let us suppose that $p$ and $\lambda$ have conjugate prior distributions, beta $\mathcal{B}(a,b)$, $a, b \in \mathbb{N}$, and gamma $\Gamma(\alpha, \beta)$, $\alpha \in \mathbb{N}, \beta > 0$, with the following joint density:

$$\pi(p, \lambda) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)} \frac{\lambda^{\alpha-1}\beta^{\alpha} e^{-\beta\lambda}}{\Gamma(\alpha)}, \quad p \in (0,1), \lambda > 0.$$

Then the joint posterior density given the sample $(\mathbf{x}, \mathbf{y})$, or, equivalently, given the sufficient statistics $(t_X, t_Y)$ is

$$\pi(p, \lambda | t_X, t_Y) = K p^{a-1+n} (1-p)^{t_X-n+b-1} \lambda^{\alpha-1+t_Y} e^{-\lambda(\beta+m)}, \; p \in (0,1), \lambda > 0,$$

where

$$K = \left( \int_0^1 \int_0^\infty p^{a-1+n} (1-p)^{t_X-n+b-1} \lambda^{\alpha-1+t_Y} e^{-\lambda(\beta+m)} d\lambda \, dp \right)^{-1}$$

is the proportionality constant.

Denote, for simplicity, $A = a + n - 1$, $B = t_X - n + b - 1$, $C = \alpha - 1 + t_Y$ and $D = \beta + m$. Since $R = 1 - e^{-\lambda p}$, we get that $p = -\frac{\ln(1-R)}{\lambda}$. Using the transformation of random variables $(p, \lambda)$ to $(R, \lambda)$ we get

$$\begin{aligned}
\pi(r, \lambda | t_X, t_Y) &= \pi\big(p(r,\lambda), \lambda(r,\lambda) | t_X, t_Y\big) \begin{vmatrix} \frac{\partial p}{\partial r} & \frac{\partial p}{\partial \lambda} \\ \frac{\partial \lambda}{\partial r} & \frac{\partial \lambda}{\partial \lambda} \end{vmatrix} \\
&= \pi\big(p(r,\lambda), \lambda(r,\lambda) | t_X, t_Y\big) \begin{vmatrix} \frac{1}{\lambda}\frac{1}{1-r} & \frac{\ln(1-r)}{\lambda^2} \\ 0 & 1 \end{vmatrix} \\
&= K \left( -\frac{\ln(1-r)}{\lambda} \right)^A \left( 1 + \frac{\ln(1-r)}{\lambda} \right)^B \lambda^C e^{-\lambda D} \frac{1}{\lambda} \frac{1}{1-r} \\
&= K \left( -\ln(1-r) \right)^A \left( 1 + \frac{\ln(1-r)}{\lambda} \right)^B \lambda^{C-A-1} \frac{e^{-\lambda D}}{1-r}, \\
&\quad r \in (0,1), \lambda > -\ln(1-r).
\end{aligned}$$

Then the marginal posterior density of $R$ is

$$\pi_R(r|t_X,t_Y) = K\int\limits_{-\ln(1-r)}^{\infty}(-\ln(1-r))^A\sum_{j=0}^{B}\binom{B}{j}\left(\frac{\ln(1-r)}{\lambda}\right)^j\lambda^{C-A-1}\frac{e^{-\lambda D}}{1-r}d\lambda$$

$$= K\sum_{j=0}^{B}\binom{B}{j}(-1)^j\frac{(-\ln(1-r))^{A+j}}{1-r}\int\limits_{-\ln(1-r)}^{\infty}\lambda^{C-A-j-1}e^{-\lambda D}d\lambda$$

$$= K\sum_{j=0}^{B}\binom{B}{j}(-1)^j\frac{(-\ln(1-r))^{A+j}}{(1-r)D^{C-A-j}}\int\limits_{-D\ln(1-r)}^{\infty}t^{C-A-j-1}e^{-t}dt,\quad r\in(0,1).$$

The Bayes estimator $\check{R}$ of $R$ for mean square loss function is the posterior mean. After some calculations (see Appendix) we obtain

$$\check{R} = 1-K\left[I_{\{C-A>0\}}\sum_{j=0}^{\min\{C-A-1,B\}}W_1 + I_{\{0\le C-A\le B\}}W_2 + I_{\{C-A<B\}}\sum_{j=\max\{0,C-A+1\}}^{B}W_3\right], \tag{4.1}$$

where

$$\begin{aligned}
W_1 &= (-1)^j\binom{B}{j}\frac{(C-1)!}{\binom{C-1}{A+j}}\sum_{i=0}^{C-A-j-1}\frac{\binom{A+j+i}{i}}{D^{C-A-j-i}(D+1)^{A+j+i+1}},\\
W_2 &= (-1)^{C-A-1}\binom{B}{C-A}C!\left[\ln D+\sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(D^i\binom{C+i}{i}-\binom{C}{i}\right)\right],\\
W_3 &= \binom{B}{j}(C-1)!\sum_{i=1}^{A-C+j}(-1)^{i+j+1}\frac{D^{i-1}\binom{C+i-1}{i}}{(D+1)^{C+i}\binom{A-C+j}{i}}+\frac{(-1)^{A-C+1}}{D^{C-A-j}}\\
&\times\binom{B}{j}\binom{A+j}{C}C!\left[\ln D+\sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(D^i\binom{A+j+i}{i}-\binom{A+j}{i}\right)\right].
\end{aligned}$$

It is possible to generalize this estimator for real values of the hyperparametres, but it would be much more complicated and not practical for presentation.

## 5. Simulation Study

In this section we perform a simulation study for various sample sizes and different values of unknown parameters.

For fixed values of $n$, $m$, $p$ and $\lambda$ we do the following procedure. We choose a sample and calculate the MLE and its variance using (2.1) and (2.2), and the UMVUE and its variance using (3.1), (3.3) and (3.4). Since we do not know the prior distributions and to get better comparison with other types of estimates, we obtain Bayes estimates using non-informative Jeffreys' priors where $\pi(p)\sim p^{-1}(1-p)^{-\frac{1}{2}}$ and $\pi(\lambda)\sim\lambda^{-\frac{1}{2}}$. We find the estimates from posterior distribution for $R$ using Monte Carlo method with 5000 replicates.

We also calculate 95% asymptotic confidence interval using (2.3) and 95% bootstrap-t confidence interval using (2.4) with $N=1000$ boot times.

This procedure is repeated for 500 samples and the averages for each estimate are calculated.

In table 1 we present point estimates for $R$ and their standard errors. In table 2 we present 95% asymptotic and bootstrap-t confidence intervals as well as 95% Bayes credible intervals based on a Monte Carlo method mentioned above. The coverage percentages of these intervals (the percentage of intervals that contain true value of $R$) are also shown.

In table 1 we can notice that in most cases the UMVUE has the value closest to $R$ as expected due to its unbiasedness. However, its standard error is the largest. For most values of $R$ the standard error of Bayes estimate is the smallest, while for larger values of $R$, the standard error of MLE has that property. In the last case ($R = 0.7981$), the standard error of Bayes estimate is even larger than the UMVUE one.

From table 2 we can see that in almost all cases the asymptotic intervals have the worst coverage percentages, which is expected because we have small sample sizes, while Bayes credible intervals and bootstrap-t confidence intervals both perform very well.

**Table 1.** Point estimates for $R$ and their standard errors

| samples | | parameters | | reliability | MLE | | UMVUE | | Bayes | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $m$ | $p$ | $\lambda$ | $R$ | $\widetilde{R}$ | $\sigma(\widetilde{R})$ | $\widehat{R}$ | $\widehat{\sigma}(\widehat{R})$ | $\check{R}$ | $\sigma(\check{R})$ |
| 10 | 15 | 0.5 | 0.5 | 0.2212 | 0.2226 | 0.0813 | 0.2176 | 0.0825 | 0.2133 | 0.0778 |
| | | 0.25 | 1 | 0.2212 | 0.2324 | 0.0742 | 0.2215 | 0.0746 | 0.2257 | 0.0719 |
| | | 0.3 | 1 | 0.2592 | 0.2634 | 0.0809 | 0.2527 | 0.0821 | 0.2552 | 0.0782 |
| | | 0.5 | 1 | 0.3935 | 0.3986 | 0.1003 | 0.3929 | 0.1047 | 0.3824 | 0.0966 |
| | | 0.8 | 1 | 0.5507 | 0.5398 | 0.1024 | 0.5454 | 0.1069 | 0.5146 | 0.1007 |
| | | 0.4 | 2 | 0.5507 | 0.5550 | 0.1045 | 0.5478 | 0.1106 | 0.5353 | 0.1023 |
| | | 0.67 | 1.5 | 0.6340 | 0.6415 | 0.0972 | 0.6343 | 0.1023 | 0.6162 | 0.0975 |
| | | 0.5 | 2 | 0.6340 | 0.6369 | 0.1011 | 0.6346 | 0.1067 | 0.6137 | 0.1006 |
| | | 0.8 | 1.5 | 0.6988 | 0.6918 | 0.0891 | 0.6987 | 0.0964 | 0.6648 | 0.0915 |
| | | 0.6 | 2 | 0.6988 | 0.7019 | 0.0927 | 0.7008 | 0.0982 | 0.6767 | 0.0945 |
| | | 0.8 | 2 | 0.7981 | 0.7868 | 0.0737 | 0.7940 | 0.1043 | 0.7611 | 0.0790 |
| 20 | 15 | 0.5 | 0.5 | 0.2212 | 0.2236 | 0.0762 | 0.2228 | 0.0774 | 0.2174 | 0.0735 |
| | | 0.25 | 1 | 0.2212 | 0.2229 | 0.0625 | 0.2202 | 0.0625 | 0.2190 | 0.0612 |
| | | 0.3 | 1 | 0.2592 | 0.2648 | 0.0706 | 0.2605 | 0.0712 | 0.2596 | 0.0689 |
| | | 0.5 | 1 | 0.3935 | 0.3914 | 0.0899 | 0.3921 | 0.0923 | 0.3818 | 0.0872 |
| | | 0.8 | 1 | 0.5507 | 0.5597 | 0.0970 | 0.5573 | 0.0998 | 0.5324 | 0.0949 |
| | | 0.4 | 2 | 0.5507 | 0.5561 | 0.0881 | 0.5547 | 0.0910 | 0.5436 | 0.0864 |
| | | 0.67 | 1.5 | 0.6340 | 0.6260 | 0.0895 | 0.6312 | 0.0920 | 0.6093 | 0.0884 |
| | | 0.5 | 2 | 0.6340 | 0.6374 | 0.0864 | 0.6392 | 0.0891 | 0.6226 | 0.0854 |
| | | 0.8 | 1.5 | 0.6988 | 0.6894 | 0.0834 | 0.6977 | 0.0848 | 0.6712 | 0.0835 |
| | | 0.6 | 2 | 0.6988 | 0.6975 | 0.0813 | 0.7021 | 0.0832 | 0.6812 | 0.0813 |
| | | 0.8 | 2 | 0.7981 | 0.7895 | 0.0668 | 0.7975 | 0.0668 | 0.7725 | 0.0688 |
| 20 | 20 | 0.5 | 0.5 | 0.2212 | 0.2256 | 0.0680 | 0.2240 | 0.0688 | 0.2200 | 0.0660 |
| | | 0.25 | 1 | 0.2212 | 0.2277 | 0.0579 | 0.2225 | 0.0580 | 0.2241 | 0.0569 |
| | | 0.3 | 1 | 0.2592 | 0.2665 | 0.0649 | 0.2616 | 0.0654 | 0.2617 | 0.0636 |
| | | 0.5 | 1 | 0.3935 | 0.3965 | 0.0817 | 0.3948 | 0.0836 | 0.3872 | 0.0796 |
| | | 0.8 | 1 | 0.5507 | 0.5449 | 0.0865 | 0.5500 | 0.0887 | 0.5297 | 0.0850 |
| | | 0.4 | 2 | 0.5507 | 0.5469 | 0.0823 | 0.5491 | 0.0848 | 0.5358 | 0.0809 |
| | | 0.67 | 1.5 | 0.6340 | 0.6350 | 0.0802 | 0.6352 | 0.0823 | 0.6201 | 0.0798 |
| | | 0.5 | 2 | 0.6340 | 0.6348 | 0.0798 | 0.6341 | 0.0822 | 0.6215 | 0.0793 |
| | | 0.8 | 1.5 | 0.6988 | 0.6951 | 0.0738 | 0.7010 | 0.0749 | 0.6791 | 0.0744 |
| | | 0.6 | 2 | 0.6988 | 0.6924 | 0.0751 | 0.6949 | 0.0770 | 0.6779 | 0.0754 |
| | | 0.8 | 2 | 0.7981 | 0.7960 | 0.0593 | 0.8018 | 0.0593 | 0.7809 | 0.0614 |
| 50 | 50 | 0.5 | 0.5 | 0.2212 | 0.2214 | 0.0432 | 0.2207 | 0.0434 | 0.2192 | 0.0427 |
| | | 0.25 | 1 | 0.2212 | 0.2226 | 0.0364 | 0.2205 | 0.0364 | 0.2212 | 0.0362 |
| | | 0.3 | 1 | 0.2592 | 0.2605 | 0.0410 | 0.2585 | 0.0411 | 0.2586 | 0.0406 |
| | | 0.5 | 1 | 0.3935 | 0.3960 | 0.0522 | 0.3953 | 0.0527 | 0.3923 | 0.0517 |
| | | 0.8 | 1 | 0.5507 | 0.5489 | 0.0552 | 0.5509 | 0.0558 | 0.5426 | 0.0548 |
| | | 0.4 | 2 | 0.5507 | 0.5494 | 0.0528 | 0.5483 | 0.0535 | 0.5449 | 0.0524 |
| | | 0.67 | 1.5 | 0.6340 | 0.6355 | 0.0513 | 0.6348 | 0.0518 | 0.6294 | 0.0512 |
| | | 0.5 | 2 | 0.6340 | 0.6327 | 0.0514 | 0.6338 | 0.0521 | 0.6273 | 0.0512 |
| | | 0.8 | 1.5 | 0.6988 | 0.6964 | 0.0472 | 0.6997 | 0.0475 | 0.6908 | 0.0474 |
| | | 0.6 | 2 | 0.6988 | 0.6952 | 0.0482 | 0.6962 | 0.0487 | 0.6892 | 0.0483 |
| | | 0.8 | 2 | 0.7981 | 0.7974 | 0.0379 | 0.7977 | 0.0379 | 0.7912 | 0.0385 |

## 6. Conclusion

In this paper we considered the estimation of the probability $P\{X \leq Y\}$ when $X$ and $Y$ are two independent random variables from geometric and Poisson distribution respectively. We determined MLE, UMVUE and Bayes point estimator. The asymptotic and bootstrap-t confidence intervals were constructed.

A simulation study was performed. The obtained point estimates were compared and in most cases UMVUEs have the smallest bias, while Bayes estimates have the smallest standard error. Comparison of interval estimates was also done and we concluded that bootstrap-t and Bayes intervals had notably higher coverage percentages than asymptotic ones.

