HACETTEPE UNIVERSITY
FACULTY OF SCIENCE
TURKEY

HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS

A Bimonthly Publication
Volume 44 Issue 1
2015

ISSN 1303 5010

HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS

Volume 44 Issue 1
February 2015

A Peer Reviewed Journal
Published Bimonthly by the
Faculty of Science of Hacettepe University

Abstracted/Indexed in

SCI-EXP, Journal Citation Reports, Mathematical Reviews,
Zentralblatt MATH, Current Index to Statistics,
Statistical Theory & Method Abstracts,
SCOPUS, Tübitak-Ulakbim.

ISSN 1303 5010

This Journal is typeset using LaTeX.

Hacettepe Journal of Mathematics and Statistics

Cilt 44 Sayı 1 (2015)
ISSN 1303 – 5010

KÜNYE
YAYININ ADI: HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS
YIL : 2015 SAYI : 44 - 1 AY : Şubat
YAYIN SAHİBİNİN ADI : H. Ü. Fen Fakültesi Dekanlığı adına Prof. Dr. Bekir Salih
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HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS

A Bimonthly Publication – Volume 44 Issue 1 (2015)

ISSN 1303 – 5010

PUBLISHED BY HACETTEPE UNIVERSITY
FACULTY OF SCIENCE

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 1 – 13

Maximal, irreducible and prime soft ideals of BCK/BCI-algebra

U. ACAR* and Y. ÖZTÜRK †

Abstract

In this paper, the notions of soft irreducible, prime and maximal soft ideals,irreducible, prime and maximal soft idealistic over on BCK/BCI-algebras are introduced, and several examples are given to illustrate. Relations between irreducible, prime and maximal idealistic soft BCK/BCI-algebras are investigated.

2000 AMS Classification: 06D72; 06F35; 46J20.

Keywords: Maximal soft ideals; irreducible soft ideals; prime soft ideals; soft BCK/BCI algebra; soft set; BCK/BCI algebra.

Received 08/04/2012 : Accepted 23/01/2014 Doi : 10.15672/HJMS.2015449109

1. Introduction

Dealing with uncertainties is a main problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with classical methods. Because, these classical methods have their inherent difficulties. To overcome these kinds of difficulties, Molodtsov [10] proposed a completely new approach, which is called soft set theory, for modeling uncertainty. Molodtsov [10], Jun[6] and Park[7] pointed out several directions for the applications of soft sets. Maji at al[9] studied several operations on the theory of soft sets. Besides, Aktaş and Çağman [1] defined soft groups and obtained the main properties of these groups. Jun[6] applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras. The notion of soft BCK/BCI-algebras and subalgebras introduced, and their basic properties have derived. Then, Jun and Park[7] presented the soft ideals and idealistic soft BCK/BCI-algebras, and their basic properties have given. In this paper we apply the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras. We introduce the notions of irreducible soft ideals, maximal soft ideals and prime soft ideals of

*Department of Mathematics, Muğla Sıtkı Koçman University, Muğla, Turkey.
Email: uacar@mu.edu.tr
Corresponding author.
†Ula Ali Koçman Vocational Scholl, Muğla Sıtkı Koçman University, Muğla, Turkey.

BCK/BCI-algebra and irreducible idealistic, prime idealistic and maximal idealistic soft BCK/BCI-algebras and we derive their basic properties.

2. Preliminaries

2.1. Basic results on BCK/BCI-algebras. A BCK- algebra is an important class of logical algebras introduced by K. Iséki[5] and was extensively investigated by several researchers. In this section we give some basic definitions and notions to be used in our work for an easy reference by readers.

We start with a well known definition.

2.1. Definition. Let X be a set with a binary operation $*$ and a constant 0. Then $(X; *, 0)$ is called *a BCI-algebra* if it satisfies the following conditions:

BCI-1: $(\forall x, y, z \in X)\ (((x*y)*(x*z))*(z*y) = 0$,
BCI-2: $(\forall x, y, z \in X)\ ((x*(x*y))*y = 0$,
BCI-3: $(\forall x \in X)\ (x*x = 0)$,
BCI-4: $(\forall x, y, z \in X)\ (x*y = 0, y*x = 0 \Rightarrow x = y)$.

2.2. Definition. Let X be a BCI-algebra. If X satisfies the following identity:

BCK-5: $(\forall x \in X)\ (0*x = 0)$,

then X is called *a BCK-algebra*.

Any BCK-algebra X satisfies the following axioms:

(a1): $(\forall x \in X)\ (x*0 = x)$
(a2): $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x*y \leq y*z, z*y \leq z*x$
(a3): $(\forall x, y, z \in X)\ ((x*y)*z = (x*z)*y)$,
(a4): $(\forall x, y, z \in X)\ ((x*z)*(y*z) \leq x*y)$,

where $x \leq y$ if and only if $x*y = 0$. A BCK-algebra X is said to be commutative if x ∧ y = y ∧ x for all $\forall x, y \in X$, where $x \wedge y = y*(y*x)$ is a lower bound of x and y. For any element x of a BCI-algebra X, we define the order of x, denoted by $o(x)$, as

$$o(x) = min\{n \in \mathbb{N} \ \mid \ 0*x^n = 0\}$$

2.3. Definition. [8] A nonempty subset S of a BCK/BCI-algebra X is called *a subalgebra* of X if $x*y \in S$ for all $x, y \in S$.

2.4. Definition. [8] A subset H of a BCK/BCI-algebra X is called *an ideal* of X if it satisfies the following axioms:

(I1): $0 \in H$,
(I2): $(\forall x \in X)\ (\forall y \in H)\ (x*y \in H \Rightarrow x \in H)$.

Any ideal H of a BCK/BCI-algebra X satisfies the following implication:
$(\forall x \in X)\ (\forall y \in H)(x \leq y \Rightarrow x \in H)$.
In this study, we present the definition of irreducible soft ideal, prime soft ideal and maximal soft ideal, irreducible, maximal and prime idealistic soft BCK-algebras. On this account, we first recall the definitions of irreducible, prime and maximal ideals on BCK/BCI-algebras.

2.5. Definition. [8] A proper ideal I of a BCK-algebra X is called to be *irreducible* if $I = A \cap B$ implies $I = A$ or $I = B$ for any A, B ideal of X.

2.6. Definition. [8] A BCK-algebra $(X; *, 0)$ is said to be a lower BCK-semilattice if X is a lower semilattice with respect to BCK order $\leq$ and denote $x \wedge y = inf\{x, y\}$

2.7. Definition. [8] Let X be a lower BCK-semilattice. A proper ideal I of X is called to be *a prime ideal* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$ for any x, $y \in X$.

2.8. Definition. [8] Given a BCK-algebra $(X;*,0)$, an ideal I of X is called to be *a maximal* ideal if I is a proper ideal of X and not a proper subset of any proper ideal of X.

It is well known that in a commutative BCK-algebra every maximal ideal is a prime ideal, and irreducible and prime ideals coincide. In the next chapter, up to the present, we introduce and investigate soft ideal.

2.2. Basic results on soft sets. Molodtsov defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subset E$.

2.9. Definition. [10] A pair (F, A) is called *a soft set over* U, where F is a mapping given by $F : A \longrightarrow P(U)$.

In other words , a soft set over U is a parameterized family of subset of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [3]

2.10. Definition. [10] Let (F, A) and (G, B) be two soft sets over a common universe U. *The intersection* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i): $C = A \cap B$,
(ii): $(\forall\, e \in C)$ $(H(e) = F(e)$ or $G(e)$, (as both are same sets)).

In this case, we write $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

2.11. Definition. [10] Let (F, A) and (G, B) be two soft sets over a common universe U. *The union* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i): $C = A \cup B$

(ii): for all e $\in$ C, $H(e) = \begin{cases} \text{F(e)}, & \text{if e} \in \text{A} \setminus \text{B}; \\ \text{G(e)}, & \text{if e} \in \text{B} \setminus \text{A}; \\ \text{F(e)} \cup \text{G(e)}, & \text{if e} \in A \cap B. \end{cases}$

In this case, we write $(F, A) \widetilde{\cup} (G, B) =(H, C)$.

2.12. Definition. [10] If (F, A) and (G, B) be two soft sets over a common universe U, then "(F, A) AND (G, B)" is denoted by $(F, A) \widetilde{\wedge} (G, B)$ is defined by
$(F, A) \widetilde{\wedge} (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$
for all $(\alpha, \beta) \in A \times B$.

2.13. Definition. [3] *The bi-intersection* of two soft sets (F, A) and (G, B) over common universe U is defined to be the soft set (H, C), where $C = A \cap B$ and $H : C \to P(U)$ is a mapping given by $H(x) = F(x) \cap G(x)$ for all $x \in C$. This is denoted by $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$

2.14. Definition. [10] If (F, A) and (G, B) be two soft sets over a common universe U, then "(F, A) OR (G, B)" is denoted by $(F, A) \widetilde{\vee} (G, B)$ is defined by $(F, A) \widetilde{\vee} (G, B) = (H, A \times B)$,
where $H(\alpha, \beta) = F(\alpha) \cup G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

2.15. Definition. [10] If (F, A) and (G, B) be two soft set over a common universe U, we say that (F, A) is *a soft subset* of (G, B), denoted by $(F, A) \widetilde{\subset} (F, A)$, if it satisfies:

i): $A \subset B$,

ii): For every $\varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are same identical approximations.

2.16. Definition. [9] A soft set (F, A) over U is said to be a NULL soft set denoted by Φ, if $\forall\ e \in A,\ F(e) = \emptyset$, (null-set).

2.17. Definition. [9] A soft set over (F, A) over is said to be absolute soft set denoted by $\widetilde{A}$, if $\forall\ e \in A,\ F(e) = U$.

2.18. Definition. [6] Let X be a BCK/BCI-algebra and let (F, A) be a soft set over X. Then (F, A) is called *a soft BCK/BCI-algebra* over X if $F(x)$ is a subalgebra of X for all $x \in A$.

2.19. Definition. [7] Let S be a subalgebra of X. A subset I of X is called *an ideal* of X related to S(briefly, S-ideal of X), denoted by $I_S \lhd S_S$, if it satisfies:

i): $0 \in I$,
i): $(\forall x \in S)(\forall y \in I)\ (x * y \in I) \Rightarrow x \in I$.

2.20. Definition. [7] Let (F, A) be a soft BCK/BCI-algebra over X. A soft set (G, I) over X is called *a soft ideal* of (F, A), denoted by $(G, I)\ \widetilde{\lhd}\ (F, A)$, if it satisfies:

i): $I \subset A$,
ii): ($\forall$ x $\in$ I) $(G(x) \lhd F(x))$.

2.21. Definition. [7] Let (F, A) be a soft set over X. Then (F, A) is called *an idealistic soft BCK/BCI-algebra over X* if $F(x)$ is an ideal of X for all $x \in A$ respectively.

3. Maximal, Irreducible and Prime Soft Ideals

We first define soft BCK/BCI subalgebra to complete gap between ideals and algebras. In this section X will be BCK/BCI-algebra.

3.1. Definition. Let (F, A) and (G, B) be two soft BCK/BCI-algebras over X. (G, B) is called *soft BCK/BCI-subalgebra* of (F, A) over X if $B \subseteq A$ and $G(x)$ is a subalgebra of $F(x)$ for each $x \in B$.

3.2. Definition. Let (F, A) be a soft BCK/BCI-algebra over X and (G, I) is a non-whole soft ideal of (F, A).

i) (G, I) is called *an maximal soft ideal* of (F, A) if $G(x)$ is an maximal ideal of $F(x)$ for all $x \in I$.
ii) Then (G, I) is called *an irreducible soft ideal* of (F, A) if $G(x)$ is an irreducible ideal of $F(x)$ for all $x \in I$.

Let (G, I) be a maximal soft ideal of a soft BCK/BCI-algebra (F, A) over X. It is clear that (G, I) is an irreducible soft ideal of (F, A) over X since every maximal ideal of a BCK/BCI-algebra is an irreducible ideal [4, Proposition 3].

3.3. Example. Let $X = \{0, a, b, c, d, e, f, g\}$ be a BCI-algebra with the following Cayley table:

*	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Let (F, A) be a soft set over X where $A = \{0, a, b, c\}$ and $F : A \to P(X)$ is set-valued function defined by $F(x) = \{0\} \cup \{y \in X \mid o(x) = o(y)\}$ for all $x \in A$.
Since $F(0) = F(a) = F(b) = F(c) = \{0, a, b, c\} \leq X$, (F, A) is soft BCI-algebra over X.
Let (G, I) be a soft set over X where $I = \{a\}$ and $G : I \longrightarrow P(X)$ is set-valued function defined by $G(x) = \{0, x\}$ for all $x \in I$. Hence $G(a) = \{0, a\} \lhd F(a)$ and ideal $\{0, a\}$ is unique proper ideal of $F(a)$. Thus $G(a)$ is a maximal ideal of $F(a)$ and (G, I) is a maximal soft ideal of (F, A).

3.4. Example. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let (F, A) be a soft set over X, where $A = X$ and $F : A \to P(X)$ is set-valued function defined by $F(x) = \{y \in X \mid y * x = 0\}$ for all $x \in A$. Then $F(0) = \{0\}$, $F(a) = \{0, a\}$, $F(b) = \{0, a, b\}$, $F(c) = \{0, a, b, c\}$, $F(d) = X$. Then (F, A) is a soft BCK-algebra over X.
Let (G, I) be a soft set over X, where $I = \{c\}$ and $G : I \to P(X)$ is set-valued function defined by $G(x) = \{y \in X \mid y * (y * x) \in \{0, a\}\}$. Then $G(c) = \{0, a\} \lhd F(c)$. Hence (G, I) is a soft ideal of (F, A). Moreover,$I_1 = \{0, a\}$ and $I_2 = \{0, a, b\}$ are ideals of $F(c)$. Then

$$G(c) = I_1 \cap I_2 \;\Rightarrow\; G(c) = I_1.$$

Hence $G(c)$ is an irreducible ideal of $F(c)$. Therefore (G, I) is an irreducible soft ideal of (F, A).

3.5. Example. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Let (F, A) be a soft set over X where $A = \{1, 2\}$ and $F : A \longrightarrow P(X)$ defined by

$$F(x) = \{y \in X \mid y * x \in \{0, 2, 3\}\}$$

Then we can easly show that

$$\begin{aligned} F(1) &= \{y \in X \mid y * 1 \in \{0, 2, 3\}\} = \{0, 1, 2, 3\} \\ F(2) &= \{y \in X \mid y * 2 \in \{0, 2, 3\}\} = \{0, 2, 3\} \end{aligned}$$

Since $F(1)$ and $F(2)$ are subalgebras of X, (F, A) is a soft BCK-algebra over X. We define soft set (G, B) with $G(x) = \{y \in X \mid y * x = 0\}$ where $B = \{1\}$ and $G : B \longrightarrow P(X)$. So

$$G(1) = \{y \in X \mid y * 1 = 0\} = \{0, 1\}$$

and $G(1)$ is an ideal of $F(1)$. The set of all ideals of $F(1)$ is
$\{I_0 = \{0\}, I_1 = \{0, 1\}, I_2 = \{0, 2\}, I_3 = \{0, 3\}, I_4 = \{0, 1, 2\}, I_5 = \{0, 1, 3\},$
$I_6 = \{0, 2, 3\}, I_7 = F(1)\}$. Thus we see $G(1) = I_4 \cap I_5$ but $G(1) \neq I_4$ and $G(1) \neq I_5$.
That is $G(1)$ is not irreducible ideal of $F(1)$. Hence (G, B) is not irreducible soft ideal of (F, A).

The authors Jun et al.[7, Theorem 4.6] proved that intersection of soft ideals is a soft ideal. The following theorem shows that intersection of soft irreducible ideals is a soft irreducible ideal.

3.6. Theorem. *Let (F, A) be a soft BCK/BCI-algebra over X. Let (G_1, I_1) and (G_2, I_2) be two irreducible ideals of (F, A). If $I_1 \cap I_2 \neq \emptyset$, then $(G_1, I_1) \mathbin{\widetilde{\cap}} (G_2, I_2)$ is an irreducible ideal of (F, A) .*

Proof. Using Definition 2.10, we can write $(G_1, I_1) \mathbin{\widetilde{\cap}} (G_2, I_2) = (G, I)$ where $I = I_1 \cap I_2$ and $G(x) = G_1(x)$ or $G(x) = G_2(x)$ for all $x \in I$. We know that $G_1(x) \lhd F(x)$ or $G_2(x) \lhd F(x)$ for all $x \in I$. Hence

$$(G_1, I_1) \mathbin{\widetilde{\cap}} (G_2, I_2) = (G, I) \widetilde{\lhd} (F, A).$$

Since (G_1, I_1) *and* (G_2, I_2) are irreducible ideals of (F, A), we have that $G(x) = G_1(x)$ is an irreducible ideal of $F(x)$ or $G(x) = G_2(x)$ is an irreducible ideal of $F(x)$ for all $x \in I$. Hence $(G_1, I_1) \mathbin{\widetilde{\cap}} (G_2, I_2) = (G, I)$ is an irreducible soft ideal of (F, A). This complete the proof. □

3.7. Theorem. Let (F, A) be a soft BCK/BCI-algebra over X. Let (G, B) *and* (H, C) be two irreducible ideals of (F, A). If $B \cap C = \emptyset$ then $(G, B) \mathbin{\widetilde{\cup}} (H, C)$ is an irreducible ideal of (F, A).

Proof. Let $D = B \cup C$. Define T on D by

$$T(x) = \begin{cases} \text{G(x)}, & \text{if x} \in \text{B} \setminus \text{C} \\ \text{H(x)}, & \text{if x} \in \text{C} \setminus \text{B} \end{cases}$$

Then it is easily checked that $(G, B) \mathbin{\widetilde{\cup}} (H, C) = (T, D)$ and $(T, D) \widetilde{\lhd} (F, A)$. Since $B \cap C = \emptyset$, either $x \in B \backslash C$ or $x \in C \backslash B$ for all $x \in D$. Let $x \in D$. If $x \in B \backslash C$, then $T(x) = G(x)$ is an irreducible ideal of $F(x)$ since (G, B) is an irreducible soft ideal of (F, A). If $x \in C \backslash B$, $T(x) = H(x)$ is an irreducible ideal of $F(x)$ since (H, C) is an irreducible soft ideal of (F, A). Hence $T(x)$ is an irreducible ideal of $F(x)$ for all $x \in D$, and so $(G, B) \mathbin{\widetilde{\cup}} (H, C)$ is an irreducible ideal of (F, A). □

4. Irreducible idealistic soft BCK/BCI-algebras

4.1. Definition. Let (F, A) be a non-whole an idealistic soft BCK/BCI-algebra over X. Then (F, A) is called *an irreducible idealistic soft BCK/BCI-algebra over X* if $F(x)$ is an irreducible ideal of X for all $x \in A$.

4.2. Example. Let $X = \{0, a, b, c, d, e, f, g\}$ and consider the following Cayley table:

*	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Then $(X; 0, *)$ is a BCI-algebra (see [2]). Let (F, A) be a soft set over X, where $A = \{0, a, b, c\}$ and $F : A \to P(X)$ is a set-valued function defined by :

$$F(x) = \{0\} \cup \{y \in X \mid o(x) = o(y)\} \text{ for all } x \in A.$$

Then $F(0) = F(a) = F(b) = F(c) = \{0, a, b, c\}$ is an irreducible ideal of X. Hence (F, A) is an irreducible idealistic soft BCI-algebra over X.

4.3. Example. Let $X = \{0, 1, 2, 3\}$ be a BCK-algebras with Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Let (F, A) be a soft set over X, where $A = \{0, 1, 2\}$ and $F : A \to P(X)$ be a set-valued function defined by $F(x) = \{y \in X \mid y * x = 0\}$ for all $x \in A$. Then $F(0) = \{0\}$, $F(1) = \{0, 1\}$, $F(2) = \{0, 2\}$ which are irreducible ideals of X. Hence (F, A) is an irreducible idealistic soft BCK-algebra over X.

4.4. Example. Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	2
3	3	3	3	0

Let (F, A) be a soft set over X , where $A = X$ and $F : A \longrightarrow P(X)$ is set-valued function defined by

$$F(x) = \{y \in X \mid y * x = 0\}$$

for all $x \in A$. Then $F(0) = \{0\}$, $F(1) = \{0, 1\} \lhd X$, $F(2) = \{0, 1, 2\} \lhd X$, $F(3) = \{0, 1, 3\} \lhd X$.Therefore (F, A) is idealistic soft BCK-algebra over X. Moreover $F(1) = \{0, 1, 2\} \cap \{0, 1, 3\}$ but $F(1) \neq \{0, 1, 2\}$ and $F(1) \neq \{0, 1, 3\}$, and so $F(1)$ is not irreducible ideal of X. Hence (F, A) is not irreducible idealistic soft BCK-algebra over X.

Note that "AND" of two irreducible soft ideals may not be an irreducible soft ideal.To prove this idea, we examine the following example.

4.5. Example. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

(i) Let (H, B) be a soft set over X, where $B = \{1, 2, 4\}$ and $H : B \longrightarrow P(X)$ be a set-valued function defined by

$$H(x) = \{y \in X \mid y * (y * x) = 0\}$$

for all $x \in B$. Then $H(1) = \{0, 2, 4\}$, $H(2) = \{0, 1, 4\}$ and $H(4) = \{0, 1, 2, 3\}$ which are irreducible ideal of X. Therefore (H, B) is an irreducible idealistic soft BCK-algebra over X.

(ii) Let (K, C) be a soft set over X, where $C = \{2\}$ and $K : C \longrightarrow P(X)$ be a set-valued function defined by

$$K(x) = \{y \in X \mid y * x \in \{0, 1\}\}$$

for all $x \in C$. Then $K(2) = \{0, 1, 2, 3\}$ which is irreducible ideal of X. Hence (K, C) is an irreducible idealistic soft BCK-algebra over X.
Use ii) and iii), $(H, B)\widetilde{\wedge}(K, C) = (F, D)$, $D = B \times C = \{(1, 2), (2, 2), (4, 2)\}$ and $F(1, 2) = H(1) \cap K(2) = \{0, 2\} \lhd X$, $F(2, 2) = H(2) \cap K(2) = \{0, 1\} \lhd X$, $F(2, 4) = H(4) \cap K(2) = \{0, 1, 2, 3\} \lhd X$. Then
$F(1, 2) = \{0, 2\} = \{0, 1, 2, 3\} \cap \{0, 2, 4\}$ and $F(1, 2) \neq \{0, 1, 2, 3\}$ and $F(1, 2) \neq \{0, 2, 4\}$. Hence $(H, B)\widetilde{\wedge}(K, C) = (F, D)$ is not irreducible idealistic soft BCK-algebra over X.

4.6. Theorem. *Let (F, A) and (G, B) be two irreducible idealistic soft BCK/BCI-algebras over X. If $A \cap B \neq \emptyset$, then the intersection $(F, A) \widetilde{\cap} (G, B)$ is an irreducible idealistic soft BCK/BCI-algebras over X.*

Proof. Let $x \in C = A \cap B$. By Definition 2.10, for all $x \in C$, we have $H(x) = F(x)$ or $H(x) = G(x)$ or $H(x) = F(x) = G(x)$ if $F(x) = G(x)$. For all $x \in C$, $F(x)$ and $G(x)$ are irreducible, so $H(x)$ is irreducible too. This completes the proof. □

4.7. Theorem. *Let (F, A) and (G, B) be two irreducible idealistic soft BCK/BCI-algebras over X. If A and B are disjoint, then the union $(H, C) = (F, A) \widetilde{\cup} (G, B)$ is an irreducible idealistic soft BCK/BCI-algebras over X.*

Proof. Let $C = A \cup B$, $A \cap B = \phi$ be null set and $(H, C) = (F, A)\widetilde{\cup}(G, B)$. By hypothesis $F(a)$ is an irreducible ideal of X for all $a \in A$ and $G(b)$ is an irreducible ideal of X for all $b \in B$. Let $c \in C$. Since $A \cap B = \phi$, $c \in A \setminus B$ or $c \in B \setminus A$. By Definition 2.11, if $c \in A \setminus B$, then $H(c) = F(c)$ is irreducible ideal. If $c \in B \setminus A$, then $H(c) = G(c)$ is an irreducible ideal of X. By definition $H(c)$ is an irreducible ideal of X also. This completes the proof. □

5. Prime Idealistic Soft BCK/BCI-algebras

In this section, we introduce the definition of prime soft idealistic over lower BCK-semilattice and several example are given. X will be a lower BCK-semilattice throughout this section. We start some definitions

5.1. Definition. Let X be lower BCK-semilattice and (F, A) be a soft BCK/BCI-algebra over X and (G, I) is a soft ideal of (F, A). (G, I) is a non-whole called *a prime soft ideal* of (F, A) if $G(x)$ is a prime ideal of $F(x)$ for all $x \in I$.

5.2. Definition. Let (F, A) be a soft idealistic BCK/BCI-algebra over X. (F, A) is called *a prime idealistic soft BCK/BCI-algebra over X* if $F(x)$ is a prime ideal of X for all $x \in A$.

5.3. Example. Let X={0,a,b,c,d} be a lower BCK-semilattice with following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let (F, A) be a soft over X, where $A = \{b, c, d\}$ and $F : A \to P(X)$ is a set-valued function defined by $F(x) = \{y \in X \mid y \in x^{-1}I\}$ for all $x \in A$ where $I = \{0, a\} \subset X$ and $x^{-1}I = \{y \in X \mid x \wedge y \in I\}$. Then $F(b) = \{0, a, c, d\}$, $F(c) = \{0, a, b, d\}$ and $F(d) = \{0, a, b, c\}$ are prime ideals of X. Therefore (F, A) is a prime soft idealistic BCK/BCI-algebra over X.

5.4. Theorem. *Let (F, A) and (G, B) be two prime idealistic soft BCK/BCI-algebras over X. If $A \cap B \neq \emptyset$, then the intersection $(H, C) = (F, A) \,\widetilde{\cap}\, (G, B)$ is a prime idealistic soft BCK/BCI-algebras over X.*

Proof. Let $x \in C = A \cap B$. By Definition 2.10, for all $x \in C$, we have $H(x) = F(x)$ or $H(x) = G(x)$ or $H(x) = F(x) = G(x)$ if $F(x) = G(x)$. Since, for all $x \in C$, $F(x)$ and $G(x)$ are prime, it follows that for all $x \in C$, $H(x)$ is prime. This completes the proof. □

5.5. Theorem. *Let (F, A) and (G, B) be two prime idealistic soft BCK/BCI-algebras over X. If A and B are disjoint, then the union $(H, C) = (F, A) \,\widetilde{\cup}\, (G, B)$ is a prime idealistic soft BCK/BCI-algebras over X.*

Proof. Let $C = A \cup B$, $A \cap B = \phi$ be null set and $(F, A)\widetilde{\cup}(G, B)$. By hypothesis $F(a)$ is a prime ideal of X for all $a \in A$ and $G(b)$ is a prime ideal of X for all $b \in B$. Let $c \in C$. Then $c \in A$ or $c \in B$ but not c belongs to $A \cap B$. By Definition 2.11, if $c \in A \setminus B$, then $H(c) = F(c)$ is a prime ideal. If $c \in B \setminus A$, then $H(c) = G(c)$ is a prime ideal of X. Hence $H(x)$ is a prime ideal of X for all $x \in C$. This completes the proof. □

Example 5.6 shows that the bi-intersection of two prime idealistic soft BCK/BCI-algebras (F, A) and (G, B) over a set X need not be a prime idealistic soft BCK/BCI-algebra.

5.6. Example. Let $X = \{0, a, b, c\}$ be a lower BCK-semilattice with following Cayley table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Let $A = \{b, c\}$, $B = \{c\}$ and $I = \{0, a\}$, $I' = \{0, b\}$. Define $F : A \to \mathcal{P}(X)$ by, for $x \in A$, $F(x) = \{y \mid x \wedge y \in I\}$; similarly $G : B \to \mathcal{P}(X)$ by for $x \in B$, $G(x) = \{y \mid x * y \in I'\}$. Then $F(b) = F(c) = \{0, a\}$ and $G(c) = \{0, c\}$. By construction of F, $F(b)$, $F(c)$ and $G(c)$ are prime idealistic soft BCK-algebra then $F(c) \cap G(c) = 0$ and $a \wedge b = b * (b * a) = b * b = 0 \in F(c) \cap G(c)$. But neither a nor b belongs to $F(c) \cap G(c)$. It follows that bi-intersection of two prime idealistic soft BCK/BCI-algebras (F, A) and (G, B) over X need not be a prime idealistic soft BCK/BCI-algebra.

6. Maximal Idealistic Soft BCK/BCI-algebra

In this section, we introduce the definition of maximal idealistic soft BCK/BCI-algebra and several example are given. Moreover, we construct some basic properties using the definition of irreducible idealistic soft algebra, irreducible idealistic soft BCK/BCI-algebra and maximal idealistic soft BCK/BCI-algebra over X.

6.1. Definition. Let (F, A) be a soft idealistic BCK/BCI-algebra over X. (F, A) is called *a maximal idealistic soft BCK/BCI-algebra over X* if $F(x)$ is a maximal ideal of X for all $x \in A$.

6.2. Example. Let $X = \{0, a, b, c, d, e, f, g\}$ and consider the following Cayley table:

*	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Then $(X;0,*)$ is a BCI-algebra (see[2]). Let (F,A) be a soft set over X, where $A = \{0,a,b,c\}$ and $F : A \to P(X)$ is a set-valued function defined as follows: $F(x) = \{0\} \cup \{y \in X \mid o(x) = o(y)\}$ for all $x \in$ A.
Then $F(0) = F(a) = F(b) = F(c) = \{0,a,b,c\}$ is an maximal ideal of X. Hence (F,A) is a maximal idealistic soft BCI-algebra over X.

We know that every maximal ideal is an irreducible ideal on BCK-algebra[4], and prime ideals and irreducible ideals are the same in a commutative BCK-algebra which is clear from [8, Theorem 8.10]. We will give the same properties on soft BCK/BCI-algebras.

6.3. Theorem. Every maximal idealistic soft BCK/BCI-algebra over X is an irreducible idealistic soft BCK/BCI-algebra over X.

Proof. Let (F,A) be a maximal soft idealistic BCK/BCI-algebra over X and let I and J be ideals of X such that $F(x) = I \cap J$ for all $x \in A$. Then $F(x) \subseteq I$ and $F(x) \subseteq J$. Then $F(x) = I$ or $F(x) = J$ since $F(x)$ is a maximal ideal of X. Hence $F(x)$ is an irreducible ideal of X. Therefore (F,A) is an irreducible soft idealistic BCK/BCI-algebra over X. □

6.4. Example. Let X={0,a,b,c,d} be a BCK-algebra with following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let (F,A) be a soft over X, where $A = \{b,c,d\}$ and $F : A \to P(X)$ is a set-valued function defined by
$F(x) = \{y \in X \mid y \in x^{-1}I\}$ for all $x \in A$ where $I = \{0,a\} \subset X$ and $x^{-1}I = \{y \in X \mid x \wedge y \in I\}$. Then $F(b) = \{0,a,c,d\}$, $F(c) = \{0,a,b,d\}$ and $F(d) = \{0,a,b,c\}$ are maximal ideals of X. Therefore (F,A) is a maximal soft idealistic BCK/BCI-algebra over X.

6.5. Theorem. Let X be a commutative BCK/BCI-algebra. Then (F,A) is a prime idealistic soft BCK/BCI-algebra over X if and only if it is an irreducible idealistic soft BCK/BCI-algebra over X.

Proof. Let (F,A) be a prime idealistic soft BCK/BCI-algebra over X. , $F(x)$ is a prime ideal for each $x \in A$ if and only if $F(x)$ is an irreducible ideal since X is commutative by [8, Theorem 8.10]. □

Every prime idealistic soft BCK-algebra need not be a maximal idealistic soft BCK-algebra over commutative algebra X.

6.6. Example. Let $X = \{0,1,2,3,4\}$ be commutative BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Let (F, A) be a soft set over X where $A = \{1, 2\}$ and $F : A \longrightarrow P(X)$ is defined by, for $x \in A$

$$F(x) = \{y \in X \mid y * x \in \{0, 1\}\}$$

Therefore $F(1) = \{0, 1, 2\}$ and $F(2) = \{0, 1, 2\}$ are prime ideals of X but not maximal ideals. Hence (F, A) is prime soft idealistic but not maximal soft idealistic. In fact $\{0, 1, 2, 3\}$ is an ideal and properly contain $\{0, 1, 2\}$.

7. Direct Products

Let $(X; *_1; 0)$ and $(Y; *_2; 0)$ be two algebras of type (2,0) and $X \times Y$ denote the direct product set of sets X and Y. Define $*$ operation on $X \times Y$ with

$(a, b) * (c, d) = (a *_1 c,\ b *_2 d)$ where $(a, b),\ (c, d) \in X \times Y$ .... (*)

Let 0 denote $(0, 0) \in X \times Y$. Then it is easy to check that $(X \times Y; *; 0)$ becomes a BCK/BCI-algebra. Namely $(X \times Y; *; 0)$ satisfies **BCI-1** : Let $(x, y), (a, b), (u, v) \in X \times Y$. Then
$(((x, y) * (a, b)) * ((x, y) * (u, v)) * ((u, v) * (a, b)) =$
$(((x *_1 a, y *_2 b) * (x *_1 u, y *_2 v)) * (u *_1 a, v *_2 b) =$
$((((x *_1 a) *_1 (x *_1 u), (y *_2 b) *_2 (y *_2 v)) * (u *_1 a, v *_2 b) =$
$((((x *_1 a) *_1 (x *_1 u)) *_1 (u *_1 a), ((y *_2 b) *_2 (y *_2 v)) *_2 (v *_2 b)) = 0$. The other conditions for $(X \times Y; *; 0)$ to be a BCK/BCI-algebra are satisfied similarly.

7.1. Definition. The BCK/BCI-algebra $(X \times Y; *; 0)$ with (*) operation is called *direct product BCK/BCI-algebra* of BCK/BCI-algebras X and Y.

7.2. Lemma. [8]. *Let S be a subalgebra of $(X; *_1; 0)$ and T be a subalgebra of $(Y; *_2; 0)$. Then $S \times T$ is a subalgebra of $X \times Y$*

Note that from now on we use xy for the element $x*y$ obtained by the binary operation $*$.

7.3. Definition. Let $X \times Y$ be a direct product BCK/BCI-algebra of BCK/BCI-algebras X and Y, and let (F, A) be a soft set over X and (G, B) be a soft set over Y. Define

$$F \times G : A \times B \to P(U) \times P(U)$$

$$\text{by } (F \times G)(a, b) = F(a) \times G(b) \text{ where } (a, b) \in A \times B.$$

It is easy to check that $F \times G$ is well-defined. We call $(F \times G, A \times B)$ *direct product soft set* of the soft sets (F, A) and (G, B).

7.4. Theorem. *Let (F, A) be a soft BCK/BCI-algebra over X and (G, B) be a soft BCK/BCI-algebra over Y. Then $(F \times G, A \times B)$ is a soft BCK/BCI-algebra over $X \times Y$.*

Proof. Let $(a, b) \in A \times B$. By definition $(F \times G)(a, b) = F(a) \times G(b)$ is a direct product of subalgebras $F(a)$ and $G(b)$ of X and Y respectively. Hence $F(a) \times G(b)$ is a subalgebra of $X \times Y$ by Lemma 7.2. □

7.5. Lemma. *Let X and Y be BCK/BCI-algebras and H_1 and H_2 be ideals of BCK/BCI-algebras X and Y respectively. Then $H_1 \times H_2$ are ideals of BCK/BCI-algebra $X \times Y$.*

Proof. Let $(x,y) \in X \times Y$ and $(h_1,h_2) \in H_1 \times H_2$. Assume that $(x,y)(h_1,h_2) \in H_1 \times H_2$. Then $(x,y)(h_1,h_2) = (xh_1, yh_2) \in H_1 \times H_2$ implies $xh_1 \in H_1$ and $yh_2 \in H_2$. Then $x \in H_1$ and $y \in H_2$ since H_1 and H_2 are ideals of BCK/BCI-algebras X and Y respectively. Hence $(x,y) \in X \in H_1 \times H_2$. This completes the proof. □

7.6. Theorem. *Let (F,A) be a soft BCK/BCI-algebra over X and (G,B) be a soft BCK/BCI-algebra over Y. If (F_1,A_1) is a soft ideal of (F,A) and (G_1,B_1) is a soft ideal of (G,B), then $(F_1 \times G_1, A_1 \times B_1)$ is a soft ideal of $(F \times G, A \times B)$.*

Proof. Let (F_1,A_1) be a soft ideal of (F,A) and (G_1,B_1) be a soft ideal of (G,B). Then $A_1 \subseteq A$ and $B_1 \subseteq B$ and $F_1(x)$ is ideal of $F(x)$ for all $x \in A_1$ and $G_1(y)$ is ideal of $G(y)$ for all $y \in B_1$. We first fix $(x,y) \in A_1 \times B_1$ and let $(a,b) \in F(x) \times G(y)$ and $(m,n) \in F_1(x) \times G_1(y)$ where $a \in F(x), b \in G(y), m \in F_1(x), n \in G_1$. Assume that $(a,b)(m,n) \in F_1(x) \times G_1(y)$. By definition $am \in F_1(x), bn \in G_1(y)$. Since $F_1(x)$ and $G_1(y)$ are ideals of $F(x)$ and $G(y)$, we have $a \in F_1(x)$ and $b \in G_1(y)$. Hence $(a,b) \in F_1(x) \times G_1(y)$. □

7.7. Theorem. *Let (F,A) be a idealistic soft BCK/BCI-algebra over X and (G,B) be a idealistic soft BCK/BCI-algebra over Y. Then $(F \times G, A \times B)$ is a idealistic soft BCK/BCI-algebra over $X \times Y$.*

Proof. By Definition 2.21, to complete the proof we show that, for any $(a,b) \in A \times B$, $F(a) \times G(b)$ is a ideal of $X \times Y$ since, by definition $(F \times G)(a,b) = F(a) \times G(b)$. $F(a)$ and $G(b)$ is an ideal of X and Y respectively. So for any $x \in X$, $y \in Y$ and $m \in F(a)$, $n \in G(b)$, $xm \in F(a)$ implies $x \in F(a)$. Similarly $yn \in G(b)$ implies $y \in G(b)$. Hence Assume that $(x,y)(m,n) \in (F \times G)(a,b)$. Then $(xm, yn) = (x,y)(m,n) \in (F \times G)(a,b) = (F(a), G(b))$ implies $xm \in F(a)$ and $yn \in G(b)$. Since $F(a)$ and $G(b)$ are ideals in X and Y respectively, we have $x \in F(a)$ and $y \in G(b)$. Hence $(x,y) \in F(a) \times G(b) = (F \times G)(a,b)$. This completes the proof. □

7.8. Theorem. *Let (F,A) be an irreducible idealistic soft BCK/BCI-algebra over X and (G,B) be an irreducible idealistic soft BCK/BCI-algebra over Y. If, for any $(a,b) \in A \times B$, $F(a) \times G(b)$ are irreducible ideals of $X \times Y$, then $F(a)$ and $G(b)$ are irreducible ideals of X and Y. The converse is true if for any ideal I of $F(a) \times G(b)$ has the form $I_1 \times I_2$ for some ideals I_1 of $F(a)$ and I_2 of $G(b)$ for all $(a,b) \in A \times B$.*

Proof. Let I_1, I_2 be two ideals of X and I_1', I_2' be two ideals of Y. We first prove that $(I_1 \cap I_2) \times (I_1' \cap I_2') = (I_1 \times I_1') \cap (I_2 \times I_2')$. For $(a,b) \in (I_1 \cap I_2) \times (I_1' \cap I_2')$ if and only if $a \in I_1 \cap I_2$, $b \in I_1' \cap I_2'$ if and only if $(a,b) \in (I_1 \times I_1') \cap (I_2 \times I_2')$.
Assume that $F(a) \times G(b)$ are irreducible ideals of $X \times Y$. Let $F(a) = I_1 \cap I_2$ and $G(b) = I_1' \cap I_2'$. Then $F(a) \times G(b) = (I_1 \cap I_2) \times (I_1' \cap I_2')$. From what we have proved preceding, it follows that $F(a) \times G(b) = (I_1 \times I_1') \cap (I_2 \times I_2')$. Then $F(a) \times G(b) = I_1 \times I_1'$ or $F(a) \times G(b) = I_2 \times I_2'$. Hence $(F(a) = I_1,\ G(b) = I_1')$ or $(F(a) = I_2,\ G(b) = I_2')$. It implies that $F(a) \times G(b) = I_1 \times I_1'$ or $F(a) \times G(b) = I_2 \times I_2'$.
Conversely, assume that $F(a)$ and $G(b)$ are irreducible ideals for all $a \in A$ and $b \in B$. Let $F(a) \times G(b) = I \cap J$ for some ideals I and J. By hypothesis $I = I_1 \times I_1'$ and $J = J_2 \times J_2'$. Then $F(a) \times G(b) = (I_1 \times I_1') \cap (J_2 \times J_2') = (I_1 \cap J_2) \times (I_1' \cap J_2')$. Then $F(a) = I_1 \cap J_2$ and $G(b) = I_1' \cap J_2'$. By assumption $F(a) = I_1 = J_2$ and $G(b) = I_1' = J_2'$. Hence $F(a) \times G(b) = I = J$ □

Example 7.9 shows that the converse of Theorem 7.8 need not be true in general.

7.9. Example. Let $X = \{0,1,2,3\}$ be the algebra and the irreducible ideals $F(1) = \{0,1\}$ and $F(2) = \{0,2\}$ considered in Example 4.3. Consider the sets

$I_1 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$ and $I_2 = \{(0,0),(0,2),(0,3),(1,0),(1,2),(1,3)\}$. It is easy to check that I_1 and I_2 are ideals and $F(1)\times F(2) = \{(0,0),(0,1),(1,0),(1,2)\} = I_1 \cap I_2$. But $F(1) \times F(2) \neq I_1$ and $F(1) \times F(2) \neq I_2$.

8. Conclusion

In this article, we introduced the notion of soft irreducible, prime and maximal soft ideals, irreducible, prime and maximal soft idealistic over BCK/BCI-algebras. Moreover, we presented to prove the important theorems of classical algebra for soft BCK/BCI-algebras in Theorem6.3 and Theorem6.5.

Acknowledgements
The authors are highly grateful to referees and Professor Yücel Tıraş, Editor-in-Chief, for their careful reading and valuable comments helpful in improving this paper.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 15 – 22

On the structure of finite groups with the given numbers of involutions

N. Ahanjideh* and M. Foroudi Ghasemabadi†

Abstract

Let G be a finite non-solvable group. In this paper, we show that if $1/8$ of elements of G have order two, then G is either a simple group isomorphic to $PSL_2(q)$, where $q \in \{7, 8, 9\}$ or $G \cong GL_2(4).\mathbb{Z}_2$. In fact in this paper, we answer Problem 132 in [1].

2000 AMS Classification: 20D06, 20D20, 20E45.

Keywords: Involution, classification of finite groups with

Received 17/02/2013 : Accepted 29/11/2013 Doi : 10.15672/HJMS.2015449083

1. Introduction

An involution in a group G is an element of order two. For a finite group G, let $T(G)$ denote the set of involutions of a group G, $t(G) = |T(G)|$ and let $t_0(G) = t(G)/|G|$. It is an elementary fact in group theory that a group G in which all of its non-identity elements have order two, is abelian. We can also see that every finite group which at least $3/4$ of its elements have order two is abelian. But this result can not be extended for the case $t_0(G) < 3/4$. So finding the structure of the finite groups according to the number of their involutions can be an interesting question. This problem has received some attention in existing literature. For instance, Wall [8] classified all finite groups in which more than half of the elements are involutions.After that, Berkovich in [1] described the structure of all finite non-solvable groups which at least $1/4$ of its elements have order two. Also, in addition to classifying all finite groups G with $t_0(G) = 1/4$, he put forward he following problem (Problem 132 in [1]):
Problem. What is the structure of finite groups G with $t_0(G) = 1/8$?
In this paper, we show that:
Main Theorem. If G is a finite non-solvable group with $t(G) = |G|/8$, then either G is a simple group isomorphic to $PSL_2(q)$, where $q \in \{7, 8, 9\}$ or $G \cong GL_2(4).\mathbb{Z}_2$.

*Department of Mathematics, Shahrekord University, Shahrekord, Iran
Email: ahanjideh.neda@sci.sku.ac.ir
†Department of Mathematics, Tarbiat Modares University, Tehran, Iran
Email:foroudi@modares.ac.ir

Actually, in [4], Davoudi Monfared has shown that the non-abelian finite simple group G with $t_0(G) = 1/8$ is isomorphic to $PSL_2(q)$, where $q \in \{7, 8, 9\}$. In [4], the author has completed the proof using the simplicity of the considered groups. In this paper, we prove the main theorem using the classification of groups with given 2-Sylow subgroups of order 8 and the relation between the set of involutions of the given group and its normal subgroups.

2. Notation and preliminary results

Throughout this paper, we use the following notation: for a finite group G, by $O(G)$ and $O'(G)$ we denote the maximal normal subgroup of odd order in G and the maximal normal subgroup of odd index in G, respectively. Also, the set of all p-Sylow subgroups of G is denoted by $\mathrm{Syl}_p(G)$, and the set of all involutions of G by $T(G)$. The p-part of G, denoted by $|G|_p$, is the order of any $P \in \mathrm{Syl}_p(G)$. We indicate by $n_p(G)$ the number of p-Sylow subgroups of G. Let $t(G) = |T(G)|$ and let $t_0(G) = t(G)/|G|$. It is evident that for every finite group G, $t_0(G) < 1$. If $H \leq G$ and $x, g \in G$, then for simplicity of notation, we write H^g and x^g instead of $g^{-1}Hg$ and $g^{-1}xg$, respectively. All further unexplained notation is standard and can be found in [3].

We start with some known facts about the structure of the finite group G with given 2-Sylow subgroups and some facts about the Frobenius groups:

2.1. Lemma. *Let G be a finite group and $S \in \mathrm{Syl}_2(G)$. Then:*

- (i) [7] *If S is cyclic, then G is 2-nilpotent (i.e. G has a normal complement to a 2-Sylow subgroup). In particular, G is solvable.*
- (ii) [9] *If S is abelian, then $O'(G/O(G))$ is a direct product of a 2-group and simple groups of one of the following types:*
  - (a) *$PSL_2(2^n)$, where $n > 1$;*
  - (b) *$PSL_2(q)$, where $q \equiv 3$ or $5 \pmod 8$ and $q > 3$;*
  - (c) *a simple group S such that for each involution J of S, $C_S(J) = \langle J \rangle \times R$, where R is isomorphic to $PSL_2(q)$, where $q \equiv 3$ or $5 \pmod 8$.*
- (iii) [7, P. 462] (The dihedral theorem) *If S is a dihedral group, then $G/O(G)$ is isomorphic to one of the following groups:*
  - (a) *a 2-Sylow subgroup of G;*
  - (b) *the alternating group A_7;*
  - (c) *a subgroup of $\mathrm{Aut}(PSL_2(q))$ containing $PSL_2(q)$, where q is odd.*

  In particular if G is simple, then G is isomorphic to either A_7 or $PSL_2(q)$, where $q > 3$ is odd.
- (iv) [2] *If G is a non-abelian simple group, S is an elementary belian group of order $q = 2^n$ and for every $x \in S - \{1\}$, $C_G(x) = S$, then $G \cong PSL_2(q)$.*

2.2. Remark. [7, P. 480] In Lemma 2.1(ii), for the simple groups of third type, Janko and Thompson showed that q is either 5 or 3^{2n+1}. In the case where $q = 5$, R has been shown to be a particular group of order $2^3.3.5.7.11.19$, by Janko, which is named the Janko group J_1. If $q = 3^{2n+1}$, then there is one other infinite family of simple groups, ${}^2G_2(3^{2n+1})$, for $n > 0$, discovered by Ree. It has been proved that these groups are the only examples.

2.3. Lemma. [7, Theorem 3.1, P. 339] *Let G be a Frobenius group with the kernel K and the complement H. Then K is nilpotent. In particular, if $|H|$ is even, then K is abelian.*

In 1993, Berkovich proved the following lemmas:

2.4. Lemma. [1] *If G is a non-solvable group, then $t_0(G) \geq 1/4$ if and only if $G = PSL(2,5) \times E$, where $\exp(E) \leq 2$;*

In the following two lemmas, we collect some facts about finite groups and the number of their involutions which are known and we just give hints of their proofs.