**Table 2.** Interval estimates for $R$ and their coverage percentages

| samples $n$ | $m$ | parameters $p$ | $\lambda$ | reliability $R$ | asymptotic CI | cov. | bootstrap CI | cov. | Bayes CI | cov. |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 15 | 0.5 | 0.5 | 0.2212 | (0.06, 0.38) | 91.4 | (0.09, 0.40) | 93.2 | (0.09, 0.39) | 94.0 |
| | | 0.25 | 1 | 0.2212 | (0.09, 0.38) | 92.0 | (0.12, 0.41) | 93.4 | (0.11, 0.38) | 94.0 |
| | | 0.3 | 1 | 0.2592 | (0.11, 0.42) | 93.0 | (0.14, 0.45) | 95.2 | (0.12, 0.42) | 94.8 |
| | | 0.5 | 1 | 0.3935 | (0.20, 0.60) | 91.4 | (0.22, 0.61) | 93.8 | (0.21, 0.58) | 94.2 |
| | | 0.8 | 1 | 0.5507 | (0.34, 0.74) | 92.6 | (0.32, 0.76) | 91.6 | (0.32, 0.71) | 93.8 |
| | | 0.4 | 2 | 0.5507 | (0.35, 0.76) | 89.6 | (0.36, 0.76) | 93.2 | (0.33, 0.73) | 93.8 |
| | | 0.67 | 1.5 | 0.6340 | (0.45, 0.83) | 90.6 | (0.44, 0.81) | 94.8 | (0.41, 0.79) | 94.8 |
| | | 0.5 | 2 | 0.6340 | (0.44, 0.83) | 92.8 | (0.44, 0.83) | 96.4 | (0.40, 0.79) | 94.6 |
| | | 0.8 | 1.5 | 0.6988 | (0.52, 0.87) | 92.2 | (0.50, 0.84) | 94.6 | (0.47, 0.83) | 93.4 |
| | | 0.6 | 2 | 0.6988 | (0.52, 0.88) | 90.2 | (0.51, 0.86) | 94.2 | (0.47, 0.84) | 94.4 |
| | | 0.8 | 2 | 0.7981 | (0.64, 0.93) | 92.8 | (0.62, 0.90) | 94.0 | (0.58, 0.89) | 94.4 |
| 20 | 15 | 0.5 | 0.5 | 0.2212 | (0.07, 0.37) | 91.6 | (0.09, 0.38) | 93.0 | (0.09, 0.38) | 93.4 |
| | | 0.25 | 1 | 0.2212 | (0.10, 0.35) | 94.6 | (0.12, 0.36) | 95.2 | (0.11, 0.35) | 95.4 |
| | | 0.3 | 1 | 0.2592 | (0.13, 0.40) | 93.6 | (0.14, 0.42) | 93.8 | (0.14, 0.41) | 94.4 |
| | | 0.5 | 1 | 0.3935 | (0.22, 0.57) | 93.0 | (0.22, 0.57) | 94.8 | (0.22, 0.56) | 94.0 |
| | | 0.8 | 1 | 0.5507 | (0.35, 0.73) | 93.6 | (0.33, 0.74) | 92.0 | (0.34, 0.71) | 94.8 |
| | | 0.4 | 2 | 0.5507 | (0.38, 0.73) | 93.2 | (0.39, 0.73) | 95.0 | (0.37, 0.71) | 94.4 |
| | | 0.67 | 1.5 | 0.6340 | (0.45, 0.80) | 95.2 | (0.44, 0.78) | 97.4 | (0.43, 0.77) | 96.4 |
| | | 0.5 | 2 | 0.6340 | (0.47, 0.81) | 94.2 | (0.46, 0.79) | 95.6 | (0.45, 0.78) | 95.8 |
| | | 0.8 | 1.5 | 0.6988 | (0.53, 0.85) | 93.4 | (0.50, 0.83) | 94.2 | (0.50, 0.82) | 95.0 |
| | | 0.6 | 2 | 0.6988 | (0.54, 0.86) | 93.8 | (0.52, 0.84) | 95.6 | (0.51, 0.83) | 94.6 |
| | | 0.8 | 2 | 0.7981 | (0.66, 0.92) | 92.4 | (0.63, 0.89) | 93.0 | (0.62, 0.89) | 93.6 |
| 20 | 20 | 0.5 | 0.5 | 0.2212 | (0.09, 0.36) | 92.8 | (0.10, 0.37) | 94.0 | (0.11, 0.36) | 93.8 |
| | | 0.25 | 1 | 0.2212 | (0.12, 0.34) | 94.0 | (0.13, 0.36) | 93.8 | (0.13, 0.35) | 94.2 |
| | | 0.3 | 1 | 0.2592 | (0.14, 0.39) | 93.0 | (0.16, 0.41) | 92.4 | (0.15, 0.40) | 93.4 |
| | | 0.5 | 1 | 0.3935 | (0.24, 0.56) | 94.2 | (0.24, 0.56) | 95.2 | (0.24, 0.55) | 95.4 |
| | | 0.8 | 1 | 0.5507 | (0.37, 0.71) | 93.8 | (0.36, 0.70) | 95.0 | (0.36, 0.69) | 94.8 |
| | | 0.4 | 2 | 0.5507 | (0.40, 0.71) | 92.6 | (0.39, 0.71) | 93.8 | (0.38, 0.69) | 93.2 |
| | | 0.67 | 1.5 | 0.6340 | (0.48, 0.79) | 95.0 | (0.47, 0.78) | 95.8 | (0.46, 0.77) | 96.8 |
| | | 0.5 | 2 | 0.6340 | (0.48, 0.79) | 91.6 | (0.47, 0.78) | 93.2 | (0.50, 0.77) | 94.0 |
| | | 0.8 | 1.5 | 0.6988 | (0.55, 0.84) | 91.6 | (0.53, 0.82) | 93.0 | (0.52, 0.81) | 94.0 |
| | | 0.6 | 2 | 0.6988 | (0.55, 0.84) | 90.8 | (0.53, 0.83) | 93.4 | (0.52, 0.81) | 92.6 |
| | | 0.8 | 2 | 0.7981 | (0.68, 0.91) | 91.8 | (0.66, 0.89) | 94.6 | (0.65, 0.89) | 95.2 |
| 50 | 50 | 0.5 | 0.5 | 0.2212 | (0.14, 0.31) | 95.4 | (0.14, 0.31) | 95.8 | (0.14, 0.31) | 95.8 |
| | | 0.25 | 1 | 0.2212 | (0.15, 0.30) | 94.6 | (0.16, 0.31) | 95.0 | (0.16, 0.30) | 95.0 |
| | | 0.3 | 1 | 0.2592 | (0.18, 0.34) | 92.6 | (0.19, 0.35) | 94.2 | (0.18, 0.34) | 94.2 |
| | | 0.5 | 1 | 0.3935 | (0.29, 0.50) | 93.2 | (0.30, 0.50) | 94.2 | (0.29, 0.50) | 94.0 |
| | | 0.8 | 1 | 0.5507 | (0.44, 0.66) | 96.0 | (0.44, 0.65) | 95.2 | (0.43, 0.65) | 95.6 |
| | | 0.4 | 2 | 0.5507 | (0.45, 0.65) | 93.6 | (0.45, 0.65) | 94.8 | (0.44, 0.65) | 94.4 |
| | | 0.67 | 1.5 | 0.6340 | (0.53, 0.74) | 93.6 | (0.53, 0.73) | 93.4 | (0.53, 0.73) | 93.2 |
| | | 0.5 | 2 | 0.6340 | (0.53, 0.73) | 94.0 | (0.53, 0.73) | 95.4 | (0.52, 0.72) | 94.4 |
| | | 0.8 | 1.5 | 0.6988 | (0.60, 0.79) | 93.6 | (0.60, 0.78) | 94.0 | (0.59, 0.78) | 94.4 |
| | | 0.6 | 2 | 0.6988 | (0.60, 0.79) | 94.2 | (0.60, 0.78) | 94.2 | (0.59, 0.78) | 94.8 |
| | | 0.8 | 2 | 0.7981 | (0.72, 0.87) | 93.0 | (0.72, 0.86) | 93.6 | (0.71, 0.86) | 94.4 |

## Appendix

$$
\begin{aligned}
\check{R} &= E(R|t_X,t_Y) = 1 - E(1-R|t_X,t_Y) = 1 - \int_0^1 (1-r)\pi_R(r|t_X,t_Y)dr \\
&= 1 - \int_0^1 (1-r)K\sum_{j=0}^{B}\binom{B}{j}(-1)^j \frac{(-\ln(1-r))^{A+j}}{(1-r)D^{C-A-j}} \int_{-D\ln(1-r)}^{\infty} t^{C-A-j-1}e^{-t}dtdr \\
&= 1 - K\sum_{j=0}^{B}\binom{B}{j}\frac{(-1)^j}{D^{C-A-j}} \int_0^1 (-\ln(1-r))^{A+j} \int_{-D\ln(1-r)}^{\infty} t^{C-A-j-1}e^{-t}dtdr \\
&= 1 - K\sum_{j=0}^{B}\binom{B}{j}\frac{(-1)^j}{D^{C-A-j}} \int_0^{\infty} s^{A+j}e^{-s} \int_{Ds}^{\infty} t^{C-A-j-1}e^{-t}dtds. \qquad (.1)
\end{aligned}
$$

We need to calculate the integral $L_q(z) = \int_z^\infty t^{q-1}e^{-t}dt$, $z > 0$, $q \in \mathbb{Z}$. Depending on $q$ we have the following three possibilities:

(1) $q > 0$

$$L_q(z) = \Gamma(q,z) = (q-1)!e^{-z}\sum_{i=0}^{q-1}\frac{z^i}{i!},$$

where $\Gamma(q,z)$ is the incomplete gamma function.

(2) $q = 0$

$$L_q(z) = -\mathrm{Ei}(-z) = -\gamma - \ln z + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{z^i}{i \cdot i!},$$

where $\mathrm{Ei}(x)$ is the exponetial integral and $\gamma$ is Euler's constant.

(3) $q < 0$

Using integration by parts $|q|$ times we get

$$L_q(z) = e^{-z}\sum_{i=1}^{-q}(-1)^{i+1}\frac{z^{i+q-1}}{(-q)!}(-q-i)! + \frac{(-1)^{-q}}{(-q)!}L_0(z).$$

Thus, the summands in (.1) can be expressed as

$$\binom{B}{j}\frac{(-1)^j}{D^{C-A-j}}\int_0^\infty s^{A+j}e^{-s}L_{C-A-j}(Ds)ds,$$

and depending on $j$, we have three types of summands:

(1) $j < C - A$

$$\begin{aligned}
W_1 &= \binom{B}{j}\frac{(-1)^j}{D^{C-A-j}}\int_0^\infty s^{A+j}e^{-s}(C-A-j-1)!e^{-Ds}\sum_{i=0}^{C-A-j-1}\frac{(Ds)^i}{i!}ds \\
&= (-1)^j\binom{B}{j}\frac{(C-A-j-1)!}{D^{C-A-j}}\sum_{i=0}^{C-A-j-1}\frac{D^i}{i!}\int_0^\infty s^{A+j+i}e^{-(D+1)s}ds \\
&= (-1)^j\binom{B}{j}\frac{(C-1)!}{\binom{C-1}{A+j}}\sum_{i=0}^{C-A-j-1}\frac{\binom{A+j+i}{i}}{D^{C-A-j-i}(D+1)^{A+j+i+1}}.
\end{aligned}$$

This type of summand appears in (.1) whenever $C - A > 0$.

(2) $j = C - A$

$$\begin{aligned}
W_2 &= \binom{B}{C{-}A}(-1)^{C-A}\int_0^\infty s^Ce^{-s}(-\gamma - \ln(Ds) + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{(Ds)^i}{i \cdot i!})ds \\
&= \binom{B}{C{-}A}(-1)^{C-A-1}\Bigg((\gamma + \ln D)\int_0^\infty s^Ce^{-s}ds + \int_0^\infty \ln s\ s^Ce^{-s}ds \\
&+ \sum_{i=1}^{\infty}(-1)^i\frac{D^i}{i \cdot i!}\int_0^\infty s^{C+i}e^{-s}ds\Bigg) \\
&= \binom{B}{C{-}A}(-1)^{C-A-1}\Bigg((\gamma + \ln D)C! + \psi(C+1)C!
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \sum_{i=1}^{\infty}(-1)^i \frac{D^i}{i\cdot i!}(C+i)!\Big) \\
&= \binom{B}{C{-}A}(-1)^{C-A-1}\Big((\gamma+\ln D)C! + C!\Big(-\gamma-\sum_{i=1}^{\infty}\frac{(-1)^i}{i}\binom{C}{i}\Big) \\
&+ \quad \sum_{i=1}^{\infty}(-1)^i \frac{D^i}{i\cdot i!}(C+i)!\Big) \\
&= (-1)^{C-A-1}\binom{B}{C{-}A}C!\left[\ln D + \sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(D^i\binom{C{+}i}{i}-\binom{C}{i}\right)\right],
\end{aligned}
$$

where $\psi(x)$ is digamma function.

This type of summand appears in (.1) whenever $0 \le C-A \le B$.

(3) $j > C-A$

$$
\begin{aligned}
W_3 \quad &= \binom{B}{j}\frac{(-1)^j}{D^{C-A-j}}\int_0^{\infty} s^{A+j}e^{-s} \\
&\times \Big(e^{-Ds}\sum_{i=1}^{A-C+j}(-1)^{i+1}\frac{(Ds)^{C-A-j+i-1}}{(A-C+j)!}(A-C+j-i)! \\
&+ \frac{(-1)^{A-C+j}}{(A-C+j)!}\Big(-\gamma-\ln(Ds)+\sum_{i=1}^{\infty}(-1)^{i+1}\frac{(Ds)^i}{i\cdot i!}\Big)\Big)ds \\
&= \binom{B}{j}\sum_{i=1}^{A-C+j}\frac{(-1)^{i+j+1}D^{i-1}(A-C+j-i)!}{(A-C+j)!} \\
&\times \int_0^{\infty} s^{C+i-1}e^{-(D+1)s}ds + \binom{B}{j}\frac{(-1)^{A-C+1}}{(A-C+j)!}\frac{1}{D^{C-A-j}} \\
&\times \Big((\gamma+\ln D)\int_0^{\infty} s^{A+j}e^{-s}ds + \int_0^{\infty}\ln s\; s^{A+j}e^{-s}ds \\
&+ \sum_{i=1}^{\infty}(-1)^i\frac{D^i}{i\cdot i!}\int_0^{\infty} s^{A+j+i}e^{-s}ds\Big) \\
&= \binom{B}{j}\sum_{i=1}^{A-C+j}\frac{(-1)^{i+j+1}D^{i-1}(A-C+j-i)!}{(A-C+j)!}\frac{(C+i-1)!}{(D+1)^{C+i}} \\
&+ \binom{B}{j}\frac{(-1)^{A-C+1}}{(A-C+j)!}\frac{1}{D^{C-A-j}}\Big((\gamma+\ln D)(A+j)! \\
&+ \psi(A+j+1)(A+j)! + \sum_{i=1}^{\infty}(-1)^i\frac{D^i}{i\cdot i!}(A+j+i)!\Big) \\
&= \binom{B}{j}(C-1)!\sum_{i=1}^{A-C+j}(-1)^{i+j+1}\frac{D^{i-1}\binom{C+i-1}{i}}{(D+1)^{C+i}\binom{A-C+j}{i}} + \frac{(-1)^{A-C+1}}{D^{C-A-j}} \\
&\times \binom{B}{j}\binom{A+j}{C}C!\left[\ln D + \sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(D^i\binom{A+j+i}{i}-\binom{A+j}{i}\right)\right].
\end{aligned}
$$

This type of summand appears in (.1) whenever $C-A < B$.

Expressing (.1) via these summands we get (4.1).

## Acknowledgement

We would like to thank the anonymous referees for their useful comments and suggestions that improved our paper.

## References

[1] Ahmad, K. E., Fakhry, M. E. and Jaheen Z. F. *Bayes estimation of $P(Y > X)$ in the geometric case*, Microelectronic Reliability **35**(5), 817-820, 1995.

[2] Baklizi, A. and Quader El-Masri, A. E. *Shrinkage estimation of $P(X < Y)$ in the exponential case with common location parameter*, Metrika **59**, 163-171, 2004.

[3] Belyaev, Y. and Lumelskii, Y. *Multidimensional Poisson Walks*, Journal of Mathematical Sciences **40**, 162-165, 1988.

[4] Birnbaum, Z. W. *On a Use of Mann-Whitney Statistics*, Proc. Third Berkeley Symp. in Math. Statist. Probab. **1** (University of California Press, Berkeley, 1956), 13-17.

[5] Constantine, K. and Carson, M. *The estimation of $P(Y < X)$ in the gamma case*, Communications in Statistics: Simulation and Computation **15**, 365-388, 1986.

[6] Constantine, K., Carson, M. and Tse, S.K. *Confidence Interval Estimation of $P(Y < X)$ in the Gamma Case*, Communications in Statistics: Simulation and Computation **19**(1), 225-244, 1990.

[7] Downtown, F. *On the Estimation of $Pr(Y < X)$ in the Normal Case*, Technometrics **15**, 551-558, 1973.

[8] Genç, A. I. *Estimation of $P(X > Y)$ with Topp-Leone Distribution*, Journal of Statistical Computation and Simulation **83**(2), 326-339, 2013.

[9] Govidarajulu, Z. *Two sided confidence limits for $P(X > Y)$ based on normal samples of $X$ and $Y$*, Sankhyã **29**, 35-40, 1967.

[10] Hall, P. *Theoretical comparison of bootstrap confidence intervals*, Annals of Statistics **16**, 927-953, 1988.

[11] Hogg, R. V., McKean, J. W. and Craig, A. T. *Introduction to Mathematical Statistics*, Sixth Edition (Pearson Prentice Hall, New Jersey, 2005), 348-349.

[12] Ismail, R., Jeyaratnam, S. S. and Panchapakesan, S. *Estimation of $Pr[X > Y]$ for Gamma Distributions*, Journal of Statistical Computation and Simulation **26**(3-4), 253-267, 1986.

[13] Ivshin, V. V. and Lumelskii, Ya. P. *Statistical Estimation Problems in "Stress-Strength" Models* (Perm University Press, Perm, Russia, 1995).

[14] Kakade, C. S., Shirke, D. T. and Kundu, D. *Inference for $P(Y < X)$ in exponentiated Gumbel distribution*, Journal of Statistics and Applications **3**(1-2), 121-133, 2008.

[15] Kotz, S., Lumelskii, Y. and Pensky, M. *The Stress-Strength Model and its Generalizations* (World Scientific Press, Singapore, 2003).

[16] Kundu, D. and Gupta, R. D. *Estimation of $P[Y < X]$ for Generalized Exponential Distribution*, Metrika **61**, 291-308, 2005.

[17] Kundu, D. and Gupta, R. D. *Estimation of $P[Y < X]$ for Weibull Distributions*, IEEE Transactions on Reliability **55**(2), 270-280, 2006.

[18] Lumelskii, Ya. P. *Unbiased sufficient estimators of probabilities in the case of multivariate normal distribution*, Vest. MGU, Mathematics **6** (in Russian), 14-17, 1968.

[19] Maiti, S. S. *Estimation of $P(X \leq Y)$ in the geometric case*, Journal of Indian Statistical Association **33**, 87-91, 1995.

[20] Owen, D. B., Craswell, K. J. and Handson, D. L. *Non-parametric upper confidence bounds for $P(Y < X)$ and confidence limits for $P(Y < X)$ when $X$ and $Y$ are normal*, Journal of American Statistical Association **59**, 906-924, 1964.

[21] Rezaei, S., Tahmasbi, R. and Mahmoodi, M. *Estimation of $P[Y < X]$ for Generalized Pareto Distribution*, Journal of Statistical Planning and Inference **140**, 480-494, 2010.

[22] Saraçoglu, B., Kaya, M. F. and Abd-Elfattah, A. M. *Comparison of Estimators for Stress-Strength Reliability in the Gompertz Case*, Hacettepe Journal of Mathematics and Statistics **38**(3), 339-349, 2009.

[23] Sathe, Y. S. and Dixit, U. J. *Estimation of $P(X \leq Y)$ in the negative binomial distribution*, Journal of Statistical Planning and Inference **93**, 83-92, 2001.

[24] Tong, H. *A Note on the Estimation of $P(Y < X)$ in the Exponential Case*, Technometrics **16**, 625. Errata: Technometrics **17**, 395, 1974.

[25] Tong, H. *On the Estimation of $P(Y < X)$ for Exponential Families*, IEEE Transactions in Reliability **26**, 54-56, 1977.

[26] Woodward, W. A. and Kelley, G. D. *Minimum variance unbiased estimation of $P(X < Y)$ in the normal case*, Technometrics **19**, 95-98, 1977.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 965 – 973

# A multiset based forecasting model for fuzzy time series

V.Vamitha * and S.Rajaram †

**Abstract**

Since the pioneering work of Song and Chissom ($1993a, b$) on fuzzy time series to model and forecast processes whose values are described by linguistic values, a number of techniques have been proposed by researchers for forecasting. In most of the realistic situation the duplicates of data are significant. This paper presents a new fuzzy time series method, which employs multiset theory. The historical data of daily average temperature in Taipei, Taiwan (central weather bureau 1996) are adopted to illustrate the forecasting process of the proposed method.

**Keywords:** Multiset, Forecasting, Fuzzy time series, Multiset relation, List.

Received 17/01/2013 : Accepted 01/03/2013 Doi : 10.15672/HJMS.2015113314

## 1. Introduction

Forecasting using fuzzy time series has been widely used in many activities. It arises in forecasting the weather, earthquakes, stock fluctuations and any phenomenon indexed by variables that change unpredictably in time. The classical time series methods can not deal with forecasting problems in which the values of time series are linguistic terms represented by fuzzy sets. Therefore in 1993, Song and Chissom proposed the concepts of fuzzy time series and outlined equations and approximate reasoning based on the fuzzy set theory introduced by Zadeh (1965). They also presented the models on time - variant and time invariant fuzzy time series [21], [22], [23] to deal with forecasting problems in which the historical data is represented by linguistic values. They asserted that all traditional forecasting methods fail when the historical enrollment data are composed of linguistic values.Sullivan and Woodall (1994) described a Markov model using linguistic values directly but with membership function of the fuzzy approach replaced by analogus probability function. Instead of complicated maximum - minimum composition operations Chen (1996) used a simple arithmetic operation for time series forecasting.

*Department of Mathematics, B.C.M.W.G.Polytehnic, Ettayapuram, Tamil Nadu, India. Email: vamithav@ymail.com

†Department of Mathematics, Sri S.R.N.M.College, Sattur - 626 203, Tamil Nadu, India. Email:rajaram_srnm@yahoo.co.in

Hwang, Chen and Lee (1998) presented a method of forecasting enrollments using fuzzy time series based on the concept that the variation of enrollment of this year is related to the trend of the enrollments of the past years. Huarng (2001) pointed out that the length of intervals will affect the forecasting accuracy rate and a proper choice of length of intervals can greatly improve the forecasting result. He presented the distribution based length approach and the average based length approach to deal with forecasting problems based on the intervals with different lengths. In recent years researchers focused on the research topic of using fuzzy time series for handling forecasting problems. A number of works have been reported on order one [2],[3],[7]-[9], [24],[25], high order [4],[5],[10],[12]-[14],single factor [2], [3], [8], [10], [24],[25], two factor [5], [7], [12]-[14] and multifactor [9] models. The formulation of fuzzy relation is one of the key issues affecting forecasting results. Li and Chen [15] proposed a forecasting model based on the hidden Markov model by enhancing Sullivan and woodall's [24] work to allow handling of two factor forecasting problems. In most of the realistic situation, we have to deal with collection of objects in which repetition of elements is significant. In this situation bags (multisets) are very useful structures. Multiset theory (Bag) was introduced by Cerf et al. [1] in 1971.Peterson [19] and Yager [26] made contributions to it. Further study was carried out by Jena et al. [11]. In this paper, we propose an efficient fuzzy time series forecasting model based on the concept of characterization of Bag to form the intervals of different length. Then based on the obtained intervals and the concept of list we present a method to deal with the temperature prediction. The remainder of this paper is organised as follows. In section 2, the basic concept of fuzzy time series is briefly introduced. In section 3, the concept of multiset is introduced and in section 4, the new forecasting model based on multiset (Bag) is proposed. Section 5 presents a performance evaluation of the model and a comparison of the results. The conclusions are discussed in section 6.

## 2. Fuzzy Time Series :

In the following, we briefly review some basic concepts of fuzzy time series and its forecasting frame work. The definition of fuzzy time series used in this paper was first proposed by Song and Chissom [20].

**2.1. Definition.** Let $Y(t)$ $(t = \dots, 0, 1, 2\dots)$, a subset of $R$, be the universe of discourse on which fuzzy sets $f_i(t)$ $(i = 1, 2\dots)$ are defined, and let $F(t)$ be a collection of $f_i(t)$. Then, $F(t)$ is called a fuzzy time series on $Y(t)$ $(t = \dots, 0, 1, 2\dots)$.

Song and Chissom employed a fuzzy relational equation to develop their forecasting model under the assumption that the observations at time t are dependent only upon the accumulated results of the observation at previous times, which is defined as follows.

**2.2. Definition.** If, for any $f_j(t) \in F(t)$ , where $j \in J$, there exist an $f_i(t-1) \in F(t-1)$, where $i \in I$, and a fuzzy relation $R_{ij}(t, t-1)$, such that $f_j(t) = f_i(t-1) \circ \mathrm{R}_{ij}(t, t-1)$, let $R(t, t-1) = \bigcup_{i,j} R_{ij}(t, t-1)$, where " $\cup$ " is the union operator and "$\circ$" is the composition. $R(t, t-1)$ is called the fuzzy relation between $F(t)$ and $F(t-1)$, which can be represented using the following fuzzy relational equation: $F(t) = F(t-1) \circ R(t, t-1)$.

**2.3. Definition.** If we suppose that $F(t)$ is caused by $F(t-1)$, $F(t-2)\dots$ or $F(t-m)(m > 0)$, then the first - order model of $F(t)$ can be expressed as $F(t) = F(t-1) \circ R(t, t-1)$(or) $F(t) = (F(t-1) \cup F(t-2) \cup \dots \cup F(t-m)) \circ R_\circ(t, t-m)$

where " $\cup$ " is the union operator and "$o$" is the composition. $R(t, t-1)$ is called the fuzzy relation between $F(t)$ and $F(t-1)$, and $R_\circ(t, t-k)$ is the fuzzy relation that joins $F(t)$ with $F(t-1), F(t-2)\dots, or F(t-k)$, where the subscript " $\circ$ " denotes the relationship "*or*". In the literature, the fuzzy relation $R_{ij}(t, t-1)$ is usually represented by a fuzzy logical relationship rule.

## 3. Multiset(Bag) :

In the following, we briefly review some basic concepts on Multiset (Bag) [11].

**3.1. Definition.** A collection of elements which may contain duplicates is called a Multiset (bag). Formally if $X$ is a set of elements, a bag drawn from the set $X$ is represented by a function count $B$ or $C_B$ defined as $C_B : X \to N$, where $N$ represents the set of non -negative integers. For each $x \in X$, $C_B(x)$ is a **characteristic value** of $x$ in $B$ and indicates the number of occurrences of the elements $x$ in $B$.

**3.2. Definition.** A **list** $L$ drawn from a set $X$ is represented by a position function $P_L$ defined as $P_L : X \to P(N)$, where $P(N)$, denotes the power set of non-negative integers subject to the following conditions:
1. $P_L(x) = \emptyset$ iff $x \notin L$
2. $P_L(x) \cap P_L(y) = \emptyset$ for all $x \neq y \in L$
3. $P_{[\,]}(x) = \emptyset$ for each $x \in X$, where$[\,]$ is an empty set.

**3.3. Definition.** For any finite list $L$ drawn from $X$, we define $|P_L(x)|$ is the number of occurrences of the element $x$ in $L$.

**Note :** The notion of list can be considered as a generalization of notion of bag in the sense that the order of occurrence of elements is unimportant in the case of bag whereas incase of a list, it is significant.

**3.4. Notation.** Let $M$ be a multiset from which $x$ appearing $n$ times in $M$ and denoted by $x \in {}^n M$. The counts of the members of the domain and codomain vary in relation to the counts of the $x$ co-ordinate and $y$ co-ordinate in $(m|x, n|y)|k$, where $(m|x, n|y)|k$ denotes that $x$ is repeated $m$-times, $y$ is repeated $n$-times and the pair $(x, y)$ is repeated $k$ times. Let $C_1(x, y)$ and $C_2(x, y)$ be the count of the first and second co-ordinate in the ordered pair $(x, y)$ respectively.

**3.5. Definition.** Let $M_1$ and $M_2$ be two multisets drawn from a set $X$. Then the **Cartesian product** of $M_1$ and $M_2$ is defined as
$M_1 \times M_2 = \{(m|x, n|y)|mn : x \in {}^m M_1, y \in {}^n M_2\}$.

**3.6. Definition.** A sub multiset $R$ of $M \times M$ is said to be a **multiset relation** on $M$ if every member $(m|x, n|y)$ of $R$ has a count $C_1(x, y)$. $C_2(x, y)$. We denote $m|x$ related to $n|y$ by $m|xRn|y$ .

**3.7. Example.** For example, let $M_1 = \{1, 2, 2\}$ and $M_2 = \{3, 3, 3, 3\}$. $M_1 \times M_2 = \{(1|1, 4|3)|4, (2|2, 4|3)|8\}$. Consider $(1|1, 4|3)$. $C_1(x, y) = 1$ and $C_2(x, y) = 4$. Then $(1|1, 4|3)$ has a count 4.

## 4. The Multiset Based Forecasting Model :

In this section we present a new method for forecasting daily average temperature based on the multiset concepts. The proposed method is now presented as follows :
**Step 1 :** Let $U$ be the universe of discourse, defined by $U = [D_{min} - D_1, D_{max} + D_2]$ where $D_1$ and $D_2$ are two proper positive numbers. Partition the universe of discourse into seven intervals $u_1, u_2, \ldots, u_7$ with equal length [17].
**Step 2 :** Collect the historical data into a crisp set $X$. Construct a multiset $B$ with all entries in ascending order. Find the characteristic value of each $x \in B$.
**Step 3 :** Choose $u_i$, $i = 1$ to 7 and $x_j \in X$ which lies in $u_i$. If $C_B(x_j) = 1$then there

is no change in the interval where $x_j$ lies. If $C_B(x_j) > 1$ and $k = \sum_j C_B(x_j)$ then re divide the interval where $x_j$ lies into $k$ intervals with equal length. Rename the obtained intervals as $v_1, v_2, \ldots, v_n$ where $v_1, v_2, \ldots, v_n$ are of different length.

**Step 4 :** Construct the fuzzy sets $A_i$ in accordance with the intervals in step 3. Fuzzify the historical data. For $n$ fuzzy sets, $A_1, A_2, \ldots, A_n$ can be defined on U [20] as follows:

$A_i = \sum_{j=1}^{n} \frac{\mu_{ij}}{v_j}$ where $\mu_{ij}$ is the membership degree of $A_i$ belonging to $v_j$ and is defined by

$$\mu_{ij} = \begin{cases} 1, & \text{if } j = i \\ 0.5, & \text{if } j = i-1 \text{ or } i+1 \\ 0, & \text{if otherwise} \end{cases}$$

Then, for a given historical datum $Y_t$, its membership degree belonging to interval $v_i$ is determined by the following heuristic rules.

**Rule 1 :** if $Y_t$ is located at $v_1$, the membership degrees are 1 for $v_1$, 0.5 for $v_2$ and 0 otherwise.

**Rule 2 :** if $Y_t$ belongs to $v_i$, $1 < i < n$, then the degrees are 1, 0.5 and 0.5 for $v_i$, $v_{i-1}$ and $v_{i+1}$, respectively and 0 otherwise.