2.5. Lemma. *Let G_1 and G_2 be two finite groups and let H be a normal subgroup of G_1. Then the following hold:*

(i) *$t(G_1 \times G_2) = t(G_1)t(G_2) + t(G_1) + t(G_2)$;*
(ii) *if $[G_1 : H]$ is an odd number, then $t(G_1) = t(H)$;*
(iii) *if $|G_1|$ is even, then $t(G_1)$ is an odd number. In particular if $t_0(G_1) = 1/8$, then $|G_1|_2 = 8$;*
(vi) *let $x \in G_1$ and $cl_H(x) = \{h^{-1}xh : h \in H\}$. Then $|cl_H(x)| = [H : C_H(x)]$.*

Proof. Since $T(G_1 \times G_2) = \{(x,y) : x \in T(G_1),\ y \in T(G_2)\} \cup \{(x,1) : x \in T(G_1)\} \cup \{(1,y) : y \in T(G_2)\}$, the proof of (i) is straightforward. For the proof of (ii), it is easy to see that if $[G_1 : H]$ is an odd number, then $\mathrm{Syl}_2(G_1) = \mathrm{Syl}_2(H)$ and hence, (ii) follows. For the proof of (iii), since for every $x \in G_1 - T(G_1)$, x and x^{-1} are distinct elements in $G_1 - T(G_1)$, and $1 \in G_1 - T(G_1)$, we deduce that $|G_1 - T(G_1)|$ is an odd number and hence, $t(G_1) = |G_1| - |G_1 - T(G_1)|$ is an odd number, too. Also if $t_0(G_1) = 1/8$, then $|G_1| = 8t(G_1)$ and hence, $|G_1|_2 = 8$, so (iii) follows. It remains to prove (iv). We can see that the function $\varphi : \{hC_H(x) : h \in H\} \longrightarrow cl_H(x)$ such that for every $hC_H(x)$, $\varphi(hC_H(x)) = hxh^{-1}$ is a bijection and hence, $|cl_H(x)| = [H : C_H(x)]$, as claimed in (iv). □

2.6. Lemma. *Let G be a finite group with a normal subgroup H of odd order and $\bar{G} = G/H$. If by $\bar{x}$, we denote the image of an element x of G in $\bar{G}$, then we have the following:*

(i) *For every involution $\bar{x} \in T(\bar{G})$, there exists an involution $y \in G$ such that $\bar{x} = \bar{y}$;*
(ii) *if $H \neq 1$ and $P \in \mathrm{Syl}_2(G)$ is an abelian non-cyclic group or a dihedral group of order 8, then $t(\bar{G})|H| \gneqq t(G)$;*
(iii) *if $cl_{\bar{G}}(\bar{x}_1), ..., cl_{\bar{G}}(\bar{x}_n)$ are distinct conjugacy classes of involutions in $\bar{G}$, then we can assume that $x_1, ..., x_n$ are distinct involutions in G and there exist natural number $t_1, ..., t_n$ such that $t_1, ..., t_n \leq |H|$ and $\sum_{i=1}^{n} |cl_{\bar{G}}(\bar{x}_i)|t_i = t(G)$.*

Proof. For the proof of (i), it suffices to establish that for the case where $\bar{x}$ is an arbitrary element in $T(\bar{G})$ and x is not an involution. Since $\bar{x} \in T(\bar{G})$ and $|H|$ is odd, we can see that $x^2 \in H$ and $O(x) = 2m$, where m is an odd number greater than 1. Thus there exist the integer numbers r and $s = 2k+1$ (for some integer k) such that $2r+sm = 1$ and hence, $x = (x^2)^r(x^2)^{mk}x^m$, which implies that $\bar{x} = \bar{x}^m$ and hence, (i) follows, because $y = x^m$ is an involution in G. For the proof of (ii), suppose, contrary to our claim, that $t(G) \geq t(\bar{G})|H|$. A routine argument shows that $T(G) = \{xh : \bar{x} \in T(\bar{G}),\ h \in H\}$. According to the structure of P, we can choose distinct involutions x and y in P such that $O(xy) = 2$. Since $x,\ y,\ xy \in T(G)$, for every $h \in H$, we have $O(xh) = O(yh) = O(xyh) = 2$. This implies that $h^{-1} = xyhyx = xyhxhyh = xyxyh = h$. But $|H|$ is odd, so $H = 1$. Now, we are going to prove (iii). By (i), we can assume that $x_1, ..., x_n$ are distinct involutions in G such that $T(\bar{G}) = \bigcup_{i=1}^{n} cl_{\bar{G}}(\bar{x}_i)$ and for every $1 \leq i, j \leq n$ such that $i \neq j$, we have $cl_{\bar{G}}(\bar{x}_i) \cap cl_{\bar{G}}(\bar{x}_j) = \emptyset$. For $x \in T(G)$, put $\mathrm{Inv}(x, H) := \{h \in H : O(xh) = 2\}$ and for every $1 \leq i \leq n$, put $t_i := |\mathrm{Inv}(x_i, H)|$. Obviously, for every $x \in T(G)$, $1 \in \mathrm{Inv}(x, H) \subseteq H$. Thus $t_1, ..., t_n \leq |H|$ are natural numbers. Also, it is evident that for every $g \in G$ and $x \in T(G)$, $|\mathrm{Inv}(x, H)| = |\mathrm{Inv}(x^g, H)|$ and for every $y \in T(G)$, there exists $1 \leq i \leq n$ such that $\bar{y} \in cl_{\bar{G}}(\bar{x}_i)$, so there exists $\bar{x}_i^{\bar{g}} \in cl_{\bar{G}}(\bar{x}_i)$ such that $(x_i^g)^{-1}y \in \mathrm{Inv}(x_i^g, H)$ and hence,

$T(G) = \{x_i^g h : h \in \mathrm{Inv}(x_i^g, H)\}$. Now, we can see that $t(G) = \sum_{i=1}^n |cl_{\bar{G}}(\bar{x}_i)|t_i$, which is the desired conclusion. □

2.7. Lemma. *Let q be an odd number and $q > 3$. Then*

(i) $t(PSL_2(q)) = \begin{cases} q(q-1)/2, & \text{if } q \equiv 3 \pmod 4 \\ q(q+1)/2, & \text{if } q \equiv 1 \pmod 4 \end{cases}$;

(ii) *there exist involutions $x_1, x_2 \in T(PGL_2(q))$ such that $t(PGL_2(q)) = |cl_{PGL_2(q)}(x_1)| + |cl_{PGL_2(q)}(x_2)|$ and, $|cl_{PGL_2(q)}(x_1)| = q(q-1)/2$ and $|cl_{PGL_2(q)}(x_2)| = q(q+1)/2$.*

Proof. The Dickson's result about the maximal subgroups of $PSL_2(q)$ (see [5]) shows that the dihedral group D_{q+1} of order $(q+1)$, where $q \neq 7,\ 9$ and the dihedral group D_{q-1} of order $(q-1)$, where $q \geq 13$ are maximal subgroups of $PSL_2(q)$. First let $q \geq 13$. Obviously, if $q \equiv 3 \pmod 4$, then $(q+1)/2$ is even and hence, the center of D_{q+1} contains an involution x^+ and if $q \equiv 1 \pmod 4$, then $(q-1)/2$ is even and hence, the center of D_{q-1} contains an involution x^-. This shows that if $q \equiv 3 \pmod 4$, then $C_{PSL_2(q)}(x^+) = D_{q+1}$ and if $q \equiv 1 \pmod 4$, then $C_{PSL_2(q)}(x^-) = D_{q-1}$. But $\gcd(2, q) = 1$ and hence, every involution of $PSL_2(q)$ is a semi-simple element of $PSL_2(q)$. Thus for every involution x of $PSL_2(q)$, there exists a maximal torus containing x. It is known that the maximal torus of $PSL_2(q)$ are cyclic groups of order $(q \pm 1)/\gcd(2, q-1)$. Thus in the case where $q \equiv 3 \pmod 4$, the involutions are contained in the maximal torus of order $(q+1)/2$ and in the case where $q \equiv 1 \pmod 4$, the involutions are contained in the maximal torus of order $(q-1)/2$. On the other hand, it is known that the maximal torus of $PSL_2(q)$ of the same order are conjugate and hence, all involutions in $PSL_2(q)$ are conjugate in $PSL_2(q)$. Therefore, if $q \equiv 3 \pmod 4$, then $t(PSL_2(q)) = |cl_{PSL_2(q)}(x^+)| = q(q-1)/2$ and if $q \equiv 1 \pmod 4$, then $t(PSL_2(q)) = |cl_{PSL_2(q)}(x^-)| = q(q+1)/2$, as claimed in (i). If $q \leq 11$, then ATLAS [3], completes the proof of (i). For the proof of (ii), since the dihedral group $D_{2(q+1)}$ of order $2(q+1)$ and the dihedral group $D_{2(q-1)}$ of order $2(q-1)$, where $q > 5$ are the maximal subgroups of $PGL_2(q)$, and the maximal torus of $PGL_2(q)$ are cyclic groups of order $(q \pm 1)$, the same argument as in the proof of (i) shows that $T(PGL_2(q)) = cl_{PGL_2(q)}(x^+) \cup cl_{PGL_2(q)}(x^-)$, where x^+ and x^- are central involutions of $D_{2(q+1)}$ and $D_{2(q-1)}$, and also $C_{PGL_2(q)}(x^+) = D_{2(q+1)}$ and $C_{PGL_2(q)}(x^-) = D_{2(q-1)}$. Thus $t(PGL_2(q)) = |cl_{PGL_2(q)}(x^+)| + |cl_{PGL_2(q)}(x^-)| = q(q-1)/2 + q(q+1)/2$. This completes the proof of (ii). Also, if $q = 5$, then according to ATLAS [3], the result is obvious. □

3. Proof of the main theorem

Let $P \in \mathrm{Syl}_2(G)$. Under the assumption of the main theorem, $t_0(G) = 1/8$ and hence, Lemma 2.5(iii) shows that $|P| = 8$. Thus, the proof falls naturally into three cases: P is an abelian group, the quaternion group or the dihedral group of order 8:

Case 1. If P is abelian, then since G is non-solvable, we deduce from Lemma 2.1(i) that P is not cyclic. Also, Lemma 2.1(ii) implies that $O'(G/O(G))$ is a direct product of a 2-group and simple groups of one of the types mentioned in 2.1(ii)(a-c). For abbreviation, put $H := O(G)$, $K/H := O'(G/O(G))$ and let $\bar{x}$ be the image of an element x of G in G/H. Obviously $|H|$ and $[G : K]$ are odd numbers. So Lemma 2.5(i) guarantees that $t(G) = t(K)$. On the other hand, $t(G) = |G|/8$ and hence, $t(K) = |G|/8$. This implies that $t_0(K) = [G : K]/8$. If $[G : K] \neq 1$, then since $[G : K]$ is an odd number and $t_0(K) < 1$, we deduce that $[G : K] \in \{3, 5, 7\}$. This forces $t_0(K)$ to be greater than 1/4. Now, we apply Lemma 2.4 to conclude that $K \cong PSL_2(5) \times E$, where $\exp(E) \leq 2$. Since $|G|_2 = 8$ and $[G : K]$ is odd, we have $|K|_2 = 8$ as well and hence, $|E| = 2$ and $|K| = 120$. Thus Lemmas 2.5(i) and 2.7(i) allow us to conclude that $t(K) = 31$. But as was obtained above, $t(K) = t_0(K)|K| = [G : K]|K|/8 \in \{3|K|/8, 5|K|/8, 7|K|/8\} = \{45, 75, 105\}$, which is a contradiction. This shows that $[G : K] = 1$. Thus $G/H = O'(G/H)$. Now according to Lemma 2.1(ii) and Remark 2.2, we have the following possibilities for G/H:

(i) $G/H \cong F \times PSL_2(2^n)$, where $n > 1$ and F is a 2-group. Then since $|G|_2 = 8$, we obtain that either $(n, |F|) = (2, 2)$ or $(n, |F|) = (3, 1)$. If $(n, |F|) = (2, 2)$, then by [3], all involutions in $PSL_2(4)$ are conjugate and hence, Lemma 2.6(iii) leads us to find involutions x_1, x_2, x_3 in $T(G)$ and natural numbers r, s, t such that $1 \leq r, s, t \leq |H|$ and $t(G) = r|cl_{G/H}(\bar{x}_1)| + s|cl_{G/H}(\bar{x}_2)| + t|cl_{G/H}(\bar{x}_3)|$, so $r/15 + s + t = |H|$. We claim that either $s > |H|/3$ or $t > |H|/3$. Suppose, contrary to our claim, that $s \leq |H|/3$ and $t \leq |H|/3$. Thus, we have $r/15 \geq |H|/3$, which implies that $r \geq 5|H|$, a contradiction. Therefore, without loss of generality, we can assume that $s > |H|/3$. According to the proof of Lemma 2.6(iii), $s = |H_2|$, where $H_2 = \{h \in H : O(x_2h) = 2\}$. Put $G_2 := \langle x_2, H\rangle$. Then since for every $h \in H_2$, $x_2hx_2h = 1$, we have $h^{-1}x_2h = h^{-2}x_2$. So $s = |H_2| \leq |cl_H(x_2)| = [H : C_H(x_2)]$ and hence, $|C_H(x_2)| < 3$, which forces $C_H(x_2) = 1$. Thus G_2 is a Frobenius group with the kernel H. It follows from Lemma 2.3, H is abelian. Thus it is easy to see that H_2 is a subgroup of H and hence, $|H_2|$ divides $|H|$. Moreover, since $s > |H|/3$, we have $|H| = |H_2| = s$. Thus $r/15+t+s = |H|$ forces $r = t = 0$, which is a contradiction. It remains to consider the case where $(n, |F|) = (3, 1)$. Since ATLAS [3] shows that $t(G/H) = |PSL_2(8)|/8$, $|H|t(G/H) = t(G)$. Thus by Lemma 2.6(ii), $H = 1$ and hence, in this case, $G \cong PSL_2(8)$.

(ii) $G/H \cong F \times PSL_2(q)$, where F is a 2-group and, $q \equiv 3$ or $5 \pmod 8$ and $q > 3$. Then since $|G/H|_2 = |G|_2 = 8$, we obtain that $|F| = 2$. Since $PSL_2(5) \cong PSL_2(4)$, as mentioned in (i), we can see that $G/H \ncong PSL_2(5) \times F$. This allows us to assume that $q > 5$. Thus by Lemmas 2.5(i) and 2.7(i), $t(G/H) = 2(|cl_{PSL_2(q)}(x)|) + 1 = 2q(q \pm 1)/2 + 1$, where $x \in T(PSL_2(q))$. Thus Lemma 2.6(ii) gives that either $H \neq 1$ and $2(|cl_{PSL_2(q)}(x)|)+1 > 2|PSL_2(q)|/8$ or $H = 1$ and $2(|cl_{PSL_2(q)}(x)|) + 1 = 2|PSL_2(q)|/8$. Obviously, $2(|cl_{PSL_2(q)}(x)|) + 1 \neq 2|PSL_2(q)|/8$. Therefore, $H \neq 1$ and hence, $4 \mid |C_{PSL_2(q)}(x)| \leq 8$. This ensures that $|C_{PSL_2(q)}(x)| = 4$. Now applying Lemma 2.1(iv) to $PSL_2(q)$ shows that $q = 4$, which is a contradiction.

(iii) G/H is isomorphic to the Janko group J_1. Then by ATLAS [3], $t(G/H) = 7.11.19$ and hence, $t(G/H) < \frac{|J_1|}{8}$, which is a contradiction with Lemma 2.6(ii).

(iv) G/H is isomorphic to the Ree group ${}^2G_2(q)$, where $q = 3^{2n+1}$, for $n > 0$. Then applying Lemma 2.1(ii)(c) and Remark 2.2 show that for each involution J of ${}^2G_2(q)$, $C_{{}^2G_2(q)}(J) = \langle J\rangle \times R$, where R is isomorphic to $PSL_2(q)$ and the 2-Sylow subgroups of ${}^2G_2(q)$ are 2-elementary abelian. Thus the 2-Sylow subgroup PH/H of G/H contains 7 elements of order 2. On the other hand, all 2-Sylow subgroups of G/H are conjugate in G/H and hence, every involution of G/H is conjugate with one of the involutions in PH/H. Thus $t(G/H) \leq |\bigcup_{\bar{x} \in PH/H - \{\bar{1}\}} cl_{G/H}(\bar{x})| \leq \frac{7|G/H|}{2|PSL_2(q)|} = \frac{7|G/H|}{q(q^2-1)}$, which is less than $\frac{|G/H|}{8}$, a contradiction with Lemma 2.6(ii).

Consequently, the above results show that if P is abelian, then G is a simple group isomorphic to $PSL_2(8)$.

Case 2. Let P be a quaternion group of order 8. Then since $P \leq N_G(P)$, $8 \mid |N_G(P)|$. Thus $n_2(G) \leq |G|/8$. If there exist $P_1, P_2 \in \mathrm{Syl}_2(G)$ such that $P_1 \cap P_2 \neq 1$, then $P_1 \cap P_2$ contains an element of order 2. On the other hand, the quaternion group has exactly one element of order 2. This implies that $t(G) < n_2(G) \le |G|/8$, which is a contradiction. Thus the intersection of every two 2-Sylow subgroups of G is trivial and hence, $t(G) = n_2(G)$. So by our assumption, $n_2(G) = |G|/8$, which forces $|N_G(P)| = 8$ and hence, $N_G(P) = P$. This implies that for every $x \in G - P$, $x^{-1}Px \cap P = 1$ and hence, G is a Frobenius group and P is its complement. It follows immediately from the fact which the kernel of G and P are nilpotent that G is solvable, which is a contradiction with our assumption.

Case 3. Let P be a dihedral group of order 8. Put $H := O(G)$. Then by Lemma 2.1(iii), we have the following possibilities for G/H:

(i) G/H is a 2-group. Thus G/H is solvable. But $H = O(G)$ is a solvable group, because by our assumption $|H|$ is odd. We thus get that G is solvable, which is a contradiction with our assumption.

(ii) $G/H \cong A_7$. Then it is easy to check that $t(G/H) = t(A_7) = \frac{7.6.5.4}{8} < \frac{|A_7|}{8}$, which is a contradiction with Lemma 2.6(ii).

(iii) G/H is isomorphic to a subgroup of $\mathrm{Aut}(PSL_2(q))$ containing $PSL_2(q)$, where q is odd. It is known that $\mathrm{Aut}(PSL_2(q)) \cong PGL_2(q).\mathbb{Z}_n$, where $q = p^n$, for a prime p. In the following, we will consider the cases $q \equiv \pm 1 \pmod 8$ and $q \equiv \pm 3 \pmod 8$, separately:

(a) Let $q \equiv \pm 1 \pmod 8$. Then since $|PSL_2(q)|_2 = |G|_2 = 8$, we deduce that $G/H = K/H.\mathbb{Z}_m$, where $K/H \cong PSL_2(q)$ and m is an odd divisor of n and hence, the same conclusion as that of in the proof of Case 1 can be drawn to conclude that $[G : K] = 1$. Thus $G/H \cong PSL_2(q)$. So from Lemma 2.7(i), we obtain that

$$t(G/H) = \begin{cases} q(q-1)/2, & \text{if } q \equiv -1 \pmod 8 \\ q(q+1)/2, & \text{if } q \equiv 1 \pmod 8 \end{cases}.$$

If $H \neq 1$, then Lemma 2.6(ii) yields $t(G/H) > |G/H|/8$ and hence,

$$\begin{cases} q+1 < 8, & \text{if } q \equiv -1 \pmod 8 \\ q-1 < 8, & \text{if } q \equiv 1 \pmod 8 \end{cases},$$

which is impossible. This forces H to be a trivial group and hence, $G \cong PSL_2(q)$. Thus Lemma 2.7(i) leads to

$$|PSL_2(q)|/8 = t(PSL_2(q)) = \begin{cases} q(q-1)/2, & \text{if } q \equiv -1 \pmod 8 \\ q(q+1)/2, & \text{if } q \equiv 1 \pmod 8 \end{cases},$$

so

$$\begin{cases} q+1 = 8, & \text{if } q \equiv -1 \pmod 8 \\ q-1 = 8, & \text{if } q \equiv 1 \pmod 8 \end{cases}.$$

Consequently, $q \in \{7, 9\}$ and hence, $G \cong PSL_2(7)$ or $PSL_2(9)$.

(b) Let $q \equiv \pm 3 \pmod 8$. Since $q = p^n \equiv \pm 3 \pmod 8$, an easy computation shows that n is odd and hence, according to $|PSL_2(q)|_2 = 4$, we deduce that $G/H = K_0/H.\mathbb{Z}_m$, where $K_0/H \cong PGL_2(q)$ and m is an odd divisor of n. Now, as in the proof of Case 1, we see that $m = 1$ and hence, $G/H = K_0/H = K/H.\mathbb{Z}_2$, where $K/H \cong PSL_2(q)$. Now we conclude from Lemmas 2.6(ii) and 2.7(ii) that $t(PGL_2(q)) = q(q-1)/2 + q(q+1)/2 = q^2 \geq \frac{|G|}{8|H|} = \frac{|PGL_2(q)|}{8}$. This shows that $q^2 \geq q(q^2-1)/8$, so $q \in \{3, 5\}$. If $q = 3$, then $PGL_2(3) \cong S_4$ and hence, G/H is solvable. Also the fact that $|H|$ is odd, forces H to be solvable and hence, G is solvable, which is a contradiction with our assumption.

It remains to consider the case where $q = 5$. Let $q = 5$. For abbreviation assume that $G/H = PGL_2(5)$ and $K/H = PSL_2(5)$. According to Lemma 2.7(ii), $t(PGL_2(5)) \neq |PGL_2(5)|/8$ and hence, $H \neq 1$. Let $\bar{x}$ be the image of an element x of G in G/H. Note that $PSL_2(q)$ is a normal subgroup of $PGL_2(q)$ and hence, every 2-Sylow subgroup $\bar{P}$ of $PGL_2(q)$ contains an involution $\bar{x} \in \bar{P} \cap PSL_2(q)$. Thus, we conclude from Lemmas 2.6(iii) and 2.7(ii) that there exist involutions $\bar{x}_1 \in T(PSL_2(q))$ and $\bar{x}_2 \in T(PGL_2(q))$, and natural numbers s, t such that $1 \leq s, t \leq |H|$, $|cl_{PGL_2(q)}(\bar{x}_1)| = \frac{|PGL_2(5)|}{2(5-1)}$ and $|cl_{PGL_2(q)}(\bar{x}_2)| = \frac{|PGL_2(5)|}{2(5+1)}$, and $\frac{s|PGL_2(5)|}{2(5-1)} + \frac{t|PGL_2(5)|}{2(5+1)} = s|cl_{PGL_2(q)}(\bar{x}_1)| + t|cl_{PGL_2(q)}(\bar{x}_2)| = t(G) = |G|/8$. This gives

$$3s + 2t = 3|H|. \tag{3.1}$$

By Lemma 2.6(i), we have $x_1, x_2 \in T(G)$. For $i \in \{1, 2\}$, put $H_i := \{h \in H : O(x_i h) = 2\}$. Then as mentioned in the proof of Lemma 2.6(iii), $|H_1| = s$ and $|H_2| = t$. Put $G_1 := \langle x_1, H \rangle$. Since $t \leq |H|$, we see $|H| \leq 3|H| - 2t = 3s$ and hence, $s \geq |H|/3$. Thus one of the following possibilities holds:

(I) $s > |H|/3$. Obviously if $h \in H_1$, then $x_1 h x_1^{-1} = h^{-1}$, so $h x_1 h^{-1} = x_1 h^{-2}$ and hence, $|H|/3 < s = |H_1| \leq |cl_{G_1}(x_1)|$. Thus $[G_1 : C_{G_1}(x_1)] > |H|/3$. This shows that $|C_{G_1}(x_1)| < 6$. On the other hand, $2 \mid |C_{G_1}(x_1)|$, $|G_1| = 2|H|$ and $|H|$ is an odd number. This gives $|C_{G_1}(x_1)| = 2$ and hence, $C_{G_1}(x_1) = \langle x_1 \rangle$. It follows that $C_H(x_1) = 1$. This forces G_1 to be a Frobenius group with the kernel H. Thus H is abelian, by Lemma 2.3 and hence, it follows easily that H_1 is a subgroup of H, so $|H_1|$ divides $|H|$. But $|H|$ is an odd number and $|H_1| = s > |H|/3$. Thus $|H_1| = |H|$ and hence, (3.1) shows that $t = 0$, which is impossible.

(II) $s = |H|/3$. Then (3.1) shows that $|H| + 2t = 3|H|$ and hence, $t = |H|$. This gives that $H_2 = H$. Thus for every $h \in H$, $O(x_2 h) = 2$, so $h^{x_2} = h^{-1}$. Let $h_1, h_2 \in H$, then $h_2^{-1} h_1^{-1} = (h_1 h_2)^{x_2} = h_1^{x_2} h_2^{x_2}$ and hence, $h_2^{-1} h_1^{-1} = h_1^{-1} h_2^{-1}$. This shows that H is abelian. Thus for every $k_1, k_2 \in H_1$, $(k_1 k_2)^{x_1} = k_1^{x_1} k_2^{x_1} = k_1^{-1} k_2^{-1} = (k_1 k_2)^{-1}$, so $O(x_1 k_1 k_2) = 2$ and hence, $k_1 k_2 \in H_1$. Obviously, $k_1^{-1} \in H_1$ and hence, $H_1 \leq H$. As above, we can assume that there exist $g, y \in K$ and $h \in H$ such that $x_1 x_1^g = x_1^y h$, because there exists $\bar{g} \in PSL_2(q)$ such that $\bar{x}_1 \bar{x}_1^{\bar{g}}$ is an involution in $PSL_2(q)$ and hence, since $T(PSL_2(q)) = cl_{PSL_2(q)}(\bar{x}_1)$, we deduce that there exists $y \in K$ such that $x_1 x_1^g \in x_1^y H$. First let $H_1 H_1^g \lneqq H$, then we see that $|H|/3$ divides $3|H_1 \cap H_1^g|$ and $|H_1 \cap H_1^g|$ divides $|H_1| = |H|/3$. Thus either $|H_1 \cap H_1^g| = |H|/9$ or $|H_1 \cap H_1^g| = |H|/3$. If $|H_1 \cap H_1^g| = |H|/9$, then $|H_1 H_1^g| = |H|$ and hence, $H_1 H_1^g = H$, which is a contradiction. Thus $|H_1 \cap H_1^g| = |H|/3$. This forces $[H_1 : H_1 \cap H_1^g] = 1$, which shows that $H_1 = H_1 \cap H_1^g$. So $H_1 = H_1^g$. It follows immediately that for every $h \in H_1$, there exists $h_1 \in H_1$ such that $h = h_1^g$ and hence, $(x_1 x_1^g)^{-1} h (x_1 x_1^g) = (x_1^g)^{-1} h^{-1} x_1^g = (x_1^{-1} h_1^{-1} x_1)^g = h_1^g = h$, because if $h \in H_1$, then $O(x_1 h) = 2$, which shows that $x_1^{-1} h x_1 = h^{-1}$. This implies that $H_1 \leq C_H(x_1 x_1^g)$ and hence, $|cl_H(x_1 x_1^g)| \leq 3$. But as mentioned above, $x_1 x_1^g = x_1^y h$. Thus, since H is an abelian group of an odd order, we deduce that for every $u \in H_1^y$, $u^{-1} x_1 x_1^g u = u^{-1} x_1^y h u = u^{-1} x_1^y u h = u^{-2} x_1^y h$, so $|H|/3 = |H_1^y| \leq |cl_H(x_1 x_1^g)|$. From this, we conclude that $|H|/3 \leq 3$ and hence, $|H| \in \{3, 9\}$, because $3 \mid |H|$ and $2 \nmid |H|$. If $|H| = 9$, then $|cl_H(x_1 x_1^g)| = 3$. But H is an abelian normal subgroup of G and $x_1 x_1^g = x_1^y h$.

Thus $C_H(x_1) = C_H(x_1^y)^{y^{-1}} = C_H(x_1^y h)^{y^{-1}}$ and hence, $|C_H(x_1)| = 3$. Thus

$$x_1 \notin C_G(H). \tag{3.2}$$

Also, $H \leq C_G(H)$ and hence, $\frac{C_G(H)}{H} \trianglelefteq \frac{G}{H} = PGL_2(5)$. Thus $\frac{C_G(H)}{H} = PGL_2(5)$, $\frac{C_G(H)}{H} = PSL_2(5)$ or $\frac{C_G(H)}{H} = 1$. If $\frac{C_G(H)}{H} = PSL_2(5) = K/H$ or $\frac{C_G(H)}{H} = PGL_2(5) = G/H$, then $K \leq C_G(H)$ and hence, $x_1 \in C_G(H)$, contrary to (3.2). Thus $\frac{C_G(H)}{H} = 1$, which means that $C_G(H) = H$. But $|H| = 9$ and hence, $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $H \cong \mathbb{Z}_9$. So by $N - C$-theorem, we obtain $PGL_2(5) = G/H = N_G(H)/C_G(H) \lesssim \mathrm{Aut}(H) \cong GL_2(3)$ or $\mathbb{Z}_8$. This forces 5 to divide $|GL_2(3)|$ or $|\mathbb{Z}_8|$, a contradiction. If $|H| = 3$, then applying the same reasoning as above shows that $\frac{C_G(H)}{H} = PSL_2(5) \cong SL_2(4)$, which leads us to see that $G \cong (\mathbb{Z}_3 \times SL_2(4)).\mathbb{Z}_2 \cong GL_2(4).\mathbb{Z}_2$. Now, a trivial verification in GAP [6] shows that $t(GL_2(4).\mathbb{Z}_2) = 2|GL_2(4)|/8$. Thus G can be isomorphic to $GL_2(4).\mathbb{Z}_2$. Now consider the case where $H_1H_1^g = H$, then a slight change in the above statements shows that $|H| \in \{9, 27\}$ and $|H_1 \cap H_1^g| = |H|/9$. If $|H| = 9$, then applying the previous argument leads us to get a contradiction. Thus $|H| = 27$ and hence, our assumption forces H to be isomorphic to $\mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Now, the same argument as above gives that $H = C_G(H)$. Thus $PGL_2(5) = \frac{G}{H} = \frac{N_G(H)}{C_G(H)} \lesssim \mathrm{Aut}(H)$. But $|\mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)| = 11232$ and $|\mathrm{Aut}(\mathbb{Z}_9 \times \mathbb{Z}_3)| = |GL_3(3)| = 108$, so $|PGL_2(5)| \nmid |\mathrm{Aut}(H)|$, a contradiction.

Consequently, the above cases show that either G is a simple group isomorphic to $PSL_2(q)$, where $q \in \{7, 8, 9\}$ or $G \cong GL_2(4).\mathbb{Z}_2$, as desired.

Acknowledgment. The authors wish to thank the referee for carefully reading this paper and making useful suggestions. The first author would like to thank Shahrekord University for financial support.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 23 – 32

Contact CR-warped product submanifolds in Sasakian space forms

Mehmet Atçeken *

Abstract

In this paper we consider Contact CR-warped product submanifolds and we investigate the status of equality in the inequality which has been found by I. Hasegawa and I. Mihai in [8].

2000 AMS Classification: 53C25, 53C40

Keywords: Sasakian space form, Contact CR-warped product, Contact CR-product.

Received 07/02/2011 : Accepted 08/01/2014 Doi : 10.15672/HJMS.2015449082

1. Preliminaries

Recently, in [8] I. Hasegawa and I. Mihai studied contact CR-warped products in Sasakian manifolds and obtained inequalities for the squared norm of the second fundamental form in terms of the warping function for contact CR-warped products in a Sasakian space form. Afterwards, I. Mihai and K. Arslan studied warped products which are CR-submanifolds in Sasakian and Kenmotsu manifolds, respectively, and established general sharp inequalities for a CR-warped product in Sasakian and Kenmotsu space forms(see [1, 7]).

In [2], we also studied contact CR-warped product submanifolds in a cosymplectic manifold and gave a necessary and sufficient condition for a contact CR-warped product to be contact CR product. In this paper we give a necessary and sufficient condition for contact CR-warped product to be contact CR-product in a Sasakian space form.

In this section, we will give some notations used throughout this paper. We recall some necessary facts and formulas from the theory of Sasakian manifolds and their submanifolds.

*Department of Mathematics, Gaziosmanpasa University, Faculty of Arts and Sciences, Department of Mathematics, 60100 Tokat/ TURKEY
Email: mehmet.atceken382@gmail.com

A $(2m+1)$-dimensional Riemannian manifold $(\bar{M}, g)$ is said to be an almost contact metric manifold if it admits an endomorphism ϕ of its tangent bundle $T\bar{M}$, a vector field ξ and a 1-form η, satisfying;

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \ \ \eta(\xi) = 1, \quad \eta(\phi X) = 0 \tag{1.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{1.2}$$

for any vector fields X, Y tangent to $\bar{M}$. Furthermore, an almost contact metric manifold is called a Sasakian manifold if ϕ and ξ satisfy;

$$(\bar{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \quad \textit{and} \quad \bar{\nabla}_X \xi = \phi X, \tag{1.3}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$[5].

Now, let $\bar{M}$ be a $(2n+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g) and M be an m-dimensional isometrically immersed the submanifold in $\bar{M}$. Moreover, we denote the Levi-Civita connections by $\bar{\nabla}$ and ∇, respectively. Then the Gauss and Weingarten formulas for M in $\bar{M}$ are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{1.5}$$

for any vector fields X, Y tangent to M and vector V normal to M, where $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$, h and A denote the second fundamental form and shape operator of M in $\bar{M}$, respectively. The A and h are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{1.6}$$

We denote the Riemannian curvature tensors of $\bar{\nabla}$ and the induced connection ∇ by $\bar{R}$ and R, respectively. Then the Gauss and Codazzi equations are, respectively, given by

$$(\bar{R}(X, Y)Z)^{\top} = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \tag{1.7}$$

and

$$(\bar{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) \tag{1.8}$$

for any vector fields X, Y, Z tangent to M, where the covariant derivative of h is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{1.9}$$

for any vector fields X, Y, Z tangent to M, where $(\bar{R}(X, Y)Z)^{\perp}$ and $(\bar{R}(X, Y)Z)^{\top}$ denote the normal and tangent components of $\bar{R}(X, Y)Z$, respectively, with respect to the submanifold[8].

For any vector field X tangent to M, we set

$$\phi X = fX + \omega X, \tag{1.10}$$

where fX and ωX are the tangential and normal components of ϕX, respectively. Then f is an endomorphism of the TM and ω is a normal-bundle valued 1-form of TM. For the same reason, any vector field V normal to M, we set

$$\phi V = BV + CV, \tag{1.11}$$

where BV and CV are the tangential and normal components of ϕV, respectively. Then B is an endomorphism of the normal bundle $T^{\perp}M$ to TM and C is a normal-bundle valued 1-form of $T^{\perp}M$.

A plane section π in $\Gamma(T\bar{M})$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a Sasakian space form and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}(c)$ is given by

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} \qquad (1.12)\end{aligned}$$

for any vector fields X, Y, Z tangent to $\bar{M}$[6]. For more details, we refer to the references.

2. Contact CR-Warped Product Submanifolds in Sasakian Manifolds

In this section, we will define contact CR-warped product submanifolds in a Sasakian manifold, have obtained some the inequalities in a Sasakian manifold.

Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and let f be a positive smooth function on M_1. We consider the product manifold $M_1 \times M_2$ with its projections $\pi : M_1 \times M_2 \longrightarrow M_1$ and $\eta : M_1 \times M_2 \longrightarrow M_2$. The warped product $M = M_1 \times_f M_2$ is a manifold $M_1 \times M_2$ equipped with the Riemannian metric such that

$$g(X,Y) = g_1(\pi_* X, \pi_* Y) + (f o \pi)^2 g_2(\eta_* X, \eta_* Y),$$

for any $X, Y \in \Gamma(TM)$, where $*$ stand for differential of map and $\Gamma(TM)$ denote set of the differentiable vector fields on M. Thus we have $g = g_1 \otimes f^2 g_2$. The function f is called the warping function of the warped product manifold $M = M_1 \times_f M_2$. If we denote the Levi Civita connection on M by ∇, then we have the following Proposition for the warped product manifold[3].

2.1. Proposition. *Let $M = M_1 \times_f M_2$ be a warped product manifold. For $X, Y \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$, we have*

(1) $\nabla_X Y \in \Gamma(TM_1)$*, that is, M_1 is totally geodesic submanifold in M,*

(2) $\nabla_X V = \nabla_V X = X(\ln f)V$*,*

(3) $nor\nabla_Z V = -g(Z,V)grad(\ln f)$*, that is, M_2 is totally umbilical submanifold in M,*

(4) $tan\nabla_Z V = \nabla'_Z V \in \Gamma(TM_2)$ *is the lift of $\nabla'_Z V$ on M_2, where ∇' denote the Levi-Civita connection of g_2*[4].

If the warping function f is constant, then the warped product is said to be Riemannian product.

Let M be an m-dimensional Riemannian manifold with Riemannian metric g, and let $\{e_1, e_2, ..., e_m\}$ be an orthonormal basis for $\Gamma(TM)$. For a smooth function f on M, the gradient and Hessian of f are, respectively, defined by

$$X(f) = g(\text{grad}(f), X) \qquad (2.1)$$

and

$$H^f(X,Y) = X(Y(f)) - (\nabla_X Y)f = g(\nabla_X \text{grad}(f), Y) \tag{2.2}$$

for any $X, Y \in \Gamma(TM)$. The Laplacian of f is defined by

$$\triangle f = \sum_{i=1}^{m}\{(\nabla_{e_i} e_i)f - e_i(e_i(f))\} = -\sum_{i=1}^{m} g(\nabla e_i \text{grad}(f), e_i). \tag{2.3}$$

From (2.2) and (2.3), note that the Laplacian is essentially the negative of the trace of the Hessian.

From the integration theory on manifolds, if M is a compact orientable Riemannian manifold without boundary, we have

$$\int_M \Delta f dV = 0, \tag{2.4}$$

where dV is the volume element of M[2].

By analogy with submanifolds in a Kenmotsu manifold, different classes of submanifolds in a Sasakian manifold were considered by many geometers(see, references).

2.2. Definition. Let M be an isometrically immersed submanifold of a Sasakian manifold $\bar{M}$.
(1) A submanifold M is tangent to ξ is called an invariant submanifold if ϕ preserves any tangent space of M, that is, $\phi(T_M(p)) \subset T_M(p)$, for every $p \in M$.
(2) A submanifold M tangent to ξ is called an anti-invariant submanifold if ϕ maps any tangent space of M into the normal space, that is, $\phi(T_M(p)) \subset T_M^{\perp}(p)$, for every $p \in M$.
(3) A submanifold M tangent to ξ is called a contact CR-submanifold if it admits an invariant distribution whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is, $TM = D \oplus D^{\perp}$, with $\phi(D_p) \subset D_p$ and $\phi(D_p^{\perp}) \subset T_M^{\perp}(p)$, for every $p \in M$.

In this paper, we shall consider warped product manifolds which are in the form $M = M_T \times_f M_{\perp}$ in a Sasakian manifold $\bar{M}$ such that M is tangent to ξ, where M_T is an invariant submanifold tangent to ξ and $M_{\perp}$ is an anti-invariant submanifold of $\bar{M}$. We simply call such manifolds contact CR-product submanifolds.

3. Contact CR Warped Product Submanifolds in Sasakian Space Forms

In this section, we will give the main results of this paper. Firstly, we will give the following two lemmas and a theorem for later use.

3.1. Lemma. *Let $M = M_T \times_f M_{\perp}$ be a contact CR-warped product submanifold of a Sasakian manifold $\bar{M}$. Then we have*

$$g(h(X,Y), \phi Y) = [\eta(X) - \phi X(\ln f)]g(Y,Y) \tag{3.1}$$

and

$$g(h(\phi X, Y), \phi Y) = \|Y\|^2 X(\ln f) \tag{3.2}$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_{\perp})$.

Proof. For any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$, by using (1.3), (1.4) and considering Proposition 2.1(2), we have

$$\begin{aligned} g(h(X,Y),\phi Y) &= g(\bar{\nabla}_Y X,\phi Y) = -g(\phi\bar{\nabla}_Y X, Y) \\ &= -g(\bar{\nabla}_Y \phi X - (\bar{\nabla}_Y \phi)X, Y) \\ &= -g(\nabla_Y \phi X, Y) + g(-g(X,Y)\xi + \eta(X)Y, Y) \\ &= -\phi X(\ln f)g(Y,Y) + \eta(X)g(Y,Y) \end{aligned}$$

and

$$\begin{aligned} g(h(\phi X,Y),\phi Y) &= g(\bar{\nabla}_Y \phi X,\phi Y) = g((\bar{\nabla}_Y\phi)X + \phi\bar{\nabla}_Y X,\phi Y) \\ &= g(-g(X,Y)\xi + \eta(X)Y,\phi Y) - g(\bar{\nabla}_Y X,\phi^2 Y) \\ &= -g(\bar{\nabla}_Y X, -Y + \eta(Y)\xi) \\ &= g(\nabla_Y X, Y) = X(\ln f)g(Y,Y), \end{aligned}$$

which proves our assertion. □

3.2. Lemma. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a Sasakian manifold $\bar{M}$. Then we have*

$$\|h(X,Y)\|^2 = g(h(\phi X,Y),\phi h(X,Y)) + [\eta(X) - \phi X \ln f]^2 g(Y,Y), \tag{3.3}$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$.

Proof. Making use of (1.3), (1.4) and consider Proposition 2.1 and Lemma 3.1 we have

$$\begin{aligned} g(h(\phi X,Y),\phi h(X,Y)) &= g(\bar{\nabla}_Y \phi X - \nabla_Y \phi X, \phi h(X,Y)) \\ &= g((\bar{\nabla}_Y \phi)X + \phi\bar{\nabla}_Y X - \phi X(\ln f)Y, \phi h(X,Y)) \\ &= g(-g(X,Y)\xi + \eta(X)Y, \phi h(X,Y)) \\ &+ g(\phi\bar{\nabla}_Y X, \phi h(X,Y)) + \phi X(\ln f)g(h(X,Y),\phi Y) \\ &= -\eta(X)g(h(X,Y),\phi Y) + g(h(X,Y),h(X,Y)) + \phi X(\ln f)g(h(X,Y),\phi Y) \\ &= \|h(X,Y)\|^2 + [\phi X(\ln f) - \eta(X)]g(h(X,Y),\phi Y) \\ &= \|h(X,Y)\|^2 + [\phi X(\ln f) - \eta(X)][\eta(X) - \phi X(\ln f)]g(Y,Y) \end{aligned}$$

This completes the proof of the Lemma. □

3.3. Theorem. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a Sasakian space form $\bar{M}(c)$. Then we have*

$$\begin{aligned} 2\|h(X,Y)\|^2 &= \{H^{\ln f}(X,X) + H^{\ln f}(\phi X,\phi X) - 2\phi X(\ln f)\eta(X) \\ &+ 2(\phi X(\ln f))^2 + 2\eta^2(X) + \eta(\nabla_X X)\eta(grad \ln f) \\ &+ \left(\frac{c+3}{4}\right) g(\phi X,\phi X)\}g(Y,Y), \end{aligned} \tag{3.4}$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$.

Proof. By using (1.8), (1.9) and making use of $\bar{\nabla}$ being Levi-Civita connection, we have

$$
\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= g((\bar{\nabla}_X h)(\phi X,Y)-(\bar{\nabla}_{\phi X}h)(X,Y),\phi Y)\\
&= g(\bar{\nabla}_X h(\phi X,Y)-h(\nabla_X\phi X,Y)-h(\phi X,\nabla_X Y),\phi Y)\\
&- g(\bar{\nabla}_{\phi X}h(X,Y)-h(\nabla_{\phi X}X,Y)-h(\nabla_{\phi X}Y,X),\phi Y)\\
&= X[g(h(\phi X,Y),\phi Y)]-g(\bar{\nabla}_X\phi Y,h(\phi X,Y))-g(h(\nabla_X\phi X,Y),\phi Y)\\
&- g(h(\nabla_X Y,\phi X),\phi Y)-\phi X[g(h(X,Y),\phi Y)]+g(\bar{\nabla}_{\phi X}\phi Y,h(X,Y))\\
&+ g(h(\nabla_{\phi X}X,Y),\phi Y)+g(h(\nabla_{\phi X}Y,X),\phi Y).
\end{aligned}
$$

Taking into account (3.1), (3.2) and Proposition 2.1(2), we obtain

$$
\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= X[X(\ln f)g(Y,Y)]-g(h(\phi X,Y),(\bar{\nabla}_X\phi)Y+\phi\bar{\nabla}_X Y)\\
&- g(Y,Y)\{\eta(\nabla_X\phi X)-(\phi\nabla_X\phi X)(\ln f)\}-X(\ln f)g(h(Y,\phi X),\phi Y)\\
&- \phi X[\{\eta(X)-\phi X(\ln f)\}g(Y,Y)]+g(h(X,Y),(\bar{\nabla}_{\phi X}\phi)Y+\phi\bar{\nabla}_{\phi X}Y)\\
&+ g(Y,Y)\{\eta(\nabla_{\phi X}X)-(\phi\nabla_{\phi X}X)(\ln f)\}+\phi X(\ln f)g(h(X,Y),\phi Y)\\
&= X(X(\ln f))g(Y,Y)+2(X(\ln f))^2g(Y,Y)-g(h(\phi X,Y),\phi\nabla_X Y+\phi h(X,Y))\\
&- \eta(\nabla_X\phi X)g(Y,Y)+g(Y,Y)(\phi\nabla_X\phi X)(\ln f)-(X(\ln f))^2g(Y,Y)\\
&- \phi X[\eta(X)-\phi X(\ln f)]g(Y,Y)-2\phi X(\ln f)\{\eta(X)-\phi X(\ln f)\}g(Y,Y)\\
&+ g(h(X,Y),\phi\nabla_{\phi X}Y+\phi h(\phi X,Y))+\eta(\nabla_{\phi X}X)g(Y,Y)\\
&- (\phi\nabla_{\phi X}X)(\ln f)g(Y,Y)+\phi X(\ln f)\{\eta(X)-\phi X(\ln f)\}g(Y,Y)\\
&= X(X(\ln f))g(Y,Y)+(X(\ln f))^2g(Y,Y)-X(\ln f)g(h(\phi X,Y),\phi Y)\\
&- g(h(\phi X,Y),\phi h(X,Y))-\eta(\nabla_X\phi X)g(Y,Y)+(\phi\nabla_X\phi X)(\ln f)g(Y,Y)\\
&- \phi X[\eta(X)]g(Y,Y)+\phi X(\phi X(\ln f))g(Y,Y)-2\phi X(\ln f)\eta(X)g(Y,Y)\\
&+ 2(\phi X(\ln f))^2g(Y,Y)+\phi X(\ln f)g(h(X,Y),\phi Y)\\
&+ g(h(X,Y),\phi h(\phi X,Y))+\eta(\nabla_{\phi X}X)g(Y,Y)-(\phi\nabla_{\phi X}X)(\ln f)g(Y,Y)\\
&+ \phi X(\ln f)\eta(X)g(Y,Y)-(\phi X(\ln f))^2g(Y,Y)\\
&= X(X(\ln f))g(Y,Y)-2g(h(\phi X,Y),\phi h(X,Y))-\eta(\nabla_X\phi X)g(Y,Y)\\
&+ (\phi\nabla_X\phi X)(\ln f)g(Y,Y)-\phi X[\eta(X)]g(Y,Y)+\phi X(\phi X(\ln f))g(Y,Y)\\
&- \phi X(\ln f)\eta(X)g(Y,Y)+(\phi X(\ln f))^2g(Y,Y)\\
&+ \phi X(\ln f)\{\eta(X)-\phi X(\ln f)\}g(Y,Y)+\eta(\nabla_{\phi X}X)g(Y,Y)\\
&- (\phi\nabla_{\phi X}X)(\ln f)g(Y,Y). \qquad (3.5)
\end{aligned}
$$

We know that on a Sasakian manifold

$$
\begin{aligned}
\phi X[\eta(X)] &= \phi X g(X,\xi)=g(\bar{\nabla}_{\phi X}X,\xi)+g(X,\bar{\nabla}_{\phi X}\xi)\\
&= \eta(\nabla_{\phi X}X)+g(\phi^2X,X)=\eta(\nabla_{\phi X}X)-g(\phi X,\phi X), \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\eta(\nabla_X\phi X) &= g(\bar{\nabla}_X\phi X,\xi)=g((\bar{\nabla}_X\phi)X+\phi\nabla_X X,\xi)\\
&= g(-g(X,X)\xi+\eta(X)X,\xi)=-g(X,X)+\eta^2(X)=-g(\phi X,\phi X). \qquad (3.7)
\end{aligned}
$$

Furthermore, considering Proposition 2.1, M_T is totally geodesic in M and $\mathrm{grad}(\ln f)\in\Gamma(TM_T)$, by direct calculations, we obtain

$$
\begin{aligned}
(\phi\nabla_{\phi X}X)(\ln f) &= g(\phi\nabla_{\phi X}X,\mathrm{grad}(\ln f))=g(\bar{\nabla}_{\phi X}\phi X-(\bar{\nabla}_{\phi X}\phi)X,\mathrm{grad}(\ln f))\\
&= g(\nabla_{\phi X}\phi X,\mathrm{grad}(\ln f))-g(-g(\phi X,X)\xi+\eta(X)\phi X,\mathrm{grad}(\ln f))\\
&= (\nabla_{\phi X}\phi X)(\ln f)-\eta(X)\phi X(\ln f) \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
(\phi\nabla_X\phi X)(\ln f) &= g(\phi\nabla_X\phi X, \text{grad}(\ln f)) = -g(\bar{\nabla}_X\phi X, \phi\text{grad}(\ln f)) \\
&= -g((\bar{\nabla}_X\phi)X + \phi\bar{\nabla}_X X, \phi\text{grad}(\ln f)) \\
&= g(-g(X,X)\xi + \eta(X)X, \phi\text{grad}(\ln f)) - g(\phi\nabla_X X, \phi\text{grad}(\ln f)) \\
&= -(\nabla_X X)(\ln f) + \eta(\nabla_X X)\eta(\phi\text{grad}(\ln f)) + \eta(X)\phi X(\ln f). \qquad (3.9)
\end{aligned}
$$

So by substituting (3.6), (3.7), (3.8) and (3.9) into (3.5), we get

$$
\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= X(X(\ln f))g(Y,Y) - 2g(h(\phi X,Y),\phi h(X,Y)) + g(\phi X,\phi X)g(Y,Y) \\
&+ g(Y,Y)\{\eta(X)\phi X(\ln f) - (\nabla_X X)(\ln f) + \eta(\nabla_X X)\eta(\text{grad}(\ln f))\} \\
&- g(Y,Y)\{\eta(\nabla_{\phi X}X) - g(\phi X,\phi X)\} + \phi X(\phi X(\ln f))g(Y,Y) \\
&+ \eta(\nabla_{\phi X}X)g(Y,Y) - (\nabla_{\phi X}\phi X)(\ln f)g(Y,Y) + \eta(X)\phi X(\ln f)g(Y,Y) \\
&= \{X(X(\ln f)) + \phi X(\phi X(\ln f)) - (\nabla_X X)(\ln f) \\
&- (\nabla_{\phi X}\phi X)(\ln f) + 2g(\phi X,\phi X) + \eta(\nabla_X X)\eta(\text{grad}(\ln f)) \\
&+ 2\eta(X)\phi X(\ln f)\}g(Y,Y) - 2g(h(\phi X,Y),\phi h(X,Y)) \\
&= \{H^{\ln f}(X,X) + H^{\ln f}(\phi X,\phi X) + 2g(\phi X,\phi X) + 2\eta(X)\phi X(\ln f) \\
&+ \eta(\nabla_X X)\eta(\text{grad}(\ln f))\}g(Y,Y) - 2g(h(\phi X,Y),\phi h(X,Y)).
\end{aligned}
$$

Thus, from (3.3), we conclude that

$$
\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= \{H^{\ln f}(X,X) + H^{\ln f}(\phi X,\phi X) + 2g(\phi X,\phi X) - 2\eta(X)\phi X(\ln f) \\
&+ 2\eta^2(X) + 2(\phi X(\ln f))^2 + \eta(\nabla_X X)\eta(\text{grad}(\ln f))\}g(Y,Y) \\
&- 2\|h(X,Y)\|^2. \qquad (3.10)
\end{aligned}
$$

On the other hand, by using (1.12), we get

$$
g(\bar{R}(X,\phi X)Y,\phi Y) = -\left(\frac{c-1}{2}\right) g(\phi X,\phi X)g(Y,Y). \qquad (3.11)
$$

By corresponding (3.12) and (3.13), we reach at (3.4). □

Now, Let $\{e_o = \xi, e_1, e_2, ..., e_p, \phi e_1, \phi e_2, ..., \phi e_p, e^1, e^2, ..., e^q\}$ be orthonormal basis of $\Gamma(TM)$ such that $e_o, e_1, e_2, ..., e_p, \phi e_1, \phi e_2, ..., \phi e_p$, are tangent to $\Gamma(TM_T)$ and $e^1, e^2, ..., e^q$ are tangent to $\Gamma(TM_\perp)$. Moreover, we suppose that $\{\phi e^1, \phi e^2, ..., \phi e^q, N_1, N_2, ..., N_{2r}\}$ is an orthonormal basis of $\Gamma(TM^\perp)$ such that $\{\phi e^1, \phi e^2, ..., \phi e^q\}$ are tangent to $\Gamma(\phi TM_\perp)$ and $\{N_1, N_2, ..., N_{2r}\}$ are tangent to $\Gamma(\nu)$, where ν denote the orthogonal distribution of $\phi D^\perp$ in $T^\perp M$.

We can give the main theorem in the rest of this paper.

3.4. Theorem. *Let M be a compact orientable contact CR-warped product submanifold of a Sasakian space form $\bar{M}(c)$. Then M is a contact CR-product if*

$$
\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_2(e_i, e^j)\|^2 \geq (\frac{c+3}{4})pq, \qquad (3.12)
$$

where, h_2 denote the component of h in $\Gamma(\nu)$.

Proof. By using (2.3), the Laplacian of $\ln f$ is given by

$$
\begin{aligned}
-\Delta \ln f &= \sum_{i=1}^{p} g(\nabla_{e_i}\text{grad}(\ln f), e_i) + \sum_{i=1}^{p} g(\nabla_{\phi e_i}\text{grad}(\ln f), \phi e_i) + \sum_{j=1}^{q} (\nabla_{e^j}\text{grad}(\ln f), e^j) \\
&+ g(\nabla_\xi \text{grad}(\ln f), \xi).
\end{aligned}
$$

Considering ∇ being Levi-Civita connection, M_T is totally geodesic in M, $M_\perp$ is totally umbilical in M, $\mathrm{grad}(\ln f) \in \Gamma(TM_T)$ and Proposition 2.1, we have

$$\begin{aligned}
-\Delta \ln f &= \sum_{i=1}^{p}\{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} \\
&+ \sum_{j=1}^{q}\{e^j g(\mathrm{grad}(\ln f), e^j) - g(\nabla_{e^j} e^j, \mathrm{grad}(\ln f))\} + g(\nabla_\xi \mathrm{grad}(\ln f), \xi) \\
&= \sum_{i=1}^{p}\{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} \\
&- \sum_{j=1}^{q}\{-g(e^j, e^j) g(\mathrm{grad}(\ln f), \mathrm{grad}(\ln f))\} + g(\phi \mathrm{grad}(\ln f), \xi) \\
&= \sum_{i=1}^{p}\{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} + q\|\mathrm{grad}(\ln f)\|^2.
\end{aligned}$$

Let $X = e_i$ and $Y = e^j$ be in (3.4), $1 \leq i \leq p$ and $1 \leq j \leq q$. By direct calculations, we have

$$\begin{aligned}
2\sum_{i=1}^{p}\sum_{j=1}^{q}\|h(e_i, e^j)\|^2 &= \{\sum_{i=1}^{p}\{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} \\
&+ 2\sum_{i=1}^{p}(\phi e_i \ln f)^2 + (\frac{c+3}{2})p\}q \\
&= \{-\Delta \ln f - q\|\mathrm{grad}(\ln f)\|^2 + 2\sum_{i=1}^{p}(\phi e_i \ln f)^2 + (\frac{c+3}{2})p\}q.
\end{aligned}$$

Thus we get

$$-\Delta \ln f = \frac{2}{q}\sum_{i=1}^{p}\sum_{j=1}^{q}\|h(e_i, e^j)\|^2 + q\|\mathrm{grad}(\ln f)\|^2 - 2\sum_{i=1}^{p}(\phi e_i(\ln f))^2 - (\frac{c+3}{2})p. \tag{3.13}$$

Furthermore, from linear algebra rules, we know that h can be written as

$$h(e_i, e^j) = \sum_{k=1}^{p} g(h(e_i, e^j), \phi e^k)\phi e^k + \sum_{\ell=1}^{2r} g(h(e_i, e^j), N_\ell)N_\ell.$$

Also, making use of (3.1), we have

$$\begin{aligned}
\sum_{i=1}^{p}\sum_{j=1}^{q} g(h(e_i, e^j), h(e_i, e^j)) &= \sum_{i=1}^{p}\sum_{k,j=1}^{q} g(h(e_i, e^j), \phi e^k)^2 \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{\ell=1}^{2r} g(h(e_i, e^j), N_\ell)^2 \\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\{g(e^j, e^j)(\eta(e_i) - \phi e_i(\ln f))^2\} + \sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2 \\
&= q\sum_{i=1}^{p}(\phi e_i(\ln f))^2 + \sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2. \qquad (3.14)
\end{aligned}$$

Finally, substituting (3.14) into (3.13), we get

$$-q\Delta \ln f = 2\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 - (\frac{c+3}{2})pq + q^2\|\text{grad}(\ln f)\|^2.$$

From (2.4) we conclude that

$$\int_M \{\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 - (\frac{c+3}{4})pq + \frac{q^2}{2}\|\text{grad}(\ln f)\|^2\}dV = 0, \tag{3.15}$$

that is,

$$\int_M \{\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 + \frac{q^2}{2}\|\text{grad}(\ln f)\|^2\}dV = \mathtt{Vol}(M)(\frac{c+3}{4})pq.$$

Here, if

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 \geq (\frac{c+3}{4})pq,$$

it implies $\|\text{grad}(\ln f)\| = 0$ because $q.p \neq 0$, that is, the warping function f is constant. So contact CR-warped product becomes a contact CR-product. □

As a consequence, $c \leq -3$ is necessary so that $M(c)$ is the standard sphere S^{2n+1}. The natural isometric embedding

$$\mathbb{C}^{k+1} \times \mathbb{R}^{\ell+1} \longrightarrow \mathbb{C}^{n+1}$$

defines an isometric embedding.

$$i : M(k,\ell) = \mathbb{S}^{2k+1} \times \mathbb{S}^{\ell} \longrightarrow \mathbb{S}^{2n+1}$$

in which $\mathbb{S}^{2k+1}$ is the standard sphere with constant ϕ-sectional curvature c and the ℓ-dimensional sphere $\mathbb{S}^{\ell}$ with constant sectional curvature $(\frac{c+3}{4})$. i maps the tangent bundle $T\mathbb{S}^{\ell}$ into $T^{\perp}M(k,\ell) \subset T\mathbb{S}^{2n+1}$. Hence $M(k,\ell)$ is a $(2k+\ell+1)$-dimensional contact CR-warped product submanifold by the definition.