**Rule 3 :** if $Y_t$ is located at $v_n$, the membership degrees are 1 for $v_n$, 0.5 for $v_{n-1}$ and 0 otherwise. Then, $Y_t$ is fuzzified as $A_j$, where the membership degree in interval $j$ is maximal

**Step 5 :** Collect the fuzzy sets into a set $Y$. Form a list with all entries as in the fuzzy time series. Find the position function $P_L$ defined as $P_L : Y \to P(N)$ where $P(N)$ denotes the power set of non - negative integers.

**Step 6 :** Construct the multiset relation $R$ which is a subset of $L \times L$.

**Step 7 :** Forecast the values by the following principles.

*Principle 1:* If $|P_L(A_i)| = 1$, then for $P_L(A_j) = P_L(A_i) + 1$ the forecasted value of $A_j$ is the midvale of $v_j$.

*Principle 2:* Consider the weighted factor [Yu (2005)] and the between the actual data and the mid values of the intervals. When $|P_L(Ai)| > 1$ and the multiset relation is of the form $(A_i, A_j)$, $(A_i, A_k)$, $(A_i, A_l)$,. . . , $(A_i, A_n)$ assign numbers as follows. $j = a_1.k = a_2, l = a_3, \ldots, n = a_n$. Then the corresponding weights are defined as $w_j = \frac{a_1}{\sum_{i=1}^{n} a_i}$, $w_k = \frac{a_2}{\sum_{i=1}^{n} a_i}$, $w_l = \frac{a_3}{\sum_{i=1}^{n} a_i}$ ,. . . , $w_n = \frac{a_n}{\sum_{i=1}^{n} a_i}$ where $w_j + w_k + w_l + \cdots + w_n = 1$. Then the forecasted value of $A_j$ is equal to $G_1(t) + G_2(t))$, where $G_1(t) = [m_j, m_k, \ldots, m_n]$ $[w_j, w_k, \ldots w_n]^T$ and $\mathrm{G}_2(t) =$ [Auctual value of $\mathrm{A}_j-$Average of midvalues of $m_j, m_k, \ldots, m_n$] Similarly the forecasted values for $A_k, A_l, \ldots, A_n$.

## 5. Model Verification :

The experiment consisted of forecasting temperature in Taipei, Taiwan, to verify the forecasting performance of the proposed model. We compare with some existing models. In the following, we apply the proposed method to forecast the daily average temperature based on multiset context. Based on the daily average temperature from June 1, 1996 to June 30, 1996 shown in Table 5, the universe of discourse of the daily average temperature is

$U$ = [26, 31] and the seven intervals with equal length are as follows.
$u_1$ = [26, 26.71) ; $u_2$ = [26.71, 27.42) ; $u_3$ = [27.42, 28.13)
$u_4$ = [28.13, 28.84); $u_5$ = [28.84, 29.55); $u_6$ = [29.55, 30.26)
$u_7$ = [30.26, 31].
We have $X$ = {26.1, 27.1, 27.4, 27.5, 27.6, 27.7, 27.8, 28.4, 28.5, 28.7, 28.8, 29, 29.3, 29.4, 29.5, 29.7, 30, 30.2, 30.3, 30.5, 30.8, 30.9}
and the bag
$B$ = {26.1, 27.1, 27.4, 27.5, 27.6, 27.7, 27.8, 27.8, 28.4, 28.5, 28.7, 28.7, 28.8, 28.8, 29, 29, 29, 29.3, 29.4, 29.4,29.5, 29.5, 29.7, 30, 30.2, 30.2, 30.3, 30.5, 30.8, 30.9}
The characteristic values of $x_j$ in $X$ are given in Table 1. By step 3, the intervals of different length and the midpoints of the intervals are given in Table 2. Now the fuzzy sets are defined and the time series is fuzzified by step 4 are given in Table 3.
Based on the fuzzy time series presented in Table 3 and step 5
we have $Y = \{A_1, A_2, A_3, A_4, A_6, A_7, A_8, A_{10}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}\}$ and the list as
$L = [A_1, A_3, A_{10}, A_{18}, A_{17}, A_{15}, A_{16}, A_{14}, A_8, A_{14}, A_{13}, A_7, A_8, A_3, A_{15}, A_8, A_{10}, \mathrm{A}_{18}, A_{17}, A_{18}, A_{18}, A_8, A_4, A_2, A_3, A_2, A_6, A_4, A_{10}, A_{17}]$
The position functions $P_L$ are as follows:
$P_L(A_1) = \{1\}$; $P_L(A_2) = \{24, 26\}$; $P_L(A_3) = \{2, 14, 25\}$;
$P_L(A_4) = \{23, 28\}$; $P_L(A_6) = \{27\}$; $P_L(A_7) = \{12\}$; $P_L(A_8) = \{9, 13, 16, 22\}$; $P_L(A_{10}) = \{3, 17, 29\}$; $P_L(A_{13}) = \{11\}$; $P_L(A_{14}) = \{8, 10\}$
$P_L(A_{15}) = \{6, 15\}$; $P_L(A16) = \{7\}$; $P_L(A_{17}) = 5, 19, 30$;
$P_L(A_{18}) = \{4, 18, 20, 21\}$
The multiset relation of order 1 obtained from step 6 is

$$R = \{(\mathrm{A}_1, A_3), (A_2, A_6), (A_2, A_3), (A_3, A_{10}), (A_3, A_2), (A_3, A_{15}), (A_4, A_2), (\mathrm{A}_4, A_{10}), (A_6, A_4), (A_7, A_8), (A_8, A_{14}), (A_8, A_3), (A_8, A_{10}), (A_8, A_4), (\mathrm{A}_{10}, A_{18}), (A_{10}, A_{18}), (A_{10}, A_{17}), (A_{13}, A_7), (A_{14}, A_8), (A_{14}, A_{13}), (\mathrm{A}_{15}, A_{16}), (A_{15}, A_8), (A_{16}, A_{14}), (A_{17}, A_{15}), (A_{17}, A_{18}), (A_{18}, A_{17}), (\mathrm{A}_{18}, A_{17}), (A_{18}, A_{18}), (A_{18}, A_8)\}.$$

By the principles in step 7, the forecasted values are given in Table 3.

Evaluate the performance of the proposed fuzzy time series model with the forecasting results by predicting the temperature and comparing with the models of Lee et al's (2006) and Li et al (2010). The average forecasting error rate (AFER) and Mean square error (M.S.E) are used in this section to compare the forecasted accuracy rate of the daily average temperature of the proposed method with the existing models where the historical data is shown in Table 5.

Average forecasting error rate (AFER) $= \frac{\frac{|A_i - F_i|}{A_i}}{n}$*100

Mean square error (M.S.E) $= \frac{\sum_{i=1}^{n}[A_i - F_i]^2}{n}$ where $A_i$ denotes the actual temperature of day $i$ , $F_i$ denotes the forecasting temperature of day $i$ and $n$ is the number of errors respectively.

**Table 3 :** Forecasting Results of the Proposed Model for June 1 to June 30,1996.

| Day | Actual Temp. | Fuzzified Temp. | Forecasted Temp. | $(\mathrm{A}_i - F_i)^2$ | $\frac{\lvert A_i-F_i \rvert}{A_i}$*100 |
|---|---|---|---|---|---|
| 1 | 26.1 | $A_1$ | - | - | - |
| 2 | 27.6 | $A_3$ | 27.5975 | 0.000006 | 0.00905 |
| 3 | 29.0 | $A_{10}$ | 29.6151 | 0.378469 | 0.12137 |
| 4 | 30.5 | $A_{18}$ | 30.5068 | 0.000047 | 0.02258 |
| 5 | 30.0 | $A_{17}$ | 30.1797 | 0.032300 | 0.59908 |
| 6 | 29.5 | $A_{15}$ | 29.5514 | 0.002647 | 0.17440 |
| 7 | 29.7 | $A_{16}$ | 29.8627 | 0.026484 | 0.54795 |
| 8 | 29.4 | $A_{14}$ | 29.3978 | 0.000005 | 0.00726 |
| 9 | 28.8 | $A_8$ | 28.8649 | 0.004212 | 0.22536 |
| 10 | 29.4 | $A_{14}$ | 29.8212 | 0.177451 | 1.43282 |
| 11 | 29.3 | $A_{13}$ | 29.3649 | 0.004212 | 0.22151 |
| 12 | 28.5 | $A_7$ | 28.5737 | 0.005439 | 0.25877 |
| 13 | 28.7 | $A_8$ | 28.7512 | 0.002627 | 0.17857 |
| 14 | 27.5 | $A_3$ | 27.9212 | 0.177451 | 1.53182 |
| 15 | 29.5 | $A_{15}$ | 30.1151 | 0.378469 | 2.08542 |
| 16 | 28.8 | $A_8$ | 28.9627 | 0.026485 | 0.56507 |
| 17 | 29.0 | $A_{10}$ | 29.4212 | 0.177451 | 1.45259 |
| 18 | 30.3 | $A_{18}$ | 30.3068 | 0.000047 | 0.02273 |
| 19 | 30.2 | $A_{17}$ | 30.3797 | 0.032301 | 0.59511 |
| 20 | 30.9 | $A_{18}$ | 30.9514 | 0.002647 | 0.16650 |
| 21 | 30.8 | $A_{18}$ | 30.9797 | 0.032301 | 0.58352 |
| 22 | 28.7 | $A_8$ | 28.8797 | 0.032301 | 0.62621 |
| 23 | 27.8 | $A_4$ | 28.2212 | 0.177451 | 1.51529 |
| 24 | 27.4 | $A_2$ | 28.0423 | 0.412585 | 2.34426 |
| 25 | 27.7 | $A_3$ | 27.8331 | 0.017729 | 0.48069 |
| 26 | 27.1 | $A_2$ | 27.7151 | 0.378469 | 2.27010 |
| 27 | 28.4 | $A_6$ | 28.5331 | 0.017729 | 0.46884 |
| 28 | 27.8 | $A_4$ | 27.9525 | 0.023256 | 0.54856 |
| 29 | 29.0 | $A_{10}$ | 29.6423 | 0.412585 | 2.21492 |
| 30 | 30.2 | $A_{17}$ | 30.2068 | 0.000047 | 0.02280 |
| | | | | $M.S.E$ = 0.10115 | $AFER$ = 0.80323 |

In Table 4 the forecasted daily average temperatures of the proposed method is compared with the existing methods.

**Table 4 :**(AFER)(In percentage)

| Month | Lee et.al(2006) | Li.et.al(2010) | Proposed Method |
|---|---|---|---|
| June | 1.44 | 2.2669 | 0.8052 |
| July | 1.59 | 2.5855 | 0.8196 |
| August | 1.26 | 2.5352 | 1.0243 |
| September | 1.89 | 2.9656 | 0.8163 |

**Table 1:** Characteristic value of $x_j \in X$

| $x_j$ | $C_B(x_j)$ | $x_j$ | $C_B(x_j)$ |
|---|---|---|---|
| 26.1 | 1 | 29.0 | 3 |
| 27.1 | 1 | 29.3 | 1 |
| 27.4 | 1 | 29.4 | 2 |
| 27.5 | 1 | 29.5 | 2 |
| 27.6 | 1 | 29.7 | 1 |
| 27.7 | 1 | 30.0 | 1 |

| $x_j$ | $C_B(x_j)$ | $x_j$ | $C_B(x_j)$ |
|---|---|---|---|
| 27.8 | 2 | 30.2 | 2 |
| 28.4 | 1 | 30.3 | 1 |
| 28.5 | 1 | 30.5 | 1 |
| 28.7 | 2 | 30.8 | 1 |
| 28.8 | 2 | 30.9 | 1 |

**Table 2 :**The intervals of different length and the midpoints of the intervals

| Intervals | Midpoints | Intervals | Midpoints |
|---|---|---|---|
| $v_1$=[26,26.71) | $m_1$=26.355 | $v_{10}$=[28.94143,29.04286) | $m_{10}$=28.992145 |
| $v_2$=[27.71,27.42) | $m_2$=27.065 | $v_{11}$=[29.04286,29.14429) | $m_{11}$=29.093575 |
| $v_3$=[27.42,27.775) | $m_3$=27.5975 | $v_{12}$=[29.14429,29.24572) | $m_{12}$=29.195005 |
| $v_4$=[27.775,28.13) | $m_4$=27.9525 | $v_{13}$=[29.24572,29.34715) | $m_{13}$=29.296435 |
| $v_5$=[28.13,28.3075) | $m_5$=28.21875 | $v_{14}$=[29.34715,29.44858) | $m_{14}$=29.397865 |
| $v_6$=[28.3075,28.485) | $m_6$=28.39625 | $v_{15}$=[29.44858,29.55) | $m_{15}$=29.49929 |
| $v_7$=[28.485,28.6625) | $m_7$=28.57375 | $v_{16}$=[29.55,29.905) | $m_{16}$=29.7275 |
| $v_8$=[28.6625,28.84) | $m_8$=28.75125 | $v_{17} = [29.905, 30.26)$ | $m_{17}$=30.0825 |
| $v_9$=[28.84,28.94143) | $m_9$=28.890715 | $v_{18}$=[30.26,31) | $m_{18}$=30.63 |

## 6. Conclusion

In this paper, we have presented a new method for forecasting the daily average temperature of the Taipei, Taiwan in which duplicates of data are significant, given in Table 5, based on the characterization of bag and multiset relations. First we compute the intervals of different length using characterization of bag. Then the daily average temperature value using list and multiset relation are forecasted. From the experimental results the proposed method provides the smallest AFER and MSE and improves on other methods using fuzzy times series forecasting methods.

**Table 5 :** Historical data of the daily average temperature from June 1, 1996 to September 30, 1996 in Taipei, Taiwan($unit :^0 C$)

(Central weather bureau, 1996)

| | Month | | | |
|---|---|---|---|---|
| Day | June | July | August | September |
| 1 | 26.1 | 29.9 | 27.1 | 27.5 |
| 2 | 27.6 | 28.4 | 28.9 | 26.8 |
| 3 | 29.0 | 29.2 | 28.9 | 26.4 |
| 4 | 30.5 | 29.4 | 29.3 | 27.5 |
| 5 | 30.0 | 29.9 | 28.8 | 26.6 |
| 6 | 29.5 | 29.6 | 28.7 | 28.2 |
| 7 | 29.7 | 30.1 | 29.0 | 29.2 |
| 8 | 29.4 | 29.3 | 28.2 | 29.0 |
| 9 | 28.8 | 28.1 | 27.0 | 30.3 |
| 10 | 29.4 | 28.9 | 28.3 | 29.9 |
| 11 | 29.3 | 28.4 | 28.9 | 29.9 |
| 12 | 28.5 | 29.6 | 28.1 | 30.5 |
| 13 | 28.7 | 27.8 | 29.9 | 30.2 |
| 14 | 27.5 | 29.1 | 27.6 | 30.3 |
| 15 | 29.5 | 27.7 | 26.8 | 29.5 |
| 16 | 28.8 | 28.1 | 27.6 | 28.3 |
| 17 | 29.0 | 28.7 | 27.9 | 28.6 |
| 18 | 30.3 | 29.9 | 29.0 | 28.1 |
| 19 | 30.2 | 30.8 | 29.2 | 28.4 |
| 20 | 30.9 | 31.6 | 29.8 | 28.3 |
| 21 | 30.8 | 31.4 | 29.6 | 26.4 |
| 22 | 28.7 | 31.3 | 29.3 | 25.7 |
| 23 | 27.8 | 31.3 | 28.0 | 25.0 |
| 24 | 27.4 | 31.3 | 28.3 | 27.0 |
| 25 | 27.7 | 28.9 | 28.6 | 25.8 |
| 26 | 27.1 | 28.0 | 28.7 | 26.4 |
| 27 | 28.4 | 28.6 | 29.0 | 25.6 |
| 28 | 27.8 | 28.0 | 27.7 | 24.2 |
| 29 | 29.0 | 29.3 | 26.2 | 23.3 |
| 30 | 30.2 | 27.9 | 26.0 | 23.5 |
| 31 | - | 26.9 | 27.7 | - |

## References


[1] Cerf, V., Fernandez, E., Gostelow, K., Volausky, S. *Formal control and low properties of a model of computation*, Report ENG 7178, Computer Science Department ,University of California, Los Angeles,CA, p.81, December, 1971.