From the integral formula (3.15) we derive the following corollaries.

3.5. Corollary. *Let M be a compact orientable contact CR-warped product submanifold of a Sasakian space form $\bar{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 = (\frac{c+3}{4})pq. \tag{3.16}$$

Proof. If (3.16) is satisfied, then (3.15) implies that $f = constant$, that is, M is a contact CR-product.

Conversely, M is a contact CR-product , from (3.2) we know that $h(X,Y) \in \Gamma(\nu)$, for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_{\perp})$. So the equality (3.16) is satisfied □

3.6. Corollary. *There exist no compact orientable contact CR products in a Sasakian space form $\bar{M}(c)$ such that $c < -3$.*

Acknowledgement:

I would like to thank to reviewer(s) for the detailed and diligently prepared suggestions which improved the paper

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 33 – 39

Inequalities with conjugate exponents in grand Lebesgue spaces

René Erlín Castillo* and Humberto Rafeiro†

Abstract

In this paper we want to show the validity of the generalized double-parametric Hilbert inequality with conjugate exponents in the framework of grand Lebesgue spaces as a particular case of a more general result involving an homogeneous kernel. We also study the boundedness of an integral operator with the aforementioned kernel.

2000 AMS Classification: Primary 46E30; Secondary 26D15

Keywords: Lebesgue spaces, grand Lebesgue spaces, Hilbert inequality

Received 02/11/2013 : Accepted 13/12 /2013 Doi : 10.15672/HJMS.2015449087

1. Introduction

The classical Hilbert inequality on double series is the following

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_n b_m}{n+m} < \pi\csc(\pi/p)\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}b_m^{p'}\right)^{\frac{1}{p'}} \tag{1.1}$$

where p and p' are conjugate exponents and $a_n, b_m \geqslant 0$. In accordance to [10], this inequality was included by Hilbert for $p = p' = 2$ in his lectures, and it was published by H. Weyl [25]. The estimate was later on improved by Schur [24] obtaining the sharp constant π. It is possible to show that the constant $\pi\csc(\pi/p)$ is sharp for the case $p > 1$. The integral analogue for (1.1) is the so-called *double-parametric Hilbert inequality*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{f(y)g(x)}{x+y}\,\mathrm{d}x\,\mathrm{d}y < \pi\csc(\pi/p)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^{p'}(\mathbb{R}_+)}, \tag{1.2}$$

which was shown in [10].

*Universidad Nacional de Colombia, Departamento de Matemáticas, Cra. 45 No. 26–85, Bogotá, Colombia.
Email: recastillo@unal.edu.co

†Pontificia Universidad Javeriana, Departamento de Matemáticas, Cra. 7 No. 43–82, Bogotá, Colombia.
Email:silva-h@javeriana.edu.co

We can generalize (1.2) substituting $1/(x+y)$ by a suitable kernel k, in this case the constant will depend on k and p. Another generalization is to drop the requirement that the exponents are conjugate, this leads to the so-called *inequalities with non-conjugate exponents.*

In 1992 T. Iwaniec and C. Sbordone [12], in their studies related with the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces.* A generalized version of them, $L^{p),\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone [9]. Harmonic analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during last years due to various applications, we mention e.g. [1, 3, 4, 5, 6, 7, 14, 17, 22] and continue to attract attention of various researchers.

For example, in the theory of PDE's, it turned out that these are the right spaces in which some nonlinear equations have to be considered (see [8, 9]). Also noteworthy to mention the extension of the ideas regarding grand Lebesgue spaces into the framework of the so-called grand Morrey spaces, e.g. [16, 15, 18, 20, 21].

In this paper we show the validity of the generalized double-parametric Hilbert inequality with conjugate exponents in the framework of grand and small Lebesgue spaces as a particular case of a more general result involving an homogeneous kernel. We also study the boundedness in grand Lebesgue spaces of an integral operator with the aforementioned kernel. Finally, we state an open problem related with the so-called *inequalities with non-conjugate exponents.*

2. Grand and Small Lebesgue Spaces

In this section we will introduce grand and small Lebesgue spaces, for more properties, see e.g. [2, 4, 14].

2.1. Definition. By Φ we denote the class of continuous positive functions on $(0,p-1)$ such that $\lim_{x\to 0+}\varphi(x)=0$.

2.2. Definition. Let $(\mathsf{X},\mathscr{A},\mu)$ be a finite measure space and $\varphi\in\Phi$. The *grand Lebesgue space*, denoted by $L^{p),\varphi}(\mathsf{X})$, is the set of all real-valued measurable functions for which

$$\|f\|_{L^{p),\varphi}(\mathsf{X})} := \sup_{0<\varepsilon<p-1}\left(\varphi(\varepsilon)\fint_{\mathsf{X}}|f(x)|^{p-\varepsilon}\,\mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where $\fint_\omega f\,\mathrm{d}\mu(x) := \frac{1}{\mu(\omega)}\int_\omega f\,\mathrm{d}\mu(x)$ stands for the integral average of the function f in ω.

Taking $\varphi:(0,p-1)\to(0,+\infty)$ with $x\mapsto x$ and the induced Lebesgue measure in a bounded subset of the Euclidean space, we recover the space introduced by T. Iwaniec and C. Sbordone in [12] and we get the space introduce in [9] when $x\mapsto x^\theta$.

2.3. Definition. Let $(\mathsf{X},\mathscr{A},\mu)$ be a finite measure space and $1<p<\infty$. If $\varphi\in\Phi$, we define the *small Lebesgue space* $L^{p)',\varphi}(\mathsf{X})$ as

$$L^{p)',\varphi}(\mathsf{X}) = \left\{g\in\mathcal{M}_0 \mid \|g\|_{L^{p)',\varphi}}(\mathsf{X})<+\infty\right\}$$

where $\mathcal{M}_0$ is the set of all measurable functions, whose values lie in $[-\infty,\infty]$ finite a.e. in X, and

$$\|g\|_{L^{p)',\varphi}}(\mathsf{X}) = \sup_{0\leqslant\psi\leqslant|g|;\psi\in L^{(p'}(\mathsf{X})} \|\psi\|_{L^{(p',\varphi}(\mathsf{X})}$$

with

$$\|g\|_{L^{(p',\varphi}(\mathsf{X})} = \inf_{g=\sum_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \frac{1}{\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}}} \left(\fint_{\mathsf{X}} |g_k|^{(p-\varepsilon)'} \,\mathrm{d}\mu(x) \right)^{\frac{1}{(p-\varepsilon)'}} \right\}.$$

We want to mention that there is another characterization of grand and small Lebesgue spaces, using rearrangement functions, see e.g. [6].

3. Hilbert integral inequality

In this section we show the validity of the generalized double-parametric Hilbert integral inequality.

We need some definitions, namely:

3.1. Definition. Let g be a function with domain in $[a,b]\subset\mathbb{R}$ and range $[\alpha,\beta]\subset\mathbb{R}$. If $\mu(E)=0$ implies that $\mu(g(E))=0$ for all $E\subset[a,b]$, then g is said to be an N-function or to satisfy the condition N.

3.2. Definition. We will call to a measure μ, a *N-dilation measure*, if $\mu(E)=0$ implies that $\mu(tE)=0$, $\mathbb{R}\ni t>0$, where tE is the dilation of the set E.

We will also need the following proposition, see e.g. [11, Corollaries 20.4 and 20.5].

3.3. Proposition. *Let $[a,b]$ be an interval in $\mathbb{R}$ and let φ be a monotone continuous N-function with domain $[a,b]$ and range $[\alpha,\beta]$ $(\alpha<\beta)$. Thus for $f\in L^1([\alpha,\beta],\mathscr{A},\mu)$, we have $(f\circ\varphi)|\varphi'|\in L^1([a,b],\mathscr{A},\mu)$, and*

$$\int_\alpha^\beta f(y)\,\mathrm{d}\mu(y) = \int_a^b f\circ\varphi(x)|\varphi'(x)|\,\mathrm{d}\mu(x).$$

We now prove the validity of the generalized double-parametric Hilbert inequality.

3.4. Theorem. *Let μ be a N-dilation finite measure on $[0,\infty)$, $\varphi\in\Phi$ and $k:[0,\infty)\times[0,\infty)\to[0,\infty)$ be homogeneous of degree -1. If $f\in L^{p),\varphi}(\mathbb{R}_+,\mu)$ and $g\in L^{p)',\varphi}(\mathbb{R}_+,\mu)$ then*

$$\left|\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} k(x,y)f(y)g(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right| \leqslant c\,\|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)}\|g\|_{L^{p)',\varphi}(\mathbb{R}_+,\mu)}. \tag{3.1}$$

with

$$c = \inf_{0<\varepsilon<p-1}\int_{\mathbb{R}_+} w^{-\frac{1}{p-\varepsilon}}k(1,w)\,\mathrm{d}\mu(w). \tag{3.2}$$

Proof. We first prove the theorem for $g\in L^{(p',\varphi}(\mathbb{R}_+,\mu)$ and then bootstrap the result to $g\in L^{p)',\varphi}(\mathbb{R}_+,\mu)$.

Let $g \in L^{p)',\varphi}(\mathbb{R}_+,\mu)$ and $|g| = \sum_{k\in\mathbb{N}} g_k$ be any decomposition with $g_k \geq 0$ for all $k \in \mathbb{N}$. Taking

$$I_{M,N} = \left| \int_0^M \int_0^N f(y) g_k(x) k(x,y) \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \right|,$$

then we have

$$\begin{aligned} I_{M,N} &\leqslant \int_0^M \mathrm{d}\mu(x) \int_0^N \left| f(y) g_k(x) k(x,y) \right| \mathrm{d}\mu(y) \\ &\leq \int_{\mathbb{R}_+} k(1,w) \, \mathrm{d}\mu(w) \int_0^M \left| f(wx) g_k(x) \right| \mathrm{d}\mu(x) \end{aligned}$$

where we used an appropriate change of variables, the fact that the kernel k is of degree -1 and also the Proposition 3.3 together with the assumption that μ is a N-dilation measure.

For each $0 < \varepsilon < p-1$, we have

$$I_{M,N} \leqslant \mu(\mathbb{R}_+) \int_{\mathbb{R}_+} k(1,w) \left(\frac{1}{\mu(\mathbb{R}_+)} \int_0^M |f(wx)|^{p-\varepsilon} \, \mathrm{d}\mu(x) \right)^{\frac{1}{p-\varepsilon}} \mathrm{d}\mu(w) \left(\frac{1}{\mu(\mathbb{R}_+)} \int_0^M |g_k(x)|^{(p-\varepsilon)'} \, \mathrm{d}\mu(x) \right)^{\frac{1}{(p-\varepsilon)'}}$$

which gives

$$I_{\infty,\infty} \lesssim \int_{\mathbb{R}_+} \frac{k(1,w)}{w^{\frac{1}{p-\varepsilon}}} \, \mathrm{d}\mu(w) \left(\varphi(\varepsilon) \fint_{\mathbb{R}_+} |f(x)|^{p-\varepsilon} \, \mathrm{d}\mu(x) \right)^{\frac{1}{p-\varepsilon}} \frac{1}{\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}}} \left(\fint_{\mathbb{R}_+} |g_k(x)|^{(p-\varepsilon)'} \, \mathrm{d}\mu(x) \right)^{\frac{1}{(p-\varepsilon)'}} . \tag{3.3}$$

Since

$$\left| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(y) g(x) k(x,y) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \right| \leqslant \int_{\mathbb{R}_+} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}_+} |f(y) g_k(x) k(x,y)| \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \tag{3.4}$$

we get from (3.3), the definition of grand and small Lebesgue spaces that

$$\left| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(y) g(x) k(x,y) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \right| \leqslant \int_{\mathbb{R}_+} \frac{k(1,w)}{w^{\frac{1}{p-\varepsilon}}} \, \mathrm{d}\mu(w) \|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)} \|g\|_{L^{(p',\varphi}(\mathbb{R}_+,\mu)} . \tag{3.5}$$

Taking now infimum in (3.5), we obtain (3.1) for the case of $g \in L^{(p',\varphi}(\mathbb{R}_+, \mu)$.

Let now $g \in L^{p'),\varphi}(\mathbb{R}_+, \mu)$ and let us define the truncated function

$$\psi_n(x) = \begin{cases} |g(x)| & \text{if } |g(x)| \leqslant n, \\ n & \text{if } |g(x)| > n, \end{cases}$$

which is in $L^{(p'}(\mathsf{X})$ due to the embedding $L^{\infty}(\mathsf{X}) \subset L^{(p'}(\mathsf{X})$.

Moreover, for all $0 \leqslant \psi \leqslant |g|$ we have $\|\psi\|_{L^{(p'}}(\mathsf{X}) \leqslant \|g\|_{L^{p)'}}(\mathsf{X})$ which yields

$$\begin{aligned} \left| \int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} k(x,y) f(y) \psi(x) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \right| &\leqslant c \|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)} \|\psi\|_{L^{(p',\varphi}(\mathbb{R}_+,\mu)} \\ &\leqslant c \|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)} \|g\|_{L^{p)',\varphi}(\mathbb{R}_+,\mu)}. \end{aligned}$$

Since $g(x) = \lim_{n\to\infty} \psi_n(x)$, by the Beppo-Levi theorem we get

$$\begin{aligned} \left| \int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} k(x,y) f(y) g(x) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \right| &\leqslant \int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} |k(x,y) f(y) g(x)| \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &= \lim_{n\to\infty} \int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} |k(x,y) f(y) \psi_n(x)| \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &\leqslant c \|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)} \|g\|_{L^{p)',\varphi}(\mathbb{R}_+,\mu)}. \end{aligned}$$

□

3.5. Remark. As an example, taking $\mathrm{d}\mu(x) = \mathrm{e}^{-x} \, \mathrm{d}x$ with the usual Lebesgue measure, and $k(x,y) = 1/(x+y)$ we get

$$\int\limits_{\mathbb{R}_+} \frac{x^{\frac{1}{p-\varepsilon}}}{1+x} \mathrm{e}^{-x} \, \mathrm{d}x \leqslant \int\limits_{\mathbb{R}_+} \frac{x^{\frac{1}{p-\varepsilon}}}{1+x} \, \mathrm{d}x = \pi \csc(\pi/(p-\varepsilon)).$$

thus satisfying the conditions in the Theorem 3.4.

We now prove a theorem regarding the boundedness of an integral operator based on the kernel k.

3.6. Theorem. *Let μ be a N-dilation finite measure on $[0,\infty)$, $\varphi \in \Phi$ and $k : [0,\infty) \times [0,\infty) \to [0,\infty)$ be homogeneous of degree -1. If $f \in L^{p),\varphi}(\mathbb{R}_+, \mu)$ and define the integral operator K has*

$$Kf(y) = \int\limits_{\mathbb{R}_+} k(x,y) f(x) \, \mathrm{d}\mu(x).$$

Then

$$\|Kf\|_{L^{p,\varphi)}(\mathbb{R}_+,\mu)} \leqslant c \, \|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)} \tag{3.6}$$

with

$$c = \sup_{0<\varepsilon<p-1} \int\limits_{\mathbb{R}_+} w^{-\frac{1}{p-\varepsilon}} k(1,w) \, \mathrm{d}\mu(w). \tag{3.7}$$

Proof. For fixed $0 < \varepsilon < p-1$, we have

$$\begin{aligned}
\|Kf\|_{L^{p-\varepsilon}(\mathbb{R}_+,\mu)} &= \left(\int\limits_{\mathbb{R}_+} \left| \int\limits_{\mathbb{R}_+} k(z,1) f(zy)\,\mathrm{d}\mu(z) \right|^{p-\varepsilon} \mathrm{d}\mu(y) \right)^{\frac{1}{p-\varepsilon}} \\
&\leqslant \int\limits_{\mathbb{R}_+} k(z,1) \left(\int\limits_{\mathbb{R}_+} |f(zy)|^{p-\varepsilon}\,\mathrm{d}\mu(y) \right)^{\frac{1}{p-\varepsilon}} \mathrm{d}\mu(z) \\
&\leqslant \int\limits_{\mathbb{R}_+} k(z,1) z^{-\frac{1}{p-\varepsilon}}\,\mathrm{d}\mu(z) \|f\|_{L^{p-\varepsilon}(\mathbb{R}_+,\mu)}
\end{aligned} \tag{3.8}$$

where we have used the Minkowski integral inequality, a proper change of variable and the homogeneity of the kernel k. From (3.8) we get

$$\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|Kf\|_{L^{p-\varepsilon}(\mathbb{R}_+,\mu)} \leqslant \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\mathbb{R}_+,\mu)} \int\limits_{\mathbb{R}_+} k(z,1) z^{-\frac{1}{p-\varepsilon}}\,\mathrm{d}\mu(z)$$

and now (3.6) follows taking (3.7) into account applying supremum over $0 < \varepsilon < p-1$ in both sides. □

Open Problem: The case of non-conjugate exponents is an open problem. For example, what is the analogue in grand Lebesgue spaces for the following result.

Let $p > 1$, $r > 1$ and $\lambda = \frac{1}{p} + \frac{1}{r}$, then

$$\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} k^{\lambda}(x,y) f(x) g(y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \left(\int\limits_0^{\infty} u^{\frac{1}{\lambda q}} k(1,u)\,\mathrm{d}u \right)^{\lambda} \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^r(\mathbb{R}_+)}$$

where $k(x,y)$ is a non-negative homogeneous kernel of degree -1.

Acknowledgements

The authors would like to thank the anonymous referee for careful reading the paper. The second named author was partially supported by research project *Inequalities in grand Lebesgue spaces*, `ID PRY: 6054`, of the Faculty of Sciences of the *Pontificia Universidad Javeriana*, Bogotá, Colombia.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 41 – 49

Clean property in amalgamated algebras along an ideal

Mohamed Chhiti* , Najib Mahdou† and Mohammed Tamekkante‡

Abstract

Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. In this paper, we give a characterization for the amalgamation of A with B along J with respect to f (denoted by $R \bowtie^f J$) to be (uniquely) clean.

2000 AMS Classification: 13D05, 13D02.

Keywords: Amalgamated algebra along an ideal, (uniquely) clean rings, neat rings.

Received 17/10/2013 : Accepted 31/01/2014 Doi : 10.15672/HJMS.2015449103

1. Introduction

Throughout this paper, all rings are commutative with identity. We denote respectively by Nilp(A), Rad(A), and Idem(A) the ideal of all nilpotent elements of the ring A, Jacobson radical of A, and the set of all idempotent of A.

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

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*Faculty of Economics and Social Sciences of Fez, University S. M. Ben Abdellah Fez, Morocco.
Email: chhiti.med@hotmail.com
†Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S. M. Ben Abdellah Fez, Morocco.
Email:mahdou@hotmail.com
‡Department of Mathematical and Computer Sciences, Faculty of sciences, University Moulay Ismail, Meknes, Morocco.
Email:tamekkante@yahoo.fr

called *the amalgamation of A with B along J with respect to f* (introduced and studied by D'Anna, Finocchiaro, and Fontana in [10, 11]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [12, 13, 14]). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([10, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealization (cf. [17, page 2]), and the CPI extensions (in the sense of Boisen and Sheldon [6]) are strictly related to it ([10, Example 2.7 and Remark 2.8]).

On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by Anderson in [1] and motivated by a classical construction due to Dorroh [9], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [10, Section 2]. Also, the authors consider the iteration of the amalgamation process, giving some geometrical applications of it.

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [10, Section 4]. This point of view allows the authors in [10, 11] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [10], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [11], they pursue the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

Various authors have studied clean rings and related conditions (cf. [2, 3, 16]). Recall that a ring R is called *(uniquely) clean* if each element in R can be written (uniquely) as the sum of a unit and an idempotent. The concept of clean rings was introduced by Nicholson [18]. Examples of clean rings (uniquely clean rings) include all commutative von Neumann regular rings (Boolean rings), local rings (with residue field $\mathbb{Z}_2$). A basic property of clean rings is that any homomorphic image of a clean ring is again clean. This leads to the definition of a neat rings [15]. A ring R is called *neat* if every proper homomorphic image of R is clean.

In this paper, we give a characterization for $A \bowtie^f J$ to be (uniquely) clean.

2. Main Results

We begin with the following result:

2.1. Proposition. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B.*

(1) *If $A \bowtie^f J$ is a clean (resp., uniquely clean) ring then A is a clean (resp., uniquely clean) ring and $f(A)+J$ is a clean ring.*

(2) *Assume that $\dfrac{f(A)+J}{J}$ is uniquely clean. Then $A \bowtie^f J$ is a clean ring if and only if A and $f(A)+J$ are clean rings.*

Proof. (1) The cases $J=(0)$ and $f^{-1}(J)=(0)$ follow easily from [10, Proposition 5.1 (3)]. Otherwise, by the same reference, the rings A and $f(A)+J$ are proper homomorphic images of $A \bowtie^f J$. Then, they are clean. Assume now that $A \bowtie^f J$ is uniquely clean and consider $u+e=u'+e'$ where $u,u' \in U(A)$ and $e,e' \in \mathrm{Idem}(A)$. Then, $(u,f(u))+(e,f(e))=(u',f(u'))+(e',f(e'))$ and clearly $(u,f(u)),(u',f(u')) \in U(A \bowtie^f J)$ and $(e,f(e)),(e',f(e')) \in \mathrm{Idem}(A \bowtie^f J)$. Then, $(u,f(u))=(u',f(u'))$ and $(e,f(e))=(e',f(e'))$. Hence, $u=u'$ and $e=e'$. Consequently, A is uniquely clean.

(2) If $A \bowtie^f J$ is a clean ring, then so A and $f(A)+J$ by 1). Conversely, assume that A and $f(A)+J$ are clean rings and consider $(a,j) \in A \times J$. Since A is clean, we can write $a = u + e$ with $(u,e) \in U(A) \times \text{Idem}(A)$. On the other hand, since $f(A)+J$ is clean, $f(a)+j = f(x)+j_1+f(y)+j_2$ with $f(x)+j_1$ and $f(y)+j_2$ are respectively unit and idempotent element of $f(A)+J$. It is clear that $\overline{f(x)} = \overline{f(x)+j_1}$ (resp. $\overline{f(u)}$) and $\overline{f(y)} = \overline{f(y)+j_2}$ (resp. $\overline{f(e)}$) are respectively unit and idempotent element of $\dfrac{f(A)+J}{J}$, and we have $\overline{f(a)} = \overline{f(u)} + \overline{f(e)} = \overline{f(x)} + \overline{f(y)}$. Thus, $\overline{f(u)} = \overline{f(x)}$ and $\overline{f(e)} = \overline{f(y)}$ since $\dfrac{f(A)+J}{J}$ is uniquely clean. Consider $j_1', j_2' \in J$ such that $f(x) = f(u)+j_1'$ and $f(y) = f(e)+j_2'$. We have, $(a, f(a)+j) = (u, f(u)+j_1'+j_1)+(e, f(e)+j_2'+j_2)$, and it is clear that $(e, f(e)+j_2'+j_2)$ is an idempotent element of $A \bowtie^f J$. Hence, we have only to prove that $(u, f(u)+j_1'+j_1)$ is invertible in $A \bowtie^f J$. Since $f(u)+j_1'+j_1$ is invertible in $f(A)+J$, there exists an element $f(\alpha)+j_0$ such that $(f(u)+j_1'+j_1)(f(\alpha)+j_0) = 1$. Thus, $\overline{f(u)f(\alpha)} = \overline{1}$. Then, $\overline{f(\alpha)} = \overline{f(u^{-1})}$. So, there exists $j_0' \in J$ such that $f(\alpha) = f(u^{-1})+j_0'$. Hence, $(u, f(u)+j_1'+j_1)(u^{-1}, f(u^{-1})+j_0'+j_0) = (u, f(u)+j_1'+j_1)(u^{-1}, f(\alpha)+j_0) = (1,1)$. Accordingly, $(u, f(u)+j_1'+j_1)$ is invertible in $A \bowtie^f J$. Consequently, $A \bowtie^f J$ is clean.

2.2. Remarks. Let $f : A \to B$ be a ring homomorphism and J an ideal of B.

(1) If $B = J$ then, $A \bowtie^f B$ is clean if and only if A and B are clean since $A \bowtie^f B = A \times B$.

(2) If $f^{-1}(J) = \{0\}$ then, $A \bowtie^f J$ is clean if and only if $f(A)+J$ is clean (by [10, Proposition 5.1(3)]).

2.3. Corollary. *Let A be a ring and I an ideal such that A/I is uniquely clean. Then, $A \bowtie I$ is clean if and only if A is clean.*

Contrary to the previous proposition, in what follows we show that, if $J \subseteq \text{Rad}(B)$, the characterization for $A \bowtie^f J$ to be clean does not depend to the choice of f.

2.4. Theorem. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B such that $f(u)+j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$. Then, $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if A is clean (resp., uniquely clean).*

More generally, if $J \cap \text{Idem}(B) = 0$ then, the following are equivalent:

(1) *$A \bowtie^f J$ is clean (resp., uniquely clean).*

(2) *A is clean (resp., uniquely clean) and $J \subseteq \text{Rad}(B)$.*

We need the following lemma.

2.5. Lemma. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B such that $J \cap \text{Idem}(B) = 0$. Then $\text{Idem}(A \bowtie^f J) = \{(e, f(e)) \mid e \in \text{Idem}(A)\}$.*

Proof. Let $(e, f(e)+j)$ be an idempotent element of $A \bowtie^f J$. It is clear that e must be an idempotent element of A. On the other hand, $(f(e)+j)^2 = f(e)+j$. Thus, $j - j^2 = 2f(e)j$. Then, $f(e)(j-j^2) = 2f(e)^2 j = 2f(e)j$. Therefore, $-f(e)j^2 = f(e)j$. Thus, $(f(e)j)^4 = (f(e)^2 j^2)^2 = (f(e)j^2)^2 = (-f(e)j)^2 = (f(e)j)^2$. Consequently, $f(e)j^2 = (f(e)j)^2 \in J \bigcap \text{Idem}(B) = \{0\}$. Hence, $f(e)j = -f(e)j^2 = 0$. Thus, $j - j^2 = 2f(e)j = 0$. Then, $j \in J \bigcap \text{Idem}(B) = \{0\}$. Consequently, $j = 0$. Accordingly, $\text{Idem}(A \bowtie^f J) \subseteq \{(e, f(e)) \mid e \in \text{Idem}(A)\}$. The converse is clear.

Proof of Theorem 2.4. Note in first that if $f(u)+j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$ then $J \cap \text{Idem}(B) = (0)$. Indeed, if $j \in J \cap \text{Idem}(B)$ then, $1 - j = -(1+j) \in \text{Idem}(B) \cap U(B) = 1$. Thus, $j = 0$. Moreover, if $A \bowtie^f J$ is (uniquely) clean then so is A (by Proposition 2.1 (1)).

Assume that A is clean and $f(u)+j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$. Consider $(a,j) \in A \times J$. Since A is clean, $a = u + e$ where u and e are respectively unit and idempotent in A. Moreover, $f(u)+j$ is invertible in B. Then, there exists $v \in B$ such that $(f(u)+j)v = 1$. Hence, $(f(u)+j)(f(u^{-1}) - vf(u^{-1})j) = f(u)f(u^{-1}) + jf(u^{-1}) - (f(u)+j)vf(u^{-1})j = 1 + jf(u^{-1}) - f(u^{-1})j = 1$. Thus, $(u, f(u)+j)$ is invertible in $A \bowtie^f J$ (since $(u, f(u)+j)(u^{-1}, f(u^{-1}) - vf(u^{-1})j) = (1,1)$). Hence, $(a, f(a)+j) = (u, f(u)+j) + (e, f(e))$ is the sum of a unit and an idempotent element in $A \bowtie^f J$. Consequently, $A \bowtie^f J$ is clean.

Assume moreover that A is uniquely clean. Since $J \cap \mathrm{Idem}(B) = (0)$ (as we see in the first lines of the proof) and by lemma 2.5, $\mathrm{Idem}(A \bowtie^f J) = \{(e, f(e)) \mid e \in \mathrm{Idem}(A)\}$. Suppose now that we have $(u, f(u)+j) + (e, f(e)) = (u', f(u')+j') + (e', f(e'))$ where $(u, f(u)+j), (u', f(u')+j') \in U(A \bowtie^f J)$ and $e, e' \in \mathrm{Idem}(A)$. Clearly, $u, u' \in U(A)$, and $u + e = u' + e'$. Since A is uniquely clean $u = u'$ and $e = e'$. Thus, $(e, f(e)) = (e', f(e'))$ and $(u, f(u)+j) = (u', f(u')+j')$. Consequently, $A \bowtie^f J$ is uniquely clean, as desired.

Assume that $J \cap \mathrm{Idem}(B) = (0)$.

(1) $\Rightarrow$ (2) We have only to prove that $J \subseteq \mathrm{Rad}(B)$. Consider $j \in J$ and $x \in B$. Since $A \bowtie^f J$ is clean and by (1) above, $(0, xj) = (u, f(u)+xj) + (e, f(e))$ where $(u, f(u)+xj)$ is unit in $A \bowtie^f J$ and $e \in \mathrm{Idem}(A)$. We have $0 = u + e$ and so $u = -1$ and $e = 1$. Therefore, $1 - xj = -(f(-1) + xj)$ is invertible in B and then $j \in \mathrm{Rad}(B)$.

(2) $\Rightarrow$ (1) Follows from above since for each $J \subseteq \mathrm{Rad}(B)$, it is clear that $f(u) + j = f(u)(1 + f(u^{-1})j)$ is invertible in B.

2.6. Corollary. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B such that $J \subseteq \mathrm{Rad}(B)$. Then $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if A is clean (resp., uniquely clean).*

Proof. Let $x \in J \cap \mathrm{Idem}(B)$. Since $J \subseteq \mathrm{Rad}(B)$, there exists a positive integer n such that $x^n = 0$. On the other hand, x is an idempotent element, and then $x = x^n = 0$. Thus, $J \cap \mathrm{Idem}(B) = (0)$. Consequently, the result follows directly from Theorem 2.4.

2.7. Example. Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, ..., X_n\}$ a finite set of indeterminates over B. Set the subring $A + XB[[X]] := \{h \in B[[X]] \mid h(0) \in A\}$ of the ring of power series $B[[X]]$. Then, $A + XB[[X]]$ is clean if and only if A is clean.

Proof. By [10, Example 2.5], $A + XB[[X]]$ is isomorphic to $A \bowtie^{\sigma} J$, where $\sigma : A \hookrightarrow B[[X]]$ is the natural embedding and $J := XB[[X]]$. By [4, p. 11, Exercise 5], $\mathrm{Rad}(B[[X]]) = \{g \in B[[X]] \mid g(0) \in \mathrm{Rad}(A)\}$. Thus, $J \subseteq \mathrm{Rad}(B[[x]]$. Hence, by Corollary 2.6, $A \bowtie^{\sigma} J$ is clean if and only if A is clean. Thus, we have the desired result.

2.8. Example. Let T be a ring and $J \subseteq \mathrm{Rad}(T)$ an ideal of T and let D be a subring of T such that $J \cap D = (0)$. The ring $D + J$ is clean if and only if D is clean.

Proof. By [10, Proposition 5.1 (3)], $D + J$ is isomorphic to the ring $D \bowtie^{\iota} J$ where $\iota : D \hookrightarrow T$ is the natural embedding. Thus, by Corollary 2.6, $D + J$ is clean if and only if A is clean.

The next result shows that the characterization for $A \bowtie^f J$ to be (uniquely) clean can be reconducted to the case where A is a reduced ring and $J \cap \mathrm{Nilp}(B) = (0)$.

2.9. Theorem. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Set $\overline{A} = A/\mathrm{Nilp}(A)$, $\overline{B} = B/\mathrm{Nilp}(B)$, $\pi : B \to \overline{B}$ the canonical projection, and $\overline{J} = \pi(J)$. Consider the ring homomorphism $\overline{f} : \overline{A} \to \overline{B}$ defined by setting $\overline{f}(\overline{a}) = \overline{f(a)}$. Then, $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is clean (resp., uniquely clean).*

To prove this theorem, we need the following lemma.

2.10. Lemma. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then,*

$$\mathrm{Nilp}(A \bowtie^f J) = \{(a, f(a) + j) \mid a \in \mathrm{Nilp}(A), j \in \mathrm{Nilp}(B) \cap J\}.$$

Proof. Consider $(a, f(a) + j) \in \mathrm{Nilp}(A \bowtie^f J)$. Then, there exists a positive integer n such that $(a, f(a) + j)^n = 0$. Thus, $a^n = 0$, and so $a \in \mathrm{Nilp}(A)$ and $f(a) \in \mathrm{Nilp}(B)$. On the other hand, $(f(a) + j)^n = 0$. Thus, $f(a) + j \in \mathrm{Nilp}(B)$. Accordingly, $j \in \mathrm{Nilp}(B)$ since $f(a) \in \mathrm{Nilp}(B)$. Hence, $j \in \mathrm{Nilp}(B) \cap J$.

Conversely, consider $a \in \mathrm{Nilp}(A)$ and $j \in \mathrm{Nilp}(B) \cap J$. It is clear that $f(a) \in \mathrm{Nilp}(B)$. Then, $f(a) + j \in \mathrm{Nilp}(B)$. Hence, $(a, f(a) + j)$ is a nilpotent element of $R \bowtie^f J$.

Proof of Theorem 2.9. It is easy to see that $\overline{f}$ is well defined and it is clearly a ring homomorphism. Consider the map:

$$\begin{array}{rccc} \psi : & A \bowtie^f J/\mathrm{Nilp}(A \bowtie^f J) & \to & \overline{A} \bowtie^{\overline{f}} \overline{J} \\ & \overline{(a, f(a) + j)} & \mapsto & (\overline{a}, \overline{f}(\overline{a}) + \overline{j}) \end{array}$$

The map ψ is well defined. Indeed, if $\overline{(a, f(a) + j)} = \overline{(b, f(b) + j')}$ then, $(a - b, f(a - b) + j - j') \in \mathrm{Nilp}(A \bowtie^f J)$. Hence, by Lemma 2.10, $a - b \in \mathrm{Nilp}(A)$ and $j - j' \in \mathrm{Nilp}(B)$. Then, $\overline{a} = \overline{b}$ and $\overline{j} = \overline{j'}$. It is also easy to check that ψ is a ring homomorphism. Moreover, $(\overline{a}, \overline{f}(\overline{a}) + \overline{j}) = (0, 0)$ implies that $a \in \mathrm{Nilp}(A)$ and $j \in \mathrm{Nilp}(B)$. Consequently, $(a, f(a) + j) \in \mathrm{Nilp}(A \bowtie^f J)$, that is $\overline{(a, f(a) + j)} = (0, 0)$. Accordingly, ψ is injective. Clearly, ψ is surjective by construction. Thus, it is an isomorphism. Consequently, the desired result follows directly from [2, Theorems 9 and 23 (3)].

The next result is a consequence of the previous theorem and it is also a particular case of Corollary 2.6.

2.11. Corollary. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. If $J \subseteq \mathrm{Nilp}(B)$ then, $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if A is clean (resp., uniquely clean).*

Proof. With the notation of Theorem 2.9, $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if $\overline{A} \bowtie^{\overline{f}} 0$ is clean (resp., uniquely clean) since $\overline{J} = (0)$. On the other hand, by [10, Proposition 5.1 (3)], $\overline{A} \bowtie^{\overline{f}} 0 \cong \overline{A}$. Consequently, the desired result follows from [2, Theorems 9 and 23 (3)].

Let E be an A-module and set $B := A \propto E$. Let $\iota : A \hookrightarrow B$ be the canonical embedding. After identifying E with $0 \propto E$, E becomes an ideal of B. It is non straightforward but it is known that $A \propto E$ coincides with $A \bowtie^\iota E$ (cf. [10, Remark 2.8]).

2.12. Corollary. *With the above notation, the ring $A \propto E$ is clean (resp., uniquely clean) if and only if A is clean (resp., uniquely clean).*

Proof. This result follows immediately from Corollary 2.11 since $(0 \propto E)^2 = (0)$.

The study of clean property over the ring $A \bowtie^f J$ allows us to provide a new proof of a characterization for $A \bowtie^f J$ to be local, already obtained in [10].

2.13. Theorem. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then, $A \bowtie^f J$ is a local ring if and only if A is a local ring and $J \subseteq \mathrm{Rad}(B)$.*

Proof. Note in first that a commutative ring is local if and only if it is an indecomposable clean ring (that is a clean ring where $\{0, 1\}$ is the set of all idempotent elements), by [2, Theorem 3].

Assume that $A \bowtie^f J$ is a local ring. Then, $A \bowtie^f J$ is an indecomposable clean ring. Clearly, A must be clean. Moreover, if $e \in \mathrm{Idem}(A)$ then $(e, f(e)) \in \mathrm{Idem}(A \bowtie^f J) = \{(0,0),(1,1)\}$. Then, $\mathrm{Idem}(A) = \{0,1\}$. Accordingly, A is an indecomposable clean ring, and so local ring.

Consider $j \in J$ and $x \in B$. Since $A \bowtie^f J$ is clean and by above, $(0, xj) = (-1, -1 + xj) + (1,1)$ where $(-1, -1+xj)$ is unit in $A \bowtie^f J$ as $\mathrm{Idem}(A \bowtie^f J) = \{(0,0),(1,1)\}$. Thus, $1 - xj = -(-1+xj)$ is invertible in B. Thus, $j \in \mathrm{Rad}(B)$.
Conversely, Assume that A is a local ring and $J \subseteq \mathrm{Rad}(B)$. By Theorem 2.4, $A \bowtie^f J$ is a clean ring. On the other hand, by Lemma 2.5, $\mathrm{Idem}(A \bowtie^f J) = \{(e, f(e)) \mid e \in \mathrm{Idem}(A)\} = \{(0,0),(1,1)\}$. Thus, $A \bowtie^f J$ is an indecomposable clean ring. Consequently, $A \bowtie^f J$ is a local ring.

2.14. Example. Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, ..., X_n\}$ a finite set of indeterminates over B. Then, $A + XB[[X]]$ is local if and only if A is local.

Proof. By [10, Example 2.5], $A+XB[[X]]$ is isomorphic to $A \bowtie^\sigma J$, where $\sigma : A \hookrightarrow B[[X]]$ is the natural embedding and $J := XB[[X]]$. By [4, p. 11, Exercise 5], $\mathrm{Rad}(B[[X]]) = \{g \in B[[X]] \mid g(0) \in \mathrm{Rad}(A)\}$. Thus, $J \subseteq \mathrm{Rad}(B[[x]]$. Hence, by Theorem 2.13, $A \bowtie^\sigma J$ is local if and only if A is local. Thus, we have the desired result.

2.15. Example. Let T be a ring, J an ideal of T, and D a subring of T such that $J \cap D = (0)$. The ring $D + J$ is local if and only if D is local and $J \subseteq \mathrm{Rad}(T)$.

Proof. By [10, Proposition 5.1 (3)], $D + J$ is isomorphic to the ring $D \bowtie^\iota J$ where $\iota : D \hookrightarrow T$ is the natural embedding. Thus, by Theorem 2.13, $D + J$ is clean if and only if A is clean.

2.16. Corollary. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. The following are equivalent:*

(1) *$A \bowtie^f J$ is local and uniquely clean.*
(2) *A is local, uniquely clean and $J \subseteq \mathrm{Rad}(B)$.*

In particular, if A is a ring and I an ideal of A then $A \bowtie I$ is local and uniquely clean if and only if A is local and uniquely clean.

Proof. (1) $\Rightarrow$ (2) Follows from Proposition 2.1 (1) and Theorem 2.13.
(2) $\Rightarrow$ (1) Follows from Theorem 2.13 and Corollary 2.6.

When the ideal J is generated by an idempotent element gives a different particular case since $J \cap \mathrm{Idem}(B) \neq 0$. However, it allows a more easy study of the transfer of the clean property between A and $A \bowtie^f J$ more easily, with respect to f.

2.17. Proposition. *Let $f : A \to B$ be a ring homomorphism and let (e) be an ideal of B generated by the idempotent element e. Then $A \bowtie^f (e)$ is clean if and only if A and $f(A) + (e)$ are clean.*
In particular, if e is an idempotent element of A then, $A \bowtie (e)$ is clean if and only if A is clean.

Proof. From Proposition 2.1(1), we have only to show that $A \bowtie^f (e)$ is clean provided A and $f(A) + (e)$ are clean. Let $(a, f(a) + re)$ be an element of $A \bowtie^f (e)$ (with $a \in A$ and $r \in B$). Since A and $f(A) + (e)$ are clean, there exists u and v (resp. u' and v') in A (resp. $f(A) + (e)$) which are respectively unit and idempotent such that $a = u + v$ and

$f(a)+re=u'+v'$. We have

$$(a,f(a)+re)=(u,f(u)+(u'-f(u))e)+(v,f(v)+(v'-f(v))e)$$

On the other hand,

$$[f(u)+(u'-f(u))e][f(u^{-1})+(u'^{-1}-f(u^{-1})e]=[f(u)(1-e)+u'e][f(u^{-1})(1-e)+u'^{-1}e]=1$$

and

$$\begin{aligned}[f(v)+(v'-f(v))e]^2 &= [f(v)(1-e)+v'e]^2\\ &= f(v)(1-e)+v'e\\ &= f(v)+(v'-f(v))e\end{aligned}$$

Then, $(u,f(u)+(u'-f(u))e)$ and $(v,f(v)+(v'-f(v))e)$ are respectively unit and idempotent in $A\bowtie^f(e)$. Consequently, $A\bowtie^f(e)$ is clean, as desired.
Finally, if $A=B$ and $f=\mathrm{id}_A$ then $A\bowtie^f(e)=A\bowtie(e)$ and $f(A)+(e)=A$. Thus, the particular case is a direct consequence of what is above.

2.18. Example. For each ring homomorphism $f:\mathbb{Z}/6\mathbb{Z}\to\mathbb{Z}/6\mathbb{Z}$, the ring $\mathbb{Z}/6\mathbb{Z}\bowtie^f(\overline{4})$ is clean and reduced since $\mathbb{Z}/6\mathbb{Z}$ is clean, reduced and $\overline{4}$ is an idempotent element of $\mathbb{Z}/6\mathbb{Z}$ and since $(\overline{4})\bigcap \mathrm{Nilp}(\mathbb{Z}/6\mathbb{Z})=\{0\}$ (by Proposition 2.17 and [10, Proposition 5.4]).

Recall that a Boolean ring R is a ring (with identity) for which $x^2=x$ for all x in R; that is, R consists only of idempotent elements. It is proved that a ring R is Boolean if and only if it is uniquely clean with $\mathrm{Rad}(R)=(0)$ if and only if R is clean with characteristic equal to 2 and 1 is the only unit in R; cf. [19, Theorem 19].

The next result gives a characterization for $A\bowtie^f J$ to be a Boolean ring.

2.19. Proposition. *Let $f:A\to B$ be a ring homomorphism and let J be an ideal of B.*

(1) *If $J\subseteq \mathrm{Idem}(B)$ then, $A\bowtie^f J$ is clean if and only if A is clean.*
(2) *The ring $A\bowtie^f J$ is Boolean if and only if A is Boolean and $J\subseteq\mathrm{Idem}(B)$.*

Proof. Note first that if $J\subseteq\mathrm{Idem}(B)$ then $2J=(0)$. Indeed, let $j\in J$. Clearly, $2j\in J\subseteq\mathrm{Idem}(B)$. Then, $j+j=(j+j)^2=j^2+2j^2+j^2=j+2j+j$. Hence, $2j=0$.

(1) Let $(a,f(a)+j)$ be an element of $A\bowtie^f J$ (with $a\in A$ and $j\in J$). We have $a=u+e$ where u and e are respectively unit and idempotent in A. We have $(f(e)+j)^2=f(e)^2+j^2+2f(e)j=f(e)+j$ since $2j=0$. Hence, $(u,f(u))$ and $(e,f(e)+j)$ are respectively unit and idempotent in $A\bowtie^f J$, and we have $(a,f(a)+j)=(u,f(u))+(e,f(e)+j)$. Consequently, $A\bowtie^f J$ is clean.
The converse implication is clear.

(2) If $A\bowtie^f J$ is Boolean, for each $a\in A$, $(a,f(a))=(a,f(a))^2=(a^2,f(a)^2)$. Then, $a=a^2$. Hence, A is Boolean. Moreover, for each $j\in J$, $(0,j)=(0,j)^2=(0,j^2)$. Thus, $j=j^2$. Hence, $J\subseteq\mathrm{Idem}(B)$.
Now, assume that A is Boolean and $J\subseteq\mathrm{Idem}(B)$. We have just proved that $2J=(0)$. Hence, for each $a\in A$ and $j\in J$, $(a,f(a)+j)^2=(a^2,f(a)^2+j^2+2f(a)j)=(a,f(a)+j)$. Thus, $A\bowtie^f J$ is Boolean.

2.20. Remark. Given a ring R, there are two cases where $R=U(R)\cup\mathrm{Idem}(R)$. Namely, a field or a Boolean ring; cf. [2, Theorem 14]. In the previous proposition, we provide a characterization of when $A\bowtie^f J$ is Boolean. On the other hand, it is easy to prove that $A\bowtie^f J$ is a field if and only if A is a field and $J\in\{(0),B\}$ where B is a field.

It is well known that von Neumann rings are particular cases of clean rings. The following result can be easily obtained from the characterization of the reduced property and from the evaluation of the dimension given in [9] and [10].

2.21. Proposition. *Let $f:A\to B$ be a ring homomorphism and let J be an ideal of B.*

(1) *If A and $f(A)+J$ are von Neumann regulars then so is $A \bowtie^f J$.*
(2) *If $A \bowtie^f J$ is von Neumann regular then A is von Neumann regular and $J \cap \mathrm{Nilp}(B) = (0)$ where the equivalence holds if f is surjective.*

2.22. Corollary. *Let R be a commutative ring and let I be a proper ideal of R. Then R is a von Neumann regular ring if and only if $R \bowtie I$ is a von Neumann regular ring.*

A ring R is called neat if every proper homomorphic image of R is clean. For instance, any clean ring is neat but the converse is false (for example, the ring of integers is neat but not clean).
Now, we construct a class of rings such that the neat and clean properties coincides. Recall that a ring R is said to be indecomposable when the only idempotents of A are 0 and 1. Otherwise, the ring is called decomposable.

2.23. Proposition. *Let $f : A \to B$ be a ring homomorphism and let J be a proper ideal of B. Consider the following conditions:*

(1) *A is a decomposable ring.*
(2) *$J \cap \mathrm{Idem}(B) \neq (0)$.*
(3) *$f(u)+j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$.*

If one of the three above conditions is satisfied then, $A \bowtie^f J$ is a clean if and only if $A \bowtie^f J$ is neat.

Proof. Assume that A is a decomposable ring, and consider $e \in \mathrm{Idem}(A)\backslash\{0,1\}$. Then, $(e, f(e)) \in \mathrm{Idem}(A \bowtie^f J)\backslash\{(0,0),(1,1)\}$.
Also, if $j \in J \cap \mathrm{Idem}(B) \neq (0)$, then, $(0,j) \in \mathrm{Idem}(A \bowtie^f J)\backslash\{(0,0),(1,1)\}$. Hence, if one of the conditions (1) or (2) is satisfied, the ring $A \bowtie^f J$ is decomposable. Thus, by [15, Proposition 2.3], $A \bowtie^f J$ is clean if and only if it is neat.
Suppose that (3) is satisfied and that $A \bowtie^f J$ is neat. By the definition of neat rings, $A \cong A \bowtie^f J/(\{0\} \times J)$ is clean. Thus, by Theorem 2.4, $A \bowtie^f J$ is clean. The opposite implication is clear.

ACKNOWLEDGEMENTS. The authors would like to express their sincere thanks for the referee for his/her helpful suggestions and comments, which have greatly improved this paper.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 51 – 57

On centralizing automorphisms and Jordan left derivations on σ-prime gamma rings

Kalyan Kumar Dey* , Akhil Chandra Paul† and Bijan Davvaz‡

Abstract

Let M be a 2-torsion free σ-prime Γ-ring and U be a non-zero σ-square closed Lie ideal of M. If $T : M \to M$ is an automorphism on U such that $T \neq 1$ and $T\sigma = \sigma T$ on U, then we prove that $U \subseteq Z(M)$. We also study the additive maps $d : M \to M$ such that $d(u\alpha u) = 2u\alpha d(u)$, where $u \in U$ and $\alpha \in \Gamma$, and show that $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

2000 AMS Classification: 16W10, 16W25, 16W20, 16U80.

Keywords: σ-prime Γ-ring, centralizing automorphisms, Lie ideals, left derivations, Jordan left derivations.

Received 24/11/2013 : Accepted 28/01/2014 Doi : 10.15672/HJMS.2015449098

1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [8] as a generalization of a classical ring and afterwards Barnes [2] improved the concepts of Nabusawa's Γ-ring and developed the more general Γ-ring in which all the classical rings are contained in this Γ-ring. Throughout this paper, we consider M as a Γ-ring in the sense of Barnes [2] and we denote the center of M by $Z(M)$. In [3], Ceven proved that every Jordan left derivation on a completely prime Γ-ring is a left derivation. Halder and Paul [5] extended this result in a Lie ideal of a Γ-ring. In Γ-rings, Paul and Uddin [13, 14] studied the Lie and Jordan structures and developed a few number of significant results made by Herstien [6] in Γ-rings. In [15] Paul and Uddin initiated the involution mapping in Γ-rings and studied characterizations of simple Γ-rings by means of involution. In [4], Halder and Paul studied the commutativity properties of σ-prime Γ-rings with a non-zero derivation. Hoque and Paul [7] studied on centralizers of semiprime Γ-rings and proved

*Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh.
Email: kkdmath@yahoo.com
†Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh.
Email:acpaulrubd_math@yahoo.com
‡Department of Mathematics, Yazd University, Yazd, Iran.
Emails:davvaz@yazd.ac.ir, bdavvaz@yahoo.com

that every Jordan left centralizer on M is a left centralizer on M if M is a 2-torsion free semiprime Γ-ring. They also proved that every Jordan centralizer of a 2-torsion free semiprime Γ-ring is a centralizer. A number of papers have been developed by Oukhtite and Salhi [10, 11, 12] on σ-prime rings made characterizations of σ-prime rings by means of Lie ideals, derivations and centralizers. By the motivation of the works of Oukhtite and Salhi we initiate to work on σ-prime Γ-rings and generalize the remarkable results of classical ring theories in Γ-ring theories. In the present paper, we work on centralizing automorphisms and Jordan left derivations on σ-prime Γ-rings. We consider M to be a 2-torsion free σ-prime Γ-ring and U to be a non-zero σ-square closed Lie ideal of M. If $T \neq 1$ is an automorphism on U of M which commutes with σ on U, then we show that U is central. We also prove that every Jordan left derivation on U of M is a left derivation on U of M.