[2] Chen, S.M, and Hau.C.C. *A new method to forecast enrollments using fuzzy time series*, Int.J. Appl. Sci. Eng., vol.2, no.3, pp.234-244, 2004.

[3] Chen,S.M., *Forecasting enrollments based on Fuzzy time series*, "Fuzzy Sets Syst.,vol.81,no.3,pp.311-319,Aug.1996.

[4] Chen,S.M., *Forecasting enrollments based on high-order fuzzy time series*, Cybern.Syst.,Int.J.,vol.33,no.1,pp.1-16,Jan.2002.

[5] Chen,S.M., and Hwang,J,R. *Temperature prediction using fuzzy time series*,"IEEE Transactions System, Man, Cybern.B.Cybern.,vol.30,no.2,pp.263-275, Apr.2000.

[6] Girish, K.P., Sunil Jacob john. *Relations and functions in multiset context*, "Information sciences 179,pp.758-768, 2009.

[7] Hsu,Y.Y., Tse,S.M., and Wu,B. *A new approach of bivariate fuzzy time series analysis to the forecasting of a stock index*, Int.J. Uncertain Fuzziness Knowl. Based Syst., vol.11,no.6, pp.671-690, 2003.

[8] Huarng, K. *Heuristic models of fuzzy time series for forecasting*, Fuzzy Sets Syst.,vol.123,no.3,pp.369-386,Nov.2001

[9] Huarng,K.H., Yu,T.H.K., and Hsu,Y.W. *A multi variate heuristic model for fuzzy times series forecasting*, IEEE Trans. Sys., Man, Cybern. B. Cybern., vol.37, no.4, pp.836-846, Aug.2007.

[10] Hwang, J.R., Chen,S.M., and Lee,C.H. *Handling forecasting problems using fuzzy time series*, Fuzzy Sets Syst.,vol.100,no.1-3,pp.217-228,Nov.1998.

[11] Jena,S.P., Ghosh,S.K. Tripathy,B.K, *On the theory of bags and lists*, Information Sciences 132, pp 241-254, 2001.

[12] Lee,L.W., Wang,L.H., Chen,S.M., and Leu,Y.H. *Handling forecasting problems based on two-factors high-order fuzzy time series*, IEEE Trans. Fuzzy Syst., vol.14, no.3, pp.468-477, Jun.2006

[13] Lee,L.W., Wang,L.H., and Chen,S.M. *Temperature prediction and TAIFEX forcasting based on fuzzy logical relationships and genetic algorithms*, Expert Syst.Appl.,vol.33,no.3,pp.539-550,Oct.2007.

[14] Li,S.T.,and Cheng,Y.C. *Deterministic fuzzy time series model for forecasting enrollments*, Comput.Math.Appl.,vol.53,no.12,pp.1904-1920, Jun.2007.

[15] Li - Wei Lee, Li - Hui Wang, Shyi - Ming Chen, and Yung - Ho Leu, *Handling forecasting problems Based on Two- factors High - order Fuzzy Time series*, IEEE Transactions on fuzzy systems, vol. 14, No. 3, June 2006.

[16] Li,Y, *Hidden markov models with states depending on observations*, Pattern Recognit.Lett.,vol.26, no.7, pp.977-984, May2005.

[17] Miller,G.A. *The magical number seven, plus or minus two some limits on our capacity of processing information*, The Psychological Review 63, 81 - 97, 1956.

[18] Nai - Yi Wang , Shyi - Ming chen. *Temperature prediction and TAIFEX fore casting based on automatic clustering techniques and two - factors high - order Fuzzy time series*, Expert systems with Applications, 36, pp. 2143 - 2154, 2009.

[19] Peterson,J. *Computation sequence sets*, Journal of Computer System Science 13(1) pp 1-24., 1976.

[20] Sheng - Tun Li and Yi - Chung cheng. *A stochastic HMM Based Forcasting model for fuzzy time series*, IEEE transactions on systems, man, and cybernetics - PART B., cybernetics, vol 40, No. 5, 2010.

[21] Song, Q and Chissom, B.S. *Forecasting enrollments with fuzzy time series - part I*, Fuzzy sets syst., Vol.54, no.1, pp1-9, Feb 1993.

[22] Song,Q and Chissom B,S. *Fuzzy time series and its models*, Fuzzy sets syst., vol.54, no.3, pp.269-277, Mar. 1993.

[23] Song, Q and Chissom B,S. *Forecasting enrollments with fuzzy time series - part II*, Fuzzy sets Syst., vol.62, no.1 pp.1-8, Feb.1994.

[24] Sullivan,J and Woodall,W.H. *A comparison of fuzzy forecasting and Markov modeling*, Fuzzy Sets Syst., vol,64, no.3, pp.279-293, Jun.1994.

[25] Tsaur,R.C.,Yang,J.C.,and Wang,H.-F. *Fuzzy relation analysis in fuzzy time series model*, Compute.Math.Appl.,vol.49,no.4, pp.539-548,Feb.2005.

[26] Yager,R.R. *On the theory of bags*, International Journal General System 13, pp 23-37, 1986.

[27] Yu,H.K. *A refined fuzzy time - series model for forecasting*, Phys. A, vol.346, no.3/4, pp 657-681, Feb.2005.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 975 – 990

# Intersection local time of subfractional Ornstein-Uhlenbeck processes

Xiuwei Yin [*] , Guangjun Shen [†] and Dongjin Zhu [‡]

**Abstract**

In this paper, we consider Ornstein-Uhlenbeck process

$$\mathrm{d}X_t^H = -X_t^H \mathrm{d}t + v\mathrm{d}S_t^H, \quad X_0^H = x,$$

driven by a subfractional Brownian motion $S^H$. We prove that the subfractional Ornstein-Uhlenbeck process $X^H$ is local nondeterministic and give some properties of this process. As an application, assume $d \geq 2$, we prove that the intersection local time of two independent, $d-$dimensional subfractional Ornstein-Uhlenbeck process, $X^H$ and $\widetilde{X}^H$, exists in $L^2$ if and only if $Hd < 2$.

*2000 AMS Classification:* 60G15, 60G18, 60H07.

**Keywords:** subfractional Brownian motion, Ornstein-Uhlenbeck process, intersection local time.

Received 31/05/2014 : Accepted 03/08/2014 Doi : 10.15672/HJMS.2014257476

## 1. Introduction

The classical Ornstein-Uhlenbeck process (see Revuz and Yor[23]) has a remarkable history in physics. It was introduced to model the velocity of the particle diffusion process and later it has been heavily used in finance, and thus in econophysics. It can be constructed as the unique strong solution of Itô stochastic differential equation

(1.1) $$\mathrm{d}X_t = -X_t \mathrm{d}t + v\mathrm{d}B_t, \quad X_0 = x,$$

where $B$ is a standard Brownian motion starting at 0.

On the other hand, extensions of the classical Ornstein-Uhlenbeck process have been suggested mainly on demand of applications. The fractional Ornstein-Uhlenbeck process

[*]Department of Mathematics, Anhui Normal University, Wuhu 241000, China.
[†]Department of Mathematics, Anhui Normal University, Wuhu 241000, China. Corresponding author.
Email:gjshen@163.com
[‡]Department of Mathematics, Anhui Normal University, Wuhu 241000, China.
Guangjun Shen is partially supported by the National Natural Science Foundation of China (11271020), the Natural Science Foundation of Anhui Province (1208085MA11).

was an extension of the classical Ornstein-Uhlenbeck process, where fractional Brownian motion $B^H$ was used as integrator

$$\mathrm{d}X_t^H = -X_t^H \mathrm{d}t + v\mathrm{d}B_t^H, \quad X_0 = x. \tag{1.2}$$

The equation (1.2) has a unique solution $X_t^H = \{X_t^H, 0 \le t \le T\}$, which can be expressed as

$$X_t^H = e^{-t}(x + v\int_0^t e^s \mathrm{d}B_s^H), \tag{1.3}$$

and the solution was called the fractional Ornstein-Uhlenbeck process. Recall that fractional Brownian motion $B^H$ with Hurst index $H \in (0,1)$ is a central Gaussian process with $B_0^H = 0$ and the covariance function

$$\mathrm{E}[B_t^H B_s^H] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \tag{1.4}$$

for all $t, s \geqslant 0$. This process was first introduced by Kolmogorov and studied by Mandelbrot and Van Ness [19]. Clearly, when $H = \frac{1}{2}$ the fractional Ornstein-Uhlenbeck process is the classical Ornstein-Uhlenbeck process $X$ with parameter $v$ starting at $x \in \mathbb{R}$. A class of superpositions of Ornstein-Uhlenbeck type processes is constructed in terms of integrals with respect to independently scattered random measures in Barndorff-Nielsen [3]. Barndorff-Nielsen and Shephard [4] construct continuous time stochastic volatility models for financial assets where the volatility processes are superpositions of positive Ornstein-Uhlenbeck processes, and they study these models in relation to financial data and theory. Recently, Habtemicael and SenGupta [12] shown that the Gamma-Ornstein-Uhlenbeck process is a possible candidate for earthquake data modeling. SenGupta [25] uses Ornstein-Uhlenbeck process in forming a partial integro differential equations in finance. More works for the fractional Ornstein-Uhlenbeck process can be found in Cheridito *et al.* [9], Hu and Nualart [15], Es-Sebaiy [11], Yan *et al.* [32, 33].

The intersection properties of Brownian motion paths have been investigated since the forties (see lévy [17]), and since then a large number of results on intersection local times of Brownian motion have been accumulated (see Albeverio *et al.* [1] and the references therein). The intersection local time of independent fractional Brownian motions has been studied by Chen and Yan [8], Jiang and Wang [16], Nualart and Ortiz-Latorre [22], Rosen [24], Wu-Xiao [30] and the references therein.

Motivated by all these results, in this paper, we will study the Ornstein-Uhlenbeck process

$$\mathrm{d}X_t^H = -X_t^H \mathrm{d}t + v\mathrm{d}S_t^H, \quad X_0^H = x,$$

driven by a subfractional Brownian motion $S^H$(see section 2 for a precise definition). The solution

$$X_t^H = e^{-t}(x + v\int_0^t e^s \mathrm{d}S_s^H), \tag{1.5}$$

is called the subfractional Ornstein-Uhlenbeck process (see Mendy [20]).

The rest of this paper is organized as follows. In section 2 we briefly recall the subfractional Brownian motion and the related Wiener-Itô integral. In section 3 we show that the subfractional Ornstein-Uhlenbeck process $X^H$ is local nondeterministic and establish some estimates for the increments of the process, that is, there exist two constant $c_{H,T}, C_{H,T} > 0$ depending on $H, T$ only which may not be the same in each occurrence such that the estimates

$$c_{H,T} v^2 (t-s)^{2H} \le E(X_t^H - X_s^H)^2 \le C_{H,T} v^2 (t-s)^{2H},$$

and

$$c_{H,T} v^2 G(t,s) \le E(X_t^H X_s^H) \le C_{H,T} v^2 G(t,s),$$

hold for all $0 < s < t < T$, where $G(t,s) = t^{2H} + s^{2H} - \frac{1}{2}[(t+s)^{2H} + (t-s)^{2H}]$. In section 4 we consider the intersection local time of two independent subfractional Ornstein-Uhlenbeck process $X^H = \{X_t^H, 0 \le t \le T\}$ and $\widetilde{X}^H = \{\widetilde{X}_t^H, 0 \le t \le T\}$ on $\mathbb{R}^d, d \ge 2$ with the same indices $H \in (0,1)$. The intersection local time is formally defined as

$$(1.6) \qquad \ell_T = \int_0^T \int_0^T \delta(X_t^H - \widetilde{X}_s^H) \mathrm{d}s\mathrm{d}t,$$

where $\delta$ denotes the Dirac delta function. It is a measure of the amount of time that the trajectories of the two processes, $X^H$ and $\widetilde{X}^H$, intersect on the time interval $[0,T]$. In order to give a rigorous meaning to $\ell_T$ we approximate the Dirac function by the heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}}, x \in \mathbb{R}^d.$$

Then, we can consider the following family of random variables indexed by $\varepsilon > 0$

$$(1.7) \qquad \ell_{\varepsilon,T} = \int_0^T \int_0^T p_\varepsilon(X_t^H - \widetilde{X}_s^H) \mathrm{d}s\mathrm{d}t.$$

We get the convergence of $\ell_{\varepsilon,T}$ as $\varepsilon$ tends to zero in the $L^2(\Omega)$.

## 2. Preliminaries

In this section, we briefly recall the definition and properties of the Wiener-Itô integer with respect to the subfractional Brownian motion. As an extension of Brownian motion, Bojdecki *et al.* [6] introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of the fractional Brownian motion, which is called the subfractional Brownian motion. This process arised from occupation time fluctuations of branching particle systems with Poisson initial condition, and it also appeared independently in a different context in Dzhaparidze and Van Zanten[10]. The so-called subfractional Brownian motion (subfBm in short) with index $H \in (0,1)$ is a mean zero Gaussian process $S^H = \{S_t^H, t \ge 0\}$ with $S_0^H = 0$ and

$$(2.1) \qquad E\left[S_t^H S_s^H\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]$$

for all $s, t \ge 0$. For $H = 1/2$, $S^H$ coincides with the standard Brownian motion $B$. $S^H$ is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with $S^H$. The subfBm has properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence, Hölder paths). However, in comparison with fractional Brownian motion, the subfBm has non-stationary increments and the increments over non-overlapping intervals are more weakly correlated and their covariance decays polynomially as a higher rate in comparison with fractional Brownian motion (for this reason in Bojdecki *et al.* [6] it is called subfBm). The properties mentioned above make the subfBm a possible candidate for models which involve long-range dependence, self-similarity and non-stationary increment. More studies on the subfBm can be found in Bardina and Bascompte [2], Bojdecki *et al.* [7], Liu and Yan [18], Shen *et al.* [26, 27, 28], Yan and Shen [31] and the references therein.

Consider the integral representation of the subfBm $S_t^H$ of the form

$$(2.2) \qquad S_t^H = \int_0^t K_H(t,u) \mathrm{d}B_u, \quad 0 \le t \le T,$$

where $K_H(t,u)$ is the kernel

$$(2.3) \quad K_H(t,s) = \frac{c_H\sqrt{\pi}}{2^H\Gamma(H+\frac{1}{2})} s^{3/2-H}\left(\frac{(t^2-s^2)^{H-\frac{1}{2}}}{t} + \int_s^t \frac{(u^2-s^2)^{H-\frac{1}{2}}}{u^2}\mathrm{d}u\right)1_{(0,t)}(s).$$

In particular, when $\frac{1}{2} < H < 1$, the kernel $K_H(t,s)$ can be written in a less complicated form:

$$(2.4) \qquad K_H(t,s) = \frac{c_H\sqrt{\pi}}{2^{H-1}\Gamma(H-\frac{1}{2})} s^{3/2-H}\int_s^t (u^2-s^2)^{H-\frac{3}{2}}\mathrm{d}u 1_{(0,t)}(s),$$

where $c_H^2 = \frac{\Gamma(1+2H)sin\pi H}{\pi}$. Using the idea in Hu [14], the kernel $K_H(t,s)$ defines an operator $\Gamma_{H,T}$ in $L^2([0,T])$ given by

$$\Gamma_{H,T}h(t) = \int_0^t K_H(t,u)h(u)\mathrm{d}u, \qquad h \in L^2([0,T]),$$

and the function $\Gamma_{H,T}h(t)$ is continuous and vanishes at zero. The transpose $\Gamma^*_{H,t}$ of $\Gamma_{H,T}$ restricted to the interval $[0,t](0 \le t \le T)$ is

$$\Gamma^*_{H,t}g(s) = C_H s^{3/2-H}[(t^2-s^2)^{H-\frac{1}{2}}t^{-1}g(t) - \int_s^t (u^2-s^2)^{H-\frac{1}{2}}u^{-1}g'(u)\mathrm{d}u$$
$$+ \int_s^t (u^2-s^2)^{H-\frac{1}{2}}u^{-2}g(u)\mathrm{d}u],$$

for $g \in \mathbf{S}$, the set of all smooth functions on $[0,T]$ with bounded derivatives, where $C_H = \frac{c_H\sqrt{\pi}}{2^{H-1}\Gamma(H-\frac{1}{2})}$.