2. Preliminaries and Notations

In this section, we give some definitions and preliminary results that we shall use.

2.1. Definition. Let R and $\mathit{\Gamma}$ be two additive abelian groups. If for all $a, b, c \in R$ and $\alpha, \beta \in \mathit{\Gamma}$, the following conditions are satisfied:

(1) $a\alpha b \in R$,
(2) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + a\alpha c$,
(3) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then R is called a $\mathit{\Gamma}$-ring in the sense of Barnes.

Throughout the paper, M will represent a Γ-ring in the sense of Barnes [2] with center $Z(M)$. Then, M is called a 2-torsion free if $2a = 0$ with $a \in M$, then $a = 0$. As usual the commutator $a\alpha b - b\alpha a$ of a and b with respect to α will be denoted by $[a, b]_\alpha$. We make the basic commutator identities

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a[\alpha, \beta]_c b + a\alpha[b, c]_\beta,$$
$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b[\alpha, \beta]_a c + b\alpha[a, c]_\beta,$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Let us assume the condition

(2.1) $\quad a\alpha b\beta c = a\beta b\alpha c$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

According to the condition (2.1), the above two identities reduce to

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta,$$
$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta,$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, which are used extensively in our paper. An additive mapping satisfying $\sigma : M \to M$ is called an involution on M if $\sigma(a+b) = \sigma(a) + \sigma(b)$, $\sigma(a\alpha b) = \sigma(b)\alpha\sigma(a)$, and $\sigma(\sigma(a)) = a$ are satisfied for all $a, b \in M$ and $\alpha \in \Gamma$. Given an involutorial Γ-ring M with an involution σ, we define $Sa_\sigma(M) = \{m \in M \ : \ \sigma(m) = \pm m\}$, which are known as symmetric and skew symmetric elements of M. Recall that M is σ-prime if

(2.2) $\quad a\Gamma M\Gamma b = a\Gamma M\Gamma\sigma(b) = 0$

implies that $a = 0$ or $b = 0$. It is clear that every prime Γ-ring having an involution is a σ-prime Γ-ring but the converse is in general not true. An additive subgroup U is called a Lie ideal if $[u, m]_\alpha \in U$, for all $u \in U$, $m \in M$ and $\alpha \in \Gamma$. A Lie ideal U of M is called a σ-Lie ideal, if $\sigma(U) = U$. If U is a σ-Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$, then U is said to be a σ-square closed Lie ideal of M. For $u, v \in U$, $\alpha \in \Gamma$,

$$(u\alpha v + v\alpha u) = (u+v)\alpha(u+v) - (u\alpha u + v\alpha v)$$

and so $u\alpha v+v\alpha u\in U$. Also, we have $u\alpha v-v\alpha u\in U$. Moreover, from these two relations we obtain $2u\alpha v\in U$. This remark will be freely used in the whole paper. An additive mapping $T:M\to M$ is called centralizing on a subset A of M if $[a,T(a)]_\alpha\subseteq Z(M)$, for every $a\in A$, $\alpha\in\Gamma$. In particular, if T satisfies $[a,T(a)]_\alpha=0$, for all $a\in A$, $\alpha\in\Gamma$, then T is called commuting on A. An additive mapping $d:M\to M$ is said to be a left derivation if $d(a\alpha b)=a\alpha d(b)+b\alpha d(a)$, for all $a,b\in M$, $\alpha\in\Gamma$. And $d:M\to M$ is said to be a Jordan left derivation if $d(a\alpha a)=2a\alpha d(a)$ is satisfied for all $a,b\in M$, $\alpha\in\Gamma$. It is clear that every left derivation is a Jordan left derivation, but the converse need not be true in general.

3. Centralizing automorphisms on σ-square closed Lie ideals

Let M be a 2-torsion free σ-prime Γ-ring and U be a Lie ideal of M such that $U\not\subset Z(M)$. In [5], Halder and Paul proved a lemma (Lemma 2.2) that if $a,b\in M$ such that $a\alpha U\beta b=a\alpha U\beta\sigma(b)=0$, for all $\alpha,\beta\in\Gamma$, then $a=0$ or $b=0$. This lemma is the key of the intensive study of the relationship between several maps (especially derivations and automorphisms) and Lie ideals of σ-prime Γ-rings and by this lemma many results can be extended to σ-prime Γ-rings. In this section, we are primarily interested in centralizing automorphisms on Lie ideals. This lemma will also play an important role in the last section of the present paper. The proof of the following lemma is similar to the proof of Lemma 1.5 in [9]. We give the proof for the sake of completeness.

3.1. Lemma. *Let $U\neq 0$ be a σ-ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition (2.1). If $[U,U]_\Gamma=0$, then $U\subseteq Z(M)$.*

Proof. Let $u\in U\cap Sa_\sigma(M)$. From $[U,U]_\Gamma=0$, it follows that $[u,[u,m]_\alpha]_\alpha=0$, for all $x\in M$, $\alpha\in\Gamma$. Let $d_u(x)=[u,x]_\alpha$, for all $x\in M$ and $\alpha\in\Gamma$. Then, d_u is a derivation and by the condition (2.1), $d_u(d_u(x))=[u,[u,x]_\alpha]_\alpha=0$. Hence, $d_u^2(x)=0$, for all $x\in M$. Now, replacing x by $x\beta y$, we have

$$\begin{aligned}0&=d_ud_u(x\beta y)\\&=d_u(d_u(x)\beta y+x\beta d_u(y))\\&=d_u^2(x)\beta y+d_u(x)\beta d_u(y)+d_u(x)\beta d_u(y)+x\beta d_u^2(y)\\&=2d_u(x)\beta d_u(y).\end{aligned}$$

Since M is 2-torsion free, we have

(3.1) $\quad d_u(x)\beta d_u(y)=0.$

For every $z\in M$ we replace x by $x\gamma z$ in (3.1), we obtain

$$\begin{aligned}0&=d_u(x\gamma z)\beta d_u(y)\\&=d_u(x)\gamma z\beta d_u(y)+x\gamma d_u(z)\beta d_u(y)\\&=d_u(x)\gamma z\beta d_u(y),\end{aligned}$$

for all $x,y,z\in M$, $\beta,\gamma\in\Gamma$. That is $d_u(x)\gamma M\beta d_u(y)=0$, for all $x,y,z\in M$, $\beta,\gamma\in\Gamma$. As $d_u\sigma=\pm\sigma d_u$, then $d_u(x)\Gamma M\Gamma d_u(y)=0=\sigma(d_u(x))\Gamma M\Gamma d_u(y)$. Since M is σ-prime, $d_u=0$ and hence $[u,x]_\alpha=0$, i.e., $u\in Z(M)$. Therefore, $U\cap Sa_\sigma(M)\subseteq Z(M)$. Let $u\in U$, as $u+\sigma(u)$ and $u-\sigma(u)$ are in $U\cap Sa_\sigma(M)$. Therefore, $u+\sigma(u)$ and $u-\sigma(u)$ are in $Z(M)$, so that $2u\in Z(M)$. Consequently, u in $Z(M)$ proving that $U\subseteq Z(M)$. □

3.2. Lemma. *([3], Lemma 2.2). If $U\not\subset Z(M)$ is a Lie ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition (2.1) and $a,b\in M$ such that $a\alpha U\beta b=a\alpha U\beta\sigma(b)=0$, for all $\alpha,\beta\in\Gamma$, then $a=0$ or $b=0$.*

3.3. Lemma. *Let U be a σ-square closed Lie ideal of a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1) having a non-trivial automorphism T centralizing on U and commuting with σ on U. If u in $U \cap Sa_\sigma(M)$ is such that $T(u) \neq u$, then $u \in Z(M)$.*

Proof. If $U \subseteq Z(M)$, then $u \in Z(M)$. So, let $U \not\subseteq Z(M)$. Then, by Lemma 3.1, $[U,U]_\Gamma \neq 0$. Since $[u,T(u)]_\alpha \in Z(M)$, after linearization of it, we obtain $[u,T(v)]_\alpha + [v,T(u)]_\alpha \in Z(M)$, for all $u,v \in U$, $\alpha \in \Gamma$. In particular, we have $[u,T(u\beta u)]_\alpha + [u\beta u, T(u)]_\alpha \in Z(M)$. Hence,

$$\begin{aligned}&[u,T(u)\beta T(u)]_\alpha + [u\beta u,T(u)]_\alpha\\&= T(u)\beta[u,T(u)]_\alpha + [u,T(u)]_\alpha\beta T(u) + u\beta[u,T(u)]_\alpha + [u,T(u)]_\alpha\beta u\\&= T(u)\beta[u,T(u)]_\alpha + T(u)\beta[u,T(u)]_\alpha + u\beta[u,T(u)]_\alpha + u\beta[u,T(u)]_\alpha\\&= 2(u+T(u))\beta[u,T(u)]_\alpha \in Z(M).\end{aligned}$$

Since M is 2-torsion free, we obtain $(u+T(u))\beta[u,T(u)]_\alpha \in Z(M)$. Hence,

$$0 = [u,(u+T(u))\beta[u,T(u)]_\alpha]_\alpha = [u,T(u)]_\alpha\beta[u,T(u)]_\alpha,$$

since $[u,T(u)]_\alpha \in Z(M)$. Thus, $[u,T(u)]_\alpha = 0$, for all $u \in U$ and $\alpha \in \Gamma$. Again, linearizing this equality, we obtain

$$(3.2) \qquad [u,T(v)]_\alpha = [T(u),v]_\alpha.$$

Let $u \in U \cap Sa_\sigma(M)$ with $T(u) \neq u$. By replacing v by $2u\beta v$ in (3.2) we obtain $0 = (u - T(u))\beta[T(u),v]_\alpha$, for all $v \in U$. By putting $2w\gamma v$ instead of v, we obtain $(u-T(u))\beta w\gamma[T(u),v]_\alpha = 0$, for all $w \in U$ and $\gamma \in \Gamma$. This shows that $(u - T(u))\Gamma U\Gamma[T(u),v]_\alpha = 0$. Therefore,

$$(u-T(u))\Gamma U\Gamma[T(u),v]_\alpha = (u-T(u))\Gamma U\Gamma\sigma([T(u),v]_\alpha) = 0,$$

for all $v \in U$. Since $T(u) \neq u$, by Lemma 3.2, $[T(u),v]_\alpha = 0$, for all $v \in U$. Hence, for all $m \in M$, $[T(u),[v,m]_\beta]_\alpha = 0$, and so $[T(u),m\beta v]_\alpha = [T(u),v\beta m]_\alpha$. Thus, $[T(u),m]_\alpha\beta v = v\beta[T(u),m]_\alpha$, for all $m \in M$ and $\beta \in \Gamma$. Replacing m by $m\gamma u$, where $u \in U$, we find that $[T(u),m]_\alpha\gamma u\beta v = v\beta[T(u),m]_\alpha\gamma u = [T(u),m]_\alpha\beta v\gamma u$. Now, by using (2.1), we obtain $[T(u),m]_\alpha\gamma[u,v]_\beta = 0$. This implies that $[T(u),m]_\alpha\delta y\gamma[u,v]_\beta = 0$, for all $y \in M$ and $\delta \in \Gamma$. Hence, $[T(u),m]_\alpha\delta M\gamma[U,U]_\beta = 0$. Since $\sigma(U) = U$,

$$[T(u),m]_\alpha\delta M\gamma[U,U]_\Gamma = 0 = [T(u),m]_\alpha\delta M\gamma\sigma([U,U]_\Gamma).$$

Since $[U,U]_\Gamma \neq 0$, the σ-primeness of M yields $[T(u),m]_\alpha = 0$. This gives that $T(u) \in Z(M)$. As T is an automorphism, it then follows that $u \in Z(M)$. □

Now, we have in position to prove the main result.

3.4. Theorem. *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1). Let $T : M \to M$ be an automorphism centralizing on a σ-square closed Lie ideal U of M such that $T \neq 1$ and $T\sigma = \sigma T$ on U. Then, $U \subseteq Z(M)$.*

Proof. If $[U,U]_\Gamma = 0$, then $U \subseteq Z(M)$ by Lemma 3.1. So, let us assume that $[U,U]_\Gamma \neq 0$. If T is the identity on U, then for all $m \in M$, $u \in U$, $\alpha \in \Gamma$,

$$(3.3) \qquad T([m,u]_\alpha) = [m,u]_\alpha = [T(m),u]_\alpha.$$

Replacing m by $m\beta v$ in (3.3), for $v \in U$ and $\beta \in \Gamma$, we obtain,

$$\begin{aligned}&[m\beta v,u]_\alpha = [T(m\beta v),u]_\alpha\\&\Rightarrow m\beta[v,u]_\alpha + [m,u]_\alpha\beta v = [T(m)\beta T(v),u]_\alpha\\&\Rightarrow m\beta[v,u]_\alpha + [m,u]_\alpha\beta v = T(m)\beta[T(v),u]_\alpha + [T(m),u]_\alpha\beta T(v)\\&\Rightarrow m\beta[v,u]_\alpha + [m,u]_\alpha\beta v = T(m)\beta[v,u]_\alpha + [m,u]_\alpha\beta v\end{aligned}$$

Thus,

(3.4) $\quad m\beta[v,u]_\alpha = T(m)\beta[v,u]_\alpha.$

For any $y \in M$ and $\gamma \in \Gamma$, we write $m\gamma y$ instead of m in (3.4), we obtain $m\gamma y\beta[v,u]_\alpha = T(m)\gamma T(y)\beta[v,u]_\alpha$ which implies that $m\gamma y\beta[v,u]_\alpha = T(m)\gamma y\beta[v,u]_\alpha$. Thus, $(T(m) - m)\gamma y\beta[v,u]_\alpha = 0$ for all $u, v \in U$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since $\sigma(U) = U$, we obtain

$$(T(m) - m)\gamma M\beta[v,u]_\alpha = 0 = (T(m) - m)\gamma M\beta\sigma([v,u]_\alpha).$$

By the fact that $[U,U]_\Gamma \neq 0$, it yields that $T(m) - m = 0$, for all $m \in M$, which is impossible. So, T is non-trivial on U. By the 2-torsion freeness of M, T is also non-trivial on $U \cap Sa_\sigma(M)$. Therefore, there is an element u in $U \cap Sa_\sigma(M)$ such that $u \neq T(u)$ and $u \in Z(M)$ by Lemma 3.3. Let $v \neq 0$ be in $U \cap Sa_\sigma(M)$ and not be in $Z(M)$. Again in view of Lemma 3.3, we conclude that $T(v) = v$. But we have $T(u\alpha v) = T(u)\alpha v = u\alpha v$, so that $(T(u) - u)\alpha v = 0$. Since $v \in Sa_\sigma(M) \cap U$, it yields that $(T(u) - u)\beta m\alpha v = (T(u) - u)\beta\alpha\sigma(v) = 0$, for all $m \in M$, $\beta \in \Gamma$. Since M is σ-prime and $T(u) \neq u$, we obtain that $v = 0$, a contradiction. Therefore, for all v in $U \cap Sa_\sigma(M)$, v must be in $Z(M)$. Now, let $u \in U$, the fact that $u - \sigma(u)$ and $u + \sigma(u)$ are in $U \cap Sa_\sigma(M)$ gives that both $u - \sigma(u)$ and $u + \sigma(u)$ are in $Z(M)$ and therefore $2u \in Z(M)$. Consequently, $u \in Z(M)$ which proves $U \subseteq Z(M)$. □

4. Jordan left derivations on σ-square closed Lie ideals

In this section M will always denote a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1) and U a σ-square closed Lie ideal of M. For proving the main result, we first state a few known results which will be used in subsequent discussion.

4.1. Lemma. *([5], Lemma 3) Let M be a 2-torsion free Γ-ring and let U be a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \to M$ is an additive mapping satisfying $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$, $\alpha \in \Gamma$ and M satisfies the condition (2.1), then*

(1) $d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u)$,
(2) $d(u\alpha v\beta u) = u\alpha u\beta d(v) + 3u\alpha v\beta d(u) - v\alpha u\beta d(u)$,
(3) $d(u\alpha v\beta w + w\alpha v\beta u) = (u\alpha w + w\alpha u)\beta d(v) + 3u\alpha v\beta d(w) + 3w\alpha v\beta d(u) - v\alpha u\beta d(w) - v\alpha w\beta d(u)$,
(4) $[u,v]_\alpha\gamma u\beta d(u) = u\gamma[u,v]_\alpha\beta d(u)$,
(5) $[u,v]_\alpha(d(u\alpha v) - u\alpha d(v) - v\alpha d(u)) = 0$,

for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.

4.2. Lemma. *([5], Lemma 4). Let M be a 2-torsion free Γ-ring satisfying the condition (2.1) and let U be a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \to M$ is an additive mapping satisfying $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$, then*

(1) $[u,v]_\alpha\beta d([u,v]_\alpha) = 0$,
(2) $(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\beta u)\beta d(v) = 0$,

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

Now, similar to the proof of Theorem in [1] and Theorem 1.6 in [9], the main result of this section is given as follows:

4.3. Theorem. *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1) and let U be a σ-square closed Lie ideal of M. If $d : M \to M$ is an additive mapping which satisfies $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.*

Proof. Let $[U,U]_\Gamma = 0$. Then, we get $U \subseteq Z$ by Lemma 3.1. Using Lemma 4.1, we obtain that $d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u)$, for all $u, v \in U$. Since $u \in Z(M)$ and M is 2-torsion free, we arrive at $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$. So, we suppose that $[U,U]_\Gamma \neq 0$. We have

$$(4.1) \qquad (u\alpha u\gamma v - 2u\alpha v\gamma u + v\gamma u\alpha u)\beta d(u) = 0.$$

Writing $[u,w]_\delta$ in place of u in (4.1), where $w \in U$ and $\delta \in \Gamma$. We obtain

$$([u,w]_\delta\alpha[u,w]_\delta\gamma v - 2[u,w]_\delta\alpha v\gamma[u,w]_\delta + v\gamma[u,w]_\delta\alpha[u,w]_\delta)\beta d([u,w]_\delta) = 0,$$

which implies that

$$\begin{aligned}&[u,w]_\delta\alpha[u,w]_\delta\gamma v\beta d([u,w]_\delta) - 2[u,w]_\delta\alpha v\gamma[u,w]_\delta\beta d([u,w]_\delta)\\ &\qquad +v\gamma[u,w]_\delta\alpha[u,w]_\delta\beta d([u,w]_\delta) = 0.\end{aligned}$$

In view of Lemma 4.2(1), we have $[u,w]_\delta\alpha[u,w]_\delta\gamma v\beta d([u,w]_\delta) = 0$. This implies that $[u,w]_\delta\alpha[u,w]_\delta\gamma U\beta d([u,w]_\delta) = 0$, for all $u, w \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Let $a, b \in Sa\sigma(M) \cap U$. We have $[a,b]_\delta\alpha[a,b]_\delta\gamma U\beta d([a,b]_\delta) = 0 = \sigma([a,b]_\delta\alpha[a,b]_\delta)\gamma U\beta d([a,b]_\delta)$ and by virtue of Lemma 3.2 either $[a,b]_\delta\alpha[a,b]_\delta = 0$ or $d([a,b]_\delta) = 0$. If $d([a,b]_\delta) = 0$, then by using Lemma 4.2(1) and the 2-torsion freeness of M, we have seen that $d(a\delta b) = a\delta d(b) + b\delta d(a)$, for all $\delta \in \Gamma$. Now, assume that $[a,b]_\delta\alpha[a,b]_\delta = 0$. From Lemma 4.2(2), it follows that $(u\alpha u\gamma v - 2u\alpha v\gamma u + v\gamma u\alpha u)\beta d(u) = 0$, for all $u, v \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Linearizing this relation in u, we obtain

$$((u+w)\alpha(u+w)\gamma v - 2(u+w)\alpha v\gamma(u+w) + v\alpha(u+w)\gamma(u+w))\beta d(v) = 0,$$

for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$. After calculation we obtain

$$(u\alpha w\gamma v + w\alpha u\gamma v - 2u\alpha v\gamma w - 2w\alpha v\gamma u + v\alpha u\gamma w + v\alpha w\gamma u)\beta d(v) = 0,$$

for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing v by $[a,b]_\delta$ and using Lemma 4.2(1), we have

$$(4.2) \qquad (-2u\alpha[a,b]_\delta\gamma w - 2w\alpha[a,b]_\delta\gamma u + [a,b]_\delta\alpha u\gamma w + [a,b]_\delta\alpha w\gamma u)\beta d([a,b]_\delta) = 0.$$

Putting $u\mu[a,b]_\delta$ instead of u in (4.2), applying $[a,b]_\delta\alpha[a,b]_\delta = 0$ and Lemma 4.2(1), we obtain $[a,b]_\delta\alpha u\mu[a,b]_\delta\gamma w\beta d([a,b]_\delta) = 0$ by using (2.1), for all $u, w \in M$ and $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$. Accordingly, $[a,b]_\delta\alpha u\mu[a,b]_\delta\gamma U\beta d([a,b]_\delta) = 0$, for all $u \in M$ and $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$. Since $[a,b]_\delta \in U \cap Sa_\sigma(M)$ and $\sigma(U) = U$, we obtain

$$[a,b]_\delta\alpha u\mu[a,b]_\delta\gamma U\beta d([a,b]_\delta) = 0 = \sigma([a,b]_\delta\alpha u\mu[a,b]_\delta)\gamma U\beta d([a,b]_\delta),$$

for all $u \in U$ and $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$. By Lemma 3.2, we obtain $d([a,b]_\delta) = 0$ or $[a,b]_\delta\alpha u\mu[a,b]_\delta = 0$, for all $u \in U$ and $\alpha, \gamma, \delta, \mu \in \Gamma$. If $[a,b]_\delta\alpha u\mu[a,b]_\delta = 0$, then we have $[a,b]_\delta\alpha u\mu\sigma([a,b]_\delta) = 0$, since $\sigma(u) = u$ for all $u \in U$ and $[a,b]_\delta \in U$ for all $a, b \in U$, $\delta \in \Gamma$. Therefore, by Lemma 3.2, $[a,b]_\delta = 0$, which shows that $[U,U]_\Gamma = 0$, which is contradiction to our assumption that $[U,U]_\Gamma \neq 0$. So, let $d([a,b]_\delta) = 0$. Then, by previous arguments, we have $d(a\delta b) = a\delta d(b) + b\delta d(a)$. Therefore, in the both cases we find that

$$(4.3) \qquad d(a\alpha b) = a\alpha d(b) + b\alpha d(a),$$

for all $a, b \in U \cap Sa_\sigma(M)$ and $\alpha \in \Gamma$. Now, let us assume that $u, v \in U$, set $u_1 = u + \sigma(u)$, $u_2 = u - \sigma(u)$, $v_1 = v + \sigma(v)$, $v_2 = v - \sigma(v)$. Then, we have $2u = u_1 + u_2$ and $2v = v_1 + v_2$. For the fact that $u_1, u_2, v_1, v_2 \in U \cap Sa_\sigma(M)$, and application of (4.3) gives

$$\begin{aligned} d(2u\alpha 2v) &= d((u_1+u_2)\alpha(v_1+v_2))\\ &= d(u_1\alpha v_1 + u_1\alpha v_2 + u_2\alpha v_1 + u_2\alpha v_2)\\ &= u_1\alpha d(v_1) + v_1\alpha d(u_1) + u_1\alpha d(v_2) + v_2\alpha d(u_1) + u_2\alpha d(v_1) + v_1\alpha d(u_2)\\ &\qquad + u_2\alpha d(v_2) + v_2\alpha d(u_2)\\ &= (u_1+u_2)\alpha d(v_1+v_2) + (v_1+v_2)\alpha d(u_1+u_2)\\ &= 2u\alpha d(2v) + 2vd(2u).\end{aligned}$$

This implies that $4d(u\alpha v) = 4(u\alpha d(v) + v\alpha d(u))$. Since M is 2-torsion free, we have $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$. □

4.4. Corollary. *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1). Then, every Jordan left derivation on M is a left derivation on M.*

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 59 – 74

Some relationships between intrinsic and extrinsic invariants of submanifolds in generalized S-space-forms

L.M. Fernández* and A.M. Fuentes†

Abstract

We establish some inequalities of Chen's type between certain intrinsic invariants (involving sectional, Ricci and scalar curvatures) and the squared mean curvature of submanifolds tangent to the structure vector fields of a generalized S-space-form and we discuss the equality cases of them. We apply the obtained results to slant submanifolds.

2000 AMS Classification: Primary 53C25, Secondary 53C40

Keywords: Generalized S-space-form, intrinsic invariant, slant submanifold.

Received 13/12/2011 : Accepted 14/12/2013 Doi : 10.15672/HJMS.201544993

1. Introduction

Intrinsic and extrinsic invariants are very powerful tools to study submanifolds of Riemannian manifolds and to establish relationships between them is one of the most fundamental problems in submanifolds theory. In this context, B.-Y. Chen [10, 11, 12] proved some basic inequalities for submanifolds of a real space-form. Corresponding inequalities have been obtained for different kinds of submanifolds (invariant, anti-invariant, slant) in ambient manifolds endowed with different kinds of structures (mainly, real, complex and Sasakian space-forms).

Moreover, it is well known that the sectional curvatures of a Riemannian manifold determine the curvature tensor field completely. So, if (M, g) is a connected Riemannian manifold with dimension greater than 2 and its curvature tensor field R has the pointwise expression

$$R(X,Y)Z = \lambda \{g(X,Z)Y - g(Y,Z)X\},$$

*Departmento de Geometría y Topología, Facultad de Matemáticas Universidad de Sevilla, Apartado de Correos 1160, 41080 Sevilla, Spain
Author is partially supported by the PAI group FQM-327 (Junta de Andalucía, Spain, 2011) and by the MEC project MTM 2011-22621 (MEC, Spain, 2011).
Email: lmfer@us.es

†Departmento de Geometría y Topología, Facultad de Matemáticas Universidad de Sevilla, Apartado de Correos 1160, 41080 Sevilla, Spain

where λ is a differentiable function on M, then M is a space of constant sectional curvature, that is, a real-space-form and λ is a constant function.

Further, when the manifold is equipped with some additional structure, it is sometimes possible to obtain conclusions from a special form of the curvature tensor field for this structure too. Thus, for almost-Hermitian manifolds, F. Tricerri and L. Vanhecke [25] introduced *generalized complex-space-forms* and, for almost contact metric manifolds, P.Alegre, D.E. Blair and A. Carriazo [1] defined and studied *generalized Sasakian-space-forms*, which generalize complex and Sasakian space-forms, respectively.

More in general, K. Yano [27] introduced the notion of f-structure on a $(2m+s)$-dimensional manifold as a tensor field f of type (1,1) and rank $2m$ satisfying $f^3 + f = 0$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of f-structures. The case $s = 2$ appeared in the study of hypersurfaces in almost contact manifolds [3, 17]. A Riemannian manifold endowed with an f-structure compatible with the Riemannian metric is called a metric f-manifold. For $s = 0$ we have almost Hermitian manifolds and for $s = 1$, metric almost contact manifolds. In this context, D.E. Blair [2] defined K-manifolds (and particular cases of S-manifolds and C-manifolds) as the analogue of Kaehlerian manifolds in the almost complex geometry and of quasi-Sasakian manifolds in the almost contact geometry and he showed that the curvature of either S-manifolds or C-manifolds is completely determined by their f-sectional curvatures. Later, M. Kobayashi and S. Tsuchiya [21] got expressions for the curvature tensor field of S-manifolds and C-manifolds when their f-sectional curvature is constant depending on such a constant. Such spaces are called S-space-forms and C-space-forms and they generalize complex and Sasakian space-forms and cosymplectic space-forms, respectively.

For metric f-manifolds, the authors and A. Carriazo [6] and, independently, M. Falcitelli and A.M. Pastore [15], have introduced a notion of *generalized S-space-form*. The first ones limited their research to the case $s = 2$, even though their definition is easily adaptable to any $s > 2$ [24], giving some non-trivial examples [5, 6]. Consequently, generalized S-space-forms make a more general framework to study the geometry of certain metric f-manifolds. Moreover, their definition generalized those ones of generalized complex-space-forms ($s = 0$) and Sasakian-space-forms ($s = 1$). Actually, a generalized S-space-form is a $(2m+2)$-dimensional metric f-manifold $(\widetilde{M}, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ with two structure vector fields in which there exist seven differentiable functions F_1, F_2, F_3 and $F_{11}, F_{12}, F_{21}, F_{22}$ on $\widetilde{M}$ such that the curvature tensor field of $\widetilde{M}$ is given by

$$\widetilde{R} = \sum_{i=1}^{3} F_i \widetilde{R}_i + \sum_{i,j=1}^{2} F_{ij} \widetilde{R}_{ij},$$

where $\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, \widetilde{R}_{11}, \widetilde{R}_{12}, \widetilde{R}_{21}, \widetilde{R}_{22}$ are basic constant sectional curvature tensorial parts. It easy to show that S-space-forms and C-space-forms become particular cases of generalized S-space-forms (see [6])

For these reasons and since some inequalities of Chen's type, involving sectional, scalar and Ricci curvatures and squared mean curvature, have been proved for different kinds of submanifolds in S-space-forms [7, 16, 19, 20], the purpose of this paper is to establish them for generalized S-space-forms with two structure vector fields. To this end, after a preliminaries section containing basic notions of Riemannian submanifolds theory, in Section 3 we present some definitions and formulas concerning metric f-manifolds for later use and in Section 4 we bound the Ricci curvature of submanifolds in generalized S-space-form by terms involving the mean curvature and the second fundamental form of the submanifold plus certain quantities involving the set of functions $\{F_i, F_{ij}\}$. In fact, we prove that for a submanifold M of dimension $(n+2)$, tangent to both structure vector fields,

$$\mathrm{Ric(U)} \leq \frac{(n+2)^2}{4}\|H\|^2 + (n+1)F_1 + 3\|TU\|^2 F_2 - (F_{11} + F_{22}),$$

for any unit vector field U orthogonal to the structure vector fields, being TU the tangential component of fU (Theorem 4.1). Furthermore, if the submanifold is a mininimal one, a unit vector field U orthogonal to both structure vector fields satisfies the equality case of the above expression if and only if it lies in the relative null space of M (Theorem 4.2). We also present some examples to illustrate these results including a compact one and we particularize them in the case of C-manifolds (Theorem 4.6) and in the case of being the submanifold a slant one (Theorem 4.7).

Finally, since in 1995 B.-Y. Chen [10, 11] introduced a well-defined Riemannian invariant $\delta_{\widetilde{M}}$ of a Riemannian manifold $\widetilde{M}$ whose definition is given by $\delta_{\widetilde{M}}(p) = \tau(p) - (\inf K)(p)$, for any $p \in \widetilde{M}$ (where τ is the scalar curvature and $(\inf K)(p) = \inf\{K(\pi) :$ plane sections $\pi \subset T_p(\widetilde{M})\}$, $K(\pi)$ denoting the sectional curvature of $\widetilde{M}$ associated with the plane section π) and he proved, for submanifolds M in real-space-forms, a basic inequality involving the intrinsic invariant δ_M and the squared mean curvature of the immersion, in Section 5 we investigate a similar inequality for a submanifold M of a generalized S-space-form, tangent to the structure vector fields, studying the equality case (Theorem 5.2).

Moreover, we consider the well-defined function on M, $(\inf_{\mathcal{L}} K)(p) = \inf\{K(\pi) :$ plane sections $\pi \subset \mathcal{L}_p\}$, where $\mathcal{L}$ denotes the complementary orthogonal distribution to that one spanned by the structure vector fields (on M). If $\delta_M^{\mathcal{L}}$ is the difference between the scalar curvature and $\inf_{\mathcal{L}} K$, it is clear that $\delta_M^{\mathcal{L}} \leq \delta_M$ and then, we obtain some general pinching results for $\delta_M^{\mathcal{L}}$, depending on the sign of F_2 (Theorems 5.9 and 5.10), studying the equality cases and presenting some examples to illustrate them. We conclude the paper applying the obtained theorems to slant submanifolds, specially in the case of the smallest possible dimension.

We should like to point out here that all the results of the paper improve those ones proved for S-space-forms in [7, 16]

2. Preliminaries.

Let M be a Riemannian manifold isometrically immersed in a Riemannian manifold $\widetilde{M}$. Let g denote the metric tensor of $\widetilde{M}$ as well as the induced metric tensor on M. If ∇ and $\widetilde{\nabla}$ denote the Riemannian connections of M and $\widetilde{M}$, respectively, the Gauss-Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),\ \widetilde{\nabla}_X V = -A_V X + D_X V, \tag{2.1}$$

for any vector fields X, Y (resp., V) tangent (resp., normal) to M, where D is the normal connection, σ is the second fundamental form of the immersion and A_V is the Weingarten endomorphism associated with V. Then, A_V and σ are related by $g(A_V X, Y) = g(\sigma(X,Y), V)$.

The curvature tensor fields R and $\widetilde{R}$ of ∇ and $\widetilde{\nabla}$, respectively, satisfies the Gauss equation

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)), \tag{2.2}$$

for any X, Y, Z, W tangent to M.

The mean curvature vector H is defined by

$$H = \frac{1}{m}\mathrm{trace}\ \sigma = \frac{1}{m}\sum_{i=1}^{m}\sigma(e_i, e_i),$$

where $\dim M = m$ and $\{e_1, \ldots, e_m\}$ is a local orthonormal basis of tangent vector fields to M. In this context, M is said to be *minimal* if H vanishes identically or, equivalently, if $\mathrm{trace} A_V = 0$, for any vector field V normal to M. Moreover, M is said to be *totally geodesic* in $\widetilde{M}$ if $\sigma \equiv 0$ and *totally umbilical* if $\sigma(X,Y) = g(X,Y)H$, for any X, Y tangent to M. Moreover, the *relative null space* of M is defined by:

$$\mathcal{N} = \{X \text{ tangent to } M : \ \sigma(X,Y) = 0, \text{ for all } Y \text{ tangent to } M\}.$$

3. Submanifolds of metric f-manifolds.

A $(2m+s)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ with an f-structure f (that is, a tensor field f of type (1,1) and rank $2m$ satisfying $f^3 + f = 0$ [27]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1, \ldots, \xi_s$ on $\widetilde{M}$ (called *structure vector fields*) such that, if $\eta_1, \ldots, \eta_s$ are their dual 1-forms, then

$$(3.1) \qquad f\xi_\alpha = 0; \ \eta_\alpha \circ f = 0; \ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha;$$

$$g(X,Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y),$$

for any X, Y tangent to $\widetilde{M}$. Let F be the 2-form on $\widetilde{M}$ defined by $F(X,Y) = g(X, fY)$. Since f is of rank $2m$, then $\eta_1 \wedge \cdots \wedge \eta_s \wedge F^m \neq 0$ and, particularly, $\widetilde{M}$ is orientable. The f-structure f is said to be *normal* if

$$[f,f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f,f]$ denotes the Nijenhuis tensor of f.

A metric f-manifold is said to be a K-manifold [2] if it is normal and $\mathrm{d}F = 0$. In a K-manifold $\widetilde{M}$, the structure vector fields are Killing vector fields [2]. Furthermore, a K-manifold is called an S-manifold if $F = \mathrm{d}\eta_\alpha$ and a C-manifold if $\mathrm{d}\eta_\alpha = 0$, for any $\alpha = 1, \ldots, s$. Note that, for $s = 0$, a K-manifold is a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold, an S-manifold is a Sasakian manifold and a C-manifold is a cosymplectic manifold. When $s \geq 2$, non-trivial examples can be found in [2, 18]. Moreover, a K-manifold $\widetilde{M}$ is an C-manifold if and only if

$$(3.2) \qquad \widetilde{\nabla}_X \xi_\alpha = 0, \ \alpha = 1, \ldots, s,$$

for any tangent vector field X.

A plane section π on a metric f-manifold $\widetilde{M}$ is said to be an *f-section* if it is determined by a unit vector X, orthogonal to the structure vector fields and fX. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold (resp., a C-manifold) is said to be an *S-space-form* (resp., a *C-space-form*) if it has constant f-sectional curvature (see [2, 21] for more details).

Next, let M be a isometrically immersed submanifold of a metric f-manifold $\widetilde{M}$. For any vector field X tangent to M we write

$$(3.3) \qquad fX = TX + NX,$$

where TX and NX are the tangential and normal components of fX, respectively. The submanifold M is said to be *invariant* if N is identically zero, that is, if fX is tangent to M, for any vector field X tangent to M. On the other hand, M is said to be an *anti-invariant* submanifold if T is identically zero, that is, if fX is normal to M, for any X tangent to M.

From now on, we suppose that all the structure vector fields are tangent to the submanifold M. Then, the distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$ and its complementary orthogonal distribution is denoted by $\mathcal{L}$. Consequently, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha = 1, \dots, s$ and if $X \in \mathcal{M}$, then $fX = 0$.

The submanifold M is said to be a *slant* submanifold if, for any $p \in M$ and any $X \in T_pM$, linearly independent on $(\xi_1)_p, \dots, (\xi_s)_p$, the angle between fX and T_pM is a constant $\theta \in [0, \pi/2]$, called the slant angle of M in $\widetilde{M}$. Note that this definition generalizes that one given by B.-Y. Chen [13] for complex geometry and that one given by A. Lotta [22] for contact geometry. Moreover, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively (for a general view about slant submanifolds, the survey written by A. Carriazo [4] can be consulted). A slant immersion which is neither invariant nor anti-invariant is called a *proper* slant immersion. In [9], it is proved that, a θ-slant submanifold M of a metric f-manifold $\widetilde{M}$ satisfies

$$g(NX, NY) = \sin^2\theta g(fX, fY), \tag{3.4}$$

for any vector fields X, Y tangent to M. Moreover, if we denote by $n + s$ the dimension of M, given a local orthonormal basis $\{e_1, \dots, e_{n+s}\}$ of tangent vector fields to M, it is easy to show that

$$\sum_{j=1}^{n+s} g^2(e_i, fe_j) = cos^2\theta(1 - \sum_{\alpha=1}^{s} \eta_\alpha^2(e_i)), \tag{3.5}$$

for any $i = 1, \dots, n$.

Concerning the behavior of the second fundamental of a submanifold in a metric f-manifold, we know that the study of totally geodesic or totally umbilical slant submanifolds of S-manifolds reduces to the study of invariant submanifolds [9]. It is necessary, then, to use a variation of these concepts, more related to the structure, namely *totally f-geodesic* and *totally f-umbilical* submanifolds, introduced by Ornea [23]. Thus, a submanifold of a metric f-manifold, tangent to the structure vector fields, is said to be a totally f-geodesic submanifold (resp., totally f-umbilical) if the distribution $\mathcal{L}$ is totally geodesic (resp., totally umbilical), that is, if $\sigma(X, Y) = 0$ (resp., if there exist a normal vector field V such that $\sigma(X, Y) = g(X, Y)V$), for any $X, Y \in \mathcal{L}$.

Denoting by $n + s$ (resp. $2m + s$) the dimension of M (resp. $\widetilde{M}$) and given a local orthonormal basis

$$\{e_1, \dots, e_n, e_{n+1} = \xi_1, \dots, e_{n+s} = \xi_s, e_{n+s+1}, \dots, e_{2m+s}\}$$

of tangent vector fields to $\widetilde{M}$, such that $\{e_1, \dots, e_n\}$ is a local orthonormal basis of $\mathcal{L}$, the squared norms of T and N are defined by

$$\|T\|^2 = \sum_{i,j=1}^{n} g^2(e_i, Te_j), \ \|N\|^2 = \sum_{i=1}^{n} \|Ne_1\|^2, \tag{3.6}$$

respectively, being independent of the choice of the above orthonormal basis. Moreover, we put $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$, for any $i, j = 1, \dots, n + s$ and $r = n + s + 1, \dots, 2m + s$. Then, the mean curvature vector H and the squared norm of σ can be written as:

$$H = \frac{1}{n+s} \sum_{r=n+1}^{2m} \sum_{i=1}^{n+s} \sigma_{ii}^r e_r, \tag{3.7}$$

$$\|\sigma\|^2 = \sum_{r=n+1}^{2m} \left\{ \sum_{i=1}^{n+s} (\sigma_{ii}^r)^2 + 2 \sum_{1 \le i < j \le n+s} (\sigma_{ij}^r)^2 \right\}. \tag{3.8}$$

4. Slant submanifolds of generalized S-space-forms with two structure vector fields.

The notion of generalized S-space-forms was introduced by the authors and A. Carriazo in [6], considering the case of two structure vector fields which appeared in the study of hypersurfaces in almost contact manifolds [3, 17] and which was the first motivation to investigate metric f-manifolds but, in fact, their definition is easily adaptable to any $s > 2$. Independently, M. Falcitelli and A.M. Pastore gave a slightly different definition [15]. From it, one can deduce that the distribution spanned by the structure vector fields must be flat which is the case, for instance, of S-manifolds and C-manifolds. However, in [5, 6] some non-trivial examples of generalized S-space-forms with non-flat distribution spanned by the structure vector fields are provided. Moreover, it is easy to show that both definitions coincide for metric f-manifolds such that either $\widetilde{\nabla}\xi_\alpha = -f$ or $\nabla\xi_\alpha = 0$, for any $\alpha = 1, \ldots, s$. Thus and for the purpose of this paper, we shall use the definition of [6].

Consequently, from now on, we consider a $(2m+2)$-dimensional metric f-manifold $(\widetilde{M}, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ with two structure vector fields. Then, $\widetilde{M}$ is said to be a *generalized S-space-form* [6, 26] if there exist seven differentiable functions F_1, F_2, F_3 and $F_{11}, F_{12}, F_{21}, F_{22}$ on $\widetilde{M}$ such that the curvature tensor field of $\widetilde{M}$ is given by

$$\widetilde{R} = \sum_{i=1}^{3} F_i \widetilde{R}_i + \sum_{i,j=1}^{2} F_{ij} \widetilde{R}_{ij}, \tag{4.1}$$

where

$$\begin{aligned}
\widetilde{R}_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\
\widetilde{R}_2(X,Y)Z &= g(X,fZ)fY - g(Y,fZ)fX + 2g(X,fY)fZ, \\
\widetilde{R}_3(X,Y)Z &= \eta_1(X)\eta_2(Y)\eta_2(Z)\xi_1 - \eta_2(X)\eta_1(Y)\eta_2(Z)\xi_1 \\
&\quad + \eta_2(X)\eta_1(Y)\eta_1(Z)\xi_2 - \eta_1(X)\eta_2(Y)\eta_1(Z)\xi_2, \\
\widetilde{R}_{ij}(X,Y)Z &= \eta_i(X)\eta_j(Z)Y - \eta_i(Y)\eta_j(Z)X \\
&\quad + g(X,Z)\eta_i(Y)\xi_j - g(Y,Z)\eta_i(X)\xi_j, \; i,j = 1,2,
\end{aligned}$$

for any X, Y, Z tangent to M. Some examples of generalized S-space-forms are given in [5, 6]. In particular, S-space-forms and C-space-forms are generalized S-space-forms.

Let M be a submanifold isometrically immersed in $\widetilde{M}$, tangent to both structure vector fields and suppose that $\dim(M) = n+2$. As above, let us consider a local orthonormal basis

$$\{e_1, \ldots, e_n, e_{n+1} = \xi_1, e_{n+2} = \xi_2, e_{n+3}, \ldots, e_{2m+2}\} \tag{4.2}$$

of tangent vector fields to $\widetilde{M}$, such that $\{e_1, \ldots, e_n\}$ is a local orthonormal basis of $\mathcal{L}$. The scalar curvature τ of M is defined by

$$\tau = \frac{1}{2}\sum_{i \neq j} K(e_i \wedge e_j), \tag{4.3}$$

where K denotes the sectional curvature of M. From (2.2), (3.7)-(3.8) and (4.1), we obtain the following relation between the scalar curvature and the mean curvature of M:

$$\begin{aligned}
2\tau =& (n+1)(n+2)F_1 - 2(n+1)(F_{11} + F_{22}) + 2F_3 \\
&+ 3F_2\|T\|^2 + (n+2)^2\|H\|^2 - \|\sigma\|^2.
\end{aligned} \tag{4.4}$$

Now, from (3.7), (3.8) and (4.4), a straightforward computation gives:

$$\begin{aligned}\tau = & \frac{(n+2)^2}{4}\|H\|^2 + \frac{(n+1)(n+2)}{2}F_1 \\ & + \frac{3\|T\|^2}{2}F_2 + F_3 - (n+1)(F_{11}+F_{22}) \\ & - \sum_{r=n+3}^{2m+2}\left\{\sum_{1\le i<j\le n+2}(\sigma_{ij}^r)^2 - \frac{1}{4}\sum_{i=1}^{n+2}(\sigma_{ii}^r)^2 + \frac{1}{2}\sum_{1\le i<j\le n+2}\sigma_{ii}^r\sigma_{jj}^r\right\}.\end{aligned} \tag{4.5}$$

By using the above formula, we can prove the following general result:

4.1. Theorem. *Let M be an $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. Then,*

$$\mathrm{Ric(U)} \le \frac{(n+2)^2}{4}\|H\|^2 + (n+1)F_1 + 3\|TU\|^2F_2 - (F_{11}+F_{22}), \tag{4.6}$$

for any unit vector field $U \in \mathcal{L}$.

Proof. We choose a local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2) and such that $e_1 = U$. Then, from (4.3):

$$\tau = \mathrm{Ric(U)} + \sum_{2\le i<j\le n} K(e_i\wedge e_j) + \sum_{i=2}^{n}\sum_{\alpha=1}^{2} K(e_i\wedge\xi_\alpha) + K(\xi_1\wedge\xi_2). \tag{4.7}$$

Now, by using (4.1), we get

$$\begin{aligned}\sum_{2\le i<j\le n} K(e_i\wedge e_j) = & \frac{(n-1)(n-2)}{2}F_1 \\ & + \sum_{2\le i<j\le n}\left\{3F_2 g(e_1, fe_j)^2 + \sum_{r=n+3}^{2m+2}\left(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2\right)\right\},\end{aligned}$$

$$\begin{aligned}\sum_{i=2}^{n}\sum_{\alpha=1}^{2} K(e_i\wedge\xi_\alpha) = & 2(n-1)F_1 - (n-1)(F_{11}+F_{22}) \\ & + \sum_{r=n+3}^{2m+2}\sum_{i=2}^{n}\sum_{\alpha=1}^{2}\left(\sigma_{ii}^r\sigma_{n+\alpha n+\alpha}^r - (\sigma_{in+\alpha}^r)^2\right)\end{aligned}$$

and:

$$K(\xi_1\wedge\xi_2) = F_1 + F_3 - (F_{11}+F_{22}) - \|\sigma(\xi_1,\xi_2)\|^2 + g(\sigma(\xi_1,\xi_1),\sigma(\xi_2,\xi_2)).$$

Then, substituting into (4.7) and taking into account (4.5), we obtain,

$$\begin{aligned}\mathrm{Ric}(U) = & \frac{(n+2)^2}{4}\|H\|^2 + (n+1)F_1 + 3\|TU\|^2F_2 - (F_{11}+F_{22}) \\ & - \sum_{r=n+3}^{2m+2}\left\{\frac{1}{4}\left(\sigma_{11}^r - \sum_{i=2}^{n+2}\sigma_{ii}^r\right)^2 + \sum_{i=2}^{n+2}(\sigma_{1i}^r)^2\right\},\end{aligned} \tag{4.8}$$

which completes the proof. □

What about the equality case of (4.6)? If the submanifold is minimal, we can prove the following theorem.

4.2. Theorem. *Let M be a minimal $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. Then, a unit vector field U in $\mathcal{L}$ satisfies the equality case of (4.6) if and only if U lies in the relative null space $\mathcal{N}$ of M.*

Proof. If $U \in \mathcal{L}$ is a unit vector field satisfying the equality case of (4.6), then, choosing a local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2) and such that $e_1 = U$, from (4.8) we get that $\sigma^r_{1n+\alpha} = 0$, for any $r = n+3, \dots, 2m+2$ and $\alpha = 1, 2$. So, $\sigma(U, \xi_\alpha) = 0$, $\alpha = 1, 2$. Furthermore, by using (4.8) again, we obtain $\sigma^r_{1i} = 0$, for any $i = 2, \dots, n$, $r = n+3, \dots, 2m+2$ (that is, $\sigma(U, e_i) = 0$, for any $i = 2, \dots, n$) and

$$\sigma^r_{11} = \sum_{i=2}^{n+2} \sigma^r_{ii},$$

for any $r = n+3, \dots, 2m+2$. But, since $H = 0$,

$$\sigma^r_{11} = -\sum_{i=2}^{n+2} \sigma^r_{ii},$$

for any $r = n+3, \dots, 2m+2$, thus $\sigma^r_{11} = 0$ and $\sigma(U, U) = 0$. Consequently, $U \in \mathcal{N}$.

Conversely, if $U \in \mathcal{N}$, choosing a local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2) with $e_1 = U$, we have that $\sigma^r_{1i} = 0$, for any $i = 1, \dots, n+2$ and $r = n+3, \dots, 2m+2$. Again, since $H = 0$, we obtain that

$$\sum_{i=2}^{n+2} \sigma^r_{ii} = 0,$$

for any $r = n+3, \dots, 2m+2$. Then, from (4.8) we deduce the equality case of (4.6). □

The following corollary is immediate:

4.3. Corollary. *Let M be a minimal $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If the equality case of (4.6) holds for all unit vector fields in $\mathcal{L}$, then M is a totally f-geodesic manifold.*

4.4. Example. To illustrate the above results, consider $\mathbf{C}^2$ with its structure of complex-space-form with constant holomorphic curvature $c = 0$. Then, it is known ([6]) that the warped product $\widetilde{M} = \mathbf{R} \times_{h_2} (\mathbf{R} \times_{h_1} \mathbf{C}^2)$ is a generalized S-space-form ($h_1, h_2 > 0$ differentiable functions on $\mathbf{R}$) with functions:

$$F_1 = -\frac{(h_1')^2}{h_1^2 h_2^2} - \frac{(h_2')^2}{h_2^2}; \; F_2 = F_{12} = F_{21} = 0;$$

$$F_3 = F_{11} = -\frac{(h_1')^2}{h_1^2 h_2^2} + \frac{h_1''}{h_1 h_2^2};$$

$$F_{22} = -\frac{(h_1')^2}{h_1^2 h_2^2} - \frac{(h_2'^2)}{h_2^2} + \frac{h_2''}{h_2}.$$

Moreover,

$$x(t_1, t_2, u, v) = (t_1, t_2, u\cos\theta, u\sin\theta, v, 0)$$

defines a 4-dimensional totally geodesic θ-slant submanifold M in $\widetilde{M}$ [5]. Consequently, for any unit vector field $U \in \mathcal{L}$, we deduce because the equality case of (4.6) that:

$$\mathrm{Ric(U)} = 3F_1 - F_{11} - F_{22} = -\frac{(h_1')^2}{h_1^2 h_2^2} - \frac{2(h_2')^2}{h_2^2} - \frac{h_1''}{h_1 h_2^2} - \frac{h_2''}{h_2}.$$

4.5. Example. Consider the sphere S^{2m+1} ($m \geq 2$) with its usual structure (ϕ, ξ, η, g) of Sasakian-space-form with constant ϕ-sectional curvature $c = 1$. Then, it is a generalized Sasakian-space-form with functions (see [1]) $f_1 = 1$ and $f_2 = f_3 = 0$. Let $\widetilde{M}$ be an isometrically immersed (orientable) hypersurface of $S^{2m+1}M$ such that the structure vector field ξ of S^{2m+1} is always tangent to $\widetilde{M}$ and let N denote the unit normal vector field of $\widetilde{M}$ in S^{2m+1}. If we put

$$\xi_1 = \xi;\ \xi_2 = -\phi N;\ \eta_1 = \eta;\ \eta_2(X) = -g(X, \phi N);\ fX = \phi X - \eta_2(X)N,$$

for any vector field X tangent to $\widetilde{M}$, then $(\widetilde{M}, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ is a metric f-manifold [28]. Moreover, if $\widetilde{M}$ is a pseudo-umbilical hypersurface of S^{2m+1}, that is, if its shape operator A satisfies (see [28])

$$AX = \widetilde{f}_1\,(X - \eta_1(X)\xi_1) + \widetilde{f}_2\eta_2(X)\xi_2 - \eta_1(X)\xi_2 - \eta_2(X)\xi_1,$$

for any X tangent to $\widetilde{M}$, $\widetilde{f}_1$ and $\widetilde{f}_2$ being differentiable functions on $\widetilde{M}$, we know that $\widetilde{M}$ is a generalized S-space-form with functions (see [6] for more details):

$$F_1 = 1 + \widetilde{f}_1^2;\ F_2 = 0;\ F_{11} = \widetilde{f}_1^2;\ F_{22} = -\widetilde{f}_1\widetilde{f}_2;$$

$$F_{12} = F_{21} = \widetilde{f}_1;\ F_3 = -1 - \widetilde{f}_1\widetilde{f}_2.$$

In this context, from Theorem 7.1 in [28], if $\widetilde{M}$ is also a compact hypersurface of S^{2m+1}, then it is congruent to $S^{2m-1}(r_1) \times S^1(r_2)$ with $r_1^2 + r_2^2 = 1$. Consequently, we have an example of a compact generalized S-space-form. Next, let M be a $(n+2)$-dimensional submanifold of $\widetilde{M}$ satisfying the conditions of Theorem 4.2 (for instance, a totally geodesic submanifold). Thus, if $X \in \mathcal{L}$ is a unit vector field, we deduce that:

$$Ric(U) = \frac{(n+2)^2}{4}\|H\|^2 + (n\widetilde{f}_1 + \widetilde{f}_2)\widetilde{f}_1 + n + 1.$$

Next, assume that $m \geq 2$. If the ambient generalized S-space-form $\widetilde{M}$ is an S-manifold, it is known (see Proposition 7 in [15] and Theorem 3.2 in [18]) that $\widetilde{M}$ is an S-space-form. Therefore [6],

$$(4.9)\qquad F_1 = \frac{c+6}{4};\ F_2 = F_3 = \frac{c-2}{4};\ F_{11} = F_{22} = \frac{c+2}{4};\ F_{12} = F_{21} = -1,$$

where c is denoting the constant f-sectional curvature. In this case, a better (in the sense of lower) upper bound for $Ric(U)$ than the one obtained in (4.6) was got in [16]. In fact and in terms of the functions of (4.9), it was proved that:

$$(4.10)\qquad \mathrm{Ric}(U) \leq \frac{(n+2)^2}{4}\|H\|^2 + (n-1)F_1 + (3F_1 - 4)\|TU\|^2.$$

It is easy to show that both upper bounds of (4.6) and (4.10) are equal if and only if $\|NU\| = 0$ and their common value is:

$$\frac{(n+2)^2}{4}\|H\|^2 + (n+2)F_1 - 4.$$

Conditions for the equality case of (4.10) have also been given in [16].