In particular, for $\frac{1}{2} < H < 1$, we have

$$\Gamma^*_{H,t}g(s) = C_H s^{\frac{3}{2}-H}\int_s^t (u^2-s^2)^{H-\frac{3}{2}}g(u)\mathrm{d}u.$$

Now, we recall the definition of the Wiener-Itô integral with respect to the subfBm, more work can be found in Nualart[21], Tudor[29].

**2.1. Definition.** Let

$$\Theta_H = \{f \in \mathbf{S} : ||f|| = \int_0^T [\Gamma^*_{H,T}f(t)]^2\mathrm{d}t < \infty\}.$$

For $f \in \Theta_H$, we define

$$\int_0^t f(u)\mathrm{d}S^H = \int_0^t \Gamma^*_{H,t}f(u)\mathrm{d}B_u, \quad 0 \le t \le T,$$

where $B = \{B_t, 0 \le t \le T\}$ is a standard Brownian motion with $B_0 = 0$.

By applying the operator $\Gamma^*_{H,t}$, we can write the subfractional Ornstein-Uhlenbeck process $X^H = \{X^H_t, t \ge 0\}$ starting from zero as

$$X^H_t = v\int_0^t F(t,u)\mathrm{d}B_u, \quad 0 \le t \le T.$$

For $0 < u < t$,

$$(2.5) \qquad F(t,u) = C_{H,T}e^{-t}u^{\frac{3}{2}-H}\int_u^t (m^2-u^2)^{H-\frac{3}{2}}e^m\mathrm{d}m,$$

with $\frac{1}{2} < H < 1$, and

$$F(t,u) = C_{H,T} u^{\frac{3}{2}-H}[-e^{-t}\int_u^t (m^2-u^2)^{H-\frac{1}{2}} m^{-1} e^m \mathrm{d}m \tag{2.6}$$
$$+ (t^2-u^2)^{H-\frac{1}{2}} t^{-1} + e^{-t}\int_u^t (m^2-u^2)^{H-\frac{1}{2}} m^{-2} e^m \mathrm{d}m],$$

with $0 < H < \frac{1}{2}$.

## 3. Some properties of subfractional Ornstein-Uhlenbeck process

In this section, we show that the subfractional Ornstein-Uhlenbeck process $X^H$ is local nondeterministic and establish some estimates for the increments of the process.

The concept of local nondeterminism was first introduced by Berman[5] to unify and extend his methods for studying local times of real-valued Gaussion process. Define the relative prediction error:

$$V_n = \frac{\mathrm{Var}(X(t_n) - X(t_{n-1}))|X(t_1), ..., X(t_{n-1}))}{\mathrm{Var}(X(t_n) - X(t_{n-1}))}$$

which is the ratio of the conditional to the unconditional variance. We consider this to be a measure of the relative predictability of the increment $X(t_n) - X(t_{n-1})$ based on the knowledge of the finite set of data $X(t_1), ..., X(t_{n-1})$. It follows from the elementary property of conditional variance that $0 \le V_n \le 1$. If $V_n = 1$, then the increment is relatively completely unpredictable because the variance is not reduced by the information about $X(t_1), ..., X(t_{n-1})$. On the other extreme, if $V_n = 0$, then the increment is relatively predictable. The process $X$ is called locally nondeterministic on an interval $J \subset R_+$ if for every integer $n \ge 2$,

$$\lim_{\epsilon\to 0} \inf_{t_n - t_1 \le \epsilon} V_n > 0, \tag{3.1}$$

where the infimum in Eq. (3.1) is taken over all ordered points $t_1 < t_2 < ... < t_n$ in $J$ with $t_n - t_1 \le \epsilon$. This condition means that a small increment of the process $X$ is not almost relatively predictable based on a finite number of observations from the immediate past.

It is well known that Eq.(3.1) is equivalent to the following property which says that $X$ has locally approximately independent increments: for any positive integer $n \ge 2$, there exist positive constants $C_n$ and $\delta$ (both may depend on $n$) such that

$$\mathrm{Var}\left(\sum_{j=1}^n u_j\,[X(t_j) - X(t_{j-1})]\right) \ge C_n \sum_{j=1}^n u_j^2 \mathrm{Var}\,[X(t_j) - X(t_{j-1})] \tag{3.2}$$

for all ordered points $0 = t_0 < t_1 < t_2 < ... < t_n$ in $J$ with $t_n - t_1 < \delta$ and all $u_j \in \mathbb{R}(1 \le j \le n)$. Xiao [34] give the properties of local nondeterminism of Gaussion and stable random fields.

By Berman[5], a process $X_t = \int_0^t K(t,u)\mathrm{d}B_u, \quad t \in J$ is local nondeterministic if and only if

$$\lim_{c\downarrow 0} \inf_{0<t-s<c: s,t\in J} \frac{\int_s^t K^2(t,u)\mathrm{d}u}{\int_0^s [K(t,u) - K(s,u)]^2 \mathrm{d}u} > 0, \tag{3.3}$$

where $K$ is a measurable function of $(t,u)$ such that $\int_0^t K^2(t,u)\mathrm{d}u < \infty$ for all $t \in J$.

In order to prove that the subfractional Ornstein-Uhlenbeck process $X^H$ is local nondeterministic, we firstly give the following Lemma.

**3.1. Lemma.** *Let $F(\cdot,\cdot)$ be given by* (2.5)*and* (2.6). *Then we have*

$$\int_0^s [F(t,u)-F(s,u)]^2 du \le C_{H,T}(t-s)^{2H}, \quad 0 \le s \le t,$$

*for all* $0 < H < 1$.

*Proof.* Firstly, for $\frac{1}{2} < H < 1$ and $0 < s < t < T$, we have

$$\begin{aligned}|F(t,u)-F(s,u)| &\le C_{H,T}|e^{-t}-e^{-s}|u^{\frac{3}{2}-H}\int_u^s (m^2-u^2)^{H-\frac{3}{2}} e^m \mathrm{d}m \\ &\quad + C_{H,T}e^{-t}u^{\frac{3}{2}-H}\int_s^t (m^2-u^2)^{H-\frac{3}{2}} e^m \mathrm{d}m \\ &:= C_{H,T}u^{\frac{3}{2}-H}(I_1+I_2).\end{aligned}$$

It is obvious that $I_2 \le \int_s^t (m^2-u^2)^{H-\frac{3}{2}}\mathrm{d}m$, and

$$I_1 \le (t-s)\int_u^s (m^2-u^2)^{H-\frac{3}{2}}\mathrm{d}m \le C_H(t-s)u^{H-\frac{3}{2}}(s-u)^{H-\frac{1}{2}}.$$

So, we have

$$\begin{aligned}\int_0^s [F(t,u)-F(s,u)]^2 \mathrm{d}u &\le C_H\int_0^s u^{3-2H}I_1^2\mathrm{d}u + C_H\int_0^s u^{3-2H}I_2^2\mathrm{d}u \\ &\le C_H(t-s)^2\int_0^s (s-u)^{2H-1}\mathrm{d}u \\ &\quad + C_H\int_s^t\int_s^t\int_0^s u^{3-2H}(m^2-u^2)^{H-\frac{3}{2}}(n^2-u^2)^{H-\frac{3}{2}}\mathrm{d}m\mathrm{d}n\mathrm{d}u \\ &\le C_H s^{2H}(t-s)^2 + C_{H,T}\int_s^t\int_s^t |m-n|^{2H-2}\mathrm{d}m\mathrm{d}n \\ &= C_H s^{2H}(t-s)^2 + C_{H,T}(t-s)^{2H} \le C_{H,T}(t-s)^{2H}.\end{aligned}$$

In the following, we consider the case $0 < H < \frac{1}{2}$, we have

$$F(t,u)-F(s,u) = C_H u^{\frac{3}{2}-H}(M_1+M_2+M_3),$$

where

$$\begin{aligned}M_1 &:= (t^2-u^2)^{H-\frac{1}{2}}t^{-1} - (s^2-u^2)^{H-\frac{1}{2}}s^{-1}, \\ M_2 &:= e^{-s}\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m - e^{-t}\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m, \\ M_3 &:= e^{-t}\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}e^m\mathrm{d}m - e^{-s}\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-2}e^m\mathrm{d}m.\end{aligned}$$

Elementary calculus can show that

$$\int_0^s u^{3-2H}|M_1|^2\mathrm{d}u \le C_{H,T}(t-s)^{2H}. \tag{3.4}$$

For the term $M_2$, we have

$$\begin{aligned}|M_2| &\le (e^{-s}-e^{-t})\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m + e^{-t}\int_s^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m \\ &\le C_{H,T}u^{H-\frac{3}{2}}\left[(t-s)(s-u)^{H+\frac{1}{2}} + (t-s)^{H+\frac{1}{2}}\right].\end{aligned}$$

Hence,

$$\int_0^s u^{3-2H}|M_2|^2 du \le C_{H,T}(t-s)^{2H}. \tag{3.5}$$

For the term $M_3$. Noting that

$$
\begin{aligned}
|M_3| &\le (t-s)\int_u^s (m^2-u^2)^{H-\frac{1}{2}} m^{-2}\mathrm{d}m + \int_s^t (m^2-u^2)^{H-\frac{1}{2}} m^{-2}\mathrm{d}m \\
&:= (t-s)M_{3,1} + M_{3,2}
\end{aligned}
$$

On the one hand,

$$
\begin{aligned}
\int_0^s u^{3-2H}|M_{3,1}|^2\mathrm{d}u &= \int_0^s u^{3-2H}\mathrm{d}u \int_u^s\int_u^s (m^2-u^2)^{H-\frac{1}{2}} m^{-2}(n^2-u^2)^{H-\frac{1}{2}} n^{-2}\mathrm{d}m\mathrm{d}n \\
&\le \int_0^s \mathrm{d}u \int_u^s\int_u^s (m-u)^{H-\frac{3}{2}}(n-u)^{H-\frac{3}{2}}\mathrm{d}m\mathrm{d}n \\
&\le C_H\int_0^s (s-u)^{2H-1}\mathrm{d}u \le C_H s^{2H}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^s u^{3-2H}|M_{3,2}|^2\mathrm{d}u \\
&= \int_0^s u^{3-2H}\mathrm{d}u \int_s^t\int_s^t (m^2-u^2)^{H-\frac{1}{2}} m^{-2}(n^2-u^2)^{H-\frac{1}{2}} n^{-2}\mathrm{d}m\mathrm{d}n \\
&\le \int_s^t\int_s^t (mn)^{H-\frac{1}{2}}\mathrm{d}m\mathrm{d}n \int_0^{m\wedge n} u^{1-2(H+\frac{1}{2})}(m-u)^{(H+\frac{1}{2})-\frac{3}{2}}(n-u)^{(H+\frac{1}{2})-\frac{3}{2}}\mathrm{d}u \\
&\le C_{H,T}\int_s^t\int_s^t (mn)^{-\frac{1}{2}}|m-n|^{2H-1}\mathrm{d}m\mathrm{d}n \le C_{H,T}(t-s)^{2H}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_0^s u^{3-2H}|M_3|^2\mathrm{d}u &\le C_{H,T}(t-s)^{2H}\int_0^s u^{3-2H}|M_{3,1}|^2\mathrm{d}u + \int_0^s u^{3-2H}|M_{3,2}|^2\mathrm{d}u \\
&\le C_{H,T}(t-s)^{2H}.
\end{aligned}
\tag{3.6}
$$

Combing with (3.4), (3.5) and (3.6), this completes the proof. □

**3.2. Theorem.** *The subfractional Ornstein-Uhlenbeck process $X^H$ is local nondeterministic.*

*Proof.* Consider the integral representation of the subfractional Ornstein-Uhlenbeck process $X_t^H = v\int_0^t F(t,u)\mathrm{d}B_u, \quad 0\le t\le T$.

When $\frac{1}{2} < H < 1$, we get

$$
\begin{aligned}
F(t,u) &\ge C_H e^{-t+u}u^{\frac{3}{2}-H}\int_u^t (m^2-u^2)^{H-\frac{3}{2}}\mathrm{d}m \\
&\ge C_H e^{-t+u}u^{\frac{3}{2}-H}(t^2-u^2)^{H-\frac{3}{2}}(t-u) \\
&\ge C_{H,T}(t-u)^{H-\frac{1}{2}}.
\end{aligned}
$$

Hence, $\int_s^t F^2(t,u)\mathrm{d}u \ge C_{H,T}\int_s^t (t-u)^{2H-1}\mathrm{d}u \ge C_{H,T}(t-s)^{2H}$.

When $0 < H < \frac{1}{2}$, without loss of generality, one may assume $0 < T < 1$. By (2.6) we get that

$$
F(t,u) \ge C_{H,T}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}u^{\frac{3}{2}-H}.
$$

Hence,

$$
\int_s^t F^2(t,u)\mathrm{d}u \ge C_{H,T}\int_s^t u^{1-2H}(t^2-u^2)^{2H-1}\mathrm{d}u \ge C_{H,T}(t-s)^{2H}.
$$

It follows from Lemma 3.1 and (3.3) that the Theorem3.2 holds. □

Next, we will study the variance of increment of the subfractional Ornstein-Uhlenbeck process. Let $X^H = \{X_t^H, 0 \le t \le T\}$ be the subfractional Ornstein-Uhlenbeck process starting from zero. Then we have

$$X_t^H = v\int_0^t e^{-t+u}\mathrm{d}S_u^H = v\int_u^t F(t,u)\mathrm{d}B_u.$$

Hence,

$$E[X_t^H X_s^H] = v^2\int_0^{t\wedge s} F(t,u)F(s,u)\mathrm{d}u.$$

In particular, for $\frac{1}{2} < H < 1$ we have

$$EX_t^H X_s^H = v^2 e^{-t-s}\int_0^t\int_0^s e^{u+v}\phi(u,v)\mathrm{d}u\mathrm{d}v,$$

where $\phi(u,v) = H(2H-1)(|u-v|^{2H-2} - |u+v|^{2H-2})$.
First, we give the following Lemmas.