Now, we suppose that the ambient generalized S-space-form $\widetilde{M}$ is a C-manifold. Then, from Proposition 8 and Remark 2 in [15] it is known that $\widetilde{M}$ is a C-space-form and so [6],

$$(4.11)\qquad F_1 = F_2 = F_3 = F_{11} = F_{22} = \frac{c}{4};\ F_{12} = F_{21} = 0,$$

where c is denoting the constant f-sectional curvature. If M is a $(n+2)$-dimensional submanifold of $\widetilde{M}$, tangent to both structure vector fields, from (2.1), (3.2) and (3.3) it is easy to show that

$$(4.12)\qquad \sigma(X, \xi_\alpha) = 0,$$

for any X tangent to M and $\alpha = 1, 2$. Then, because of the identities (4.11) the equation (4.8) becomes to

$$\begin{aligned}\operatorname{Ric}(U) =& \frac{(n+2)^2}{4}\|H\|^2 + \{(n-1) + 3\|TU\|^2\}F_1 \\ & - \sum_{r=n+3}^{2m+2}\left\{\frac{1}{4}\left(\sigma_{11}^r - \sum_{i=2}^{n}\sigma_{ii}^r\right)^2 + \sum_{i=2}^{n}(\sigma_{1i}^r)^2\right\}\end{aligned} \tag{4.13}$$

and so,

$$\operatorname{Ric}(U) \leq \frac{(n+2)^2}{4}\|H\|^2 + \{(n-1) + 3\|TU\|^2\}F_1, \tag{4.14}$$

for any unit vector field $U \in \mathcal{L}$. To study the equality case of the above equation, we prove:

4.6. Theorem. *Let M be a $(n+2)$-dimensional submanifold ($n \geq 2$) of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If $\widetilde{M}$ is also an C-manifold, then the equality case of (4.14) holds for all unit vector field in $\mathcal{L}$ if and only if either M is a totally f-umbilical submanifold when $n = 2$ or M is a totally geodesic submanifold when $n > 2$.*

Proof. If the equality case of (4.14) is true for any unit vector field $U \in \mathcal{L}$, then, by choosing local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2) and since e_1 can be chosen to be any arbitrary unit vector field in $\mathcal{L}$, from (4.13) we get

$$\sigma_{ii}^r = \sigma_{jj}^r = \frac{1}{2}(\sigma_{11}^r + \cdots + \sigma_{nn}^r),\ i, j = 1, \ldots, n,$$

$$\sigma_{ij}^r = 0,\ i \neq j,\ i, j = 1, \ldots, n,$$

for any $r = n+3, \ldots, 2m$. Thus, we have to consider two cases. Firstly, if $n = 2$, we deduce that $\sigma_{11}^r = \sigma_{22}^r$, for any r and M is a totally f-umbilical submanifold. Secondly, if $n > 2$ we obtain that $\sigma_{ii}^r = 0$, for any $i = 1, \ldots n$ and r and so, together with (4.12), we deduce that M is a totally geodesic submanifold. The converse part is obvious from (4.13). □

To illustrate the above result we can consider the warped product of Example 4.4 with functions $h_1 = h_2 = 1$. It is easy to show that, in this case, the generalized S-space-form $\widetilde{M}$ is a C-manifold.

The above results imply the following theorem for slant submanifolds of generalized S-space-forms.

4.7. Theorem. *Let M be a $(n+2)$-dimensional ($n \geq 2$) θ-slant submanifold of a generalized S-space-form $\widetilde{M}$ and $U \in \mathcal{L}$ be any unit vector field. Then:*

(i) We have that:

$$\operatorname{Ric}(\mathrm{U}) \leq \frac{1}{4}(n+2)^2\|H\|^2 + (n+1)F_1 + 3\cos^2\theta F_2 - (F_{11} + F_{22}). \tag{4.15}$$

(ii) If $\widetilde{M}$ is also an S-manifold, we have

$$\operatorname{Ric}(U) \leq \frac{(n+2)^2}{4}\|H\|^2 + (n-1)F_1 + (3F_1 - 4)\cos^2\theta$$

and the equality holds for all unit vector field in $\mathcal{L}$ if and only if either M is a totally f-geodesic submanifold when $n > 2$ or M is a totally f-umbilical submanifold when $n = 2$.

(iii) If $\widetilde{M}$ is also a C-manifold, we have

$$\mathrm{Ric}(U) \leq \frac{(n+2)^2}{4}\|H\|^2 + \{(n-1) + 3\cos^2\theta\}F_1$$

and the equality holds for all unit vector field in $\mathcal{L}$ if and only if either M is a totally f-umbilical submanifold when $n = 2$ or M is a totally geodesic submanifold when $n > 2$.

Proof. For any unit vector field $U \in \mathcal{L}$, by using a local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2), such that $e_1 = U$, we get from (3.5) and (3.6) that $\|TU\|^2 = \cos^2\theta$ and so, from (4.6) we have (4.15). For the rest of the proof we only have to consider the results of [16] for S-manifolds and Theorem 4.6 for C-manifolds. □

5. The scalar curvature.

In the articles [10, 11], B.-Y. Chen introduced a well-defined Riemannian invariant $\delta_{\widetilde{M}}$ of a Riemannian manifold $\widetilde{M}$ whose definition is given by

$$\delta_{\widetilde{M}}(p) = \tau(p) - (\inf K)(p),$$

for any $p \in \widetilde{M}$, where τ is the scalar curvature and

$$(\inf K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset T_p(\widetilde{M})\},$$

with $K(\pi)$ denoting the sectional curvature of $\widetilde{M}$ associated with the plane section π. Moreover, for submanifolds M in a real-space form of constant sectional curvature c, Chen gave the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature of the immersion

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{(n+1)(n-2)}{2}c,$$

where n denotes the dimension of M. A similar inequality for S-space-forms, conditions for the equality case and some applications have been established in [7]. In this section, we want to study the more general case of generalized S-space-forms.

Let $\widetilde{M}$ be a generalized S-space-form with two structure vectors ξ_1, ξ_2 and M a $(n+2)$-dimensional submanifold of $\widetilde{M}$, tangent to both structure vector fields. Let $\pi \subset \mathcal{L}_p$ a plane section at $p \in M$. Then,

$$F^2(\pi) = g^2(e_1, fe_2) \tag{5.1}$$

is a real number in $[0,1]$ which is independent on the choice of the orthonormal basis $\{e_1, e_2\}$ of π. First, we recall an algebraic lemma from [14]:

5.1. Lemma. *Let $a_1, \dots, a_k, c$ be $k+1$ $(k \geq 2)$ real numbers such that:*

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)\left(\sum_{i=1}^{k} {a_i}^2 + c\right).$$

Then, $2a_1a_2 \geq c$, with the equality holding if and only if:

$$a_1 + a_2 = a_3 = \cdots = a_k.$$

Now, we can prove the following theorem.

5.2. Theorem. *Let M be a $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. Then, for any point $p \in M$ and any plane section $\pi \subset \mathcal{L}_p$, we have:*

$$\tag{5.2} \begin{aligned} \tau - K(\pi) \leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}) \\ + 3F_2\left(\frac{\|T\|^2}{2} - F^2(\pi)\right) \end{aligned}$$

The equality in (5.2) holds at $p \in M$ if and only if there exist orthonormal bases $\{e_1, \ldots, e_{n+2}\}$ and $\{e_{n+3}, \ldots, e_{2m+2}\}$ of T_pM and $T_p^{\perp}M$, respectively, such that:

(i) $e_{n+j} = (\xi_j)_p$, for $j = 1, 2$.
(ii) π is spanned by e_1 and e_2.
(iii) The shape operators $A_r = A_{e_r}$, $r = n+3, \ldots, 2m+2$, take the following forms at p:

$$\tag{5.3} A_{n+3} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ b & c-a & 0 & 0 & 0 \\ 0 & 0 & c & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix},$$

$$\tag{5.4} A_r = \begin{pmatrix} a_r & b_r & 0 \\ b_r & -a_r & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $a, b, c, a_r, b_r \in \mathbf{R}$, for any $r = n+4, \ldots, 2m+2$.

Proof. Let $\pi \subset \mathcal{L}_p$ be a plane section and choose orthonormal bases $\{e_1, \ldots, e_{n+2}\}$ of T_pM and $\{e_{n+2}, \ldots, e_{2m+2}\}$ of $T_p^{\perp}M$ such that $e_{n+j} = (\xi_j)_p$, for $j = 1, 2$, π is spanned by e_1, e_2 and e_{n+3} is in the direction of the mean curvature vector H. Then, from (4.1)

$$\tag{5.5} K(\pi) = \sigma_{11}^{n+3}\sigma_{22}^{n+3} - (\sigma_{12}^{n+3})^2 + \sum_{r=n+4}^{2m+2}(\sigma_{11}^r\sigma_{22}^r - (\sigma_{12}^r)^2) + F_1 + 3F_2F^2(\pi).$$

Now, put:

$$\tag{5.6} \varepsilon = 2\tau - \frac{n(n+2)^2}{n+1}|H|^2 - n(n+3)F_1 + 2(n+1)(F_{11}+F_{22}) - 3F_2\|T\|^2 - 2F_3.$$

Hence, (4.4) and (5.6) imply:

$$(n+2)^2\|H\|^2 = (n+1)\{\|\sigma\|^2 + \varepsilon - 2F_1\}$$

that is, respect to the above orthonormal bases:

$$\begin{aligned} \left(\sum_{i=1}^{n+2}\sigma_{ii}^{n+3}\right)^2 = &(n+1)\left\{\sum_{i=1}^{n+2}(\sigma_{ii}^{n+3})^2 + \sum_{i\neq j}^{n+2}(\sigma_{ij}^{n+3})^2\right. \\ &\left. + \sum_{r=n+4}^{2m+2}\sum_{i,j=1}^{n+2}(\sigma_{ij}^r)^2 + \varepsilon - 2F_1\right\}. \end{aligned}$$

Therefore, by applying Lemma 5.1, we get:

$$\tag{5.7} 2\sigma_{11}^{n+3}\sigma_{22}^{n+3} \geq \sum_{i\neq j}^{n+2}(\sigma_{ij}^{n+3})^2 + \sum_{r=n+4}^{2m+2}\sum_{i,j}^{n+2}(\sigma_{ij}^r)^2 + \varepsilon - 2F_1.$$

Thus, from (5.5) and (5.7), we obtain:

$$
\begin{aligned}
K(\pi) \geq & \sum_{r=n+3}^{2m+2}\sum_{j>2}^{n+2}\left\{(\sigma_{1j}^r)^2+(\sigma_{2j}^r)^2\right\}+\frac{1}{2}\sum_{i\neq j>2}^{n+2}(\sigma_{ij}^{n+3})^2+\frac{1}{2}\sum_{r=n+4}^{2m+2}\sum_{i,j>2}^{n+2}(\sigma_{ij}^r)^2 \\
& +\frac{1}{2}\sum_{r=n+4}^{2m+2}(\sigma_{11}^r+\sigma_{22}^r)^2+\frac{\varepsilon}{2}+3F_2F^2(\pi)\geq\frac{\varepsilon}{2}+3F_2F^2(\pi).
\end{aligned} \tag{5.8}
$$

Consequently, combining (5.6) and (5.8), we get (5.2). If the equality in (5.2) holds, then the inequalities in (5.7) and (5.8) become equalities. So, we have:

$$\sigma_{1j}^r=\sigma_{2j}^r=0,\ r=n+2,\dots,2m+2,j>2;$$

$$\sigma_{ij}^{n+3}=0,\ i\neq j>2;$$

$$\sigma_{ij}^r=0,\ r=n+4,\dots,2m+2;i,j>2;$$

$$\sigma_{11}^r+\sigma_{22}^r=0,\ r=n+4,\dots,2m+2;$$

$$\sigma_{11}^{n+3}+\sigma_{22}^{n+3}=\sigma_{ii}^{n+3},\ i=3,\dots,n+2.$$

Thus, with respect to the chosen orthonormal basis $\{e_1,\dots,e_{2m+2}\}$, the shape operators of M take the forms (5.3) and (5.4).

The converse follows from a direct calculation. □

Now, we consider:

$$(\inf_{\mathcal{L}}K)(p)=\inf\{K(\pi):\text{plane sections }\pi\subset\mathcal{L}_p\}.$$

Then, $\inf_{\mathcal{L}}K$ is a well-defined function on M. Let $\delta_M^{\mathcal{L}}$ denote the difference between the scalar curvature and $\inf_{\mathcal{L}}K$, that is:

$$\delta_M^{\mathcal{L}}(p)=\tau(p)-(\inf_{\mathcal{L}}K)(p).$$

It is clear that $\delta_M^{\mathcal{L}}\leq\delta_M$.

It is obvious that if the submanifold M is anti-invariant, then $\|T\|^2=F^2(\pi)=0$, for any plane section in $\mathcal{L}$. Consequently, from (5.2) we obtain:

5.3. Corollary. *Let M be a $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If either $F_2=0$ or M is an anti-invariant submanifold, then we have:*

$$\delta_M^{\mathcal{L}}\leq\frac{n(n+2)^2}{2(n+1)}\|H\|^2+\frac{n(n+3)}{2}F_1-(n+1)(F_{11}+F_{22})+F_3.$$

By using Theorem 5.2 we can obtain some general pinching results for $\delta_M^{\mathcal{L}}$ if either $F_2\geq 0$ or $F_2<0$.

5.4. Theorem. *Let M be a $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If $F_2\geq 0$, then we have:*

$$\delta_M^{\mathcal{L}}\leq\frac{n(n+2)^2}{2(n+1)}\|H\|^2+\frac{n(n+3)}{2}F_1+F_3-(n+1)(F_{11}+F_{22})+\frac{3n}{2}F_2. \tag{5.9}$$

The equality in (5.9) holds identically if and only if n is even and M is an invariant submanifold.

Proof. Since $F_2 \geq 0$, from (5.2) we deduce

$$\delta_M^{\mathcal{L}} \leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}) + 3F_2\frac{\|T\|^2}{2}$$

and, by using that $\|T\|^2 \leq n$, we get (5.9). Moreover, the equality holds if and only if $\|T\|^2 = n$, that is, if and only if M is invariant and so, n is even. □

5.5. Theorem. *Let M be a $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If $F_2 < 0$, then we have:*

$$\delta_M^{\mathcal{L}} \leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}). \tag{5.10}$$

The equality in (5.10) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n, (\xi_1)_p, (\xi_2)_p\}$ of $T_p(M)$ such that the subspace spanned by $e_3, \ldots, e_n$ is anti-invariant, that is, $Te_j = 0$, for any $j = 3, \ldots, n$.

Proof. From Theorem 5.2, we have (5.2) which implies:

$$\begin{aligned}\delta_M^{\mathcal{L}} &\leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}) \\ &\quad + 3F_2\left\{\sum_{j=3}^{n}\left(g^2(e_1,Te_j) + g^2(e_2,Te_j)\right) + \frac{1}{2}\sum_{i,j=3}^{n} g^2(e_i,Te_j)\right\} \\ &\leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}).\end{aligned} \tag{5.11}$$

If the equality in (5.10) holds, then both inequalities in (5.11) become equalities. Thus, we complete the proof. □

We observe that examples of generalized S-space-forms with $F_2 = 0$ are given in Examples 4.4 and 4.5. On the other hand, $\mathbf{R}^{2m+2}$ with its structure of S-space-form with constant f-sectional curvature $c = -6$ (see [18] for more details) is a generalized S-space-form with $F_2 = -2 < 0$ ([6]). Moreover, the warped product $\widetilde{M} = \mathbf{R} \times_{h_2} (\mathbf{R} \times_{h_1} N(c))$, $h_1, h_2 > 0$ differentiable functions on $\mathbf{R}$ and $N(c)$ a complex-space-form of constant holomorphic curvature $c > 0$, is a generalized S-space-form with ([6]):

$$F_2 = \frac{c}{4h_1^2h_2^2} > 0.$$

Finally, we are going to study inequality (5.2) when M is a slant submanifold. First, we observe that, if M is a $(n+2)$-dimensional θ-slant submanifold of a metric f-manifold, then, from (3.4), (3.5) and (3.6):

$$\|T\|^2 = n\cos^2\theta; \ \|N\|^2 = n\sin^2\theta. \tag{5.12}$$

Now, by using (5.2) and (5.12), we obtain:

5.6. Theorem. *Let M be a $(n+2)$-dimensional θ-slant submanifold of a generalized S-space-form $\widetilde{M}$. Then, for any point $p \in M$ and any plane section $\pi \subset \mathcal{L}_p$, we have:*

$$\begin{aligned}\tau - K(\pi) &\leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 \\ &\quad - (n+1)(F_{11}+F_{22}) + 3F_2\left(\frac{n}{2}\cos^2\theta - F^2(\pi)\right).\end{aligned} \tag{5.13}$$

It is well known [8] that there are no proper slant submanifolds of metric f-manifolds of dimension lower than $2+s$, being s the number of structure vector fields. Then, for $(2+2)$-dimensional slant submanifolds, we can state the following result:

5.7. Corollary. *Let M be a 4-dimensional θ-slant submanifold of a generalized S-space-form $\widetilde{M}$. Then, we have:*

$$\delta_M^{\mathcal{L}} \leq \frac{16}{3}\|H\|^2 + 5F_1 - 3(F_{11} + F_{22}) + F_3. \tag{5.14}$$

Moreover, the equality holds if and only if M is minimal.

Proof. Since $n = 2$, then it is clear that

$$\delta_M^{\mathcal{L}} = \tau - K(\mathcal{L}) \tag{5.15}$$

and $F^2(\mathcal{L}) = \cos^2\theta$. Thus, (5.14) follows directly from (5.13). On the other hand, by using (4.1) and (4.3), it easy to show that:

$$\tau - K(\mathcal{L}) = 5F_1 - 3(F_{11} + F_{22}) + F_3. \tag{5.16}$$

Hence, (5.15) and (5.16) imply the condition for the equality case in (5.14). □

This result improves that one obtained for S-space-forms in [7].

To illustrate the above result, we can consider $\mathbf{R}^{4+2}$ with it structure of S-space-form of constant f-sectional curvature $c = -6$. Then, it is a generalized S-space-form with $F_1 = 0$, $F_3 = -2$ and $F_{11} = F_{22} = -1$ ([6]). Moreover, for any $\theta \in [0, \pi/2]$,

$$x(u, v, t_1, t_2) = 2(u\cos\theta, u\sin t, v, 0, t_1, t_2)$$

defines a minimal (2+2)-dimensional θ-slant submanifold M (see [9] for more details). Consequently, from (5.14), we deduce that $\delta_M^{\mathcal{L}} = 4$.

Acknowledgements: The authors wish to express their gratitude to the referee of this paper for his/her valuable comments in order to improve it.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 75 – 81

Gelfand numbers of diagonal matrices

Z.I. Ismailov* , L. Cona† and E. Otkun Çevik‡

Abstract

In this work , a connection between Gelfand numbers of infinite diagonal matrix with linear bounded operator-elements in the direct sum of Banach spaces and its coordinate operators has been investigated.

2000 AMS Classification: 47A05

Keywords: Direct Sum of Banach Spaces and Operators; Gelfand Numbers of Operator.

Received 07/10/2013 : Accepted 13/01/2014 Doi : 10.15672/HJMS.2015449111

1. Introduction

The general theory of so-called singular numbers(singuläre zahlen) for linear compact operators has been explained in the famous book of I.Z.Gohberg and M.G.Krein [1]. But the first results in this area can be found in the papers of E.Schmidt [2] and J.von Neumann, R. Schatten [3] who used this concept in the theory of non-selfadjoint integral equations.

In recent times much attention has been separated to the study of linear bounded operators in Hilbert space and Banach space by means of geometric quantities such as approximation numbers, Gelfand numbers, Weyl numbers and etc. In the last years of 20th century research activity in this area grew considerably. Many of classical problems were solved, interesting new developments started.Deep connections between Banach space geometry and other areas of mathematics were discovered.

The axiomatic theory of Gelfand numbers has been given by A.Pietsch in [4,5]. In generally, in studies concerning to Gelfand numbers have been estimated or found for the special mapping on some functional Banach spaces or Banach spaces of sequences. For

*Karadeniz Technical University, Faculty of Sciences, Department of Mathematics, 61080 Trabzon, Turkey
Email: zameddin.ismailov@gmail.com
†Gumushane University, Department of Mathematical Engineering, 29100 Gumushane, Turkey
Email: lale_cona@hotmail.com
‡Karadeniz Technical University, Faculty of Sciences, Department of Mathematics, 61080 Trabzon, Turkey
Email:e_otkuncevik@hotmail.com

instance, A.Pietsch in [4] proved that for identity operator $id : l_p^n \longrightarrow l_q^n$, $n \in \mathbb{N}$, $1 \leq q \leq p \leq \infty$ the formula is valid that

$$c_k(id : l_p^n \longrightarrow l_q^n) = (n-k+1)^{\frac{1}{q}-\frac{1}{p}}, \ , k \geq 1$$

In classical papers mainly identity maps were considered in form

$$id : l_p^n \longrightarrow l_q^n, \ n \in \mathbb{N}, \ 1 \leq q \leq p \leq \infty$$

For example, for the infinite diagonal matrix

$$S(x_n) = (\sigma_n x_n), (x_n) \in l_u, \ 1 \leq u \leq \infty, \quad \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq ... \geq 0$$

has been established such that $c_n(S : l_u \longrightarrow l_u) = \sigma_n, n \geq 1$ and

$$c_n(S : l_u \longrightarrow l_v) = (\sum_{k=n}^{\infty} \sigma_k^r)^{\frac{1}{r}}, 1 \leq v \leq u \leq \infty, \ \frac{1}{r} = \frac{1}{v} - \frac{1}{u} \qquad [4]$$

In special case B.S. Kashin [6] and B.S. Mitiagin [7] proved that very striking result

$$c_n(id : l_1^m \longrightarrow l_2^m) \leq \rho \frac{[\log(m+1)]^{\frac{3}{2}}}{n^{\frac{1}{2}}}, \ n = 1, 2, ..., m$$

In this work these studies will be continued.

It is known that infinite direct sum of Banach spaces $\mathfrak{X}_m$, $m \geq 1$ in the sense of l_p , $1 \leq p < \infty$ and infinite direct sum of linear densely defined closed operators A_m in $\mathfrak{X}_m$, $m \geq 1$ are defined as

$$\mathfrak{X} = (\oplus_{m=1}^{\infty} \mathfrak{X}_m)_p = \left\{ x = (x_m) : \ x_m \in \mathfrak{X}_m \ , \ m \geq 1, \ \|x\|_p = (\sum_{m=1}^{\infty} \|x_m\|_{\mathfrak{X}_m}^p)^{\frac{1}{p}} < \infty \right\}$$

and

$$A = \oplus_{m=1}^{\infty} A_m \ , \ A : D(A) \subset \mathfrak{X} \longrightarrow \mathfrak{X} \ ,$$

$$D(A) = \{x = (x_m) \in \mathfrak{X} : \ x_m \in D(A_m), m \geq 1, \ Ax = (A_m x_m) \in \mathfrak{X}\} \ [8]$$

In second section the general facts concerning to the boundedness and compactness properties of direct sum operators in the direct sum of Banach spaces will be given.

In third section, some estimate formulas for the Gelfand numbers of the diagonal matrix with operator elements in form

$$S = \begin{pmatrix} S_1 & & & & & \\ & S_2 & & & 0 & \\ & & S_3 & & & \\ & & & \ddots & & \\ & 0 & & & S_m & \\ & & & & & \ddots \end{pmatrix} \quad , \quad S : \mathfrak{X} \longrightarrow \mathfrak{X},$$

where $S_m \in L(\mathfrak{X}_m)$, $m \geq 1$ and $S \in L(\mathfrak{X})$ will be investigated.

Note that many physics problems of today in the modelling of processes of multiparticle quantum mechanics, quantum field theory and in the physics of rigid bodies support to study a theory of direct sum of linear operators in the direct sum of Banach spaces [9].

In this paper, the norms $\|\cdot\|_p$ in $\mathfrak{X}$ and $\|\cdot\|_{\mathfrak{X}_m}$ in $\mathfrak{X}_m$, $m \geq 1$ will be denoted by $\|\cdot\|$ and $\|\cdot\|_m$, $m \geq 1$ respectively. In any Banach space $\mathfrak{B}$ the class of linear bounded and compact operators will be denoted by $L(\mathfrak{B})$ and $C_\infty(\mathfrak{B})$ respectively.

2. Direct sum of bounded and compact operators

In this section continuity and compactness properties of the operator $A = \oplus_{m=1}^{\infty} A_m$ in $\mathfrak{X}$ will be investigated when $A_m \in L(\mathfrak{X}_m)$ and $A_m \in C_\infty(\mathfrak{X}_m)$, $m \geq 1$ respectively.

Using the techniques of the Banach spaces l_p , $1 \leq p < \infty$ and Operator Theory the following two propositions can be proved in general.

2.1. Theorem. Let $A_m \in L(\mathfrak{X}_m)$, $m \geq 1$, $A = \oplus_{m=1}^{\infty} A_m$ in $\mathfrak{X}$. In order to $A \in L(\mathfrak{X})$ the necessary and sufficient condition is $\sup\limits_{m\geq 1} \|A_m\| < \infty$. Morever, in the case when $A \in L(\mathfrak{X})$ it is true that $\|A\| = \sup\limits_{m\geq 1} \|A_m\|$.

Note that from the definition of compactness of operators [10] it is implied that if $A \in C_\infty(\mathfrak{X}_m)$, then for each $m \geq 1$, $A_m \in C_\infty(\mathfrak{X}_m)$.

In general, the following result is true.

2.2. Theorem. Let $A_m \in C_\infty(\mathfrak{X}_m)$ for each $m \geq 1$, $A = \oplus_{m=1}^{\infty} A_m : \mathfrak{X} \longrightarrow \mathfrak{X}$. In this case $A \in C_\infty(\mathfrak{X})$ if and only if $\lim\limits_{m\longrightarrow\infty} \|A_m\| = 0$.

Proof. Assume that $\limsup\limits_{(m)} \|A_m\| > 0$. Then there exists a number $c > 0$ and a sequence $(k_m) \subset \mathbb{N}$ such that

$$\|A_{k_m}\| = \sup\left\{ \frac{\|A_{k_m} x_{k_m}\|_{k_m}}{\|x_{k_m}\|_{k_m}} : \ x_{k_m} \in \mathfrak{X}_{k_m} \diagdown \{0\}\, , m \geq 1 \right\} \geq c > 0$$

In this case there exist a sequence $(x^*_{k_m}) \in \mathfrak{X}_{k_m}$, such that $\frac{\|A_{k_m} x^*_{k_m}\|_{k_m}}{\|x^*_{k_m}\|_{k_m}} \geq c$, $m \geq 1$.

Now consider the following set in $\mathfrak{X}$ in form

$$M := \left\{ \left\{ 0, 0, ..., 0, \frac{x^*_{k_m}}{\left\|A_{k_m} x^*_{k_m}\right\|_{k_m}}, 0, ... \right\} \in \mathfrak{X} : \ m \geq 1 \right\}$$

It is clear that for $x \in M$, $\|x\| \leq \frac{1}{c} < \infty$, that is, M is a bounded set in $\mathfrak{X}$. On the other hand $AM = \left\{ \left\{ 0, 0, ..., 0, \frac{A_{k_m} x^*_{k_m}}{\left\|A_{k_m} x^*_{k_m}\right\|_{k_m}}, 0, ... \right\} \in \mathfrak{X} : \ m \geq 1 \right\}$.

From this it is easy to see that a set $\overline{AM} \subset \mathfrak{X}$ is not compact. Consequently, $\limsup\limits_{(m)} \|A_m\| = 0$, that is, it is obtained that $\lim\limits_{(m)} \|A_m\| = 0$.

On the contrary, define the following operators $K_n : \mathfrak{X} \longrightarrow \mathfrak{X}$, $n \geq 1$ in form

$$K_n := A_1 \oplus A_2 \oplus ... \oplus A_n \oplus 0 \oplus 0 \oplus ...$$

In this case for $x \in \mathfrak{X}$ we have

$$\begin{aligned} \|(A - K_n)x\|^p &\leq \sum_{m=n+1}^{\infty} \|A_m\|^p \, \|x_m\|_m^p \\ &\leq \sup_{m\geq n+1} \|A_m\|^p \sum_{m=n+1}^{\infty} \|x_m\|_m^p \\ &\leq \left(\sup_{m\geq n+1} \|A_m\|^p \right) \|x\|^p \end{aligned}$$

From this it is obtained that $\|A - K_n\| \leq \sup\limits_{m\geq n+1} \|A_m\|$, $n \geq 1$. Since $\limsup\limits_{(n)} \|A_n\| = 0$, then from last relation it is implied that sequence of operators (K_n) in $L(\mathfrak{X})$ is convergent to the operators A in operator norm. On the other hand $K_n \in C_\infty(\mathfrak{X})$, $n \geq 1$, then by the important theorem of the compact operators theory the operator A belong to the class $C_\infty(\mathfrak{X})$ [10]. $\square$

3. Gelfand numbers of direct sum operators

In this section, the relationship between the Gelfand numbers of the direct sum of operators in the direct sum of Banach spaces and its coordinate operators will be investigated.

Note that firstly the concept of s-number functions (particularly Gelfand number functions) for the operators in Banach spaces was introduced by A.Pietsch in [11].

Now give definitions of these number functions from works [4] and [12].

3.1. Definition. Let $L(E,F)$ be a Banach spaces of linear bounded operators from Banach space E to a Banach space F with operator norm. For the operator $T \in L(E,F)$ the following number

$$c_n(T) := \inf \left\{ \|T\|_Z : Z \subset E,\ co\dim Z < n \right\},\ n \geq 1$$

is called the n-th Gelfand number of the operator T.

3.2. Definition. Let E, F, E_0, F_0 be Banach spaces. A map s which to every operators $S \in L(E,F)$ a unique sequence $(s_n(S))$ is called an s-function (or s-number function) if the following conditions are satisfied:

(1) For $S \in L(E,F)$ $\|S\| = s_1(S) \geq s_2(S) \geq ... \geq 0$;

(2) For $S, T \in L(E,F)$ $s_n(S+T) \leq s_n(S) + \|T\|$;

(3) For $T \in L(E_0, E), S \in L(E,F)$ and $R \in L(F, F_0)$ $s_n(RST) \leq \|R\|\, s_n(S)\, \|T\|$;

(4) If $S \in L(E,F)$ and $dim(S) < n$, then $s_n(S) = 0$;

(5) If $id : l_2^n \longrightarrow l_2^n$ is the identity map, then $s_n(id) = 1$.

On the other hand $s_n(S), n \geq 1$ is called the n-th s-number of the operator S.

The advanced analysis of these numbers has been given in books of A.Pietsch [4,5].Particularly, in Hilbert spaces case for any $n \geq 1$ $c_n(S) = s_n(S) = \lambda_n(|S|) = \lambda_n(|S^*S|^{\frac{1}{2}})$ (for the more informations see [1]). On the other hand for $S \in C_\infty(H)$, where H is a Hilbert space, the significant method for computation of $s_n(S), n \geq 1$ has been given by Dzh.E. Allakhverdiev in [13].

3.3. Theorem. If $S = \oplus_{m=1}^{\infty} S_m$, $S \in L(\mathfrak{X})$ and for any $m \geq 1$, $n_m = \dim \mathfrak{X}_m < \infty$, then for $n > m_1 + m_2 + ... + m_k, k \geq 1$ it is true that $c_n(S) \leq \sup\limits_{m \geq k+1} \|S_m\|$.

Proof. Firstly, for any $k \in \mathbb{N}$ define the operator $P_k : \mathfrak{X} \longrightarrow \mathfrak{X}$ in following form

$$P_k(x_m) := \{x_1, x_2, ..., x_k, 0, ...\},\ \text{for } x = (x_m) \in \mathfrak{X}$$

In this case for $k \in \mathbb{N}$

$$SP_k(x_m) = \left(\oplus_{m=1}^{\infty} S_m\right)(P_k(x_m)) = \{S_1x_1, S_2x_2, ..., S_kx_k, 0, 0, 0, ...\}$$

and $SP_k \in L(\mathfrak{X})$.

Therefore, for any $x = (x_m) \in \mathfrak{X}$ it is clear that

$$\begin{aligned} \|(S - SP_k)(x_m)\| &= \|\{0, ..., 0, S_{k+1}x_{k+1}, ...\}\| = \left(\sum_{m=k+1}^{\infty} \|S_m x_m\|_m^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{m=k+1}^{\infty} \|S_m\|^p \|x_m\|_m^p \right)^{\frac{1}{p}} \leq \sup_{m \geq k+1} \|S_m\| \, \|x\| \end{aligned}$$

Hence $\|S - SP_k\| \leq \sup\limits_{m \geq k+1} \|S_m\|$. From this and definition of Gelfand numbers it is implied that for $n > m_1 + m_2 + ... + m_k$, $k \geq 1$ it is true that $c_n(S) \leq \sup\limits_{m \geq k+1} \|S_m\|$.

□

3.4. Theorem. Let $S = \oplus_{m=1}^{\infty} S_m \in L(\mathfrak{X})$ and the operator $S : \mathfrak{X} \longrightarrow \mathfrak{X}$ is invertible, i.e. there exist S^{-1} and $S^{-1} \in L(\mathfrak{X})$. Then $\inf\limits_{1\le m\le n} \frac{1}{\|S_m^{-1}\|} \le c_n(S),\ n \ge 1$.

Proof. It is known that in this case $S^{-1} = \oplus_{m=1}^{\infty} S_m^{-1}$ and $\|S^{-1}\| = \sup\limits_{m\ge1} \|S_m^{-1}\|$.

Now define

$$J_n = (\oplus_{m=1}^{n} \mathfrak{X}_m)_p \longrightarrow (\oplus_{m=1}^{\infty} \mathfrak{X}_m)_p \quad,\ n \ge 1,$$
$$Q_n = (\oplus_{m=1}^{\infty} \mathfrak{X}_m)_p \longrightarrow (\oplus_{m=1}^{n} \mathfrak{X}_m)_p \quad,\ n \ge 1,$$
$$J_n(\{x_1, ..., x_n\}) = \{x_1, x_2, ...x_n, 0, ...\},$$
$$Q_n(\{x_1, ..., x_n, x_{n+1}, ...\}) = \{x_1, x_2, ...x_n\}$$

From these definitions it is obtained that the operator

$$R_n := Q_n S J_n : (\oplus_{m=1}^{n} \mathfrak{X}_m)_p \longrightarrow (\oplus_{m=1}^{n} \mathfrak{X}_m)_p$$

is in form $R_n(\{x_1, x_2, ..., x_n\}) = \{S_1x_1, S_2x_2, ..., S_nx_n\}$, $n \ge 1$ for any $\{x_1, x_2, ..., x_n\} \in (\oplus_{m=1}^{n} \mathfrak{X}_m)_p$.

Therefore, there exist R_n^{-1} ,the inverse of the operator R_n, $n \ge 1$ and

$$R_n^{-1} = \oplus_{m=1}^{n} S_m^{-1},\quad R_n^{-1} : (\oplus_{m=1}^{n} \mathfrak{X}_m)_p \longrightarrow (\oplus_{m=1}^{n} \mathfrak{X}_m)_p\,,\quad \|R_n^{-1}\| = \sup_{1\le m\le n} \|S_m^{-1}\|\,,\ n \ge 1$$

Since the mapping $c : S \longrightarrow (c_n(S))$, $S \in L(\mathfrak{X})$ is a s-number function, then from the property of s-number function it is clear that

$$\begin{aligned} 1 \quad &= \quad c_n(id : (\oplus_{m=1}^{n} \mathfrak{X}_m)_p \longrightarrow (\oplus_{m=1}^{n} \mathfrak{X}_m)_p) = c_n(R_n R_n^{-1}) \le c_n(Q_n S J_n)\,\|R_n^{-1}\| \\ &\le \|Q_n\|\, c_n(S)\, \|J_n\|\, \|R_n^{-1}\| \le c_n(S)\, \|R_n^{-1}\|\,, n \ge 1 \end{aligned}$$

Hence $\frac{1}{\|R_n^{-1}\|} \le c_n(S)$, i.e. $\frac{1}{\sup\limits_{1\le m\le n}\|S_m^{-1}\|} \le c_n(S)$, $n \ge 1$. In other words, for each $n \ge 1$ it is true that $\inf\limits_{1\le m\le n} \frac{1}{\|S_m^{-1}\|} \le c_n(S)$.

□

3.5. Corollary. If $S = \oplus_{m=1}^{\infty} S_m$, $S_m = \alpha_m id$, $\alpha_m \in \mathbb{C}$, $id : \mathfrak{X}_m \longrightarrow \mathfrak{X}_m$ for each $m \ge 1$, then $\inf\limits_{1\le m\le n} |\alpha_m| \le c_n(S) \le \sup\limits_{m\ge n} |\alpha_m|\,,\ n \ge 1$.

3.6. Remark. In case when $\alpha_m \in \mathbb{R}$, $\alpha_1 \ge \alpha_2 \ge ... \ge \alpha_m \ge ... \ge 0$ and $\dim \mathfrak{X}_m = 1, m \ge 1$, then from Corollary 3.5 it is obtained that for every $n \ge 1$ $c_n(S) = \alpha_n,\ n \ge 1$.

This result has been obtained in [4].

Now prove the following results which explained some relation between Gelfand numbers of direct sum operator and its coordinate operators.

3.7. Theorem. Let us $S = \oplus_{m=1}^{\infty} S_m$, $S : \mathfrak{X} \longrightarrow \mathfrak{X}$. In this case for every $n \ge 1$

$$\sup_{n\ge1} c_n^{(m)}(S_m) \le c_n(S),$$

where $c_n^{(m)}(S_m)$ is denoted by the n-th Gelfand number of the operator $S_m \in L(\mathfrak{X}_m)$, $m \ge 1$.

Proof. If the following operators

$$D_m : \mathfrak{X}_m \longrightarrow \mathfrak{X}\ ,\ T_m : \mathfrak{X} \longrightarrow \mathfrak{X}_m,\ m \ge 1$$

define in forms

$$D_m x_m = \{0, 0, ..., 0, x_m, 0, ...\}\,, x_m \in \mathfrak{X}_m,$$
$$T_m(x_m) = x_m, (x_m) \in \mathfrak{X}, x_m \in \mathfrak{X}_m,\ m \ge 1,$$

then D_m, T_m are linear bounded operators and $\|D_m\| \leq 1, \|T_m\| \leq 1, m \geq 1$. Moreover, it is clear that $S_m = T_m S D_m,\ m \geq 1$.

Hence from third condition in definition of s-functions for any $n \geq 1$ and $m \geq 1$ it is established that $c_n^{(m)}(S_m) = c_n^{(m)}(T_m S D_m) \leq \|T_m\| \, c_n(S) \, \|D_m\| \leq c_n(S)$.

From last relation the validity of claim is evident.

On the other hand the following assertion is true. □

3.8. Theorem. If $S = \oplus_{m=1}^{\infty} S_m \in L(\mathfrak{X})$, then for any $n, m \geq 1$ it is valid that

$$c_n(S) \leq c_n(S_m) + \sup_{n \neq m} \|S_n\|$$

Proof. Indeed, from second condition in definition of s-functions it is established that

$$\begin{aligned} c_n(S) &= c_n(0 \oplus 0 \oplus ... \oplus 0 \oplus S_m \oplus 0 \oplus ... + S_1 \oplus S_2 \oplus ... \oplus S_{m-1} \oplus 0 \oplus S_{m+1} \oplus ...) \\ &\leq c_n(0 \oplus 0 \oplus ... \oplus 0 \oplus S_m \oplus 0 \oplus ...) + \|S_1 \oplus S_2 \oplus ... \oplus S_{m-1} \oplus 0 \oplus S_{m+1} \oplus ...\| \\ &= c_n(S_m) + \sup_{n \neq m} \|S_n\| \,,\ n \geq 1,\ m \geq 1 \end{aligned}$$

□

3.9. Theorem. If $S = \oplus_{m=1}^{\infty} S_m \in L(\mathfrak{X})$ and $S^{(p)} := S_1 \oplus S_2 \oplus ... \oplus S_p \oplus 0 \oplus ...$, $S^{(p)} : \mathfrak{X} \longrightarrow \mathfrak{X}$, $p \geq 1$, then $\left| c_n(S) - c_n(S^{(p)}) \right| \leq \sup_{m \geq p+1} \|S_m\|$, $n \geq 1$.

In particular, if $S \in C_{\infty}(\mathfrak{X})$, then $\lim_{p \longrightarrow \infty} c_n(S^{(p)}) = c_n(S)$, $n \geq 1$.

Proof. Since Gelfand number function is a s-number function, then in this case the validity of assertion is clear from inequality $\left\| c_n(S) - c_n(S^{(p)}) \right\| \leq \left\| S - S^{(p)} \right\|$, $p \geq 1$ and 2.2. Theorem. □

3.10. Remark. In Hilbert spaces case the analogous results have been obtained in [14].

Acknowledgement The authors would like to thank to Prof. C. Orhan (Ankara University, Turkey) for his interesting and encouragement discussions.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 83 – 91

Non existence of totally contact umbilical GCR-lightlike submanifolds of indefinite Kenmotsu manifolds

Varun Jain *, Rakesh Kumar† and Rakesh Kumar Nagaich ‡

Abstract

In present paper, after finding the conditions for the integrability of various distributions of a GCR-lightlike submanifold of indefinite Kenmotsu manifolds, we prove that there do not exist totally contact umbilical GCR-lightlike submanifolds of indefinite Kenmotsu manifolds other than totally contact geodesic GCR-lightlike submanifolds and moreover it is a totally geodesic GCR-lightlike submanifold.

2000 AMS Classification: 53C15, 53C40, 53C50.

Keywords: Indefinite Kenmotsu manifolds, GCR-lightlike submanifolds, totally contact umbilical GCR-lightlike submanifolds.

Received 16/03/2012 : Accepted 13/12/2013 Doi :10.5672/HJMS.20154490086

1. Introduction

Theory of lightlike submanifolds of semi-Riemannian manifolds is one of the most important topic of differential geometry since in this theory, the normal vector bundle intersects with the tangent bundle, contrary to classical theory of submanifolds. Therefore the theory of lightlike (degenerate) submanifolds becomes more interesting and remarkably different from the theory of non-degenerate submanifolds. In the development of the theory of lightlike submanifolds, Duggal and Bejancu [6] played a very crucial role. Since there is a significant use of the contact geometry in differential equations, optics, and phase spaces of a dynamical system (see Arnold [1], Maclane [11], Nazaikinskii et al. [12]). Therefore Duggal and Sahin [7] introduced contact CR-lightlike submanifolds and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds. But there does not exist any inclusion relation between invariant and screen real submanifolds so a new

*Department of Mathematics, Multani Mal Modi College, Patiala, Punjab, India.
Email: varun82jain@gmail.com

†Department of Basic and Applied Sciences, Punjabi University, Patiala. INDIA.
Email:dr_rk37@yahoo.co.in - Corresponding Author.

‡Department of Mathematics, Punjabi University, Patiala. INDIA.
Email:nagaichrakesh@yahoo.com

class of submanifolds called, Generalized Cauchy-Riemann GCR-lightlike submanifolds of indefinite Sasakian manifolds (which is an umbrella of invariant, screen real, contact CR-lightlike submanifolds) was derived by Duggal and Sahin [8]. Recently Gupta and Sharfuddin [10], defined GCR-lightlike submanifold of indefinite Kenmotsu manifolds.

In present paper we further elaborate the theory of GCR-lightlike submanifold of indefinite Kenmotsu manifolds. In section 3, we find the conditions for the integrability of various distributions and for the distributions to define totally geodesic foliation in submanifold. In section 4, we study totally contact umbilical GCR-lightlike submanifolds and prove that there do not exist totally contact umbilical GCR-lightlike submanifolds of indefinite Kenmotsu manifolds other than totally contact geodesic GCR-lightlike submanifolds and moreover it is a totally geodesic GCR-lightlike submanifold.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [6] by Duggal and Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\},$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM,$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is,

$$TM = RadTM \perp S(TM), \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$, where $\{\xi_1, ..., \xi_r\}, \{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(RadTM \mid_u)$, $\Gamma(ltr(TM) \mid_u)$ and $\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp}) \mid_u)$ and $\Gamma(S(TM) \mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have

2.1. Theorem. *([6]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM) \mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp} \mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.4}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (2.3), the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall \quad X, Y \in \Gamma(TM), \tag{2.5}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \quad \forall \quad X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, A_U is linear a operator on M, known as a shape operator.

Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then using (2.2), the Gauss and Weingarten formulae become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.7}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.8}$$

where we put $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.9}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.10}$$

where $X \in \Gamma(TM)$, $\quad N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using (2.2)-(2.3) and (2.7)-(2.10), we obtain

$$\bar{g}(h^s(X,Y), W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Let P be the projection morphism of TM on $S(TM)$. Then using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X,Y), \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively.

An odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi, V, \eta, \bar{g})$, where ϕ is a $(1,1)$ tensor field, V is a vector field called structure vector field, η is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ satisfying

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \eta(X)\eta(Y), \quad \bar{g}(X,V) = \eta(X), \tag{2.14}$$

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1, \tag{2.15}$$

for $X, Y \in \Gamma(T\bar{M})$, where $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$.

An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Kenmotsu manifold if (see [4]),

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(\phi X, Y)V + \eta(Y)\phi X, \ \textit{and} \ \bar{\nabla}_X V = -X + \eta(X)V, \tag{2.16}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}$.

3. Genralized Cauchy-Riemann (GCR)-Lightlike Submanifold

Calin[5], proved that if the characteristic vector field V is tangent to $(M, g, S(TM))$ then it belongs to $S(TM)$. We assume characteristic vector V is tangent to M throughout this paper.

3.1. Definition. Let $(M, g, S(TM), S(TM^{\perp}))$ be a real lightlike submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

(3.1) $$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM).$$

(B) There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$ such that

(3.2) $$S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V, \quad \phi(\bar{D}) = L \perp S.$$

where D_0 is invariant non degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V and L, S are vector subbundles of $ltr(TM)$ and $S(TM)^{\perp}$, respectively.

Then tangent bundle TM of M is decomposed as

(3.3) $$TM = \{D \oplus \bar{D} \oplus \{V\}\}, \quad D = Rad(TM) \oplus D_0 \oplus \phi(D_2).$$

A GCR-lightlike submanifold of indefinite Kenmotsu manifold is said to be proper if $D_0 \neq \{0\}$, $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $L_1 \neq \{0\}$.

Let Q, P_1, P_2 be the projection morphism on D, ϕL, ϕS respectively, therefore any $X \in \Gamma(TM)$ can be written as

(3.4) $$X = QX + V + P_1X + P_2X.$$

Applying ϕ to (3.4), we obtain

(3.5) $$\phi X = fX + \omega P_1 X + \omega P_2 X,$$

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(L)$ and $\omega P_2 X \in \Gamma(S)$, or, we can write (3.5), as

(3.6) $$\phi X = fX + \omega X,$$

where fX and ωX are the tangential and transversal components of ϕX, respectively. Similarly,

(3.7) $$\phi U = BU + CU, \quad U \in \Gamma(tr(TM)),$$

where BU and CU are the sections of TM and $tr(TM)$, respectively.

Differentiating (3.5) and using (2.7)-(2.10) and (3.7), we have

(3.8) $$D^l(X, \omega P_2 Y) = -\nabla^l_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y) + \eta(Y)\omega P_1 X,$$

(3.9) $$D^s(X, \omega P_1 Y) = -\nabla^s_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y) + \eta(Y)\omega P_2 X,$$

for all $X, Y \in \Gamma(TM)$. By using Kenmotsu property of $\bar{\nabla}$ with (2.7) and (2.8), we have the following lemmas.

3.2. Lemma. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then we have*

(3.10) $$(\nabla_X f)Y = A_{\omega Y} X + Bh(X, Y) - g(\phi X, Y)V + \eta(Y) fX,$$

and

(3.11) $$(\nabla^t_X \omega)Y = Ch(X, Y) - h(X, fY) + \eta(Y)\omega X,$$

where $X, Y \in \Gamma(TM)$ and

(3.12) $$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y,$$

(3.13) $$(\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y.$$

3.3. Lemma. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then we have*

(3.14) $\quad (\nabla_X B)U = A_{CU}X - fA_U X - g(\phi X, U)V,$

and

(3.15) $\quad (\nabla^t_X C)U = -\omega A_U X - h(X, BU),$

where $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$ and

(3.16) $\quad (\nabla_X B)U = \nabla_X BU - B\nabla^t_X U,$

(3.17) $\quad (\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U.$

3.4. Theorem. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then*

(A) *The distribution $D \oplus \{V\}$ is integrable, if and only if*

(3.18) $\quad h(X, fY) = h(Y, fX), \quad \forall \quad X, Y \in \Gamma(D \oplus \{V\}).$

(B) *The distribution $\bar{D}$ is integrable, if and only if*

(3.19) $\quad A_{\phi Z}U = A_{\phi U}Z, \quad \forall \quad Z, U \in \Gamma(\bar{D}).$

Proof: Using (3.8) and (3.9), we have $\omega\nabla_X Y = h(X, fY) - Ch(X, Y)$, for any $X, Y \in \Gamma(D \oplus \{V\}$. Here replacing X by Y and subtracting the resulting equation from this equation, we get $\omega[X, Y] = h(X, fY) - h(Y, fX)$, which proves (A).

Next from (3.10) and (3.12), we have $-f(\nabla_Z U) = A_{\omega U}Z + Bh(Z, U)$, for all $Z, U \in \Gamma(\bar{D})$. Then, similarly as above, we obtain $f[Z, U] = A_{\phi Z}U - A_{\phi U}Z$, which completes the proof of (B).

3.5. Theorem. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $Bh(X, \phi Y) = 0$, for any $X, Y \in D \oplus \{V\}$.*

Proof: Since $\bar{D} = \phi(L \perp S)$, therefore $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if

$$g(\nabla_X Y, \phi\xi) = g(\nabla_X Y, \phi W) = 0,$$

for any $X, Y \in \Gamma(D \oplus \{V\})$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$.
Using (2.7) and (2.16), we have

(3.20) $\quad g(\nabla_X Y, \phi\xi) = -\bar{g}(\bar{\nabla}_X \phi Y, \xi) = -\bar{g}(h^l(X, fY), \xi),$

(3.21) $\quad g(\nabla_X Y, \phi W) = -\bar{g}(\bar{\nabla}_X \phi Y, W) = -\bar{g}(h^s(X, fY), W).$

Hence, from (3.20) and (3.21) the assertion follows.

3.6. Theorem. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the distribution $\bar{D}$ does not define a totally geodesic foliation in M.*

Proof: We know that $\bar{D}$ defines a totally geodesic foliation in M, if and only if

$$g(\nabla_X Y, N) = g(\nabla_X Y, \phi N_1) = g(\nabla_X Y, V) = g(\nabla_X Y, \phi Z) = 0,$$

for $X, Y \in \Gamma(\bar{D})$, $N \in \Gamma(ltr(TM))$, $Z \in \Gamma(D_0)$ and $N_1 \in \Gamma(L)$. But using (2.5) and (2.16), we obtain $g(\nabla_X Y, V) = g(\bar{\nabla}_X Y, V) = -g(Y, \bar{\nabla}_X V) = -g(Y, X)$, which may be non zero because $\phi S \subset \bar{D}$ is non degenrate. Hence the assertion follows.

3.7. Theorem. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the induced connection ∇ is metric connection, if and only if*

$$A^*_{\phi\xi}X - \nabla^{*t}_X\phi\xi \in \Gamma(\phi D_2 \perp D_1), \quad for \quad \xi \in \Gamma(D_1),$$
$$\nabla^*_X\phi\xi + h^*(X,\phi\xi) \in \Gamma(\phi D_2 \perp D_1), \quad for \quad \xi \in \Gamma(D_2),$$
$$h(X,\phi\xi) \in \Gamma(L\perp S)^\perp \quad and \quad A^*_\xi X \in \Gamma(\bar{D}\perp D_0 \perp \phi D_2),$$

for $\xi \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$.

Proof: For any $X \in \Gamma(TM)$ and $\xi \in \Gamma Rad(TM)$, using (2.16), we have

$$\phi\bar{\nabla}_X\xi = \bar{\nabla}_X\phi\xi + g(\phi X,\xi)V,$$

applying ϕ to both sides of above equation and then using (2.13) and (2.15), we obtain

$$\nabla_X\xi + h(X,\xi) = -\phi(\nabla_X\phi\xi + h(X,\phi\xi)) + g(A^*_\xi X, V)V. \tag{3.22}$$

Let $\xi \in \Gamma(D_1)$ then again using (2.13) in (3.22), we obtain

$$\nabla_X\xi + h(X,\xi) = -\phi(-A^*_{\phi\xi}X + \nabla^{*t}_X\phi\xi) - Bh(X,\phi\xi) - Ch(X,\phi\xi) + g(A^*_\xi X,V)V.$$

Equating tangential components of above equation both sides, we get

$$\nabla_X\xi = fA^*_{\phi\xi}X - f\nabla^{*t}_X\phi\xi - Bh(X,\phi\xi) + g(A^*_\xi X,V)V,$$

therefore $\nabla_X\xi \in \Gamma(RadTM)$, if and only if, $Bh(X,\phi\xi) = 0$, $fA^*_{\phi\xi}X - f\nabla^{*t}_X\phi\xi \in \Gamma(RadTM)$ and $g(A^*_\xi X,V) = 0$ or, if and only if,

$$h(X,\phi\xi) \in \Gamma(L\perp S)^\perp, \quad A^*_{\phi\xi}X - \nabla^{*t}_X\phi\xi \in \Gamma(\phi D_2 \perp D_1), \tag{3.23}$$

and

$$A^*_\xi X \in \Gamma(\bar{D}\perp D_0\perp\phi D_2). \tag{3.24}$$

Similarly, let $\xi \in \Gamma(D_2)$ then using (2.12) in (3.22) and then compare the tangential components of the resulting equation, we obtain

$$\nabla_X\xi = -f\nabla^*_X\phi\xi - fh^*(X,\phi\xi) - Bh(X,\phi\xi) + (A^*_\xi X,V)V,$$

therefore $\nabla_X\xi \in \Gamma(RadTM)$, if and only if, $Bh(X,\phi\xi) = 0$, $f\nabla^*_X\phi\xi + fh^*(X,\phi\xi) \in \Gamma(RadTM)$ and $g(A^*_\xi X,V) = 0$ or, if and only if,

$$h(X,\phi\xi) \in \Gamma(L\perp S)^\perp, \quad \nabla^*_X\phi\xi + h^*(X,\phi\xi) \in \Gamma(\phi D_2\perp D_1), \tag{3.25}$$

and

$$A^*_\xi X \in \Gamma(\bar{D}\perp D_0\perp\phi D_2). \tag{3.26}$$

Hence from (3.23) to (3.26), the assertion follows.

4. Totally Contact Umbilical GCR-Lightlike Submanifolds

4.1. Definition. ([13]). If the second fundamental form h of a submanifold tangent to characteristic vector field V, of a Sasakian manifold $\bar{M}$ is of the form

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called a totally contact umbilical submanifold. M is called a totally contact geodesic submanifold if $\alpha = 0$ and a totally geodesic submanifold if $h = 0$.

The above definition also holds for a lightlike submanifold M. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_L + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \tag{4.2}$$

$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_S + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \tag{4.3}$$

where $\alpha_L \in \Gamma(ltr(TM))$ and $\alpha_S \in \Gamma(S(TM^{\perp}))$.

4.2. Lemma. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\nabla_X X \in \Gamma(D \oplus \{V\})$, for any $X \in \Gamma(D)$.*

Proof: Since $\bar{D} = \phi(L \perp S)$ therefore $\nabla_X X \in \Gamma(D \oplus \{V\})$, if and only if,

$$g(\nabla_X X, \phi\xi) = g(\nabla_X X, \phi W) = 0,$$

for any $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$. Since M is totally contact umbilical GCR-lightlike submanifold therefore for any $X \in \Gamma(D)$, using (2.5), (2.7), (2.16) and (4.2), we obtain

$$\begin{aligned} g(\nabla_X X, \phi\xi) &= \bar{g}(\bar{\nabla}_X X, \phi\xi) = -\bar{g}(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X, \xi) = -\bar{g}(h^l(X, \phi X), \xi) \\ &= -\bar{g}(X, \phi X)\bar{g}(\alpha_L, \xi) = 0. \qquad (4.4) \end{aligned}$$

Also

$$\begin{aligned} g(\nabla_X X, \phi W) &= \bar{g}(\bar{\nabla}_X X, \phi W) = -\bar{g}(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X, W) = -\bar{g}(h^s(X, \phi X), W) \\ &= -\bar{g}(X, \phi X)\bar{g}(\alpha_S, \xi) = 0. \qquad (4.5) \end{aligned}$$

Hence using (4.4) and (4.5), the assertion follows.

4.3. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\alpha \in \Gamma(L \perp S)$.*

Proof: Using (3.9), for any $X \in \Gamma(D_0)$, we obtain $h^s(X, fX) = \omega P_2 \nabla_X X + Ch^s(X, X)$, then using (4.3) we get $g(X, \phi X)\alpha_S = \omega P_2 \nabla_X X + g(X, X)C\alpha_S$. By virtue of the Lemma (4.2), we get $g(X, X)C\alpha_S = 0$, then the non degeneracy of the distribution D_0 implies that $C\alpha_S = 0$. Hence $\alpha_S \in \Gamma(S)$.

Similarly by using (3.8) and (4.2) we can prove $\alpha_L \in \Gamma(L)$. Hence $\alpha \in \Gamma(L \perp S)$.

4.4. Remark. Since $\alpha \in \Gamma(L \perp S)$ therefore for any $X \in D_0$ with (4.1), we have $h(X, X) = g(X, X)\alpha$, this implies that $h(X, X) \in \Gamma(L \perp S)$.

4.5. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\alpha_L = 0$.*

Proof: Since M is a totally contact umbilical GCR-lightlike submanifold then, by direct calculations, using (2.7), (2.8) and (2.16) and then taking tangential parts of the resulting equation, we obtain

$$A_{\phi Z} Z + f\nabla_Z Z + Bh^l(Z, Z) + Bh^s(Z, Z) = 0,$$

where $Z \in \phi(S)$. Hence for $\xi \in \Gamma(D_2)$, we obtain

$$\bar{g}(A_{\phi Z} Z, \phi\xi) + \bar{g}(h^l(Z, Z), \xi) = 0,$$

then using (2.11), we get $\bar{g}(h^s(Z, \phi\xi), \phi Z) + \bar{g}(h^l(Z, Z), \xi) = 0$. Therefore using (4.2) and (4.3), we obtain$g(Z, Z)\bar{g}(\alpha_L, \xi) = 0$, then the non degeneracy of ϕS implies that $\alpha_L = 0$, which completes the proof.

4.6. Lemma. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\nabla_X \phi X = \phi \nabla_X X$, for any $X \in \Gamma(D_0)$.*

Proof: For any $X \in \Gamma(D_0)$ using (3.11) and (3.13), we have $\omega\nabla_X X = h(X, fX) - Ch(X, X)$. Since M be a totally contact umbilical therefore using (4.1), we have $\omega\nabla_X X = g(X, \phi X)\alpha - Ch(X, X)$, then using remark (4.4), we get $\omega\nabla_X X = 0$. Hence $\nabla_X X \in \Gamma(D)$. Let $Y \in \Gamma(D_0)$ then using(2.14) to (2.16), we obtain

$$g(\nabla_X \phi X, Y) = \bar{g}(\bar{\nabla}_X \phi X, Y) = \bar{g}(\phi\bar{\nabla}_X X, Y) = -g(\nabla_X X, \phi Y) = g(\phi\nabla_X X, Y),$$

this implies $g(\nabla_X \phi X - \phi\nabla_X X, Y) = 0$, then the non degeneracy of the distribution D_0 gives the result.

4.7. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\alpha_S = 0$.*

Proof: Let $W \in \Gamma(S(TM^{\perp}))$ and $X \in \Gamma(D_0)$ the using (2.16), (4.1) and the Lemma (4.6), we have

$$
\begin{aligned}
\bar{g}(\phi\bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X \phi X, \phi W) \\
&= \bar{g}(\nabla_X \phi X, \phi W) + \bar{g}(h(X, \phi X), \phi W) \\
&= \bar{g}(\phi\nabla_X X, \phi W) \\
&= \bar{g}(\nabla_X X, W) \\
&= 0.
\end{aligned} \tag{4.6}
$$

Also using (4.3), we have

$$
\begin{aligned}
\bar{g}(\phi\bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X X, W) - \eta(W)\eta(\phi\bar{\nabla}_X X) \\
&= \bar{g}(\nabla_X X + h^s(X, X) + h^l(X, X), W) \\
&= \bar{g}(\nabla_X X, W) + \bar{g}(h^s(X, X), W) \\
&= g(X, X)g(\alpha_s, W).
\end{aligned} \tag{4.7}
$$

Therefore using (4.6) and (4.7), we get $g(X, X)g(\alpha_S, W) = 0$, then the non degeneracy of D_0 and $S(TM^{\perp})$ implies that $\alpha_S = 0$.

4.8. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then M is a totally contact geodesic GCR-lightlike submanifold.*

Proof: The result follows from the Theorems (4.5) and (4.7).

4.9. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ such that $\bar{\nabla}_X V \in \Gamma(TM)$ then the induced connection ∇ is a metric connection on M.*

Proof: Using the Theorem (4.5), we have $\alpha_L = 0$. Since $\bar{\nabla}_X V \in \Gamma(TM)$ therefore this implies that $h^l(X, V) = 0$, hence using (4.2) we obtain

$$h^l = 0. \tag{4.8}$$

Thus using the Theorem 2.2 in [6] at page 159, the induced connection ∇ becomes a metric connection on M.

4.10. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ such that $\bar{\nabla}_X V \in \Gamma(TM)$ then M is totally geodesic GCR-lightlike submanifold.*

Proof: Using the Theorem (4.7), we have $\alpha_S = 0$. Since $\bar{\nabla}_X V \in \Gamma(TM)$ therefore this implies that $h^s(X, V) = 0$, hence using (4.3), we obtain

$$h^s = 0. \tag{4.9}$$

Thus using (4.8) and (4.9), the assertion follows.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 93 – 99

A new efficient multi-parametric homotopy approach for two-dimensional Fredholm integral equations of the second kind

Y. Khan* and M. Fardi †

Abstract

In this paper, a new multi-parametric homotopy approach is proposed to find the approximate solution of linear and non-linear two-dimensional Fredholm integral equations of the second kind. In this framework, convergence of the proposed approach for these types of equations is investigated. This homotopy contain two auxiliary parameters that provide a simple way of controlling the convergence region of series solution. The results of present method are compared with Adomian decomposition method (ADM) results which provide confirmation for the validity of proposed approach. Two examples are presented to illustrate the accuracy and effectiveness of the proposed approach.

2000 AMS Classification: 45B05

Keywords: multi-parametric; convergence-control; Adomian decomposition method (ADM); Fredholm integral equation; averaged residual error (ARE)

Received 24/10/2013 : Accepted 12/12/2013 Doi : 10.15672/HJMS.2015449096

1. Introduction

Homotopy analysis method (HAM) has been presented by Liao [1, 2] to obtain the analytical solutions for various nonlinear problems. There are many eminent researchers that deal with the Homotopy analysis method such as, Alomari et al. [3,4] applied the HAM to study the delay differential equations and the hyperchaotic Chen system, Turkyilmazoglu [5] constructed the convergent series solutions of strongly nonlinear problems via HAM, Gupta [6] implemented the HAM to obtain the approximate analytical solution of nonlinear fractional diffusion equation, Abbasbandy applied the uni-parametric homotopy method to solve the Fredholm integral equations [7], Marinca and Herisanu [8, 9] proposed the OHAM for fluid mechanics problem and nonlinear differential equations. Recently Turkyilmazoglu [10] constructed the explicit analytic solution of the Thomas-Fermi equation thorough a new kind of homotopy analysis technique. He used new base functions

*Department of Mathematics, Zhejiang University, Hangzhou 310027, China
Email: yasirmath@yahoo.com
†Department of Mathematics, Islamic Azad University, Najafabad Branch, Najafabad, Iran

and auxiliary linear operator to form a better homotopy method.The basic motivation of the present study is proposed a new multi-parametric homotopy approach to develop an approximate solution for the linear and nonlinear two-dimensional Fredholm integral equations. The present method is much easier to implement as compared with the decomposition method where huge complexities are involved. Moreover, we prove the convergence of the solution for two-dimensional Fredholm integral equations. It is shown that the approximate solutions given by the proposed approach are more accurate than the numerical solution given by the traditional homotopy analysis method (THAM) and the Adomian decomposition method (ADM) [11, 12].

2. Description of approach

To illustrate the procedure, consider the following second kind of two-dimensional Fredholm integral equation:

$$F(u(t,x)) = u(t,x) - f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi)N(u(s,\xi))d\xi ds,\ (t,x) \in D, \tag{2.1}$$

where $f(t,x)$ and $K(t,s,x,\xi)$ are analytical functions on $D = L^2([a,b]\times[c,d])$ and $E = D\times D$, respectively.
We choose $u_0(t,x) = f(t,x)$ as initial approximation guess for simplicity, in order to obtain convergent series solutions to two-dimensional Fredholm integral equation (2.1), we first construct the zeroth order deformation equation

$$\begin{aligned}(1 - A(q;\varpi_1))[\varphi(t,x;q) &- u_0(t,x)] = B(q;\hbar)[\varphi(t,x;q) - f(t,x) \\ &- \int_a^b \int_c^d K(t,s,x,\xi)N(\varphi(s,\xi;q))d\xi ds],\end{aligned} \tag{2.2}$$

where

$$A(q;\varpi) = (1-\varpi)\sum_{j=1}^{\infty} \varpi^{j-1}q^j,\ \ |\varpi| < 1, B(q;\hbar) = q\hbar,\ \hbar \neq 0. \tag{2.3}$$

Due to Taylor's theorem, we can write

$$\varphi(t,x;q) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x)q^j, \tag{2.4}$$

where

$$u_j(t,x) = \frac{1}{j!}\frac{\partial^j \varphi(t,x;q)}{\partial q^j}\Big|_{q=0}. \tag{2.5}$$

The convergence of series (2.4) depends upon $\hbar$ and ϖ. Assume that $\hbar$ and ϖ are properly chosen so that the power series of (2.4) converges at $q = 1$, then we have under these assumption the solution series

$$u(t,x) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x). \tag{2.6}$$

By differentiating (2.2) m times with respect to q, then dividing the equation by $m!$ and setting $q = 0$, the mth-order deformation equation is formulated as follows

$$u_m(t,x) - \sum_{k=1}^{m-1}(1-\varpi)\varpi^{m-k-1}u_k(t,x) = \hbar H_m(u_0(t,x), ..., u_{m-1}(t,x)), \tag{2.7}$$

where

$$H_m = u_{m-1}(t,x) - (1-\chi_m)f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi)\frac{\partial^{m-1}N(\varphi(s,\xi;q))}{(m-1)!\partial q^{m-1}} \mid_{q=0} d\xi ds, \tag{2.8}$$

$$u_m(t,x) = \frac{\partial^m \varphi(t,x;q)}{m!\partial q^m} \mid_{q=0}, \tag{2.9}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1; \\ 1, & m > 1. \end{cases} \tag{2.10}$$

The mth-order deformation equations (2.7) are linear in principle. The code is developed by using symbolic computation software MAPLE. Then, the Nth-order approximate solution of (2.7) can be written as

$$U_N(t,x) = u_0(t,x) + \sum_{j=1}^{N} u_j(t,x). \tag{2.11}$$

If $\varpi = 0$ the mth-order deformation equation defined by (2.7) becomes

$$u_1(t,x) = \hbar[u_0(t,x) - f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi)N(u_0(s,\xi;q))d\xi ds], \tag{2.12}$$

and

$$u_m(t,x) - u_{m-1}(t,x) = \hbar[u_{m-1}(t,x) - \int_a^b \int_c^d K(t,s,x,\xi)\frac{\partial^{m-1}N(\varphi(s,\xi;q))}{(m-1)!\partial q^{m-1}} \mid_{q=0} d\xi ds], \tag{2.13}$$

Then, we can derive the following remarks instantly.

Remark1: The value $\varpi = 0$ reduces the present approach to the traditional HAM.

Remark2: The values $\hbar = -1$ and $\varpi = 0$ reduce the present approach to the ADM.

2.1. Convergence theorems.

Theorem 1. If the solution series

$$u(t,x) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x), \tag{2.14}$$

is convergent, then we have the following statement

$$\sum_{m=1}^{\infty} H_m = 0. \tag{2.15}$$

Proof.
Since the solution series

$$u(t,x) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x), \tag{2.16}$$

is convergent, we have

$$\lim_{m\to\infty} u_m(t,x) = 0. \tag{2.17}$$

Using the the left-hand side of (2.6) satisfies

$$
\begin{aligned}
\sum_{m=1}^{\infty}\big[u_m(t,x) \quad &- \quad \sum_{k=1}^{m-1}(1-\varpi)\varpi^{m-k-1}u_k(t,x)\big] \\
&= \quad u_1(t,x) \\
&+ \quad u_2(t,x)-(1-\varpi)u_1(t,x) \\
&+ \quad u_3(t,x)-(1-\varpi)\varpi u_1(t,x)-(1-\varpi)u_2(t,x) \\
&\quad \vdots \\
&= \quad (1-(1-\varpi)\sum_{j=0}^{\infty}\varpi^j)\sum_{j=0}^{\infty}u_j(t,x)=0. \qquad (2.18)
\end{aligned}
$$

Then, from (2.7) and (2.18) we have

$$\sum_{m=1}^{\infty}H_m=0. \qquad (2.19)$$

Theorem 2. Assume that the operator $N[u(t,x)]$ be contraction and the solution series

$$u(t,x)=u_0(t,x)+\sum_{j=1}^{\infty}u_j(t,x), \qquad (2.20)$$

is convergent, it must be the solution of two-dimensional Fredholm integral equation.

Proof.
Let

$$\varepsilon(t,x;q)=\varphi(t,x;q)-f(t,x)-\int_a^b\int_c^d K(t,s,x,\xi)N(\varphi(s,\xi;q))d\xi ds. \qquad (2.21)$$

Using Taylor's series around $q=0$ for $\varepsilon(t,x;1)$, we have

$$
\begin{aligned}
\varepsilon(t,x;1) \quad &= \quad \sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m\varphi(t,x;q)}{\partial q^m}|_{q=0}-f(t,x) \\
&- \quad \int_a^b\int_c^d K(t,s,x,\xi)\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m N(\varphi(s,\xi;q))}{\partial q^m}|_{q=0}d\xi ds. \qquad (2.22)
\end{aligned}
$$

If the solution series

$$u(t,x)=u_0(t,x)+\sum_{j=1}^{\infty}u_j(t,x), \qquad (2.23)$$

is convergent, then the series

$$\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m N(\varphi(t,x;q))}{\partial q^m}|_{q=0}, \qquad (2.24)$$

will converge to $N[u(t,x)]$ (see [12]).
Now, by using theorem 1 we have

$$\varepsilon(t,x;1)=u(t,x)-f(t,x)-\int_a^b\int_c^d K(t,s,x,\xi)N(u(s,\xi))d\xi ds=0. \qquad (2.25)$$

This completes the proof.

3. Main results

The Nth-order approximation of the solution $u(t,x)$ can be expressed as

$$U_N(t,x) = u_0(t,x) + \sum_{j=1}^{N} u_j(t,x), \tag{3.1}$$

which is mathematically dependent upon the convergence-control parameters $\hbar$ and ϖ. In our work for optimal values of $\hbar$ and ϖ, we use a technique that has been shown to produce a fast converging approximation. In principle, the technique seeks to minimize the exact residual error (ERE) of (2.1) at the Nth-order approximation. The ERE is given by

$$\widehat{E}_M(\hbar,\varpi) = \int_a^b \int_c^d (F(U_N(s,\xi)))^2 d\xi ds, \tag{3.2}$$

In practice, however, the evaluation of $\widehat{E}_M(\hbar,\varpi)$ tends to be time-consuming. A simpler alternative consists of calculating the averaged residual error (ARE). We use here the ARE defined by

$$E_M^n(\hbar,\varpi) = \frac{(b-a)(d-c)}{n^2} \sum_{j=0}^{n} \sum_{i=0}^{n} (F(U_N(t_i,x_j)))^2, \tag{3.3}$$

where

$$t_i = \frac{(b-a)i}{n},\ i = 1,2,...,n,\ \ x_j = \frac{(d-c)j}{n},\ j = 1,2,...,n. \tag{3.4}$$

At the Nth-order of approximation, the ARE contain two unknown convergence-control parameters, whose "optimal" values are determined by solving the nonlinear algebraic equations

$$\frac{\partial E_M^n}{\partial \hbar} = 0,\ \frac{\partial E_M^n}{\partial \varpi} = 0. \tag{3.5}$$

In this section, two different examples of two-dimensional integral equations are employed to illustrate the validity of present approach which is described in Section 2. The convergence, accuracy and efficiency of this approach are investigated by comparing it with the THAM and the ADM.

3.1. Example 1.

Consider the following nonlinear two-dimensional integral equation

$$u(t,x) = x\sin(\pi t) - \frac{x}{6} + \int_0^1 \int_0^1 (x + \cos(\pi s))u^2(s,\xi) d\xi ds, \tag{3.6}$$

with the exact solution

$$u(t,x) = x\sin(\pi t). \tag{3.7}$$

For $\hbar \neq 0$ and $\varpi = 0$, our approach gives the "optimal" value of the convergence-control parameter $\hbar \neq 0$ by solving the equation $\frac{dE_4^{20}}{dh} = 0$, which leads to $\hbar = -1.456$ with the corresponding minimum ARE $E_4^{20} = 5.483E - 7$. For $\hbar \neq 0$ and $\varpi \neq 0$, we obtain the "optimal" values of $\hbar = -1.521$ and $\varpi = -0.148$ by solving the algebraic equations $\frac{dE_4^{20}}{dh} = 0$ and $\frac{dE_4^{20}}{d\varpi} = 0$, which gives with the corresponding minimum ARE $E_4^{20} = 4.458E - 22$.
For comparison the solution series given by the present approach with the exact solution, we report the absolute error which is defined by

$$|e_N(t,x)| = |u(t,x) - U_N(t,x)|.$$

In Table 1, we compared the present approach with the ADM. The approximate solutions given by the present approach are more accurate than the solution given by the ADM, as shown in Table 1.

Table 1. Absolute errors of the proposed approach and ADM (example 1).

convergence-control parameters $(\hbar, \varpi)$			
$t = x$	$(-1.521, -0.148)$ *with seven terms*	$(-1.456, 0)$ *with seven terms*	ADM *with seven terms*
1	$1.531E-4$	$3.996E-4$	$2.503E-3$
$\frac{1}{2}$	$7.651E-5$	$1.998E-4$	$1.252E-3$
$\frac{1}{2^2}$	$3.826E-5$	$9.999E-5$	$6.258E-4$
$\frac{1}{2^3}$	$1.913E-5$	$4.995E-5$	$3.129E-4$
$\frac{1}{2^4}$	$9.567E-6$	$2.497E-5$	$1.565E-4$
$\frac{1}{2^5}$	$4.783E-6$	$1.249E-5$	$7.823E-5$
$\frac{1}{2^6}$	$2.392E-6$	$6.244E-6$	$3.911E-5$

Table 2. Absolute errors of the proposed approach and ADM (example 2).

convergence-control parameters $(\hbar, \varpi)$			
$t = x$	$(-0.756, -0.755)$ *with nine terms*	$(-1.581, 0)$ *with nine terms*	ADM *with nine terms*
1	$3.712E-6$	$1.109E-3$	$1.699E-2$
$\frac{1}{2}$	$2.520E-6$	$9.600E-4$	$1.161E-2$
$\frac{1}{2^2}$	$1.929E-6$	$8.953E-4$	$8.914E-3$
$\frac{1}{2^3}$	$1.625E-6$	$8.629E-4$	$7.567E-3$
$\frac{1}{2^4}$	$1.481E-6$	$8.467E-4$	$6.894E-3$
$\frac{1}{2^5}$	$1.401E-6$	$8.386E-4$	$6.557E-3$
$\frac{1}{2^6}$	$1.366E-6$	$8.345E-4$	$6.389E-3$

3.2. Example 2. Consider the following linear two-dimensional integral equation

$$u(t,x) = xe^{-t} + (\frac{e^{-2}}{4} - \frac{1}{4})x + \frac{e^{-2}}{6} - \frac{1}{6} + \int_0^1 \int_0^1 (x+\xi)e^{-(2t-s)}u(s,\xi)d\xi ds, \tag{3.8}$$

with the exact solution

$$u(t,x) = xe^{-t}. \tag{3.9}$$

For $\hbar \neq 0$ and $\varpi = 0$, the present approach reduces to traditional HAM and E_8^{20} has the minimum $1.393E-7$ at the "optimal" value $\hbar = -1.581$. For $\hbar \neq 0$ and $\varpi \neq 0$, the optimal convergence occurs at $\hbar = -1.155$ and $\varpi = -0.306$ and has a ARE of $E_8^{20} = 9.892E-22$.
The approximate solutions given by the present approach are more accurate than the solution given by the ADM and the THAM, as shown in Table 2.

4. Concluding remarks

In this paper, we have proposed a method for solving two-dimensional Fredholm integral equations. The results have been compared with the THAM and ADM solutions to show the efficiency of our technique. By introducing this method for two-dimensional Fredholm integral equations, the following observations have been made:
(i) This approach contains two convergence-control parameters which provide us a simple way to adjust and control the convergence region and rate of the obtained series solution.
(ii) The obtained results elucidate the very fast convergence of present approach, which does not need higher-order of approximation.
(iii) All the given examples reveal that the multi-parametric homotopy yields a very effective and convenient approach to the approximate solutions of two-dimensional Fredholm integral equations.

(iv) The ADM cannot give better results than the present approach.
(v) In fact, the THAM and ADM are special cases of present method.
In conclusion, a new multi-parametric homotopy approach may be considered as a nice refinement in existing numerical techniques.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 101 – 109

Optimizing the convergence rate of the Wallis sequence

Cristinel Mortici*

Abstract

The aim of this paper is to introduce a method for increasing the convergence rate of the Wallis sequence. Some sharp inequalities are stated.

2000 AMS Classification: 33B15; 40A25; 41A25; 40A05

Keywords: Wallis' formula; asymptotic expansion; speed of convergence

Received 29/10/2010 : Accepted 20/01/2014 Doi : 10.15672/HJMS.2015449090

1. Introduction

One of the most known formula for estimating of the number π is the Wallis' formula

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot \ldots \cdot (2n-1)(2n+1)}, \tag{1.1}$$

e.g., [1, Rel. 6.1.49, p. 258].

It was discovered in 1655 by the English mathematician John Wallis (1616-1703), while he was preoccupied to calculate the value of π by finding the area under the quadrant of a circle. The Wallis' formula is also related to the problem of estimation of the large factorials, which plays a central role in combinatorics, graph theory, special functions and other branches of science as physics or applied statistics.

The Wallis' sequence

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}$$

converges very slowly to its limit. For example,

$$W_{100} \approx \frac{\pi}{2} - 3.9026\times 10^{-3} \quad , \quad W_{10000} \approx \frac{\pi}{2} - 3.9267\times 10^{-5},$$

and in consequence, many authors are concerned to improve the speed of convergence of the Wallis formula (1.1), usually by indicate the upper and lower bounds of the Wallis sequence $(W_n)_{n\geq 1}$.

*Valahia University of Târgovişte, Dept. of Mathematics
Bd. Unirii 18, 130082, Târgovişte, Romania.
Academy of the Romanian Scientists, Splaiul Independenţei 54, 050094 Bucureşti, Romania.
Email: cristinel.mortici@hotmail.com

Here, to every approximation of the form $\pi/2 \approx f(n)$, we define the relative error sequence $(\omega_n)_{n\geq 1}$ by the formula

$$\frac{\pi}{2} = f(n) \cdot \exp \omega_n$$

and we say that the approximation $\pi/2 \approx f(n)$ is as better as $(\omega_n)_{n\geq 1}$ faster converges to zero.

The speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is computed throughout this paper, using the following basic

Lemma 1.1. *If $(\omega_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l \in \mathbb{R}, \tag{1.2}$$

with $k > 1$, then

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}.$$

We see from Lemma 1.1 that the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ increases together with the value k satisfying (1.2). This Lemma was used by Mortici [2]-[15] for constructing asymptotic expansions, or to accelerate some convergences. For proof and other details, see, *e.g.*, [2], or [4].

We use these ideas by considering new factors to accelerate the convergence of the Wallis sequence and to obtain better approximations for π.

It is well known that the speed of convergence of the sequence $\ln W_n$ toward $\ln \frac{\pi}{2}$ is n^{-1} (this also follows from Theorem 2.1, i) from the next section).

We will see that a simple change in $(W_n)_{n\geq 1}$ of the last factor $2n$ by $2n - \frac{1}{4}$ in the denumerator and of the last factor $2n+1$ by $2n + \frac{1}{4}$ in the denominator, we obtain the quicker sequence

$$t_n = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot ... \cdot 2n \cdot \left(2n - \frac{1}{4}\right)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot ...(2n-1)\left(2n + \frac{1}{4}\right)}$$

such that $\ln t_n$ converges to $\ln \frac{\pi}{2}$ with the speed of convergence n^{-3}.

Other idea to improve the speed of convergence is to consider new factors of the form

$$T_n = W_n \cdot \frac{P(n)}{Q(n)},$$

where P, Q are polynomials of equal degrees. More precisely, we define in this paper the sequences

$$\rho_n = W_n \cdot \frac{n + \frac{5}{8}}{n + \frac{3}{8}}$$

$$\sigma_n = W_n \cdot \frac{n^2 + \frac{9}{8}n + \frac{23}{64}}{n^2 + \frac{7}{8}n + \frac{15}{64}}$$

$$\tau_n = W_n \cdot \frac{n^3 + \frac{13}{8}n^2 + \frac{37}{32}n + \frac{167}{512}}{n^3 + \frac{11}{8}n^2 + \frac{29}{32}n + \frac{105}{512}}$$

$$\chi_n = W_n \cdot \frac{n^4 + \frac{17}{8}n^3 + \frac{161}{64}n^2 + \frac{389}{256}n + \frac{1473}{4096}}{n^4 + \frac{15}{8}n^3 + \frac{137}{64}n^2 + \frac{291}{256}n + \frac{945}{4096}}$$

having increasingly speed of convergence. In fact, the sequences $\ln \rho_n$, $\ln \sigma_n$, $\ln \tau_n$, $\ln \chi_n$ converge to $\ln \frac{\pi}{2}$ with the speed of convergence n^{-3}, n^{-5}, n^{-7} and n^{-9} respectively.

2. Modifying the last fraction

First we modify the last fraction from (1.1) to define the sequence

$$t_n = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot ...\cdot 2n\cdot (2n+a)}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot ...(2n-1)(2n+b)} = W_n \frac{(2n+a)(2n+1)}{2n(2n+b)}, \tag{2.1}$$

where a, b are real parameters. For $a = 0$ and $b = 1$, the Wallis sequence is obtained and we prove that for $a = -1/4$ and $b = 1/4$, the resulting sequence has a superior speed of convergence.

In this sense, let us define the sequence $(\omega_n)_{n\geq 1}$ by

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot ...\cdot 2n\cdot (2n+a)}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot ...\cdot (2n-1)(2n+b)}\cdot \exp\omega_n = t_n\cdot \exp\omega_n\ . \tag{2.2}$$

Now we are in position to state the following

Theorem 2.1. *Let $(\omega_n)_{n\geq 1}$ be the sequence defined by* (2.2)*, where a, b are real parameters. Then:*

i) *If $b - a \neq \frac{1}{2}$, then the speed of convergence of $(\omega_n)_{n\geq 1}$ is n^{-1}, since*

$$\lim_{n\to\infty} n\omega_n = \frac{1}{2}\left(b - a - \frac{1}{2}\right) \neq 0.$$

ii) *If $b - a = \frac{1}{2}$ and $(a,b) \neq \left(-\frac{1}{4}, \frac{1}{4}\right)$, then the speed of convergence of $(\omega_n)_{n\geq 1}$ is n^{-2}, since*

$$\lim_{n\to\infty} n^2\omega_n = -\frac{4a+1}{32} \neq 0.$$

iii) *If $(a,b) = \left(-\frac{1}{4}, \frac{1}{4}\right)$, then the speed of convergence of $(\omega_n)_{n\geq 1}$ is n^{-3}, since*

$$\lim_{n\to\infty} n^3\omega_n = \frac{3}{256} \neq 0.$$

Proof. As we are interested to compute limits of the form (1.2), we develop in power series of n^{-1} the sequence

$$\omega_n - \omega_{n+1} = \ln\frac{2n(2n+2)(2n+2+a)(2n+b)}{(2n+1)^2(2n+2+b)(2n+a)},$$

namely

$$\omega_n - \omega_{n+1} = \left(\frac{1}{2}b - \frac{1}{2}a - \frac{1}{4}\right)\frac{1}{n^2} + \left(\frac{1}{4}a^2 + \frac{1}{2}a - \frac{1}{4}b^2 - \frac{1}{2}b + \frac{1}{4}\right)\frac{1}{n^3} -$$
$$-\left(\frac{1}{8}a^3 + \frac{3}{8}a^2 + \frac{1}{2}a - \frac{1}{8}b^3 - \frac{3}{8}b^2 - \frac{1}{2}b + \frac{7}{32}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right)$$

and now the conclusion easily follows using Lemma 1.1.□

3. Adding new rational factors

Other interesting method to improve the speed of convergence of the Wallis sequence is to add new factors of the form

$$T_n = W_n\cdot\frac{P(n)}{Q(n)},$$

where P, Q are polynomials of equal degrees, having the leading coefficient equal to one. In this sense, we give the following results:

Theorem 3.1. *Let us define the sequence $(\mu_n)_{n\geq 1}$ by*

$$\frac{\pi}{2} = W_n\cdot\frac{n+a}{n+b}\cdot\exp\mu_n.$$

i) *If* $a-b\neq\frac{1}{4}$, *then the speed of convergence of* $(\mu_n)_{n\geq 1}$ *is* n^{-1}, *since*

$$\lim_{n\to\infty} n\mu_n = b-a+\frac{1}{4}\neq 0.$$

ii) *If* $a-b=\frac{1}{4}$ *and* $(a,b)\neq\left(\frac{5}{8},\frac{3}{8}\right)$, *then the speed of convergence of* $(\mu_n)_{n\geq 1}$ *is* n^{-2}, *since*

$$\lim_{n\to\infty} n^2\mu_n = \frac{8a-5}{32}\neq 0.$$

iii) *If* $(a,b)=\left(\frac{5}{8},\frac{3}{8}\right)$, *then the speed of convergence of* $(\mu_n)_{n\geq 1}$ *is* n^{-3}, *since*

$$\lim_{n\to\infty} n^3\mu_n = \frac{-3}{256}\neq 0.$$

Proof. We have

$$\mu_n-\mu_{n+1}=\ln\left(\frac{(2n+2)^2}{(2n+1)(2n+3)}\cdot\frac{n+1+a}{n+1+b}\cdot\frac{n+b}{n+a}\right)$$

or

$$\mu_n-\mu_{n+1}=\left(b-a+\frac{1}{4}\right)\frac{1}{n^2}+\left(a^2+a-b^2-b-\frac{1}{2}\right)\frac{1}{n^3}-$$
$$-\left(a^3+\frac{3}{2}a^2+a-b^3-\frac{3}{2}b^2-b-\frac{25}{32}\right)\frac{1}{n^4}+O\left(\frac{1}{n^5}\right),$$

and the conclusion follows using Lemma 1.1.□

Theorem 3.2. *Let us define the sequence* $(\nu_n)_{n\geq 1}$ *by*

$$\frac{\pi}{2}=W_n\cdot\frac{n^2+an+b}{n^2+cn+d}\cdot\exp\nu_n$$

where a, b, c, d *are real parameters and denote:*

$$\alpha=c-a+\frac{1}{4}$$
$$\beta=a^2+a-c^2-c-2b+2d-\frac{1}{2}$$
$$\gamma=3b-a+c-3d+3ab-3cd-\frac{3}{2}a^2-a^3+\frac{3}{2}c^2+c^3+\frac{25}{32}$$
$$\delta=a^4+2a^3-4a^2b+2a^2-6ab+a+2b^2-4b-c^4-$$
$$-2c^3+4c^2d-2c^2+6cd-c-2d^2+4d-\frac{9}{8}$$

i) *If* $\alpha\neq 0$, *then the speed of convergence of the sequence* $(\nu_n)_{n\geq 1}$ *is* n^{-1}, *since*

$$\lim_{n\to\infty} n\nu_n=\alpha\neq 0.$$

ii) *If* $\alpha=0$ *and* $\beta\neq 0$, *then the speed of convergence of* $(\nu_n)_{n\geq 1}$ *is* n^{-2}, *since*

$$\lim_{n\to\infty} n^2\nu_n=\frac{\beta}{2}\neq 0.$$

iii) *If* $\alpha=\beta=0$, *and* $\gamma\neq 0$, *then the speed of convergence of* $(\mu_n)_{n\geq 1}$ *is* n^{-3}, *since*

$$\lim_{n\to\infty} n^3\mu_n=\frac{\gamma}{3}\neq 0.$$

iv) *If* $\alpha=\beta=\gamma=0$ *and* $\delta\neq 0$, *then the speed of convergence of* $(\mu_n)_{n\geq 1}$ *is* n^{-4}, *since*

$$\lim_{n\to\infty} n^4\mu_n=\frac{\delta}{4}\neq 0.$$

v) *If* $\alpha=\beta=\gamma=\delta=0$, (*equivalent with* $a=\frac{9}{8}$, $b=\frac{23}{64}$, $c=\frac{7}{8}$, $d=\frac{15}{64}$), *then the speed of convergence of* $(\mu_n)_{n\geq 1}$ *is* n^{-5}, *since*

$$\lim_{n\to\infty} n^5\nu_n = \frac{45}{16384}.$$

Proof. We have

$$\nu_n-\nu_{n+1}=\ln\left(\frac{(2n+2)^2}{(2n+1)(2n+3)}\cdot\frac{(n+1)^2+a(n+1)+b}{(n+1)^2+c(n+1)+d}\cdot\frac{n^2+cn+d}{n^2+an+b}\right),$$

or

$$\nu_n-\nu_{n+1}=\left(c-a+\frac{1}{4}\right)\frac{1}{n^2}+\left(a^2+a-c^2-c-2b+2d-\frac{1}{2}\right)\frac{1}{n^3}+$$
$$+\left(3b-a+c-3d+3ab-3cd-\frac{3}{2}a^2-a^3+\frac{3}{2}c^2+c^3+\frac{25}{32}\right)\frac{1}{n^4}+$$
$$+\left(a^4+2a^3-4a^2b+2a^2-6ab+a+2b^2-4b-c^4-\right.$$
$$\left.-2c^3+4c^2d-2c^2+6cd-c-2d^2+4d-\frac{9}{8}\right)\frac{1}{n^5}+O\left(\frac{1}{n^6}\right)$$

and we recognize the coefficients α, β, γ, δ in this power series. The conclusion follows using Lemma 1.1.□

More accurate results can be established in case of the family of approximations of the form

$$\frac{\pi}{2}\approx W_n\cdot\frac{n^3+an^2+bn+c}{n^3+dn^2+fn+g},$$

where a, b, c, d, f, g are real parameters. As above, we introduce the sequence $(\psi_n)_{n\geq 1}$ by

$$\frac{\pi}{2}=W_n\cdot\frac{n^3+an^2+bn+c}{n^3+dn^2+fn+g}\cdot\exp\psi_n$$

and we can state the following

Theorem 3.3. *The fastest sequence of the form* $(\psi_n)_{n\geq 1}$ *is obtained for*

$$a=\frac{13}{8}\ ,\quad b=\frac{37}{32}\ ,\quad c=\frac{167}{512}\ ,\quad d=\frac{11}{8}\ ,\quad f=\frac{29}{32}\ ,\quad g=\frac{105}{512}. \tag{3.1}$$

In this case, we have

$$\lim_{n\to\infty} n^7\psi_n=-\frac{1575}{1048\,576}.$$

For every other real parameters a, b, c, d, f, g, *different from the values* (3.1), *the speed of convergence of the sequence* $(\psi_n)_{n\geq 1}$ *is at most* n^{-6}.

Proof. We have

$$\psi_n-\psi_{n+1}=\ln\frac{(2n+2)^2}{(2n+1)(2n+3)}+$$
$$+\ln\left(\frac{(n+1)^3+a(n+1)^2+b(n+1)+c}{(n+1)^3+d(n+1)^2+f(n+1)+g}\cdot\frac{n^3+dn^2+fn+g}{n^3+an^2+bn+c}\right),$$

or

$$\psi_n-\psi_{n+1}=-\left(a-d-\frac{1}{4}\right)\frac{1}{n^2}+\left(a^2+a-d^2-d-2b+2f-\frac{1}{2}\right)\frac{1}{n^3}+$$
$$+\left(3b-a-3c+d-3f+3g+3ab-3df-\frac{3}{2}a^2-a^3+\frac{3}{2}d^2+d^3+\frac{25}{32}\right)\frac{1}{n^4}-$$

$$-\left(4b-a-6c+d-4f+6g+4a^2b-4d^2f+6ab-4ac-6df+4dg-\right.$$
$$\left.-2a^2-2a^3-a^4-2b^2+2d^2+2d^3+d^4+2f^2+\frac{9}{8}\right)\frac{1}{n^5}-$$
$$-\left(a-5b+10c-d+5f-10g+5ab^2-10a^2b-5a^3b+5a^2c-5df^2+\right.$$
$$+10d^2f+5d^3f-5d^2g-10ab+10ac-5bc+10df-10dg+5fg+\frac{5}{2}a^2+$$
$$\left.+\frac{10}{3}a^3+\frac{5}{2}a^4+5b^2+a^5-\frac{5}{2}d^2-\frac{10}{3}d^3-\frac{5}{2}d^4-d^5-5f^2-\frac{301}{192}\right)\frac{1}{n^6}+$$
$$+\left(-a^6-3a^5+6a^4b-5a^4+15a^3b-6a^3c-5a^3-9a^2b^2+20a^2b-\right.$$
$$-15a^2c-3a^2-15ab^2+12abc+15ab-20ac-a+2b^3-10b^2+$$
$$+15bc+6b-3c^2-15c+d^6+3d^5-6d^4f+5d^4-15d^3f+6d^3g+$$
$$+5d^3+9d^2f^2-20d^2f+15d^2g+3d^2+15df^2-12dfg-15df+$$
$$\left.+20dg+d-2f^3+10f^2-15fg-6f+3g^2+15g+\frac{69}{32}\right)\frac{1}{n^7}+O\left(\frac{1}{n^8}\right).$$

If we impose that the first coefficients of n^{-k}, for $k=2,3,4,5,6,7$, to vanish, then the obtained system with unknowns a, b, c, d, f, g has the unique solution (3.1).

Otherwise, if p denotes the smallest element of $\{2,3,4,5,6,7\}$ such that the coefficient of n^{-p} is not zero, then the corresponding limit is non-zero:

$$\lim_{n\to\infty} n^p\left(\psi_n-\psi_{n+1}\right)\neq 0.$$

According with Lemma 1.1, the speed of convergence of the sequence $(\psi_n)_{n\geq 1}$ is n^{p-1}, which is less than n^{-7}.□

4. Concluding Remarks

Similar results, which are increasingly accurate can be obtained if we consider approximations of the form

$$\frac{\pi}{2}\approx W_n\cdot\frac{P(n)}{Q(n)},$$

where $\deg P=\deg Q\geq 4$. In case of the polynomials of fourth degree, it can be proved that the best approximation is

$$\frac{\pi}{2}\approx W_n\cdot\frac{n^4+\frac{17}{8}n^3+\frac{161}{64}n^2+\frac{389}{256}n+\frac{1473}{4096}}{n^4+\frac{15}{8}n^3+\frac{137}{64}n^2+\frac{291}{256}n+\frac{945}{4096}}.$$

We omit the proof for sake of simplicity.

As above, the coefficients of these polynomials are the solution of the system defined by the first eight coefficients from the associated power series.

Moreover, the sequence $(\zeta_n)_{n\geq 1}$ defined by

$$\frac{\pi}{2}=W_n\cdot\frac{n^4+\frac{17}{8}n^3+\frac{161}{64}n^2+\frac{389}{256}n+\frac{1473}{4096}}{n^4+\frac{15}{8}n^3+\frac{137}{64}n^2+\frac{291}{256}n+\frac{945}{4096}}\cdot\exp\zeta_n$$

satisfies

$$\zeta_n-\zeta_{n+1}=\frac{893025}{67108864}\cdot\frac{1}{n^{10}}+O\left(\frac{1}{n^{11}}\right).$$

According to Lemma 1.1, the speed of convergence of $(\zeta_n)_{n\geq 1}$ is n^{-9}, since

$$\lim_{n\to\infty} n^9\zeta_n=\frac{99225}{67108864}.$$

Our new defined sequences have great superiority over the Wallis sequence. Precisely, we tabulate the following numerical results:

n	$\frac{\pi}{2} - W_n$	$\rho_n - \frac{\pi}{2}$	$\frac{\pi}{2} - \sigma_n$	$\tau_n - \frac{\pi}{2}$	$\frac{\pi}{2} - \chi_n$
1	0.23746	5.0×10^{-3}	4.3×10^{-4}	7.2×10^{-5}	$\mathbf{1.8 \times 10^{-5}}$
10	3.7×10^{-2}	1.6×10^{-5}	3.4×10^{-8}	1.6×10^{-10}	$\mathbf{1.4 \times 10^{-12}}$
20	1.9×10^{-2}	2.1×10^{-6}	1.2×10^{-9}	1.5×10^{-12}	$\mathbf{3.6 \times 10^{-15}}$
50	7.8×10^{-3}	1.4×10^{-7}	1.3×10^{-11}	2.8×10^{-15}	$\mathbf{1.1 \times 10^{-18}}$
100	3.9×10^{-3}	1.8×10^{-8}	4.2×10^{-13}	2.3×10^{-17}	$\mathbf{2.2 \times 10^{-21}}$
300	1.3×10^{-3}	6.8×10^{-10}	1.8×10^{-15}	1.1×10^{-20}	$\mathbf{1.2 \times 10^{-25}}$
500	7.8×10^{-4}	1.5×10^{-10}	1.4×10^{-16}	3.0×10^{-22}	$\mathbf{1.1 \times 10^{-27}}$
1000	3.9×10^{-4}	1.8×10^{-11}	4.3×10^{-18}	2.4×10^{-24}	$\mathbf{7.6 \times 10^{-29}}$

4. Sharp bounds

Whenever an approximation formula of the form $f(n) \approx g(n)$ is given, there is a tendency to improve it by using a series of the form

$$f(n) \sim g(n) \exp\left(\sum_{k=1}^{\infty} \frac{a_k}{n^k}\right),$$

also called an asymptotic series. Although in asymptotic analysis, the problem of constructing asymptotic expansions is considered to be technically difficult, an elementary method for establishing the asymptotic series associated to the Wallis sequence was given by Mortici [2]

$$W_n \sim \frac{\pi}{2} \exp\left(-\frac{1}{4n} + \frac{1}{8n^2} - \frac{5}{96n^3} + \frac{1}{64n^4} - \cdots\right).$$

Even if such asymptotic series may not converge, in a truncated form, it provides approximations of any desired accuracy. We prove the following sharp bounds for the sequence $(\rho_n)_{n\geq 1}$, arising from its asymptotic expansion, which can be constructed for example, using the method from [2].

Theorem 3.1. *For every integer $n \geq 1$, it holds*

$$\frac{\pi}{2} \exp\left(\frac{3}{256n^3} - \frac{9}{512n^4}\right) < \rho_n < \frac{\pi}{2} \exp\left(\frac{3}{256n^3} - \frac{9}{512n^4} + \frac{1441}{81\,920n^5}\right).$$

Proof. The sequences

$$x_n = \frac{\pi}{2\rho_n} \exp\left(\frac{3}{256n^3} - \frac{9}{512n^4}\right), \quad y_n = \frac{\pi}{2\rho_n} \exp\left(\frac{3}{256n^3} - \frac{9}{512n^4} + \frac{1441}{81\,920n^5}\right)$$

converge to 1, and we prove that $(x_n)_{n\geq 1}$ is strictly increasing and $(y_n)_{n\geq 1}$ is strictly decreasing. In consequence, $x_n < 1$ and $y_n > 1$ and the theorem is proved.

In this sense, we denote $x_{n+1}/x_n = \exp u(n)$ and $y_{n+1}/y_n = \exp v(n)$, where

$$u(t) = \ln(r(t)) + \frac{3}{256(t+1)^3} - \frac{9}{512(t+1)^4} - \frac{3}{256t^3} + \frac{9}{512t^4}$$

$$v(t) = \ln(r(t)) + \frac{3}{256(t+1)^3} - \frac{9}{512(t+1)^4} + \frac{1441}{81\,920(t+1)^5} - \frac{3}{256t^3} + \frac{9}{512t^4} - \frac{1441}{81\,920t^5}$$

with

$$r(t) = \frac{(2t+1)(2t+3)(8t+5)(8t+11)}{4(8t+3)(8t+13)(t+1)^2}.$$

We have $u' < 0$ and $v' > 0$, since

$$u'(t) = -\frac{9P(t)}{256t^5(t+1)^5(2t+1)(2t+3)(8t+3)(8t+5)(8t+11)(8t+13)}$$

$$v'(t) = \frac{Q(t)}{16384t^6(t+1)^6(2t+1)(2t+3)(8t+3)(8t+5)(8t+11)(8t+13)},$$

where

$$P(t) = 164\,427t + 919\,901t^2 + 2964\,474t^3 + 6067\,340t^4 + 8187\,804t^5$$
$$+7305\,696t^6 + 4169\,712t^7 + 1382\,400t^8 + 202\,240t^9 + 12\,870$$

and

$$Q(t) = 124\,965\,786t + 745\,324\,705t^2 + 2597\,843\,148t^3 + 5881\,230\,025t^4$$
$$+9094\,461\,530t^5 + 9845\,176\,300t^6 + 7519\,686\,680t^7 + 4003\,302\,912t^8$$
$$+1413\,183\,488t^9 + 291\,323\,904t^{10} + 25\,165\,824t^{11} + 9272\,835.$$

Finally, u is strictly decreasing, v is strictly increasing, with $u(\infty) = v(\infty) = 0$, so $u > 0$ and $v < 0$ and the conclusion follows.

Now it is clear that our new method is suitable for establishing similar better results for all other sequences discussed here.