**3.3. Lemma.** *Let $0 < H < 1/2$. Then*

$$\int_0^s F(t,u)F(s,u)\,du \ge C_{H,T}G(t,s).$$

*Proof.* Without loss of generacity, one can assume that $0 < s < t < 1$. It follows from (2.6) that

$$F(t,u) \ge C_{H,T}u^{\frac{3}{2}-H}\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}\mathrm{d}m.$$

So,

$$\begin{aligned}
&\int_0^s F(t,u)F(s,u)\mathrm{d}u \\
&\ge C_{H,T}\int_0^s\int_u^t\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}u^{3-2H}\mathrm{d}m\mathrm{d}n\mathrm{d}u \\
&\ge C_{H,T}\int_0^s\int_0^s (mn)^{-2}\mathrm{d}m\mathrm{d}n\int_0^{m\wedge n}(m^2-u^2)^{H-\frac{1}{2}}(n^2-u^2)^{H-\frac{1}{2}}u^{3-2H}\mathrm{d}u \\
&\ge C_{H,T}\int_0^s m^{2H-4}\mathrm{d}m\int_0^m n^2\mathrm{d}n = C_{H,T}s^{2H}.
\end{aligned}$$

Using the inequality $s^{2H} \ge t^{2H} - (t-s)^{2H}$, we get

$$\begin{aligned}
\int_0^s F(t,u)F(s,u)du &\ge C_{H,T}[s^{2H} + t^{2H} - (t-s)^{2H}] \\
&\ge C_{H,T}[s^{2H} + t^{2H} - \frac{1}{2}(t-s)^{2H} - \frac{1}{2}(t+s)^{2H}] \\
&= C_{H,T}G(t,s).
\end{aligned}$$

This completes the proof. □

**3.4. Lemma.** *Let $0 < H < \frac{1}{2}$. Then for all $0 < s \le t < T$, we have*

$$\int_0^s u^{3-2H}du\int_u^t\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-1}(n^2-u^2)^{H-\frac{1}{2}}n^{-1}\,dmdn \le C_{H,T}G(t,s),$$

$$\int_0^s u^{3-2H}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}(s^2-u^2)^{H-\frac{1}{2}}s^{-1}\,du \le C_{H,T}G(t,s),$$

$$\int_0^s u^{3-2H}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}\,du\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-1}\,dm \le C_{H,T}G(t,s),$$

$$\int_0^s u^{3-2H}(s^2-u^2)^{H-\frac{1}{2}}s^{-1}du\int_u^t(m^2-u^2)^{H-\frac{1}{2}}m^{-1}dm \le C_{H,T}G(t,s),$$

$$\int_0^s u^{3-2H}du\int_u^t\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}dmdn \le C_{H,T}G(t,s).$$

*Proof.* We only prove the first and the third estimate, the other estimates can be proved similarily. On the one hand

$$\begin{aligned}&\int_0^s u^{3-2H}\mathrm{d}u\int_u^t\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-1}(n^2-u^2)^{H-\frac{1}{2}}n^{-1}\mathrm{d}m\mathrm{d}n\\ &\le C_{H,T}\int_0^s \mathrm{d}u\int_u^t\int_u^s(m-u)^{H-\frac{1}{2}}(n-u)^{H-\frac{1}{2}}\mathrm{d}n\mathrm{d}m\\ &\le C_{H,T}\int_0^s[(t+u)^{2H+1}+(t-u)^{2H+1}]\mathrm{d}u\\ &\le C_{H,T}[(t+s)^{2H}-(t-s)^{2H}]\le C_{H,T}G(t,s).\end{aligned}$$

On the other hand

$$\begin{aligned}&\int_0^s u^{3-2H}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}\mathrm{d}u\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-1}\mathrm{d}m\\ &\le \int_0^s(t-u)^{H-\frac{1}{2}}\mathrm{d}u\int_u^s(m-u)^{H-\frac{1}{2}}\mathrm{d}m\\ &\le C_{H,T}\int_0^s(t-u)^{2H-1}du\le C_{H,T}G(t,s).\end{aligned}$$

This completes the proof. □

**3.5. Proposition.** *Let $0<H<1$. Then for all $0<s<t<T$, we have*

$$c_{H,T}v^2G(t,s)\le E[X_t^HX_s^H]\le C_{H,T}v^2G(t,s). \tag{3.7}$$

*Proof.* For $0<H<1/2$, the left inequality in (3.7) follows from Lemma 3.3. Next, we prove the right estimate in (3.7) holds.

$$\begin{aligned}E[X_t^HX_s^H]&=v^2\int_0^s F(t,u)F(s,u)\mathrm{d}u\\ &\le v^2\int_0^s u^{3-2H}\mathrm{d}u\int_u^t\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-1}(n^2-u^2)^{H-\frac{1}{2}}n^{-1}\mathrm{d}m\mathrm{d}n\\ &\quad+v^2\int_0^s u^{3-2H}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}(s^2-u^2)^{H-\frac{1}{2}}s^{-1}\mathrm{d}u\\ &\quad+v^2\int_0^s u^{3-2H}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}\mathrm{d}u\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-1}dm\\ &\quad+v^2\int_0^s u^{3-2H}(s^2-u^2)^{H-\frac{1}{2}}s^{-1}\mathrm{d}u\int_u^t(m^2-u^2)^{H-\frac{1}{2}}m^{-1}\mathrm{d}m\\ &\quad+v^2\int_0^s u^{3-2H}\mathrm{d}u\int_u^t\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}\mathrm{d}m\mathrm{d}n.\end{aligned}$$

Thus, Lemma 3.4 yields the right estimate in (3.7).
For $1/2<H<1$, by an elementary calculus we have

$$\frac{1}{2}e^{-t-s}v^2G(t,s)\le EX_t^HX_s^H\le\frac{1}{2}v^2G(t,s).$$

This completes the proof. □

**3.6. Lemma.** *Let* $0 < H < 1$, *then*

$$\int_s^t F^2(t,u)du \le C_{H,T}(t-s)^{2H}, \quad 0 \le s \le t.$$

*Proof.* Let $\frac{1}{2} < H < 1$, then

$$\begin{aligned}
\int_s^t F^2(t,u)\mathrm{d}u &= C_{H,T}e^{-2t}\int_s^t u^{3-2H}\mathrm{d}u\int_u^t\int_u^t (m^2-u^2)^{H-\frac{3}{2}}(n^2-u^2)^{H-\frac{3}{2}}e^{m+n}\mathrm{d}m\mathrm{d}n \\
&\le C_{H,T}\int_s^t\int_s^t\int_s^{m\wedge n} u^{3-2H}(m^2-u^2)^{H-\frac{3}{2}}(n^2-u^2)^{H-\frac{3}{2}}\mathrm{d}u\mathrm{d}m\mathrm{d}n \\
&\le C_{H,T}\int_s^t\int_s^t\int_0^{m\wedge n} u^{3-2H}(m^2-u^2)^{H-\frac{3}{2}}(n^2-u^2)^{H-\frac{3}{2}}\mathrm{d}u\mathrm{d}m\mathrm{d}n \\
&\le C_{H,T}(t-s)^{2H}.
\end{aligned}$$

Let $0 < H < \frac{1}{2}$ , we have

$$\begin{aligned}
|F(t,u)| \le C_{H,T}u^{\frac{3}{2}-H}\Bigg(&\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}\mathrm{d}m + (t^2-u^2)^{H-\frac{1}{2}}t^{-1} \\
&+\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}\mathrm{d}m\Bigg) := C_{H,T}u^{\frac{3}{2}-H}(I+II+III).
\end{aligned}$$

Since,

$$\begin{aligned}
\int_s^t u^{3-2H}I^2\mathrm{d}u &= \int_s^t u^{3-2H}\int_u^t\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}(n^2-u^2)^{H-\frac{1}{2}}n^{-1}\mathrm{d}m\mathrm{d}n\mathrm{d}u \\
&\le \int_s^t \mathrm{d}u\left[\int_u^t (m-u)^{H-\frac{1}{2}}\mathrm{d}m\right]^2 = C_H(t-s)^{2H+2}.
\end{aligned}$$

$$\int_s^t u^{3-2H}(t^2-u^2)^{2H-1}t^{-2}\mathrm{d}u \le \int_s^t (t-u)^{2H-1}\mathrm{d}u = C_H(t-s)^{2H}.$$

and

$$\begin{aligned}
\int_s^t u^{3-2H}III^2\mathrm{d}u &= \int_s^t u^{3-2H}\int_u^t\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}\mathrm{d}m\mathrm{d}n\mathrm{d}u \\
&\le \int_s^t u^{3-2H}\int_u^t\int_u^t (m-u)^{H-\frac{3}{2}}(n-u)^{H-\frac{3}{2}}u^{2H-3}\mathrm{d}m\mathrm{d}n\mathrm{d}u \\
&= \int_s^t\left[\int_u^t (m-u)^{H-\frac{3}{2}}\mathrm{d}m\right]^2\mathrm{d}u = C_H\int_s^t (t-u)^{2H-1}\mathrm{d}u = C_H(t-s)^{2H}.
\end{aligned}$$

Hence, $\int_s^t F^2(t,u)du \le C_{H,T}(t-s)^{2H}$. This completes the proof. □

**3.7. Theorem.** *For all* $0 \le s < t < T$. *Let*

(3.8) $\quad \sigma_{t,s}^2 = E[(X_t^H - X_s^H)^2].$

*Then,*

(3.9) $\quad v^2 c_{H,T}(t-s)^{2H} \le \sigma_{t,s}^2 \le v^2 C_{H,T}(t-s)^{2H}.$

*Proof.* By Theorem 3.2, we have

$$\begin{aligned}\sigma_{t,s}^2 &= v^2 \int_0^t [F(t,u) - F(s,u)1_{[0,s]}(u)]^2 du \\ &= v^2 \int_0^s [F(t,u) - F(s,u)]^2 du + v^2 \int_s^t F^2(t,u) du \\ &\geq v^2 \int_s^t F^2(t,u) du \geq c_{H,T} v^2 (t-s)^{2H}.\end{aligned}$$

The right inequality follows from Lemma 3.1 and Lemma 3.6. This completes the proof. □

The following result show the subfractional Ornstein-Uhlenbeck process is not of long range dependence.

**3.8. Proposition.** *Let* $0 < H < 1$, *and let*

$$\rho_H(n) = E[X_1^H (X_{n+1}^H - X_n^H)],$$

*for every positive integer* $n$. *Then* $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$.

*Proof.* Let first consider $\frac{1}{2} < H < 1$. Clearly, we have

$$e^{-1} \int_u^{n+1} (m^2 - u^2)^{H-\frac{3}{2}} e^m dm - \int_u^n (m^2 - u^2)^{H-\frac{3}{2}} e^m dm \sim e^n n^{2H-3}, \qquad n \to \infty.$$

It follows that

$$|\rho_H(n)| = v^2 \left| \int_0^{n+1} F(1,u)[F(n+1,u) - F(n,u)] dB_u \right| \sim n^{2H-3}.$$

Thus,

$$\sum_{n=1}^{\infty} |\rho_H(n)| < \infty.$$

On the other hand, if $0 < H < 1/2$, we have

$$e^{-1} \int_u^{n+1} (m^2 - u^2)^{H-\frac{1}{2}} m^{-1} e^m dm - \int_u^n (m^2 - u^2)^{H-\frac{1}{2}} m^{-1} e^m dm \sim e^n n^{2H-2}, \qquad n \to \infty,$$

$$[(n+1)^2 - u^2]^{H-\frac{1}{2}} (n+1)^{-1} - (n^2 - u^2)^{H-\frac{1}{2}} n^{-1} \sim n^{2H-2}, \qquad n \to \infty,$$

$$e^{-1} \int_u^{n+1} (m^2 - u^2)^{H-\frac{1}{2}} m^{-2} e^m dm - \int_u^n (m^2 - u^2)^{H-\frac{1}{2}} m^{-2} e^m dm \sim e^n n^{2H-2}, \qquad n \to \infty.$$

So,

$$\begin{aligned}|\rho_H(n)| &= E[X_1^H (X_{n+1}^H - X_n^H)] \\ &= v^2 \left| \int_0^{n+1} F(1,u)[F(n+1,u) - F(n,u)] dB_u \right| \leq C_H v^2 n^{2H-2}.\end{aligned}$$

which leads to $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$. □

## 4. Existence of the intersection local time

The aim of this section is to prove the existence of the intersection local time of two independent subfractional Ornstein-Uhlenbeck process $X^H=\{X_t^H=(X_t^{H,1},\cdots,X_t^{H,d}),0\le t\le T\}$ and $\widetilde{X}^H=\{\widetilde{X}_t^H=(\widetilde{X}_t^{H,1},\cdot\cdot\cdot,\widetilde{X}_t^{H,d}),0\le t\le T\}$ on $\mathbb{R}^d, d\ge 2$ with the same index $H\in(0,1)$. The intersection local time is formally defined as : for every $T>0$

$$\ell_T=\int_0^T\int_0^T\delta(X_t^H-\widetilde{X}_s^H)\mathrm{d}s\mathrm{d}t, \tag{4.1}$$

where $\delta$ denotes the Dirac delta function. As we pointed out, this definition is only formal. In order to give a rigorous meaning to $\ell_T$, we approximate the Dirac delta function by the heat kernel

$$p_\varepsilon(x)=(2\pi\varepsilon)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2\varepsilon}}, x\in\mathbb{R}^d.$$

Then, we consider the following family of random variables indexed by $\varepsilon>0$

$$\ell_{\varepsilon,T}=\int_0^T\int_0^T p_\varepsilon(X_t^H-\widetilde{X}_s^H)\mathrm{d}s\mathrm{d}t. \tag{4.2}$$

Using the following classical equality

$$p_\varepsilon(x)=\frac{1}{(2\pi\varepsilon)^{d/2}}e^{-\frac{|x|^2}{2\varepsilon}}=\frac{1}{(2\pi)^d}\int_{R^d}e^{i\langle\xi,x\rangle}e^{-\frac{|\xi|^2}{2}\varepsilon}\mathrm{d}\xi,$$

we have

$$\ell_{\varepsilon,T}=\int_0^T\int_0^T p_\varepsilon(X_t^H-\widetilde{X}_s^H)\mathrm{d}s\mathrm{d}t=\frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{R^d}e^{i\langle\xi,X_t^H-\tilde{X}_s^H\rangle}e^{-\frac{|\xi|^2}{2}\varepsilon}\mathrm{d}\xi\mathrm{d}s\mathrm{d}t.$$

Let $\bar{\sigma}_{t,s}^2:=E(X_t^{H,i}-\tilde{X}_s^{H,i})^2,\sigma_t^2:=E(X_t^{H,i})^2, i=1,2$. We have

$$\begin{aligned}
E(\ell_{\varepsilon,T})&=\frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{R^d}E(e^{i\langle\xi,X_t^H-\tilde{X}_s^H\rangle})e^{-\frac{|\xi|^2}{2}\varepsilon}\mathrm{d}\xi\mathrm{d}s\mathrm{d}t\\
&=\frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{R^d}e^{-\frac{1}{2}(\varepsilon+\bar{\sigma}_{t,s}^2)|\xi|^2}\mathrm{d}\xi\mathrm{d}s\mathrm{d}t\\
&=\frac{1}{(2\pi)^{d/2}}\int_0^T\int_0^T(\varepsilon+\bar{\sigma}_{t,s}^2)^{-\frac{d}{2}}\mathrm{d}s\mathrm{d}t,
\end{aligned}$$

where we have used the fact that

$$\int_{R^d}e^{-\frac{1}{2}(\varepsilon+\bar{\sigma}_{t,s}^2)|\xi|^2}\mathrm{d}\xi=\left(\frac{2\pi}{\varepsilon+\bar{\sigma}_{t,s}^2}\right)^{d/2}.$$

We also have

(4.3)

$$E(\ell_{\varepsilon,T}^2)=\frac{1}{(2\pi)^{2d}}\int_{R^{2d}}E[e^{i\langle\xi,X_t^H-\tilde{X}_s^H\rangle+i\langle\eta,X_u^H-\tilde{X}_v^H\rangle}]\times e^{-\frac{\varepsilon(|\xi|^2+|\eta|^2)}{2}}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}s\mathrm{d}t\mathrm{d}u\mathrm{d}v.$$

Let we introduce some notations that will be used throughout this paper

$$\lambda=Var(X_t^{H,1}-\tilde{X}_s^{H,2}),\quad \rho=Var(X_{t'}^{H,1}-\tilde{X}_{s'}^{H,2}),$$

and

$$\mu=Cov(X_t^{H,1}-\tilde{X}_s^{H,2},X_{t'}^{H,1}-\tilde{X}_{s'}^{H,2}).$$

Using the above notation, we can rewrite (4.3) as followings:

$$
\begin{aligned}
E(\ell_{\varepsilon,T}^2) &= \frac{1}{(2\pi)^{2d}} \int_{[0,T]^4} \int_{R^{2d}} exp\{-\frac{1}{2}[(\lambda+\varepsilon)|\xi|^2 + (\rho+\varepsilon)|\eta|^2 \\
&\quad + 2\mu\langle \xi, \eta\rangle]\} \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t' \\
&= \frac{1}{(2\pi)^d} \int_{[0,T]^4} [(\lambda+\varepsilon)(\rho+\varepsilon) - \mu^2]^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t'.
\end{aligned} \tag{4.4}
$$

Using the local nondeterminism of subfractional Ornstein-Uhlenbeck process, we have the follows Lemmas (see also Hu [13]).