4.1. Acknowledgement. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0087. This work was completed during the author's visit at Moldova State University in Chişinău, Republic of Moldova. The author thanks Prof. dr. Hab. Andrei Perjan and Conf. dr. Valeriu Guţu for their hospitality. Computations made in this paper were performed using Maple software.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 111 – 119

Approximation of fixed points of multifunctions in partial metric spaces

Daniela Paesano * and Pasquale Vetro†

Abstract

Recently, Reich and Zaslavski [S. Reich and A.J. Zaslavski, Convergence of Inexact Iterative Schemes for Nonexpansive Set-Valued Mappings, Fixed Point Theory Appl. 2010 (2010), Article ID 518243, 10 pages] have studied a new inexact iterative scheme for fixed points of contractive multifunctions. In this paper, using the partial Hausdorff metric introduced by Aydi et al., we prove an analogous to a result of Reich and Zaslavski for contractive multifunctions in the setting of partial metric spaces. An example is given to illustrate our result.

2000 AMS Classification: 47H10, 47H09.

Keywords: Fixed point, contractive multifunctions, partial metric spaces, inexact iterative scheme.

Received 28/03/2013 : Accepted 26/12/2013 Doi : 10.15672/HJMS.2015449088

1. Introduction

The study of iterative schemes for various classes of contractive and nonexpansive mappings is a central topic in metric fixed point theory. It started with the work of Banach [3] who proved a classical theorem, known as the Banach contraction principle, for the existence of a unique fixed point for a contraction. The importance of this result is also in the fact that it gives the convergence of an iterative scheme to a unique fixed point. Since Banach's result, there has been a lot of activity in this area and many developments have been taken place. In metric fixed point theory, there are many existence and approximation results for fixed points of those nonexpansive mappings which are not necessarily strictly contractive. Some authors have also provided results dealing with

*Università degli Studi di Palermo Dipartimento di Matematica e Informatica Via Archirafi, 34 - 90123 Palermo (Italy).
Email: daniela_paesano@libero.it
†Università degli Studi di Palermo Dipartimento di Matematica e Informatica Via Archirafi, 34 - 90123 Palermo (Italy)
Email: pasquale.vetro@unipa.it

the existence and approximation of fixed points of certain classes of contractive multifunctions [4, 5, 7, 9, 12, 13]. In [12] Reich and Zaslavski introduced and studied new inexact iterative schemes for approximating fixed points of contractive and nonexpansive multifunctions. More recently, Aydi et al. [2] introduced a notion of partial Hausdoff metric type, that is a metric type associated to a partial metric. In [2] the authors using the partial Hausdorff metric proved an analogous to the well known Nadler's fixed point theorem [9]. In this paper, using the partial Hausdorff metric we prove an analogous to a result of [12] for contractive multifunctions in the setting of partial metric spaces. An example is given to illustrate our result.

2. Preliminaries

First, we recall some definitions of partial metric spaces that can be found in [6, 8, 10, 11, 14]. A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then from $(p1)$ and $(p2)$ it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of partial metric space is the pair $([0, +\infty), p)$, where $p(x, y) = \max\{x, y\}$.

Each partial metric p on X generates a T_0 topology τ_p on X, which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$, $\epsilon > 0$.

Let (X, p) be a partial metric space. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$. A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \to +\infty} p(x_n, x_m)$. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.

A sequence $\{x_n\}$ in (X, p) is called $0-$Cauchy if $\lim_{n,m \to +\infty} p(x_n, x_m) = 0$. We say that (X, p) is $0-$complete if every $0-$Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Now, we recall the definition of partial Hausdorff metric and some property that can be found in [2]. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X, p), induced by the partial metric p. Note that closedness is taken from (X, τ_p) and boundedness is given as follows: A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(x_0, x_0) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define

$$\begin{aligned} p(x, A) &= \inf\{p(x, a),\ a \in A\},\ \delta_p(A, B) = \sup\{p(a, B) : a \in A\} \text{ and} \\ \delta_p(B, A) &= \sup\{p(b, A) : b \in B\}. \end{aligned}$$

2.1. Remark (see [1]). Let (X, p) be a partial metric space and A any nonempty set in (X, p), then

(2.1) $\quad a \in \bar{A}$ if and only if $p(a, A) = p(a, a)$, where $\bar{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X,p) if and only if $A=\bar{A}$.

In the following proposition, we bring some properties of the mapping $\delta_p : CB^p(X)\times CB^p(X)\to[0,+\infty)$.

2.2. Proposition ([2], Proposition 2.2)**.** *Let (X,p) be a partial metric space. For any $A,B,C\in CB^p(X)$, we have the following:*

$$\begin{aligned}
(i) &\quad:\quad \delta_p(A,A)=\sup\{p(a,a): a\in A\};\\
(ii) &\quad:\quad \delta_p(A,A)\le\delta_p(A,B);\\
(iii) &\quad:\quad \delta_p(A,B)=0 \text{ implies that } A\subseteq B;\\
(iv) &\quad:\quad \delta_p(A,B)\le\delta_p(A,C)+\delta_p(C,B)-\inf_{c\in C}p(c,c).
\end{aligned}$$

Let (X,p) be a partial metric space. For $A,B\in CB^p(X)$, define

$$H_p(A,B)=\max\left\{\delta_p(A,B),\delta_p(B,A)\right\}.$$

In the following proposition, we bring some properties of the mapping H_p.

2.3. Proposition ([2], Proposition 2.3)**.** *Let (X,p) be a partial metric space. For all $A,B,C\in CB^p(X)$, we have*

$$\begin{aligned}
(h1) &\quad:\quad H_p(A,A)\le H_p(A,B);\\
(h2) &\quad:\quad H_p(A,B)=H_p(B,A);\\
(h3) &\quad:\quad H_p(A,B)\le H_p(A,C)+H_p(C,B)-\inf_{c\in C}p(c,c).
\end{aligned}$$

2.4. Corollary ([2], Corollary 2.4)**.** *Let (X,p) be a partial metric space. For $A,B\in CB^p(X)$ the following holds:*

$$H_p(A,B)=0 \text{ implies that } A=B.$$

2.5. Remark. The converse of Corollary 2.4 is not true in general as it is clear from the following example.

2.6. Example ([2], Example 2.6)**.** Let $X=[0,1]$ be endowed with the partial metric $p: X\times X\to\mathbb{R}^+$ defined by

$$p(x,y)=\max\{x,y\}.$$

From (i) of Proposition 2.2, we have

$$H_p(X,X)=\delta_p(X,X)=\sup\{x: 0\le x\le 1\}=1\ne 0.$$

In view of Proposition 2.3 and Corollary 2.4, we call the mapping $H_p : CB^p(X)\times CB^p(X)\to[0,+\infty)$, a partial Hausdorff metric induced by p.

2.7. Remark. It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see Example 2.6).

3. Main result

The following theorem is the main result.

3.1. Theorem. *Let (X,p) be a 0−complete partial metric space, $T: X\to CB^p(X)$ a multifunctions and $\{\varepsilon_i\}$ and $\{\delta_i\}$ two sequences in $(0,+\infty)$ such that*

$$\sum_{i=0}^{+\infty}\varepsilon_i<+\infty \quad \text{and} \quad \sum_{i=0}^{+\infty}\delta_i<+\infty. \tag{3.1}$$

Suppose that there exists $k \in [0,1)$ such that

$$H_p(Tx, Ty) \leq kp(x, y) \quad \text{for all } x, y \in X. \tag{3.2}$$

Let $T_i : X \to 2^X \backslash \{\emptyset\}$ satisfy, for each integer $i \geq 0$,

$$H_p(Tx, T_i x) \leq \epsilon_i, \quad \text{for all } x \in X. \tag{3.3}$$

Assume that $x_0 \in X$ and that for each integer $i \geq 0$,

$$x_{i+1} \in T_i x_i, \quad p(x_i, x_{i+1}) \leq p(x_i, T_i x_i) + \delta_i. \tag{3.4}$$

Then, the sequence $\{x_i\}_{i=0}^{+\infty}$ converges to a fixed point of T.

Proof. We first show that $\{x_i\}_{i=0}^{+\infty}$ is a 0-Cauchy sequence. To this end, let $i \geq 0$ be an integer. Then, we have

$$\begin{aligned} p(x_{i+1}, x_{i+2}) &\leq p(x_{i+1}, T_{i+1}x_{i+1}) + \delta_{i+1} \\ &\leq p(x_{i+1}, Tx_{i+1}) + \delta_p(Tx_{i+1}, T_{i+1}x_{i+1}) - \inf_{c \in Tx_{i+1}} p(c, c) + \delta_{i+1} \\ &\leq p(x_{i+1}, Tx_{i+1}) + \delta_p(Tx_{i+1}, T_{i+1}x_{i+1}) + \delta_{i+1} \\ &\leq p(x_{i+1}, Tx_{i+1}) + H_p(Tx_{i+1}, T_{i+1}x_{i+1}) + \delta_{i+1} \\ &\leq p(x_{i+1}, Tx_{i+1}) + \epsilon_{i+1} + \delta_{i+1} \\ &\leq H_p(T_i x_i, Tx_{i+1}) + \epsilon_{i+1} + \delta_{i+1} \\ &\leq H_p(T_i x_i, Tx_i) + H_p(Tx_i, Tx_{i+1}) - \inf_{c \in Tx_i} p(c, c) + \epsilon_{i+1} + \delta_{i+1} \\ &\leq H_p(Tx_i, Tx_{i+1}) + \epsilon_i + \epsilon_{i+1} + \delta_{i+1}. \end{aligned}$$

Hence,

$$p(x_{i+1}, x_{i+2}) \leq kp(x_i, x_{i+1}) + \epsilon_i + \epsilon_{i+1} + \delta_{i+1}. \tag{3.5}$$

Now, we show by induction that for each $n \geq 1$, we have

$$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1) + \sum_{i=0}^{n-1} k^i (\epsilon_{n-i} + \delta_{n-i} + \epsilon_{n-i-1}). \tag{3.6}$$

In view of (3.5), inequality (3.6) holds for $n = 1, 2$. Assume that $j \geq 1$ is an integer and that (3.6) holds for $n = j$. When combined with (3.5), this implies that

$$\begin{aligned} p(x_{j+1}, x_{j+2}) &\leq kp(x_j, x_{j+1}) + \epsilon_{j+1} + \delta_{j+1} + \epsilon_j \\ &\leq k^{j+1} p(x_0, x_1) + \sum_{i=0}^{j-1} k^{i+1} (\epsilon_{j-i} + \delta_{j-i} + \epsilon_{j-i-1}) + \epsilon_{j+1} + \delta_{j+1} + \epsilon_j \\ &= k^{j+1} p(x_0, x_1) + \sum_{i=0}^{j} k^i (\epsilon_{j+1-i} + \delta_{j+1-i} + \epsilon_{j-i}). \end{aligned}$$

This implies that (3.6) holds for all $n \geq 1$. From (3.6), by (3.1), we get

$$\begin{aligned}\sum_{n=1}^{+\infty} p(x_n, x_{n+1}) &\leq \sum_{n=1}^{+\infty} \Big(k^n p(x_0, x_1) + \sum_{i=0}^{n-1} k^i(\epsilon_{n-i} + \delta_{n-i} + \epsilon_{n-i-1})\Big) \\ &= \sum_{n=1}^{+\infty} \Big(k^n p(x_0, x_1) + \sum_{i=1}^{n} k^{n-i}(\epsilon_i + \delta_i + \epsilon_{i-1})\Big) \\ &= \sum_{n=1}^{+\infty} k^n p(x_0, x_1) + (k^0 + k^1 + k^2 + \cdots)(\epsilon_1 + \delta_1 + \epsilon_0) \\ &+ (k^0 + k^1 + k^2 + \ldots)(\epsilon_2 + \delta_2 + \epsilon_1) \\ &+ (k^0 + k^1 + k^2 + \cdots)(\epsilon_3 + \delta_3 + \epsilon_2) + \cdots \\ &= \sum_{n=1}^{+\infty} k^n p(x_0, x_1) + \sum_{i=1}^{+\infty} \Big(\sum_{j=0}^{+\infty} k^j\Big)(\epsilon_i + \delta_i + \epsilon_{i-1}) \\ &\leq \Big(\sum_{n=0}^{+\infty} k^n\Big)\Big[p(x_0, x_1) + \sum_{n=1}^{+\infty}(\epsilon_n + \delta_n + \epsilon_{n-1})\Big] < +\infty.\end{aligned}$$

This implies that $\lim_{i,j\to+\infty} p(x_i, x_j) = 0$ and hence $\{x_i\}_{i=0}^{+\infty}$ is a 0-Cauchy sequence and so there exists $x^* \in X$ such that

(3.7) $$\lim_{i\to+\infty} p(x_i, x^*) = p(x^*, x^*) = 0.$$

We claim that x^* is a fixed point of T, that is $x^* \in Tx^*$. From

$$H_p(Tx_i, Tx^*) \leq kp(x_i, x^*) \quad \text{for all } i \in \mathbb{N},$$

letting $i \to +\infty$, we obtain

(3.8) $$\lim_{i\to+\infty} H_p(Tx_i, Tx^*) = 0.$$

As $x_{i+1} \in T_i x_i$ for all i, we have

$$\begin{aligned}p(x_{i+1}, Tx^*) &\leq \delta_p(T_i x_i, Tx^*) \\ &\leq H_p(T_i x_i, Tx_i) + H_p(Tx_i, Tx^*) \\ &\leq \varepsilon_i + H_p(Tx_i, Tx^*).\end{aligned}$$

Letting $i \to +\infty$, by (3.1) and (3.8), we obtain

(3.9) $$\lim_{i\to+\infty} p(x_{i+1}, Tx^*) = 0.$$

Now, using (3.7) and (3.9), from

$$p(x^*, Tx^*) \leq p(x^*, x_{i+1}) + p(x_{i+1}, Tx^*) \quad \text{for all } i \in \mathbb{N},$$

as $i \to +\infty$ we deduce that $p(x^*, Tx^*) = 0$. Hence, $p(x^*, x^*) = p(x^*, Tx^*) = 0$ and so by Remark 2.1 we get that $x^* \in Tx^*$.

□

We also have the following result.

3.2. Theorem. *Let (X, p) be a $0-$complete partial metric space, $T : X \to CB^p(X)$ a multifunction and $\{\delta_i\}$ a sequence in $(0, +\infty)$ such that*

(3.10) $$\sum_{i=0}^{+\infty} \delta_i < +\infty.$$

Suppose that there exists $k \in [0,1)$ such that

(3.11) $\quad H_p(Tx, Ty) \leq kp(x,y) \quad \text{for all } x, y \in X.$

Assume that $x_0 \in X$ and that for each integer $i \geq 0$,

(3.12) $\quad x_{i+1} \in Tx_i, \quad p(x_i, x_{i+1}) \leq H_p(Tx_{i-1}, Tx_i) + \delta_i.$

Then, the sequence $\{x_i\}_{i=0}^{+\infty}$ converges to a fixed point of T.

Proof. We first show that $\{x_i\}_{i=0}^{+\infty}$ is a 0-Cauchy sequence. To this end, let $i \geq 0$ be an integer. Then, we have

$$p(x_{i+1}, x_{i+2}) \leq H_p(Tx_i, Tx_{i+1}) + \delta_{i+1}.$$

Hence,

(3.13) $\quad p(x_{i+1}, x_{i+2}) \leq kp(x_i, x_{i+1}) + \delta_{i+1}.$

Now, we show by induction that for each $n \geq 1$, we have

(3.14) $$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1) + \sum_{i=0}^{n-1} k^i \delta_{n-i}.$$

In view of (3.13), inequality (3.14) holds for $n = 1, 2$. Assume that $j \geq 1$ is an integer and that (3.14) holds for $n = j$. When combined with (3.13), this implies that

$$\begin{aligned} p(x_{j+1}, x_{j+2}) &\leq kp(x_j, x_{j+1}) + \delta_{j+1} \\ &\leq k^{j+1} p(x_0, x_1) + \sum_{i=0}^{j-1} k^{i+1} \delta_{j-i} + \delta_{j+1}. \end{aligned}$$

This implies that (3.14) holds for all $n \geq 1$. From (3.14), by (3.10), proceeding as in the proof of Theorem 3.1, we get

$$\begin{aligned} \sum_{n=1}^{+\infty} p(x_n, x_{n+1}) &\leq \sum_{n=1}^{+\infty} \Big(k^n p(x_0, x_1) + \sum_{i=0}^{n-1} k^i \delta_{n-i} \Big) \\ &= \sum_{n=1}^{+\infty} k^n p(x_0, x_1) + \sum_{i=1}^{+\infty} \Big(\sum_{j=0}^{\infty} k^j \Big) \delta_i \\ &\leq \Big(\sum_{n=0}^{+\infty} k^n \Big) \Big[p(x_0, x_1) + \sum_{n=1}^{+\infty} \delta_n \Big] < +\infty. \end{aligned}$$

This implies that $\lim_{n,m \to +\infty} p(x_i, x_j) = 0$ and hence $\{x_i\}_{i=0}^{+\infty}$ is a 0-Cauchy sequence and so there exists $x^* \in X$ such that

(3.15) $\quad \lim_{i \to +\infty} p(x_i, x^*) = p(x^*, x^*) = 0.$

We claim that x^* is a fixed point of T, that is $x^* \in Tx^*$. From

$$H_p(Tx_i, Tx^*) \leq kp(x_i, x^*) \quad \text{for all } i \in \mathbb{N},$$

letting $i \to +\infty$, we obtain

$$\lim_{i \to \infty} H_p(Tx_i, Tx^*) = 0.$$

As $x_{i+1} \in Tx_i$ for all i, we have

$$p(x_{i+1}, Tx^*) \leq \delta_p(Tx_i, Tx^*) \leq H_p(Tx_i, Tx^*).$$

Letting $i \to +\infty$, we get that

(3.16) $\quad \lim_{i \to +\infty} p(x_{i+1}, Tx^*) = 0.$

Now, using (3.15) and (3.16), from

$$p(x^*,Tx^*) \le p(x^*,x_{i+1}) + p(x_{i+1},Tx^*) \qquad \text{for all } i \in \mathbb{N},$$

as $i \to +\infty$ we deduce that $p(x^*,Tx^*) = 0$. Hence, $p(x^*,x^*) = p(x^*,Tx^*) = 0$ and so $x^* \in Tx^*$, that is x^* is a fixed point of T. □

3.3. Lemma. *Let (X,p) be a partial metric space, $A,B \in CB^p(X)$ and $\alpha > 0$. For any $a \in A$, there exists $b = b(a) \in B$ such that*

$$p(a,b) \le H_p(A,B) + \alpha. \tag{3.17}$$

Proof. Without loss of generality, we can assume that $H_p(A,B) > 0$. If we choose $h > 1$ such that $h\,H_p(A,B) = H_p(A,B) + \alpha$, the existence of $b \in B$ satisfying (3.17) follows from Lemma 3.1 of [2]. □

3.4. Lemma. *Let (X,p) be a partial metric space, $T : X \to CB^p(X)$ a multifunction. Suppose that there exists $k \in [0,1)$ such that*

$$H_p(Tx,Ty) \le kp(x,y) \qquad \text{for all } x,y \in X. \tag{3.18}$$

Then for all $x_0 \in X$ there exists a sequence $\{x_i\}_{i=0}^{+\infty}$ such that

$$x_{i+1} \in Tx_i, \quad p(x_i,x_{i+1}) \le H_p(Tx_{i-1},Tx_i) + k^i. \tag{3.19}$$

Proof. We may assume $k > 0$. Choose $x_1 \in Tx_0$. As $Tx_0, Tx_1 \in CB^p(X)$ and $x_1 \in Tx_0$, there is a point $x_2 \in Tx_1$ such that

$$p(x_1,x_2) \le H_p(Tx_0,Tx_1) + k.$$

Now, since $Tx_1, Tx_2 \in CB^p(X)$ and $x_2 \in Tx_1$ there is a point $x_3 \in Tx_2$ such that $p(x_2,x_3) \le H_p(Tx_1,Tx_2) + k^2$. Continuing in this way we produce a sequence $\{x_i\}_{i=0}^{+\infty}$ of points of X such that $x_{i+1} \in Tx_i$ and $p(x_i,x_{i+1}) \le H_p(Tx_{i-1},Tx_i) + k^i$ for all $i \ge 1$. □

From Theorem 3.2 and Lemma 3.4, we deduce the following result, which generalizes Theorem 3.2 of [2].

3.5. Theorem. *Let (X,p) be a 0-complete partial metric space. If $T : X \to CB^p(X)$ is a multifunction such that for all $x,y \in X$, we have*

$$H_p(Tx,Ty) \le k\,p(x,y) \tag{3.20}$$

where $k \in [0,1)$. Then T has a fixed point.

To illustrate the usefulness of our result, we give the following example.

3.6. Example. Let $X = [0,2]$ be endowed with the usual metric. Define the multifunctions $T, T_i : X \to CB^p(X)$ by

$$Tx = \begin{cases} [\frac{x}{8}, \frac{x}{4}] & \text{if } x \in [0,1], \\ \{0\} & \text{otherwise.} \end{cases}$$

$$T_i x = \begin{cases} [\frac{x}{8} - \frac{x}{8^{i+2}}, \frac{x}{4}] & \text{if } x \in [0,1], \\ \{0\} & \text{otherwise.} \end{cases}$$

It is easy to see that Theorem 2.1 of [12] is not applicable in this case. Indeed, for $x = \dfrac{19}{18}$ and $y = \dfrac{8}{9}$, we have

$$\begin{aligned} H(T(\tfrac{19}{18}), T(\tfrac{8}{9})) &= H(\{0\}, [\tfrac{1}{9}, \tfrac{2}{9}]) \\ &= \tfrac{2}{9} \not\le \tfrac{k}{6} = k\,d(\tfrac{19}{18}, \tfrac{8}{9}), \end{aligned}$$

for any $k \in [0,1)$.

On the other hand, if we endow X with the partial metric defined by

$$p(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,1], \\ \frac{|x-y|}{4} + \frac{\max\{x,y\}}{2} & \text{otherwise.} \end{cases}$$

Then (X,p) is a complete partial metric space and Tx is closed for all $x \in X$.

We shall show that for all $x, y \in X$, (3.20) is satisfied with $k = \frac{2}{3}$.

Consider the following cases:

- If $x \in [0,1]$ and $y \in (1,2]$, then $p(x,y) = \frac{3}{4}y - \frac{x}{4} > \frac{1}{2}$ and

$$\begin{aligned} H_p(Tx,Ty) &= H_p([\frac{x}{8},\frac{x}{4}],\{0\}) \\ &= \max\{\frac{x}{8},\frac{x}{4}\} = \frac{x}{4} \leq \frac{1}{4} < \frac{1}{3} = \frac{k}{2} < kp(x,y). \end{aligned}$$

- If $x, y \in (1,2]$, then $H_p(Tx,Ty) = H_p(\{0\},\{0\}) = 0$ and (3.20) is satisfied obviously.
- If $x, y \in [0,1]$, with $x \leq y$, then

$$\begin{aligned} H_p(Tx,Ty) &= H_p([\frac{x}{8},\frac{x}{4}],[\frac{y}{8},\frac{y}{4}]) \\ &= \max\{\frac{y-x}{8},\frac{y-x}{4}\} \\ &= \frac{y-x}{4} < \frac{2}{3}(y-x) = kp(x,y). \end{aligned}$$

It is easy to see that $H_p(T_i x, Tx) \leq 1/8^{i+2}$ for all $x \in X$. Moreover, for all $x_0 \in [0,1]$ the sequence $\{x_i\}_{i=0}^{+\infty}$ defined by $x_{i+1} = x_i/4 - x_i/4^{i+2} \in T_i x_i$ for all $i \geq 0$ is such that

$$p(x_i, x_{i+1}) = \frac{3}{4}x_i + \frac{x_i}{4^{i+2}} = p(x_i, T_i x_i) + \frac{x_i}{4^{i+2}} \leq p(x_i, T_i x_i) + \frac{1}{4^{i+2}}.$$

If $x_0 \in (1,2]$, then we choose $x_i = 0$ for all $i > 0$.

Thus, all the conditions of Theorem 3.1 are satisfied with $\varepsilon_i = 1/8^{i+2}$ and $\delta_i = 1/4^{i+2}$. Moreover, $x_i \to 0$ and $x = 0$ is a fixed point of T in X.

Acknowledgements: The authors are grateful to the anonymous Referee for his/her helpful comments.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 121 – 127

Applications of k-Fibonacci numbers for the starlike analytic functions

Janusz Sokół* , Ravinder Krishna Raina † and Nihal Yilmaz Özgür ‡

Abstract

The k-Fibonacci numbers $F_{k,n}$ $(k > 0)$, defined recursively by $F_{k,0} = 0, F_{k,1} = 1$ and $F_{k,n} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$ are used to define a new class $\mathcal{SL}^k$. The purpose of this paper is to apply properties of k-Fibonacci numbers to consider the classical problem of estimation of the Fekete–Szegö problem for the class $\mathcal{SL}^k$. An application for inverse functions is also given.

2000 AMS Classification: Primary 30C45, secondary 30C80

Keywords: univalent functions; convex functions; starlike functions; subordination; k-Fibonacci numbers.

Received 27/12/2012 : Accepted 04/12/2013 Doi : 10.15672/HJMS.2015449091

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disc on the complex plane. The class of all holomorphic functions f in the open unit disc $\mathbb{D}$ with normalization $f(0) = 0$, $f'(0) = 1$ is denoted by $\mathcal{A}$ and the class $\mathcal{S} \subset \mathcal{A}$ is the class which consists of univalent functions in $\mathbb{D}$. We say that f is subordinate to F in $\mathbb{D}$, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some $\omega \in \mathcal{A}$, $|\omega(z)| < 1$, $z \in \mathbb{D}$.

Recently, N. Yilmaz Özgür and J. Sokół [5] defined and introduced the class $\mathcal{SL}^k$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition.

*Department of Mathematics, Rzeszów University of Technology, Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
Email: jsokol@prz.edu.pl
†M.P. University of Agri. and Technology, Udaipur, Rajasthan, India
Present address: 10/11 Ganpati Vihar, Opposite Sector 5, Udaipur 313002, India
Email:rkraina_7@hotmail.com
‡Department of Mathematics, Balıkesir University, Çağış Kampüsü, 10145 Balıkesir, Turkey
Email:nihal@balikesir.edu.tr

1.1. Definition. Let k be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}^k$ if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}_k(z),\ z \in \mathbb{D}, \tag{1.1}$$

where

$$\widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2},\ \tau_k = \frac{k-\sqrt{k^2+4}}{2},\ z \in \mathbb{D}. \tag{1.2}$$

For $k = 1$, the class $\mathcal{SL}^k$ becomes the class $\mathcal{SL}$ of shell-like functions defined in [3], see also [4].

It was proved in [5] that functions in the class $\mathcal{SL}^k$ are univalent in $\mathbb{D}$. Moreover, the class $\mathcal{SL}^k$ is a subclass of the class of starlike functions $\mathcal{S}^*$, even more, starlike of order $k(k^2+4)^{-1/2}/2$. The name attributed to the class $\mathcal{SL}^k$ is motivated by the shape of the curve

$$\mathcal{C} = \left\{\widetilde{p}_k(e^{it}) : t \in [0, 2\pi) \setminus \{\pi\}\right\}.$$

The curve $\mathcal{C}$ has a shell-like shape and it is symmetric with respect to the real axis. Its graphic shape, for $k = 1$, is given below in Fig.1.

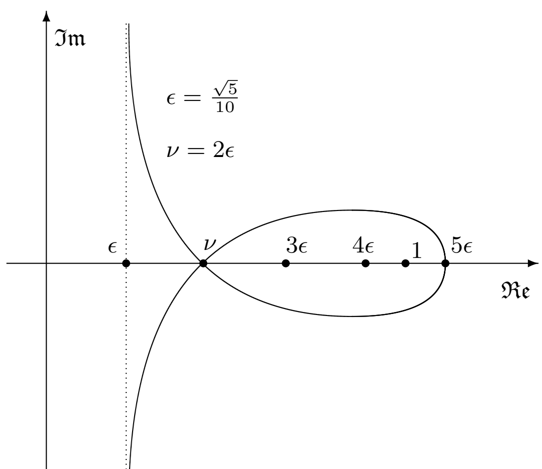

Fig. 1. $\widetilde{p}_1(e^{i\varphi}) : y^2 = \frac{(\sqrt{5}-2x)(\sqrt{5}x-1)^2}{10x-\sqrt{5}}$.

For $k \le 2$, note that we have

$$\widetilde{p}_k\left(e^{\pm i \arccos\left(k^2/4\right)}\right) = k(k^2+4)^{-1/2},$$

and so the curve $\mathcal{C}$ intersects itself on the real axis at the point $w_1 = k(k^2+4)^{-1/2}$. Thus $\mathcal{C}$ has a loop intersecting the real axis also at the point $w_2 = (k^2+4)/(2k)$. For $k > 2$, the curve $\mathcal{C}$ has no loops and it is like a conchoid, see for details [5]. Moreover, the coefficients of $\widetilde{p}_k$ are connected with k-Fibonacci numbers.

For any positive real number k, the k-Fibonacci number sequence $\{F_{k,n}\}_{n=0}^{\infty}$ is defined recursively by

$$F_{k,0} = 0,\ F_{k,1} = 1 \text{ and } F_{k,n} = kF_{k,n} + F_{k,n-1} \text{ for } n \ge 1. \tag{1.3}$$

When $k = 1$, we obtain the well-known Fibonacci numbers F_n. It is known that the nth k-Fibonacci number is given by

$$F_{k,n} = \frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}}, \tag{1.4}$$

where $\tau_k = (k - \sqrt{k^2+4})/2$. If $\widetilde{p}_k(z) = 1 + \sum_{n=1}^{\infty} \widetilde{p}_{k,n} z^n$, then we have

$$\widetilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau^n, \ n = 1, 2, 3, \ldots, \tag{1.5}$$

see also [5].

1.2. Lemma. [5] *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ belongs to the class $\mathcal{SL}^k$, then we have*

$$|a_n| \le |\tau_k|^{n-1} F_{k,n}, \tag{1.6}$$

where $\tau_k = (k - \sqrt{k^2+4})/2$. Equality holds in (1.6) for the function

$$\begin{aligned} f_k(z) &= \frac{z}{1 - k\tau_k z - \tau_k^2 z^2} \\ &= \sum_{n=1}^{\infty} \tau_k^{n-1} F_{k,n} z^n \\ &= z + \frac{(k-\sqrt{k^2+4})k}{2} z^2 + (k^2+1)\left(\frac{(k-\sqrt{k^2+4})k}{2} + 1\right) z^3 + \cdots . \end{aligned} \tag{1.7}$$

2. The classical Fekete–Szegö functional

A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. Let $\mathcal{S}$ be the class of univalent functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ mapping $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into $\mathbb{C}$ (the complex plane). The classical Fekete–Szegö functional is $\mathcal{L}_\lambda = |a_3 - \lambda a_2^2|$, $0 < \lambda \le 1$. Over the years, many results have been found for the classical functional $\mathcal{L}_\lambda$. Fekete and Szegö [1] bounded $\mathcal{L}_\lambda$ by $1 + 2\exp(-2\lambda/(1-\lambda))$, for $0 \le \lambda < 1$ and $f \in \mathcal{S}$, where $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{D}$. This inequality is sharp for each λ. In particular, for $\lambda = 1$, one has $|a_3 - a_2^2| \le 1$ if $f \in \mathcal{S}$. Note that the quantity $a_3 - a_2^2$ represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of locally univalent functions f in $\mathbb{D}$. It is interesting to consider the behavior of $\mathcal{L}_\lambda$ for subclasses of the class $\mathcal{S}$. The Fekete–Szegö problem is to determine sharp upper bound for Fekete–Szegö functional $\mathcal{L}_\lambda$ over a family $\mathcal{F} \subset \mathcal{S}$. In the literature, there exists a large number of results about inequalities for $a_3 - a_2^2$ corresponding to various subclasses of $\mathcal{S}$. In the present paper we obtain the Fekete–Szegö inequalities for the class $\mathcal{SL}^k$. Before we consider how the Taylor series coefficients of functions in the class $\mathcal{SL}^k$ might be bounded, let us first recall this problem for the Caratheodory functions. Let $\mathcal{P}$ denote the class of analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $\mathfrak{Re}\{p(z)\} > 0$.

2.1. Lemma. [2] *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z + \cdots$, then*

$$|c_n| \le 2, \quad for \quad n \ge 1. \tag{2.1}$$

If $|c_1| = 2$, then $p(z) \equiv p_1(z) = (1+xz)/(1-xz)$ with $x = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|x| = 1$, then $c_1 = 2x$. Furthermore, we have

$$|c_2 - c_1/2| \le 2 - |c_1|^2/2. \tag{2.2}$$

If $|c_1 < 2|$ and $|c_2 - c_1/2| = 2 - |c_1|^2/2$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + \overline{x}wz + z(wz+x)}{1 + \overline{x}wz - z(wz+x)}$$

and $x = c_1/2$, $w = (2c_2 - c_1^2)/(4 - |c_1|^2)$. *Conversely, if if* $p(z) \equiv p_2(z)$ *for some* $|x| < 1$ *and* $w = 1$, *then* $c_1 = 2x$, $w = (2c_2 - c_1^2)/(4 - |c_1|^2)$ *and* $|c_2 - c_1/2| = 2 - |c_1|^2/2$.

2.2. Theorem. *If* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *and*

$$p(z) \prec \widetilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2},\ \tau_k = \frac{k - \sqrt{k^2+4}}{2},\ z \in \mathbb{D},$$

then we have

$$|p_1| \le \frac{\left(\sqrt{k^2+4} - k\right) k}{2} \tag{2.3}$$

and

$$|p_2| \le (k^2 + 2)\left\{\frac{(k - \sqrt{k^2+4})k}{2} + 1\right\}. \tag{2.4}$$

The above estimations are sharp.

Proof. If $p \prec \widetilde{p}_k$, then there exists an analytic function w such that $|w(z)| \le |z|$ in $\mathbb{D}$ and $p(z) = \widetilde{p}_k(w(z))$. Therefore, the function

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z + \cdots \quad (z \in \mathbb{D})$$

is in the class $\mathcal{P}(0)$. It follows that

$$w(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \cdots \tag{2.5}$$

and

$$\begin{aligned}\widetilde{p}_k(w(z)) &= 1 + \widetilde{p}_{k,1}\left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \cdots\right\} + \widetilde{p}_{k,2}\left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \cdots\right\}^2 + \cdots \\ &= 1 + \frac{\widetilde{p}_{k,1} c_1}{2} z + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\widetilde{p}_{k,1} + \frac{1}{4} c_1^2 \widetilde{p}_{k,2}\right\} z^2 + \cdots \\ &= p(z). \qquad (2.6)\end{aligned}$$

From (1.5), we find the coefficients $\widetilde{p}_{k,n}$ of the function $\widetilde{p}_k$ given by

$$\widetilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau^n.$$

This shows the relevant connection $\widetilde{p}_k$ with the sequence of k-Fibonacci numbers

$$\begin{aligned}\widetilde{p}_k(z) &= 1 + \sum_{n=1}^{\infty} \widetilde{p}_{k,n} z^n \\ &= 1 + (F_{k,0} + F_{k,2})\tau_k z + (F_{k,1} + F_{k,3})\tau_k^2 z^2 + \cdots \\ &= 1 + k\tau_k z + (k^2 + 2)\tau_k^2 z^2 + (k^3 + 3k)\tau_k^3 z^3 + \cdots. \qquad (2.7)\end{aligned}$$

If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, then by (2.6) and (2.7), we have

$$p_1 = \frac{k\tau_k c_1}{2} \tag{2.8}$$

and

$$p_2 = \frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(k^2+2)}{4} c_1^2 \tau_k^2. \tag{2.9}$$

From (2.8) and (2.1) we directly obtain (2.3). From (2.9) and (2.2), we obtain

$$\begin{aligned}|p_2| &= \left|\frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(k^2+2)}{4}c_1^2\tau_k^2\right| \\ &\leq \left|\frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right)\right| + \left|\frac{(k^2+2)}{4}c_1^2\tau_k^2\right| \\ &\leq \frac{k|\tau_k|}{2}\left(2 - \frac{1}{2}|c_1|^2\right) + \frac{(k^2+2)}{4}|c_1|^2\tau_k^2 \\ &= k|\tau_k| + \frac{|c_1|^2}{4}\left((k^2+2)\tau_k^2 - k|\tau_k|\right). \qquad (2.10)\end{aligned}$$

Since $\tau_k = (k - \sqrt{k^2+4})/2$, so it is easily verified that

$$(k^2+2)\tau_k^2 - k|\tau_k| = \frac{(k(k-\sqrt{k^2+4}))(k^2+3)}{2} + k^2 + 2. \qquad (2.11)$$

We want to show that (2.11) is positive for $k > 0$. Notice that

$$\frac{(k-\sqrt{k^2+4})(k^3+3k)}{2} + k^2 + 2 = \frac{(k^2+2)\sqrt{k^2+4} - k^3 - 4k}{k + \sqrt{k^2+4}}. \qquad (2.12)$$

Thus, (2.11) is positive when

$$(k^2+2)\sqrt{k^2+4} > k^3 + 4k, \quad k > 0, \qquad (2.13)$$

or equivalently, when

$$\left\{(k^2+2)\sqrt{k^2+4}\right\}^2 > \left\{k^3+4k\right\}^2, \quad k > 0. \qquad (2.14)$$

The inequality (2.14) yields the inequality

$$4k^2 + 16 > 0, \quad k > 0, \qquad (2.15)$$

which is evidently true, and hence (2.11) is positive. Therefore, $(k^2+2)\tau_k^2 - |\tau_k| > 0$ and from (2.10), we obtain

$$\begin{aligned}|p_2| &\leq k|\tau_k| + \frac{|c_1|^2}{4}\left((k^2+2)\tau_k^2 - k|\tau_k|\right) \\ &\leq k|\tau_k| + (k^2+2)\tau_k^2 - k|\tau_k| \\ &= (k^2+2)\tau_k^2 \\ &= (k^2+2)\left\{\frac{(k-\sqrt{k^2+4})k}{2} + 1\right\}.\end{aligned}$$

Thus, the equality in estimations (2.3), (2.4) are attained by the coefficients of the function given by(2.7). □

2.3. Theorem. *Let λ be real. If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to $\mathcal{SL}^k$, then*

$$|a_3 - \lambda a_2^2| \leq (k(k-\sqrt{k^2+4})/2 + 1)(k^2 + 1 + k^2|\lambda|). \qquad (2.16)$$

The above estimation is sharp. If $\lambda \leq 0$, then the equality in (2.16) is attained by the function f_k given in (1.6), and by the function $-f_k(-z)$ when $\lambda \geq 0$.

Proof. For given $f \in \mathcal{SL}^k$, define $p(z) = 1 + p_1z + p_2z^2 + \cdots$ by

$$\frac{zf'(z)}{f(z)} = p(z) \quad (z \in \mathbb{D}),$$

where $p \prec \widetilde{p}_k$ in $\mathbb{D}$. Hence

$$z + 2a_2z^2 + 3a_3z^3 + \cdots = \left\{z + a_2z^2 + a_3z^3 + \cdots\right\}\left\{1 + p_1z + p_2z^2 + \cdots\right\}$$

and

$$a_2 = p_1, \quad 2a_3 = p_1 a_2 + p_2.$$

Therefore, $|a_3 - \lambda a_2| = |(p_1 a_2 + p_2)/2 + \lambda p_1^2|$. Using this and the bounds (2.3), (2.4) and (1.6), we obtain

$$\begin{aligned}
|a_3 - \lambda a_2^2| &= |(p_1 a_2 + p_2)/2 - \lambda p_1^2| \\
&\leq \frac{|p_1||a_2| + |p_2|}{2} + |\lambda||p_1^2| \\
&\leq \frac{k(k-\sqrt{k^2+4})/2 \cdot k(k-\sqrt{k^2+4})/2 + (k^2+2)(k(k-\sqrt{k^2+4})/2+1)}{2} \\
&+ |\lambda| \left\{ \frac{\left(\sqrt{k^2+4}-k\right)k}{2} \right\}^2 \\
&= \frac{k^2(k(k-\sqrt{k^2+4})/2+1) + (k^2+2)(k(k-\sqrt{k^2+4})/2+1)}{2} \\
&+ |\lambda| \left\{ \frac{\left(\sqrt{k^2+4}-k\right)k}{2} \right\}^2 \\
&= (k^2+1)(k(k-\sqrt{k^2+4})/2+1) + |\lambda| \left\{ \frac{\left(\sqrt{k^2+4}-k\right)k}{2} \right\}^2 \\
&= (k(k-\sqrt{k^2+4})/2+1)(k^2+1+k^2|\lambda|).
\end{aligned}$$

□

2.4. Corollary. *If* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $|z| < r_0(g)$, $r_0(g) \geq 1/4$, *is an inverse to* $f \in \mathcal{SL}^k$, *then we have*

$$\text{(2.17)} \qquad |b_2| \leq \frac{(k-\sqrt{k^2+4})k}{2},$$

$$\text{(2.18)} \qquad |b_3| \leq (k(k-\sqrt{k^2+4})/2+1)(3k^2+1).$$

The above estimation is sharp. The equalities are attained by the function $-if_k^{-1}(iz)$, *where* f_k *is given in* (1.6).

Proof. For each $f \in \mathcal{S}$, the Koebe one-quarter theorem ensures that the image of $\mathbb{D}$ under f contains the disc of radius 1/4. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is univalent in $\mathbb{D}$ then, f has the inverse f^{-1} with the expansion

$$\text{(2.19)} \qquad f^{-1}(z) = z - a_2 z^2 + (2a_2^2 - a_3)z^3 + \cdots, \ |z| < r_0(f), \ r_0(f) \geq 1/4.$$

It was proved in [5] that functions in the class $\mathcal{SL}^k$ are univalent in $\mathbb{D}$. From Lemma 1.2 and (2.19), we obtain the inequality (2.17). Also, from Theorem 2.3 (with $\lambda = 2$) and (2.19), we obtain the inequality (2.18). If $f \in \mathcal{SL}^k$, then the function $-if_k(iz)$ satisfies

(1.1), so it belongs to the class $\mathcal{SL}^k$ too. Moreover, from (1.6), we have

$$
\begin{aligned}
&-if_k^{-1}(iz)\\
&= z + i\frac{(k-\sqrt{k^2+4})k}{2}z^2\\
&-\left\{2\left(\frac{(k-\sqrt{k^2+4})k}{2}\right)^2 + (k^2+1)\left(\frac{(k-\sqrt{k^2+4})k}{2}+1\right)\right\}z^3+\cdots\\
&= z + i\frac{(k-\sqrt{k^2+4})k}{2}z^2 - (k(k-\sqrt{k^2+4})/2+1)(3k^2+1)z^3+\cdots.
\end{aligned}
$$

This shows that the equalities in (2.17) and (2.18) are attained by the second and third coefficients of the function $-if_k^{-1}(iz)$. □

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 129 – 152

Simple modules for some Cartan-type Lie superalgebras

Zhu Wei* , Yongzheng Zhang and Qingcheng Zhang†

Abstract

The modules are induced from the finite-dimensional Cartan-type modular Lie superalgebras W, S, H and K over a field of prime characteristic, respectively. We give the Cartan subalgebras of these modular Lie superalgebras. Using certain properties of the positive root vectors, we discuss the simplicity of these modules.

2000 AMS Classification: 17B50, 17B40

Keywords: Cartan subalgebra, weight, maximal vector, simple module.

Received : : : Accepted : : Doi :

1. Introduction

The development of representation theories of Lie algebras and Lie superalgebras over a field of characteristic 0 has shown a remarkable evolution. We know that the representation theory of Lie algebras plays a central role in the classification of simple Lie algebras of characteristic 0. Kac (see [5]) classified finite-dimensional simple Lie superalgebras over the field $\mathbb{C}$, proposing three Cartan series the Witt type, special type and Hamiltonian type, with an additional series of the classical type. Until now, the classification of finite-dimensional simple Lie superalgebras over a field of prime characteristic has not been completed. Zhang (see [11]) studied finite-dimensional Lie superalgebras of Cartan type over fields of prime characteristic.

Motivated by the connection between the representation theory of Lie algebras and modular Lie algebras, many researchers have investigated the representation theory of modular Lie algebras (see [1]-[4], [6]-[11]). Many results have also been obtained for the representative theory of modular Lie algebras, i.e., Lie algebras over a field of characteristic $p > 0$ (see [1]-[4], [7]-[9]). The modular Lie superalgebra has experienced rather

*School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

† Email:zhangqc569@nenu.edu.cn

This work is supported by the Fundamental Research Funds for the Central Universities of China (Grant No. 14QNJJ003), SRFHLJED (Grant No. 12521157), NSFC (Grant No. 10871057), and NSFJL (Grant No. 20130101068JC).

vigorous development and further Cartan-type Lie superalgebras in prime characteristic are constructed. For a restricted Cartan-type Lie algebra, the restricted simple modules have been determined in the sense that their isomorphism classes have been parameterized and their dimensions have been computed. Concrete constructions and dimensions of these modules were obtained by Shen and Holmes (see [1]-[4], [7]-[9]). Shen (see [7]-[9]) constructed the graded modules for the Witt, special and Hamiltonian Lie algebras. Shen determined the simple modules having fundamental dominant weights, except the contact algebra. Holmes (see [1]) solved the remaining problem regarding the contact algebra. With a few exceptional weights, he showed that the simple restricted modules were induced from the restricted universal enveloping algebra for the homogeneous component of degree zero extended trivially to positive components.

However, there are few results for the representation theory of Lie superalgebras over a field of characteristic $p > 0$, i.e., modular Lie superalgebras. Liu (see [6]) solved the dimension formula of induced modules and obtained the properties of induced modules. The structure of Cartan-type Lie superalgebras is not as symmetric as that of classical type Lie superalgebras. The representation theory of Cartan-type Lie superalgebras seems to be more difficult than that of classical type Lie superalgebras. Motivated by the ideas of Holmes (see [1]-[4], [7]-[9]), in this paper we construct modules of Lie superalgebras $W(m,n,\underline{1})$, $S(m,n,\underline{1})$, $H(m,n,\underline{1})$ and $K(m,n,\underline{1})$, induced from the homogeneous components of their restricted universal enveloping superalgebras. We show that the generator $1 \otimes m$ of these constructed modules belongs to their nonzero submodules, before showing that the sufficient conditions of these modules are simple modules.

2. Preliminaries

In this paper, let $\mathbb{F}$ always denote an algebraically closed field of characteristic $p > 0$.

First, we recall the definitions of $W(m,n,\underline{t}), S(m,n,\underline{t}), H(m,n,\underline{t})$ and $K(m,n,\underline{t})$.

Let $\mathbb{N}$, and $\mathbb{N}_0$ be the set of positive integers, and the set of nonnegative integers, respectively. Let $m,n \in \mathbb{N}$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}_0^m$, $\beta = (\beta_1, \cdots, \beta_m) \in \mathbb{N}_0^m$. Then we define

(1) $\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_m + \beta_m)$. $0 = (0, \cdots, 0)$. $1 = (1, \cdots, 1)$.

(2) $\binom{\alpha}{\beta} = \prod_{i=1}^{m} \binom{\alpha_i}{\beta_i}$, where $\binom{\alpha_i}{\beta_i}$ denotes the binomial coefficient.

(3) $\alpha \leqslant \beta :\Leftrightarrow \alpha_i \leqslant \beta_i$, $i = 1, \cdots, m$.

(4) $\varepsilon_i := (0, \cdots, 1, \cdots, 0)$, where 1 occurs at the ith place.

(5) For every element $(x_1, \cdots, x_m)$, we put $x^{(\alpha)} := \prod_{i=1}^{m} x_i^{\alpha_i}$, noting that (5) implies that $x^0 = 1$.

Let $\mathcal{U}(m)$ denote the $\mathbb{F}$-algebra of divided power series in the variables $x_1, \ldots, x_m$ and define the multiplication by $x^{(\alpha)} x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}$, where $\alpha, \beta \in \mathbb{N}_0^m$.

Let $\Lambda(n)$ be an exterior algebra over $\mathbb{F}$, generated by $x_{m+1}, \cdots, x_s$, where $s = m + n$. Put $\Lambda(m,n) := \mathcal{U}(m) \otimes \Lambda(n)$. We write fg for $f \otimes g$ in the following, where $f \in \mathcal{U}(m)$, $g \in \Lambda(n)$. The following identities hold in $\Lambda(m,n)$:

(1) $x^{(\alpha)} x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}$.

(2) $x_i x_j = -x_j x_i$, $i, j = m+1, \cdots, s$.

(3) $x^{(\alpha)} x_j = x_j x^{(\alpha)}$, $\forall\, \alpha \in \mathbb{N}_0^m$, $j = m+1, \cdots, s$.

For $k = 1, \cdots, n$, define $B_k := \{(i_1, \cdots, i_k) \mid m+1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant s\}$. Let $B(n) = \bigcup_{k=0}^{n} B_k$, where $B_0 = \emptyset$. If $u = (i_1, i_2, \cdots, i_r) \in B_r$, where $m+1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant s$, then we set $x^u := x_{i_1} x_{i_2} \cdots x_{i_r}$, $|u| := r$. Put $x^{\emptyset} = 1$. Note that $\Lambda(m,n)$ is $\mathbb{Z}_2$-graded by $\Lambda(m,n)_{\bar{0}} = \mathcal{U}(m) \otimes \Lambda(n)_{\bar{0}}$, $\Lambda(m,n)_{\bar{1}} = \mathcal{U}(m) \otimes \Lambda(n)_{\bar{1}}$. Note that $\Lambda(m,n)$ is $\mathbb{Z}$-

graded by $\Lambda(m,n)_i = \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid \alpha\in\mathbb{N}_0^m, u\in B(n), |\alpha|+|u|=i\}$, where $|\alpha|=\sum\limits_{i=1}^{m}\alpha_i$. Obviously, $\Lambda(m,n)$ is a $\mathbb{Z}$-graded associative superalgebra.

Put $Y_0 := \{1,2,\cdots,m\}$, $Y_1 := \{m+1,\cdots,s\}$, $Y = Y_0\cup Y_1$.

Let ∂_i be the special derivation of $\Lambda(n)$ defined by $\partial_i(x_j)=\delta_{ij}$, where $i,j\in Y_1$ and δ_{ij} is the Kronecker delta. For $i\in Y$, we consider $D_i\in \mathrm{Der}_{\mathbb{F}}(\Lambda(m,n))$ given by $D_i(x^{(\alpha)}x^u) := \begin{cases} x^{(\alpha-\varepsilon_i)}x^u, & \forall i\in Y_0 \\ x^{(\alpha)}\partial_i(x^u), & \forall i\in Y_1\end{cases}$. $D_1,\cdots,D_s$ are called the special derivations of $\Lambda(m,n)$.

We put $\underline{t} := (t_1,\cdots,t_m)\in\mathbb{N}^m$, $\pi_i := p^{t_i}-1$, $\forall i\in Y_0$. Denote

$$A(m,\underline{t}) := \{\alpha=(\alpha_1,\cdots,\alpha_m)\in\mathbb{N}_0^m \mid 0\leqslant\alpha_i\leqslant\pi_i, i\in Y_0\}.$$

$$\Lambda(m,n,\underline{t}) := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid \alpha\in A(m,\underline{t}), u\in B(n)\}.$$

$$\Lambda(m,n,\underline{t})_i := \Lambda(m,n)_i\cap\Lambda(m,n,\underline{t}).$$

Then $\Lambda(m,n,\underline{t}) = \bigoplus\limits_{i=0}^{\xi}\Lambda(m,n,\underline{t})_i$ is a $\mathbb{Z}$-graded superalgebra, where $\xi := \sum\limits_{i=1}^{m}\pi_i+n$.

Remark: In this paper, $\mathrm{hg}(L)$ denotes the set of homogeneous elements of Lie superalgebra L, i.e., $\mathrm{hg}(L)=L_{\overline{0}}\oplus L_{\overline{1}}$. If $f\in\mathrm{hg}(L)$, then $\mathrm{d}(f)$ always denotes the $\mathbb{Z}_2$-graded degree of the element f. Write $\tau(i) := \begin{cases}\overline{0}, & \forall i\in Y_0\\ \overline{1}, & \forall i\in Y_1\end{cases}$.

In the following, we illustrate the definitions of the graded Cartan-type Lie superalgebras $W(m,n,\underline{t})$, $S(m,n,\underline{t})$, $H(m,n,\underline{t})$ and $K(m,n,\underline{t})$ (see [1]).

$$W(m,n,\underline{t}) := \{\sum_{i=1}^{s} f_iD_i \mid f_i\in\Lambda(m,n,\underline{t}),\ \forall i\in Y\}.$$

$$S(m,n,\underline{t}) := \mathrm{span}_{\mathbb{F}}\{D_{ij}(f) \mid i,j\in Y, f\in\mathrm{hg}(\Lambda(m,n,\underline{t}))\},$$

where $D_{ij}(f) = (-1)^{\tau(i)\tau(j)}D_i(f)D_j - (-1)^{\mathrm{d}(f)(\tau(i)+\tau(j))}D_j(f)D_i$.