**4.1. Lemma.** (1) *For $0 < s < s' < t < t' < T$, we have*

$$\lambda\rho - \mu^2 \geq kv^2[t^{2H} + s^{2H}][(t'-t)^{2H} + (s'-s)^{2H}],$$

(2) *For $0 < s' < s < t < t' < T$, we have*

$$\lambda\rho - \mu^2 \geq kv^2[(t^{2H} + s^{2H})(t'-t)^{2H} + (t'^{2H} + s'^{2H})(s-s')^{2H}],$$

(3) *For $0 < s < t < s' < t' < T$, we have*

$$\lambda\rho - \mu^2 \geq kv^2[(t^{2H} + s^{2H})(t'-t)^{2H} + (t'^{2H} + s'^{2H})(s-s')^{2H}].$$

*where $k > 0$ is an enough small constant.*

**4.2. Lemma.** *Let*

$$A_T := \int_{[0,T]^4} (\lambda\rho - \mu^2)^{-\frac{d}{2}} ds dt ds' dt'.$$

*Then $A_T < \infty$ if and only if $Hd < 2$.*

*Proof.* First, we give the proof of sufficient condition. Let $Hd < 2$. We have

$$A_T = 2\left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right)(\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t',$$

where

$$
\begin{aligned}
I_1 &= \{(s,t,s',t') : 0 < s < s' < t < t' < T\}, \\
I_2 &= \{(s,t,s',t') : 0 < s' < s < t < t' < T\}, \\
I_3 &= \{(s,t,s',t') : 0 < s < t < s' < t' < T\}.
\end{aligned}
$$

For $(s,t,s',t') \in I_1$, we have

$$
\begin{aligned}
&\int_{I_1} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t' \\
&\leq C_{H,T} kv^2 \int_0^T \int_s^T \int_{s'}^T \int_t^T t^{-\frac{Hd}{2}} s^{-\frac{Hd}{2}} (t'-t)^{-\frac{Hd}{2}} (s'-s)^{-\frac{Hd}{2}} \mathrm{d}t' \mathrm{d}t \mathrm{d}s' \mathrm{d}s \\
&\leq C_{H,T} kv^2 \int_0^T \int_s^T \int_{s'}^T t^{-\frac{Hd}{2}} s^{-\frac{Hd}{2}} (s'-s)^{-\frac{Hd}{2}} \mathrm{d}t \mathrm{d}s' \mathrm{d}s \\
&\leq C_{H,T} kv^2 \int_0^T \int_0^{s'} s^{-\frac{Hd}{2}} (s'-s)^{-\frac{Hd}{2}} \mathrm{d}s \mathrm{d}s' \leq C_{H,T} kv^2 \int_0^T s^{1-Hd} \mathrm{d}s < \infty.
\end{aligned}
$$

By a similar way, we can prove that

$$\int_{I_2} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t' < \infty, \quad \int_{I_3} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t' < \infty.$$

Now, we turn to the proof of the necessary condition. By Proposition 3.5 one can get

$$\lambda \leq C_{H,T} v^2 (t^{2H} + s^{2H}), \quad \rho \leq C_{H,T} v^2 (t'^{2H} + s'^{2H}),$$

and

$$\mu^2 \geq c_{H,T} v^2 [t^{2H} + s^{2H} + t'^{2H} + s'^{2H} \\ - \frac{1}{2}((t+s)^{2H} + (t'+s')^{2H} + |t-s|^{2H} + |t'-s'|^{2H})].$$

So,

$$\lambda\rho - \mu^2 \leq C_{H,T} v^2 \{(t^{2H} + s^{2H})(t'^{2H} + s'^{2H}) - [t^{2H} + s^{2H} + t'^{2H} + s'^{2H} \\ - \frac{1}{2}((t+s)^{2H} + (t'+s')^{2H} + |t-s|^{2H} + |t'-s'|^{2H})]\}.$$

Hence, making a change to spherical coordinates, as the integrand is always positive, we have

$$A_T = \int_{[0,T]^4} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t' \geq \int_{D_T} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t' \\ \geq \int_0^T r^{3-2Hd} \int_\Theta \phi(\theta) d\theta,$$

where $D_T := \{(s+t+s'+t') \in R_+^4 : s^2 + t^2 + s'^2 + t'^2 \leq \varepsilon^2\}$. Note that the angular integral is different from zero thanks to the positivity of the integrand. It follows that if $A_T < \infty$, then $Hd < 2$. Thus completes the proof. □

From Lemma 4.2, we get the following Theorem.

**4.3. Theorem.** *Let $H \in (0,1)$. Then $\ell_{\varepsilon,T}$ converges in $L^2(\Omega)$ as $\varepsilon \to 0$ if and only if $Hd < 2$. Morever, if the limits denoted by $\ell_T$, then $\ell_T \in L^2(\Omega)$.*

*Proof.* A slight extension of (4.4) yields

$$E(\ell_{\varepsilon,T}\ell_{\eta,T}) = \frac{1}{(2\pi)^d} \int_{[0,T]^4} [(\lambda+\varepsilon)(\rho+\varepsilon) - \mu^2]^{-\frac{d}{2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'.$$

Consequently, a necessary and sufficient condition for the convergence in $L^2(\Omega)$ of $\ell_{\varepsilon,T}$ is that

$$A_T := \int_{[0,T]^4} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t' < \infty.$$

Thus, it is sufficient to prove that $A_T < \infty$ if and only if $Hd < 2$. By Lemma 4.2, this complete the proof. □

**Conclusions.** In this paper, we discuss and analyze the subfractional Ornstein-Uhlenbeck process and show that this process is local nondeterministic. At the same time, we establish several estimates for the increments of the process, and give the sufficient and necessary conditions for the existence of the intersection local time of two independent subfractional Ornstein-Uhlenbeck process. In a sequel of this paper we will study the Ornstein-Uhlenbeck process driven by general Gaussian process.

**Acknowledgements.** The authors are deeply grateful to the two anonymous referees for their careful reading, valuable comments and correcting some errors, which have greatly improved the quality of the paper. Dongjin Zhu is partially supported by the Key Natural Science Foundation of the Anhui Educational Committee (KJ2012ZD01) and the Philosophy and Social Science Planning Foundation of Anhui Province (AHSK11-12D128), Xiuwei Yin is partially supported by Mathematical Tianyuan Foundation of China (11326171) and the Natural Science Foundation of Anhui Province (1308085QA14).

## References

[1] S. Albeverio, M. JoÃo Oliveira and L. Streit, Intersection local times of independent Brownian motions as generalized White noise functionals. *Acta Appl. Math.* **69** (2001), 221-241.

[2] X. Bardina and D. Bascompte, Weak convergence towards two independent Gussian process from a unique poisson process. *Collect Math.* **61** (2010), 191-204.

[3] O. E. Barndorff-Nielsen, Superposition of Ornstein-Uhlenbeck type processes. *Theory Probab. Appl.* **45** (2001), 175-194.

[4] O. E. Barndorff-Nielsen and N. Shephard, Non-Gaussian Ornstein-Uhlenbeck- based models and some of their uses in financial economics. *J. R. Statist. Soc. B* **63** (2001), 167-241.

[5] S. M. Berman, Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** (1973), 69-94.

[6] T. Bojdecki, L. G. Gorostiza and A. Talarczyk, Sub-fractional Brownian motion and its relation to occupation times. *Statist. Probab. lett.* **69** (2004), 405-419.

[7] T. Bojdecki, L. G. Gorostiza and A. Talarczyk, Some extensions of fractional Brownian motion and sub-fractional Brownian motion related to particle systems. *Electron. Commun. Probab.* **12** (2007), 161-172.

[8] C. Chen and L. Yan, Remarks on the intersection local time of fractional Brownian motions. *Statist. Probab. Lett.* **81** (2011), 1003-1012.

[9] P. Cheridito, H. Kawaguchi and M. Maejima, Fractional Ornstein-Uhlenbeck processes. *Electr. J. Probab.* **8** (2003), 1-14.

[10] K. Dzhaparidze and H. Van Zanten, A series expansion of fractional Brownian motion. *Probab. Theory Relat. Fields.* **103** (2004), 39-55.

[11] K. Es-Sebaiy, Berry-Esséen bounds for the least squares estimator for discretely observed fractional Ornstein-Uhlenbeck processes. *Statist. Probab. Lett.* **83** (2013), 2372-2385.

[12] S. Habtemicael and I. SenGupta, Ornstein-Uhlenbeck processes for geophysical data analysis. *Phys. A* **399**(2014), 147-156.

[13] Y. Hu, Self intersection local time for fractional Browinian motions-Via chaos expansion. *J. Math. Kyoto Univ.* **41** (2001), 233-250.

[14] Y. Hu, Integral transformations and anticipatiove calculus for fractional Brownian motions. *Mem. Am. Math. Soc.* **175** (2005).

[15] Y. Hu and D. Nualart, Parameter estimation for fractional Ornstein-Uhlenbeck process. *Statist. Probab. Lett.* **80** (2010), 1030-1038.

[16] Y. Jiang and Y. Wang, Self-intersection local times and collision local times of bifractional Brownian motions. *Sci China Ser A*, **52** (2009), 1905-1919.

[17] P. Lévy, Le mouvement brownien plan. *Amer. J. Math.* **62** (1940), 487-550.

[18] J. Liu and L. Yan, Remarks on asymptotic behavior of weighted quadratic variation of subfractional Brownian motion. *J Korean Statist Soc*, **41** (2012), 177-187.

[19] B. B. Mandelbrot and J. W. Van Ness, Fractional Brownian motion, fractional noises and applications. *SIAM Rev.* **10** (1968), 422-37.

[20] I. Mendy, Parametric estimation for sub-fractional Ornstein-Uhlenbeck process. *J. Statist. Plann. Inference* **143** (2013), 663-674.

[21] D. Nualart, Malliavin Calculus and Related Topics. Berlin: Springer, 2006.

[22] D. Nualart and S. Ortiz-Latorre, Intersection local time for two independent fractional Brownian motions. *J. Theor. Probab.* **20** (2007), 759-767.

[23] D. Revuz and M. Yor, Continuous Martingales and Brownian Motion. Berlin: Springer,1999.

[24] J. Rosen, The intersection local time of fractional Brownian motion in the plane. *J. Multivar. Anal.* **23** (1987), 37-46.

[25] I. SenGupta, Option pricing with transaction costs and stochastic interest rate. *Applied Mathematical Finance*. (2014) , online version doi:10.1080/1350486X.2014.881263

[26] G. Shen and L. Yan, Remarks on an integral functional driven by sub-fractional Brownian motion. *J. Korean. Statist. Soc.* **40** (2011), 337-346.

[27] G. Shen and C. Chen, Stochastic integration with respect to the sub-fractional Brownian motion with $0 < H < 1/2$. *Statist. Probab. Lett.* **82** (2012), 240-251.

[28] G. Shen, Necessary and sufficient condition for the smoothness of intersection local time of sub fractional Brownian motions. *J. Inequal. Appl.*. **2011**:139.

[29] C. Tudor, Inner product spaces of integrands associated to subfractional Brownian motion. *Statist. Probab. Lett.* **78** (2008), 2201-2209.

[30] D. Wu and Y. Xiao, Regularity of intersection local times of fractional Brownian motions. *J. Theor. Probab.* **23** (2010), 972-1001.

[31] L. Yan and G. Shen, On the collision local time of sub-fractional Brownian Motions. *Statist. Probab. Lett.* **80** (2010), 296-308.

[32] L. Yan, Y. Lu and Zh. Xu, Some properties of the fractional Ornstein-Uhlenbeck process. *J. Phys. A: Math. Theor.* **41** (2008), 145007 (17pp).

[33] L. Yan and M. Tian, On the local times of fractional Ornstein-Uhlenbeck process. *Lett. Math. Phys.* **73**(2005), 209-220.

[34] Y. Xiao, Properties of local nondeterminism of Gaussion and stable random fields and their applications. *Ann. Fac. Sci. Toulouse.* **15** (2006), 157-193.

# *ADVICE TO AUTHORS*

*Hacettepe Journal of Mathematics and Statistics* publishes short to medium length research papers and occasional survey articles written in English. All papers are refereed to international standards.

**Address for Correspondence**

Editorial Office,
Hacettepe Journal of Mathematics and Statistics,
Hacettepe University,
Faculty of Science,
Department of Mathematics,
06532 Beytepe,
Ankara,
Turkey.

**E-mail**: `hjms@hacettepe.edu.tr`
**Tel**: Editors: `+ 90 312 297 7859`
`+ 90 312 297 7898`
Associate Editor: `+ 90 312 297 7880`
**Fax** : `+ 90 312 299 2017`

**Web Page** : `http//www.hjms.hacettepe.edu.tr`

**Advise to Authors :** The style of articles should be lucid but concise. The text should be preceded by a short descriptive title and an informative abstract of not more than 100 words. Keywords or phrases and the 2010 AMS Classification should also be included.The main body of the text should be divided into numbered sections with appropriate headings. Items should be numbered in the form 2.4. Lemma, 2.5. Definition. These items should be referred to in the text using the form Lemma 2.4., Definition 2.5. Figures and tables should be incorporated in the text. A numbered caption should be placed above them. References should be punctuated according to the following examples, be listed in alphabetical order according to in the (first) author' s surname, be numbered consecutively and referred to in the text by the same number enclosed in square brackets. Only recognized abbreviations of the names of journals should be used.

[1 ] Banaschewski, B. *Extensions of topological spaces*, Canad. Math. Bull. **7** (1), 1–22, 1964.

[2 ] Ehrig, H. and Herrlich, H. *The construct PRO of projection spaces: its internal structure*, in: Categorical methods in Computer Science, Lecture Notes in Computer Science **393** (Springer-Verlag, Berlin, 1989), 286–293.

[3 ] Hurvich, C. M. and Tsai, C. L. *Regression and time series model selection in small samples*, Biometrika **76** (2), 297–307, 1989.

[4 ] Papoulis, A. *Probability random variables and stochastic process* (McGraw-Hill, 1965).

## *INSTRUCTIONS FOR AUTHORS*

**Preparation of Manuscripts :** Manuscripts will be typeset using the LATEX typesetting system. Authors should prepare the article using the HJMS style before submission by e-mail. The style bundle contains the styles for the article and the bibliography as well as example of formatted document as pdf file. The authors should send their paper directly to hjms@hacettepe.edu.tr with the suggestion of area editor and potential reviewers for their submission. The editorial office may not use these suggestions, but this may help to speed up the selection of appropriate reviewers.

**Copyright :** No manuscript should be submitted which has previously been published, or which has been simultaneously submitted for publication elsewhere. The copyright in a published article rests solely with the Faculty of Science of Hacettepe University, and the paper may not be reproduced in whole in part by any means whatsoever without prior written permission.

**Notice :** The content of a paper published in this Journal is the sole responsibility of the author or authors, and its publication does not imply the concurrence of the Editors or the Publisher.