Let $m=2r$ or $m=2r+1$. Put

$$i' := \begin{cases} i+r, & 1\leqslant i\leqslant r\\ i-r, & r<i\leqslant 2r\\ i, & 2r<i\leqslant s\end{cases}.$$

$$\sigma(i) := \begin{cases} 1, & 1\leqslant i\leqslant r\\ -1, & r<i\leqslant 2r\\ 1, & 2r<i\leqslant s\end{cases}.$$

Write $D_H(f) = \sum\limits_{i=1}^{s}\sigma(i')(-1)^{\tau(i')\mathrm{d}(f)}D_{i'}(f)D_i$, $\forall i\in Y$, $f\in\mathrm{hg}(\Lambda(m,n,\underline{t}))$. Then

$$H(m,n,\underline{t}) := \mathrm{span}_{\mathbb{F}}\{D_H(f) \mid f\in\mathrm{hg}(\Lambda(m,n,\underline{t}))\},$$

where $m=2r$ is even.

Let $L=\Lambda(m,n,\underline{t})$ and $m=2r+1$ be odd. Define a multiplication on L by means of

$$\begin{aligned}[f,g] \quad = \quad & (2f-\sum_{i\in Y\setminus\{m\}}x_iD_i(f))D_m(g)\\ & -(-1)^{\mathrm{d}(f)\mathrm{d}(g)}(2g-\sum_{i\in Y\setminus\{m\}}x_iD_i(g))D_m(f)\\ & +\sum_{i\in Y\setminus\{m\}}\sigma(i)(-1)^{\tau(i)\mathrm{d}(f)}D_i(f)D_{i'}(g).\end{aligned}$$

Then L is a Lie superalgebra. $K(m,n,\underline{t}) := [L,L]$see [1] shows that

$$K(m,n,\underline{t}) := \begin{cases} L, & \text{if} \quad n-m-3 \not\equiv 0 (\text{mod } p) \\ \bigoplus_{i=0}^{\xi-1} \Lambda(m,n,\underline{t})_i, & \text{if} \quad n-m-3 \equiv 0 (\text{mod } p) \end{cases}.$$

In the following, we simply write $W := W(m,n,\underline{1})$, $S := S(m,n,\underline{1})$, $H := H(m,n,\underline{1})$ and $K := K(m,n,\underline{1})$, respectively. They inherit $\mathbb{Z}$-gradations from $\Lambda(m,n,\underline{t})$ by means of

$$W = \bigoplus_{i=-1}^{\eta-1} W_i,$$

where $W_i = \text{span}_{\mathbb{F}}\{x^{(\alpha)}x^u D_j \mid |\alpha| + |u| = i+1, j \in Y\}$, $\eta = m(p-1)+n$.

$$S = \bigoplus_{i=-1}^{\eta-2} S_i,$$

where $S_i = \text{span}_{\mathbb{F}}\{D_{ij}(x^{(\alpha)}x^u) \mid |\alpha| + |u| = i+2, i,j \in Y\}$.

$$H = \bigoplus_{i=-1}^{\eta-3} H_i,$$

where $H_i = \text{span}_{\mathbb{F}}\{D_H(x^{(\alpha)}x^u) \mid |\alpha| + |u| = i+2\}$.

$$K = \bigoplus_{i=-2}^{\lambda} K_i,$$

where $K_i = \text{span}_{\mathbb{F}}\{(x^{(\alpha)}x^u) \mid |\alpha| + \alpha_m + |u| = i+2\}$. And

$$\lambda = \begin{cases} \eta + \pi_m - 2, & n-m-3 \not\equiv 0 (\text{mod } p) \\ \eta + \pi_m - 3, & n-m-3 \equiv 0 (\text{mod } p) \end{cases}.$$

We first recall the definition of the restricted universal enveloping superalgebra. Let $(L,[p])$ be a restricted Lie superalgebra. A pair $(u(L),i)$ consisting of an associative $\mathbb{F}$-superalgebra with unity and a restricted homomorphism $i : L \to u(L)^-$, is called a restricted universal enveloping superalgebra if given any associative $\mathbb{F}$- superalgebra A with unity and any restricted homomorphism $f : L \to A^-$, there is a unique homomorphism $\overline{f} : u(L) \to A$ of associative $\mathbb{F}$-superalgebra such that $\overline{f} \circ i = f$. The category of $u(L)$-modules and that of restricted L-modules are equivalent. According to the PBW theorem, the following statements hold: Let $(L,[p])$ be a restricted Lie superalgebra. If $(u(L),i)$ is a restricted universal enveloping superalgebra and $(l_j)_{j\in J_0} \cup (f_j)_{j\in J_1}$ is an ordered basis of L over $\mathbb{F}$, where $l_j \in L_{\bar{0}}$, $f_j \in L_{\bar{1}}$, then the elements $i(l_{j_1})^{s_1} i(l_{j_2})^{s_2} \cdots i(l_{j_n})^{s_n} i(f_{i_1}) i(f_{i_2}) \cdots i(f_{i_m})$, $j_1 < \cdots < j_n$, $0 \leqslant s_k \leqslant p-1$, $1 \leqslant k \leqslant n$, $i_1 < \cdots < i_m$, consist of a basis of $u(L)$ over $\mathbb{F}$. Sometimes, with no confusion, we will identify L with its image $i(L)$ in $u(L)$. Note that $D_i^p = 0$, for $i \in Y_0$; $D_i^2 = 0$, for $i \in Y_1$.

We know that W, S, H and K are restricted Lie superalgebras. They are also simple Lie superalgebras.

2.1. Definition. Let L be a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$. Suppose that A is a Cartan subalgebra of L_0, where L_0 is the set of the 0th homogenous elements of $\mathbb{Z}$-graded Lie superalgebra L. For $\lambda \in A^*$ and a $u(L_0)$-module V, we set $V_\lambda := \{v \in V \mid h \cdot v = \lambda(h)v, \ \forall h \in A\}$. If $V_\lambda \neq 0$, then λ is called a weight and a nonzero vector v in V_λ is called a weight vector (of weight λ). A nonzero vector $v \in V_\lambda$ is called a maximal vector (of weight λ) provided $x \cdot v = 0$, where x is any positive root vector of L_0.

Let $L = \Sigma_{i\in\mathbb{Z}} L_i$ be a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$. Set $N^+ := \sum_{i>0} L_i$, where L_i denotes the homogeneous component of degree i in the $\mathbb{Z}$-graded Lie superalgebra L. Then $N^+ \triangleleft N^+ + L_0 := L^+$ and $L^+/N^+ \cong L_0$. In particular, any L_0-module becomes a L^+-module by letting $\mathbb{N}^+$ act trivially. Define $M_L(B) := u(L) \otimes_{u(L^+)} B$, where $u(L)$ and $u(L^+)$ denote the restricted universal enveloping superalgebras of L and L^+, respectively, and B is a simple $u(L_0)$-module. According to the classical theory, for each weight λ, there exists a simple $u(L_0)$-module $B(\lambda)$ which is generated by a maximal vector of weight λ.

In the following, we will discuss the simplicity of $M_L(B(\lambda)) = u(L) \otimes_{u(L^+)} B(\lambda)$, where L denotes one of four classes of Cartan-type Lie superalgebras W, S, H or K.

Remark:

(1) If C is a subset of some linear space, then $\langle C \rangle$ denotes the subspace spanned by the set C over $\mathbb{F}$.

(2) ω always denotes a maximal vector of a Cartan-type Lie superalgebra L.

(3) $\widehat{D_i}$ means that D_i is deleted.

3. The simple module of Lie superalgebra $W(m, n, \underline{1})$

3.1. Lemma. $A = \sum\limits_{i=1}^{s} \mathbb{F} x_i D_i$ is a Cartan subalgebra of W_0. The positive root vectors of W_0 are $\{x_i D_j \mid 1 \leqslant i < j \leqslant s\}$.

Proof. Let $\varphi_W : W_0 \to gl(\Lambda(m, n, \underline{1})_1)$ be a homomorphism of Lie superalgebras such that $\varphi_W(x_i D_j) = E_{ij}$, where $gl(\Lambda(m, n, \underline{1})_1)$ is the general linear Lie superalgebra and E_{ij} is the $s \times s$-matrix with 1 in the (i, j)-position and zeros elsewhere, $\forall i, j \in Y$. Note that φ_W is an isomorphism. By a straightforward computation, we obtain that the Cartan subalgebra of $gl(\Lambda(m, n, \underline{1})_1)$ is $\langle \{E_{ii} \mid i \in Y\} \rangle$, and the positive root vectors of $gl(\Lambda(m, n, \underline{1})_1)$ are $\{E_{ij} \mid 1 \leqslant i < j \leqslant s\}$. By the isomorphism φ_W, we find that $A = \sum\limits_{i=1}^{s} \mathbb{F} x_i D_i$ is a Cartan subalgebra of W_0 and $\{x_i D_j \mid 1 \leqslant i < j \leqslant s\}$ are positive root vectors of W_0. □

According to the Definition 2.1 and Lemma 3.1, the following statements hold: If $A = \sum\limits_{i=1}^{s} \mathbb{F} x_i D_i$ is a Cartan subalgebra of W_0, V is a $u(W_0)$-module and $\lambda \in A^*$, then $V_\lambda = \{v \in V \mid (x_i D_i) \cdot v = \lambda(x_i D_i) v, i \in Y\}$. Write $\lambda_i := \lambda(x_i D_i)$. A nonzero element $v \in V_\lambda$ is a maximal vector (of weight λ) provided $x_i D_j \cdot v = 0$, $\forall\ 1 \leqslant i < j \leqslant s$.

3.2. Lemma. Suppose that $M := < u(W) D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega >$, for $i \in Y_0$. If the following each situation holds, respectively,

(1) $\lambda_i \neq -1, \beta_i = p - 1$.

(2) $\lambda_i = 0, \beta_i \neq 1$.

(3) $\lambda_i = -1, \beta_i \neq p - 1$.

(4) There exists $j \in Y_0$, $1 \leqslant i < j \leqslant m$, such that $\lambda_j \neq -1$, and $\beta_j = p - 1$.

(5) There exists $j \in Y_0$, $1 \leqslant j < i \leqslant m$, such that $\lambda_j \neq 0$, and $\beta_j = 0$.

Then $D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

(6) In addition, if $\lambda_i \neq 0, \beta_i = p - 2$, then $D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

Proof. Obviously, M is a W-module. With the equality

$$[fD, gE] = fD(g)E - (-1)^{\mathrm{d}(fD)\mathrm{d}(gE)} gE(f)D + (-1)^{\mathrm{d}(D)\mathrm{d}(g)} fg[D, E],$$

where $f, g \in \mathrm{hg}(\Lambda(m,n,\underline{t}))$, $D, E \in \mathrm{hg}(\mathrm{Der}(\Lambda(m,n,\underline{t})))$, we obtain

$$\begin{aligned} & (x^{(2\varepsilon_i)}D_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & (D_i\cdot(x^{(2\varepsilon_i)}D_i)-x_iD_i)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & (D_i\cdot(x^{(2\varepsilon_i)}D_i)\cdot D_i^{\beta_i-1}-D_i^{\beta_i-1}\cdot(x_iD_i)+(\beta_i-1)D_i^{\beta_i-1})D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & D_i^{\beta_i-1}\cdot(x^{(2\varepsilon_i)}D_i)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega-\beta_iD_i^{\beta_i-1}\cdot(x_iD_i)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ & +(1+2+\cdots+(\beta_i-1))D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega. \end{aligned}$$

Since the $\mathbb{Z}$-graded degree of $x^{(2\varepsilon_i)}D_i$ is 1 and ω is a maximal vector of weight λ, the first term vanishes and the second term equals $-\beta_i\lambda_iD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega$. Then

$$(x^{(2\varepsilon_i)}D_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega=-\frac{\beta_i}{2}(2\lambda_i-\beta_i+1)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.$$

Satisfying the situation (1), (2) or (3), respectively, we can conclude that $D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

(4) According to the situation (4), there exists $j\in Y_0$, such that $\lambda_j\neq-1$ and $\beta_j=p-1$. We know that $D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$. Hence, we obtain

$$\begin{aligned} & (x_iD_j)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & [D_i\cdot(x_iD_j)-D_j]D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & D_i^{\beta_i}\cdot(x_iD_j)\cdot D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ & -\beta_iD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-1}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M. \end{aligned} \tag{3.1}$$

The maximal vector ω implies that the first term vanishes. Now (3.1) yields the desired result, $D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

(5) For $1\leqslant j<i\leqslant m$, we have

$$\begin{aligned} & (x_iD_j)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & (D_i\cdot(x_iD_j)-D_j)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & D_i^{\beta_i}\cdot(x_iD_j)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega-\beta_iD_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M. \end{aligned} \tag{3.2}$$

With the equality $[x^{(2\varepsilon_j)}D_j, x_iD_j]=-x_ix_jD_j$ and (3.2) multiplied on the left by $x^{(2\varepsilon_j)}D_j$, we have

$$\begin{aligned} & (x^{(2\varepsilon_j)}D_j)\cdot(D_i^{\beta_i}\cdot(x_iD_j)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega-\beta_iD_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega) \\ = \; & D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\cdot(x^{(2\varepsilon_j)}D_j)\cdot(x_iD_j)\otimes\omega \\ & -\beta_i(D_j\cdot(x^{(2\varepsilon_j)}D_j)-x_jD_j)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & \beta_i\lambda_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M. \end{aligned} \tag{3.3}$$

The assertion follows from (3.3).

(6) In the particular case of $\lambda_i\neq0$ and $\beta_i=p-2$, we obtain

$$\begin{aligned} & (x^{((p-1)\varepsilon_i)}D_i)\cdot D_i^{p-2}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & (D_i\cdot(x^{((p-1)\varepsilon_i)}D_i)-x^{((p-2)\varepsilon_i)}D_i)D_i^{p-3}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = \; & -\lambda_iD_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M. \end{aligned} \tag{3.4}$$

Along with (3.4), we get the result. □

3.1. Theorem. $M_W(B(\lambda))$ is simple, if one of the following situations holds:

(1) $(\lambda_1, \cdots, \lambda_m) \neq \sum\limits_{t=k+1}^{m} -\varepsilon_t$, (the empty sum being zero) for $0 \leqslant k \leqslant m$.

(2) There exist $i, j \in Y_1$ and $j < i$ such that $\lambda_i \neq 1$, $\lambda_j \neq 0$.

Proof. Let M' be a nonzero submodule of $M_W(B(\lambda)) = u(W) \otimes_{u(W^+)} B(\lambda)$. $u(W^+) \cdot B(\lambda) = u(W_0) \cdot B(\lambda)$ implies that $u(W) \cdot B(\lambda) = (u(W_0) + u(W_{-1})) \cdot B(\lambda) \subseteq u(W_{-1}) \cdot B(\lambda)$. Then, $M_W(B(\lambda)) = u(W) \otimes_{u(W^+)} B(\lambda) = (u(W_{-1}) + u(W_0)) \otimes_{u(W^+)} B(\lambda) \subseteq u(W_{-1}) \otimes_{u(W^+)} B(\lambda)$, namely,

$$M_W(B(\lambda)) = u(W_{-1}) \otimes_{u(W^+)} B(\lambda). \tag{3.5}$$

Choose $v \neq 0 \in M'$. (3.5) implies that v can be written by

$$v = \Sigma_{\beta \in I} c(\beta) i(D_1)^{\beta_1} \cdots i(D_s)^{\beta_s} \otimes b_\beta,$$

where $\beta = (\beta_1, \cdots, \beta_s)$, $I := \{(\beta_1, \cdots, \beta_s) \mid 0 \leqslant \beta_i \leqslant p-1, \text{for any } i \in Y_0; \beta_i = 0 \text{ or } 1, \text{for any } i \in Y_1\} \subset \mathbb{Z}^s$, $c(\beta) \in \mathbb{F}$, $b_\beta \in B(\lambda)$. In the following, with no confusion, we usually write D_j to instead $i(D_j)$ in $u(L)$. Then

$$v = \Sigma_{\beta \in I} c(\beta) D_1^{\beta_1} \cdots D_s^{\beta_s} \otimes b_\beta. \tag{3.6}$$

Define an order of I such that $\beta = (\beta_1, \cdots, \beta_s) < \beta' = (\beta'_1, \cdots, \beta'_s)$ if and only if there exists $k \in \{1, 2, \cdots, s\}$ such that $\beta_i = \beta'_i$ for all $i > k$ and $\beta_k < \beta'_k$. Let $\mathscr{C} := \{\beta \in I \mid c(\beta) \neq 0\}$, where $c(\beta)$ comes from the right side of the equality (3.6). According to the order of I, we choose the least element $\eta = (\eta_1, \cdots, \eta_s) \in \mathscr{C}$. Obviously, $c(\eta) \neq 0$. Put $y := \prod\limits_{t=1}^{m} D_t^{p-1-\eta_t} \prod\limits_{l=m+1}^{s} D_l^{1-\eta_l}$. Since $[D_i, D_j] = 0$, namely, $D_i D_j = (-1)^{\tau(i)\tau(j)} D_j D_i$ holds in $u(W)$. Then

$$\begin{aligned} yv &= \prod_{t=1}^{m} D_t^{p-1-\eta_t} \prod_{l=m+1}^{s} D_l^{1-\eta_l} (\Sigma_{\beta \in I} c(\beta) D_1^{\beta_1} \cdots D_s^{\beta_s} \otimes b_\beta) \\ &= \alpha c(\eta) \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes b_\beta \in M', \end{aligned}$$

where $\alpha = 1$ or -1. Consequently, $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes b_\beta \in M'$.

$\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes B(\lambda)$ is a $u(W_0)$-module. In fact, if $k \in Y_0$, a straightforward computation shows that,

$$\begin{aligned} &(x_k D_l) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda) \\ = &\prod_{t_1=1}^{k-1} D_{t_1}^{p-1} \cdot (x_k D_l) \cdot D_k^{p-1} \prod_{t_2=k+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda) \\ = &\prod_{t_1=1}^{k-1} D_{t_1}^{p-1} (D_k^{p-1} \cdot x_k D_l - (p-1) D_l D_k^{p-2}) \prod_{t_2=k+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda) \\ \subseteq &\prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda). \end{aligned}$$

Similarly, if $k \in Y_1$, then we have

$$\begin{aligned} & (x_kD_l)\cdot\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+1}^{s}D_l\otimes B(\lambda) \\ = & \prod_{t=1}^{m}D_t^{p-1}\cdot(x_kD_l)\cdot\prod_{l=m+1}^{s}D_l\otimes B(\lambda) \\ \subseteq & \prod_{t=1}^{m}D_t^{p-1}\cdot\prod_{l_1=m+1}^{k-1}D_{l_1}\cdot(-1)^{\mathrm{d}(x_kD_l)}(D_k\cdot(x_kD_l)-D_l)\cdot\prod_{l_2=k+1}^{s}D_{l_2}\otimes B(\lambda) \\ \subseteq & \prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+1}^{s}D_l\otimes B(\lambda). \end{aligned}$$

As $B(\lambda)$ is a simple $u(W_0)$-module, we see that $\prod\limits_{t=1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes B(\lambda)$ is a simple $u(W_0)$-module. It can be regarded as a $u(W)$-module. By virtue of $\prod\limits_{t=1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes b_\beta\in M'\cap\left(\prod\limits_{t=1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes B(\lambda)\right)$, we have $\prod\limits_{t=1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes B(\lambda)\subseteq M'$. Thus, there exists a maximal vector ω of weight λ such that $\prod\limits_{t=1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes\omega\in M'$.

For the case of the situation (1), without loss of generality, we assume $\prod\limits_{t=i}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes\omega\in M'$, for any $i\in Y_0$. We proceed according to several different situations.

(i) $\lambda_i\neq-1$ and $\lambda_i\neq 0$.

By (1) and (6) in Lemma 3.2, we conclude that $\prod\limits_{t=i+1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes\omega\in M'$.

(ii) $\lambda_i=-1$.

By $(\lambda_1,\cdots,\lambda_m)\neq\sum\limits_{t=i}^{m}-\varepsilon_t$, there exists $j\in Y_0$, $j>i$ such that $\lambda_j\neq-1$, or $j<i$, $\lambda_j\neq 0$. By (4) or (5) in Lemma 3.2, we have $\prod\limits_{t=i+1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes\omega\in M'$.

(iii) $\lambda_i=0$.

By $(\lambda_1,\cdots,\lambda_m)\neq\sum\limits_{t=i+1}^{m}-\varepsilon_t$, there exists $j\in Y_0$, such that $j>i$, $\lambda_j\neq-1$, or $j<i$, $\lambda_j\neq 0$. Also by (4) or (5) in Lemma 3.2, we have $\prod\limits_{t=i+1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes\omega\in M'$.

The assumption of arbitrary i implies that $\prod\limits_{l=m+1}^{s}D_l\otimes\omega\in M'$.

Furthermore, without loss of the generality, we assume $\prod\limits_{l=j}^{s}D_l\otimes\omega\in M'$. The situation (1) implies that there exists $i\in Y_0$ such that $\lambda_i\neq 0$. Then, $\prod\limits_{l=j}^{s}D_l\otimes\omega\in M'$ multiplied on the left by $x_jx_iD_i$, for $j\in Y_1$, we obtain

$$\begin{aligned} & (x_jx_iD_i)\cdot D_j\cdots D_s\otimes\omega \\ = & (-D_j\cdot(x_jx_iD_i)+x_iD_i)D_{j+1}\cdots D_s\otimes\omega \\ = & \lambda_iD_{j+1}\cdots D_s\otimes\omega. \end{aligned} \tag{3.7}$$

Consequently, $D_{j+1}\cdots D_s\otimes\omega\in M'$. Along with the choice of j, we obtain $1\otimes\omega\in M'$. Since $u(W_0)$-module $M_W(B(\lambda))$ is generated by $1\otimes\omega$, $1\otimes\omega\in M'$ indicates that $M_W(B(\lambda))=M'$. It suffices to demonstrate that $M_W(B(\lambda))$ is simple.

Situation (2) implies that $i\neq m+1\in Y_1$. Obviously, $[x_{m+1}x_iD_i,D_l]=\delta_{m+1,l}x_iD_i-\delta_{il}x_{m+1}D_i$, for $l\in Y_1$. Thus

$$(x_{m+1}x_iD_i)\cdot D_l=-D_l\cdot(x_{m+1}x_iD_i)+\delta_{m+1,l}x_iD_i-\delta_{il}x_{m+1}D_i \tag{3.8}$$

holds in $u(W)$. By virtue of (3.8), we find

$$\begin{aligned}
&(x_{m+1}x_iD_i)\cdot\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+1}^{s}D_l\otimes\omega\\
=\;&\prod_{t=1}^{m}D_t^{p-1}-D_{m+1}\cdot(x_{m+1}x_iD_i)+x_iD_i)D_{m+2}\cdots D_s\otimes\omega\\
=\;&\prod_{t=1}^{m}D_t^{p-1}(-1)^{i-1-m}D_{m+1}\cdots D_{i-1}\cdot(x_{m+1}x_iD_i)\cdot D_i\cdots D_s\otimes\omega\\
&+\prod_{t=1}^{m}D_t^{p-1}D_{m+2}\cdots D_{i-1}\cdot(x_iD_i)\cdot D_i\cdots D_s\otimes\omega\\
=\;&\prod_{t=1}^{m}D_t^{p-1}(-1)^{n}D_{m+1}\cdots D_s\cdot(x_{m+1}x_iD_i)\otimes\omega\\
&+\prod_{t=1}^{m}D_t^{p-1}(-1)^{i-m}D_{m+1}\cdots\widehat{D_i}\cdots D_s\cdot(x_{m+1}D_i)\otimes\omega\\
&+\prod_{t=1}^{m}D_t^{p-1}D_{m+2}\cdots D_s\cdot(x_iD_i)\otimes\omega\\
&-\prod_{t=1}^{m}D_t^{p-1}D_{m+2}\cdots D_s\otimes\omega.
\end{aligned}$$

As $B(\lambda)$ is a $u(W_0)$-module and ω is a maximal vector of weight λ, the first summation vanishes and so does the second for $m+1<i$. According to the assertion above, we obtain

$$(x_{m+1}x_iD_i)\cdot\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+1}^{s}D_l\otimes\omega=(\lambda_i-1)\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+2}^{s}D_l\otimes\omega.$$

As $\lambda_i\neq1$, this entails $\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+2}^{s}D_l\otimes\omega\in M'$. If $\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+2}^{s}D_l\otimes\omega$ multiplied on the left by the elements $x_{m+2}x_iD_i,x_{m+3}x_iD_i,\cdots,x_{i-1}x_iD_i$, successively, yielding $\prod_{t=1}^{m}D_t^{p-1}\prod_{l=i}^{s}D_l\otimes\omega\in M'$. By the assumption of the theorem, there exists $m+1\leqslant j<i\leqslant s$ such that $\lambda_j\neq0$. Thus we have

$$\begin{aligned}
&(x_ix_jD_j)\cdot\prod_{t=1}^{m}D_t^{p-1}\prod_{l=i}^{s}D_l\otimes\omega\\
=\;&\prod_{t=1}^{m}D_t^{p-1}(-D_i)\cdot(x_ix_jD_j)\cdot\prod_{l=i+1}^{s}D_l\otimes\omega+\prod_{t=1}^{m}D_t^{p-1}\cdot(x_jD_j)\cdot\prod_{l=i+1}^{s}D_l\otimes\omega.
\end{aligned}\tag{3.9}$$

Using the fact that the $\mathbb{Z}$-graded degree of $x_ix_jD_j$ is 1, we can see that (3.9) coincides with $\lambda_j\prod_{t=1}^{m}D_t^{p-1}\prod_{l=i+1}^{s}D_l\otimes\omega$. The assertion that $\prod_{t=1}^{m}D_t^{p-1}\prod_{l=i+1}^{s}D_l\otimes\omega\in M'$ follows from $\lambda_j \neq 0$. We multiply $\prod_{t=1}^{m} D_t^{p-1} \prod_{l=i+1}^{s} D_l \otimes \omega$ by the elements $x_{i+1}x_jD_j, \cdots, x_sx_jD_j$, consecutively. Repeating the process above yields $\prod_{t=1}^{m} D_t^{p-1} \otimes \omega \in M'$. The situation (2) ensures that there exists $j \in Y_1$ such that $\lambda_j \neq 0$. Thus, for any $i \in Y_0$, we have

$$\begin{aligned} & (x_ix_jD_j) \cdot \prod_{t=i}^{m} D_t^{p-1} \otimes \omega \\ = \quad & (D_i \cdot (x_ix_jD_j) - x_jD_j) \prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega \\ = \quad & -\lambda_j \prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega. \end{aligned}$$

$\lambda_j \neq 0$ entails $\prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega \in M'$. In the general case of i, this implies that $1 \otimes \omega \in M'$.

Hence, $M_W(B(\lambda))$ is simple. □

4. The simple module of Lie superalgebra $S(m, n, \underline{1})$

4.1. Lemma. Let

$A = \langle \{-D_{i,i+1}(x_ix_{i+1}), D_{j,j+1}(x_jx_{j+1}), D_{m,m+1}(x_mx_{m+1}) \mid i, i+1 \in Y_0; j, j+1 \in Y_1\} \rangle$.

Then A is a Cartan subalgebra of S_0. The positive root vectors of S_0 are $\{x_iD_j \mid 1 \leqslant i < j \leqslant s\}$.

Proof. The homomorphism φ_S is the restriction of the isomorphism $\varphi_W : W_0 \to gl(\Lambda(m, n, \underline{t})_1)$. Note that $S_0 \cong \mathscr{L}$, where $\mathscr{L} := \langle \{ \left(\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & D_1 \end{smallmatrix}\right), E_{m,m} + E_{m+1,m+1}, E_{ij} \mid A_1 \in Sl_m(\mathbb{F}), D_1 \in Sl_n(\mathbb{F}); i \in Y_0, j \in Y_1; \text{or } i \in Y_1, j \in Y_0 \} \rangle$.
By a straightforward computation, we get the Cartan subalgebra of $\mathscr{L}$ is $\langle \{ \left(\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & D_1 \end{smallmatrix}\right), E_{m,m} + E_{m+1,m+1} \mid A_1 \in Sl_m(\mathbb{F}), D_1 \in Sl_n(\mathbb{F}) \} \rangle$ and the positive root vectors of $\mathscr{L}$ are $\{E_{ij} \mid 1 \leqslant i < j \leqslant s\}$. By the isomorphism φ_S, we obtain that the Cartan subalgebra of S_0 is

$A = \langle \{-D_{i,i+1}(x_ix_{i+1}), D_{j,j+1}(x_jx_{j+1}), D_{m,m+1}(x_mx_{m+1}) \mid i, i+1 \in Y_0; j, j+1 \in Y_1\} \rangle$

and the positive root vectors of S_0 are $\{x_iD_j \mid 1 \leqslant i < j \leqslant s\}$. □

Our preceding results of Lemma 4.1 discuss the weight vectors and a maximal vector. The following facts hold: If A is a Cartan subalgebra of S_0, V is a $u(S_0)$-module and $\lambda \in A^*$. Then

$V_\lambda = \{v \in V \mid D_{i+1,i}(x_ix_{i+1}) \cdot v = \lambda_i v, \ 1 \leq i \leq m-1; D_{j,j+1}(x_jx_{j+1}) \cdot v = \lambda_j v, m \leqslant j \leqslant s-1\}$.

A nonzero element $v \in V_\lambda$ is a maximal vector (of weight λ) provided $x_iD_j \cdot v = 0$, whenever $1 \leqslant i < j \leqslant s$.

4.2. Lemma. Suppose that $M := < u(S)D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega >$, for $1 \leqslant i \leqslant m-1$. If the following situations hold, respectively,

(1) $\lambda_i \neq 0, \beta_i = p-1$.

(2) $\lambda_i = 1, \beta_i \neq 1$.

(3) $\lambda_i = 0, \beta_i \neq p-1$.

(4) There exists $j \in Y_0$, $1 \leqslant i < j \leqslant m$, such that $\beta_j = p-1$ and $\lambda_j \neq 0$.

(5) There exists $j \in Y_0$, $2 \leqslant j+1 < i \leqslant m-1$, such that $\beta_j = 0$ and $\lambda_j \neq 0$. In addition, for $i = j+1, \beta_i = p-1$ and $\lambda_j \neq 1$, or $\beta_i \neq p-1$ and $\lambda_j \neq 0$.

Then $D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

(6) In addition, if $\lambda_i \neq 1, \beta_i = p-2$, then $D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

Proof. By virtue of $[D_k, D_{ij}(f)] = (-1)^{\tau(k)\tau(i)} D_{ij}(D_k(f))$, for all $i, j, k \in Y$, we have

$$\begin{aligned}
& D_{i+1,i}(x^{(2\varepsilon_i)} x_{i+1}) \cdot D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & (D_i \cdot D_{i+1,i}(x^{(2\varepsilon_i)} x_{i+1}) - D_{i+1,i}(x_i x_{i+1})) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & D_i \cdot D_{i+1,i}(x^{(2\varepsilon_i)} x_{i+1}) \cdot D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
& - D_i^{\beta_i - 1} \cdot D_{i+1,i}(x_i x_{i+1}) \cdot D_{i+1}^{p-1} \cdots D_m^{p-1} + (\beta_i - 1) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & D_i \cdot D_{i+1,i}(x^{(2\varepsilon_i)} x_{i+1}) \cdot D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
& + (-\lambda_i + \beta_i) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & ((-\lambda_i + \beta_i) + (-\lambda_i + \beta_i - 1) + \cdots + (-\lambda_i + 1)) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
(4.1) = \quad & -\frac{\beta_i}{2}(2\lambda_i - \beta_i - 1) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M.
\end{aligned}$$

The foregoing equality (4.1) implies that (1), (2) or (3) can conclude $D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$, respectively.

(4) Our assumption of the situation (4) entails that $D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_{j-1}^{p-1} D_j^{p-2} D_{j+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$. Then,

$$\begin{aligned}
& D_{ij}(x^{(2\varepsilon_i)}) \cdot D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_{j-1}^{p-1} D_j^{p-2} D_{j+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & (D_i \cdot (x_i D_j) - D_j) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_{j-1}^{p-1} D_j^{p-2} D_{j+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & D_i^{\beta_i} \cdot (x_i D_j) \cdot D_{i+1}^{p-1} \cdots D_{j-1}^{p-1} D_j^{p-2} D_{j+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
(4.2) \quad & -\beta_i D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_j^{p-1} \cdots D_m^{p-1} \otimes \omega.
\end{aligned}$$

For $i < j$, the first term vanishes. Hence (4.2) implies that $D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

(5) For $i > j$, we have

$$\begin{aligned}
& D_{ij}(x^{(2\varepsilon_i)}) \cdot D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & D_i^{\beta_i} \cdot (x_i D_j) \cdot D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega - \beta_i D_j D_i^{\beta_i - 1} \cdots D_m^{p-1} \otimes \omega \in M.
\end{aligned}$$

Multiplying this equation by $D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1})$ on the left, we obtain

$$D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) \cdot (D_i^{\beta_i} \cdot (x_i D_j) \cdot D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega - \beta_i D_j D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega) \in M.$$

If $i = j+1$, then

(4.3)

$$D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) \cdot (D_{j+1}^{\beta_{j+1}} \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega - \beta_{j+1} D_j D_{j+1}^{\beta_{j+1} - 1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega) \in M,$$

where,

$$\begin{aligned}
& D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) \cdot D_{j+1}^{\beta_{j+1}} \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & (D_{j+1} \cdot D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) - D_{j+1,j}(x^{(2\varepsilon_j)})) D_{j+1}^{\beta_{j+1} - 1} \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & D_{j+1}^{\beta_{j+1}} \cdot D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
& + \beta_{j+1} D_{j+1}^{\beta_{j+1} - 1} \cdot (x_j D_{j+1}) \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \quad & \beta_{j+1} \lambda_j D_{j+1}^{\beta_{j+1} - 1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega,
\end{aligned}$$

and

$$\begin{aligned} & D_{j+1,j}(x^{(2\varepsilon_j)}x_{j+1}) \cdot (-\beta_{j+1} D_j D_{j+1}^{\beta_{j+1}-1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega) \\ = \quad & -\beta_{j+1}(D_j \cdot D_{j+1,j}(x^{(2\varepsilon_j)}x_{j+1}) - D_{j+1,j}(x_j x_{j+1})) D_{j+1}^{\beta_{j+1}-1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\ = \quad & \beta_{j+1} D_{j+1,j}(x_j x_{j+1}) \cdot D_{j+1}^{\beta_{j+1}-1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\ = \quad & \beta_{j+1}(\lambda_j + \beta_{j+1} - 1) D_{j+1}^{\beta_{j+1}-1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega. \end{aligned}$$

Hence, (4.3) coincides with

$$\beta_{j+1}(2\lambda_j + \beta_{j+1} - 1) D_{j+1}^{\beta_{j+1}-1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M.$$

If $i > j+1$, then

$$\begin{aligned} & D_{j+1,j}(x^{(2\varepsilon_j)}x_{j+1}) \cdot (D_i^{\beta_i} \cdot (x_i D_j) \cdot D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega - \beta_i D_j D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega) \\ = \quad & -\beta_i (D_j \cdot D_{j+1,j}(x^{(2\varepsilon_j)}x_{j+1}) - D_{j+1,j}(x_j x_{j+1})) D_i^{\beta_i-1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\ = \quad & \beta_i \lambda_j D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M. \end{aligned}$$

We get the desired results.

(6) In the special case, $\beta_i = p-2$, we obtain that

$$\begin{aligned} & D_{i+1,i}(x^{((p-1)\varepsilon_i)}x_{i+1}) \cdot D_i^{p-2} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\ = \quad & D_{i+1,i}(x^{((p-1)\varepsilon_i)}x_{i+1}) \cdot D_{i+1}^{p-1} D_i^{p-2} D_{i+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\ = \quad & (D_{i+1} \cdot D_{i+1,i}(x^{((p-1)\varepsilon_i)}x_{i+1}) - D_{i+1,i}(x^{((p-1)\varepsilon_i)})) D_{i+1}^{p-2} D_i^{p-2} D_{i+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\ = \quad & (D_{i+1}^{p-1} \cdot D_{i+1,i}(x^{((p-1)\varepsilon_i)}x_{i+1}) - x_i^{p-2} D_{i+1} \cdot D_{i+1}^{p-2}) D_i^{p-2} D_{i+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\ = \quad & (-\lambda_i + 1) D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega. \end{aligned}$$

From $\lambda_i \neq 1$, we obtain the desired identity, $D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$. □

4.3. Lemma. Suppose that $M := < u(S) D_m^{p-1} D_{m+1} \otimes \omega >$. If the following situations hold, respectively,

(1) $\lambda_m \neq 0, \beta_m = p-1$.

(2) $\lambda_m = 1, \beta_m \neq 1$.

(3) $\lambda_m = 0, \beta_m \neq p-1$.

(4) There exists $j \in Y_0$, $2 \leqslant j+1 < m$ such that $\beta_j = 0$ and $\lambda_j \neq 1$. In addition, for $m = j+1, \beta_m = p-1, \lambda_j \neq 1$, or $\beta_m \neq p-1, \lambda_j \neq 0$.

Then $D_m^{\beta_m - 1} D_{m+1} \otimes \omega \in M$.

(5) In addition, if $\lambda_m \neq 1, \beta_m = p-2$, then $D_{m+1} \otimes \omega \in M$.

Proof. The proof is completely analogous to the one given in Lemma 4.2. □

4.1. Theorem. $M_S(B(\lambda))$ is simple, if one of the following situations holds:

(1) $(\lambda_1, \cdots, \lambda_m) \neq \varepsilon_{i-1}$, for all $1 \leqslant i \leqslant m+1$, where $\varepsilon_i := (0, \cdots, 1, \cdots, 0) \in \mathbb{N}_0^m$, 1 occurs at the ith place, $\varepsilon_0 := (0, \cdots, 0, \cdots, 0)$.

(2) There exist $i, j \in Y_1$, such that $| j - i | > 1$ and $\lambda_i \neq 0, \lambda_j \neq 0$.

Proof. Let M' be a nonzero submodule of $M_S(B(\lambda))$. The similar discussion of the Lie superalgebra W applies to the Lie superalgebra S. There exists a maximal vector ω such that $\prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

For the case of the situation (1), in general case, we assume $\prod_{t=i}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$, for $1 \leqslant i \leqslant m-1$.

(i) $\lambda_i \neq 0$ and $\lambda_i \neq 1$.

By (1) and (6) in Lemma 4.2, we have $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(ii) $\lambda_i = 1$.

By (2) in Lemma 4.2, we have $D_i \prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$. For $(\lambda_1, \cdots, \lambda_m) \neq \varepsilon_i$, then there exists $j \in Y_0$, $j > i$ such that $\lambda_j \neq 0$, or $j < i$ such that $\lambda_j \neq 0$.

If $j > i$, $\lambda_j \neq 0$, by (4) in Lemma 4.2, we have $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

If $j + 1 < i$, $\lambda_j \neq 0$, by (5) in Lemma 4.2, we have $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

If $j + 1 = i$, $\lambda_j \neq 0$, we have

$$\begin{aligned} & D_{j+1,j}(x_j x^{(2\varepsilon_{j+1})}) \cdot D_i \prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \\ = \quad & (D_{j+1} \cdot D_{j+1,j}(x_j x^{(2\varepsilon_{j+1})}) - D_{j+1,j}(x_j x_{j+1})) \prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \\ = \quad & -\lambda_j \prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'. \end{aligned}$$

We can conclude that $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(iii) $\lambda_i = 0$.

Since $(\lambda_1, \cdots, \lambda_m) \neq \varepsilon_{i-1}$, there exists $j \in Y_0$, such that $j > i$, $\lambda_j \neq 0$, or $j + 1 < i$, $\lambda_j \neq 0$, or $j + 1 = i$, $\lambda_j \neq 1$. By (4) or (5) in Lemma 4.2, we have $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

The assumption of arbitrary of i implies that $D_m \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$. By Lemma 4.3, the similar discussion as before, we have $\prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

In general, we set $\prod_{l=j}^{s} D_l \otimes \omega \in M'$, for $j \in Y_1$. The situation (1) implies that there exists $i \in Y_0$, $1 \leqslant i \leqslant m$ such that $\lambda_i \neq 0$. Therefore, $\prod_{l=j}^{s} D_l \otimes \omega$ multiplied on the left by $D_{i+1,i}(x_i x_{i+1} x_j)$, we obtain

$$\begin{aligned} & D_{i+1,i}(x_i x_{i+1} x_j) \cdot D_j \cdots D_s \otimes \omega \\ = \quad & (-D_j \cdot D_{i,i+1}(x_i x_{i+1} x_j) + D_{i+1,i}(x_i x_{i+1})) D_{j+1} \cdots D_s \otimes \omega \\ = \quad & \lambda_i D_{j+1} \cdots D_s \otimes \omega \in M'. \end{aligned}$$

Then $D_{j+1} \cdots D_s \otimes \omega \in M'$ follows from $\lambda_i \neq 0$. According to the arbitrary $j \in Y_1$, as well as $1 \otimes \omega \in M'$, we obtain that $M_S(B(\lambda))$ is simple.

For the case of the situation (2), if $i \neq m+1 \in Y_1$, we have

$$
\begin{aligned}
& D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \\
= \; & -\prod_{t=1}^{m} D_t^{p-1} D_{m+1} \cdot D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{l=m+2}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} \cdot D_{i,i+1}(x_ix_{i+1}) \cdot \prod_{l=m+2}^{s} D_l \otimes \omega \\
= \; & \prod_{t=1}^{m} D_t^{p-1} (-1)^{i-m-1} D_{m+1} \cdots D_{i-1} \cdot D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{l=i}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{m+2} \cdots D_{i-1} \cdot D_{i,i+1}(x_ix_{i+1}) \cdot \prod_{l=i}^{s} D_l \otimes \omega \\
= \; & \prod_{t=1}^{m} D_t^{p-1} (-1)^{i-m} D_{m+1} \cdots D_i \cdot D_{i,i+1}(x_{m+1}x_{m+i}x_{i+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1} (-1)^{i+1-m} D_{m+1} \cdots D_{i-1} \cdot D_{i,i+1}(x_{m+1}x_{i+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{m+2} \cdots D_i \cdot D_{i,i+1}(x_ix_{i+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{m+2} \cdots D_{i-1} \cdot D_{i,i+1}(x_{i+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
= \; & \prod_{t=1}^{m} D_t^{p-1} (-1)^{i+1-m} D_{m+1} \cdots D_{i+1} \cdot D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1} (-1)^{i+1-m} D_{m+1} \cdots D_i \cdot D_{i,i+1}(x_{m+1}x_i) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1} (-1)^{i+2-m} D_{m+1} \cdots D_{i-1} \cdot D_{i,i+1}(x_{m+1}) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{m+2} \cdots D_{i+1} \cdot D_{i,i+1}(x_ix_{i+1}) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1} (-1)^{i+1-m} D_{m+1} \cdots \widehat{D_i} D_{i+1} \cdot D_{i,i+1}(x_{m+1}x_{i+1}) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1} D_{m+2} \cdots D_i \cdot D_{i,i+1}(x_i) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega - \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i+2}^{s} D_l \otimes \omega.
\end{aligned}
$$

Since the $\mathbb{Z}$-graded degree of $D_{i,i+1}(x_{m+1}x_ix_{i+1})$ is 1, it implies that the first term vanishes. The definition of a maximal vector ω implies that the second and the forth terms vanish. Finally, we obtain

$$\begin{aligned} & D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \\ = & -\prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+2}^{s} D_l \cdot D_{i,i+1}(x_ix_{i+1}) \otimes \omega \\ = & \lambda_i \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+2}^{s} D_l \otimes \omega. \end{aligned}$$

Since $\lambda_i \neq 0$, we have $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+2}^{s} D_l \otimes \omega \in M'$.

Multiply $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+2}^{s} D_l \otimes \omega$ by $D_{i,i+1}(x_{m+2}x_ix_{i+1}), \cdots, D_{i,i+1}(x_{i-1}x_ix_{i+1})$, in turn. The same calculation as above yields $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i}^{s} D_l \otimes \omega \in M'$, where $m+1 < i < s$.

If $m+1 \leqslant j < i-1 \leqslant s$, then we have

$$\begin{aligned} & D_{j,j+1}(x_ix_jx_{j+1}) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i}^{s} D_l \otimes \omega \\ = & \prod_{t=1}^{m} D_t^{p-1} (-1)^{s-i+1} D_iD_{i+1} \cdots D_s \cdot D_{j,j+1}(x_ix_jx_{j+1}) \otimes \omega \\ & -\prod_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_s \cdot D_{j,j+1}(x_jx_{j+1}) \otimes \omega. \end{aligned}$$

It also can be found that the first term vanishes. By the assumption of the Theorem 4.1, we have $\lambda_j \neq 0$. Hence, from the second term we can conclude that $\prod\limits_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_s \otimes \omega \in M'$. Multiplying $\prod\limits_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_s \otimes \omega$ on the left by $D_{j,j+1}(x_{i+1}x_jx_{j+1}), \cdots, D_{j,j+1}(x_sx_jx_{j+1})$, we obtain that $\prod\limits_{t=1}^{m} D_t^{p-1} \otimes \omega \in M'$.

If $m+1 \leqslant i+1 < j \leqslant s$, then we have

$$
\begin{aligned}
& D_{j,j+1}(x_i x_j x_{j+1}) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i}^{s} D_l \otimes \omega \\
= \; & \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i} D_i \cdots D_{j-1} \cdot D_{j,j+1}(x_i x_j x_{j+1}) \cdot \prod_{l=j}^{s} D_l \otimes \omega \\
& - \prod_{t=1}^{m} D_t^{p-1} \cdot D_{j,j+1}(x_j x_{j+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
= \; & \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i+1} D_i \cdots D_j \cdot D_{j,j+1}(x_i x_j x_{j+1}) \cdot \prod_{l=j+1}^{s} D_l \otimes \omega \\
& + \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i} D_i \cdots D_{j-1} \cdot D_{j,j+1}(x_i x_{j+1}) \cdot \prod_{l=j+1}^{s} D_l \otimes \omega \\
& - \prod_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_{j-1} D_j \cdot D_{j,j+1}(x_j x_{j+1}) \cdot \prod_{l=j+1}^{s} D_l \otimes \omega \\
& - \prod_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_{j-1} \cdot D_{j,j+1}(x_{j+1}) \cdot \prod_{l=j+1}^{s} D_l \otimes \omega \\
= \; & \prod_{t=1}^{m} D_t^{p-1} (-1)^{s-i+1} \prod_{l=i}^{s} D_l \cdot D_{j,j+1}(x_i x_j x_{j+1}) \cdot \otimes \omega \\
& + \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i+2} D_i \cdots \widehat{D_{j+1}} \cdots D_s \cdot D_{j,j+1}(x_i x_j) \otimes \omega \\
& + \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i+1} D_i \cdots \widehat{D_j} \cdots D_s \cdot (x_i D_j) \otimes \omega \\
& - \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i+1}^{s} D_l \cdot D_{j,j+1}(x_j x_{j+1}) \otimes \omega \\
= \; & \lambda_j \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i+1}^{s} D_l \otimes \omega .
\end{aligned}
$$

Obviously, we can obtain $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i+1}^{s} D_l \otimes \omega \in M'$, for $\lambda_j \neq 0$. Similarly, considering $D_{j,j+1}(x_{i+1} x_j x_{j+1}) \cdot \prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i+1}^{s} D_l \otimes \omega$, for $j \neq i+1$, it implies that $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i+2}^{s} D_l \otimes \omega \in M'$. Continue to multiply $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i+2}^{s} D_l \otimes \omega \in M'$ on the left by $D_{i,i+1}(x_{i+2} x_i x_{i+1}), \cdots,$ $D_{i,i+1}(x_s x_i x_{i+1})$, consecutively. Finally, we obtain $\prod\limits_{t=1}^{m} D_t^{p-1} \otimes \omega \in M'$.

If $i = m+1$, we have $D_{j,j+1}(x_{m+1} x_j x_{j+1}) \cdot \prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega = \lambda_j \prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+2}^{s} D_l \otimes \omega \in M'$, furthermore, $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+2}^{s} D_l \otimes \omega \in M'$. Imitating the process of calculation above for $i \neq m+1$, we have $\prod\limits_{t=1}^{m} D_t^{p-1} \otimes \omega \in M'$.

Using the fact that there exists $j \in Y_1$ such that $\lambda_j \neq 0$, and we see, for $i \in Y_0$,

$$\begin{aligned} & D_{j,j+1}(x_i x_j x_{j+1}) \cdot \prod_{t=i}^{m} D_t^{p-1} \otimes \omega \\ = \quad & (D_i \cdot D_{j,j+1}(x_i x_j x_{j+1}) - D_{j,j+1}(x_j x_{j+1})) D_i^{p-2} \prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega \\ = \quad & \lambda_j D_i^{p-2} \prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega. \end{aligned}$$

$(p-1)$-fold multiplication with $\prod\limits_{t=i}^{m} D_t^{p-1} \otimes \omega$ implies that $\prod\limits_{t=i+1}^{m} D_t^{p-1} \otimes \omega \in M'$. In general case of i, it entails $1 \otimes \omega \in M'$. $M_S(B(\lambda))$ is simple as desired. $\square$

5. The simple module of Lie superalgebra $H(m, n, \underline{1})$

In this section, we consider the Lie superalgebra $H(m, n, \underline{1})$. First, we suppose that $n = 2q$ is an even number.

5.1. Lemma. $A = \langle \{-D_H(x_i x_{i'}), \mu D_H(x_{j^v} x_j) \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle$ is a Cartan subalgebra of H_0, where $\mu^2 = -1$, $j^v = \begin{cases} j+q, & m+1 \leqslant j \leqslant m+q \\ j-q, & m+q+1 \leqslant j \leqslant m+2q \end{cases}$. The positive root vectors of H_0 are $\{D_H(-x_i x_{j'}), 1 \leqslant i < j \leqslant r; D_H(x_i x_j), 1 \leqslant i < j \leqslant r; D_H(x_i^2), 1 \leqslant i \leqslant r; D_H(-\frac{1}{2} x_i x_j - \frac{\mu}{2} x_i x_{j^v}), 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; D_H(-x_i x_j + \mu x_i x_{j^v}), 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; D_H(\frac{1}{2} x_j x_i + \frac{\mu}{2} x_{j^v} x_i + \frac{\mu}{2} x_{i^v} x_j + \frac{1}{2} x_{j^v} x_{i^v}), m+1 \leqslant i < j \leqslant m+q; D_H(x_j x_i + \mu x_{i^v} x_j + \mu x_i x_{j^v} + x_{i^v} x_{j^v}), m+1 \leqslant i < j \leqslant m+q\}$.

Proof. Let $\varphi : H_0 \to \mathscr{L} := \{\left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right) \in pl(m,n) \mid A_1^t G + G A_1 = 0, B_1^t G + C_1 = 0, D_1^t + D_1 = 0\}$ be a homomorphism of Lie superalgebras such that $\varphi(D_H(x_i D_j)) = \sigma(j)(-1)^{\tau(j)}(E_{ij'} + \sigma(i)\sigma(j)(-1)^{\tau(i)\tau(j)+\tau(i)+\tau(j)} E_{ji'})$, where $G = \left(\begin{smallmatrix} & I_r \\ -I_r & \end{smallmatrix}\right)$, I_r is the $r \times r$ identity matrix, $pl(m,n) := pl_{\bar{0}}(m,n) \oplus pl_{\bar{1}}(m,n)$, for $pl_{\bar{0}}(m,n) := \{\left(\begin{smallmatrix} A_1 & 0 \\ 0 & D_1 \end{smallmatrix}\right) \mid A_1$ is the $m \times m$ matrix over $\mathbb{F}$, D_1 is the $n \times n$ matrix over $\mathbb{F}\}$, $pl_{\bar{1}}(m,n) := \{\left(\begin{smallmatrix} 0 & B_1 \\ C_1 & 0 \end{smallmatrix}\right) \mid B_1$ is the $m \times n$ matrix over $\mathbb{F}$, C_1 is the $n \times m$ matrix over $\mathbb{F}\}$. It can be checked easily that $H_0 \cong \mathscr{L} \cong \mathscr{L}(P) := \{P^{-1} E P \mid E \in \mathscr{L}\}$, where $P := \left(\begin{smallmatrix} I_m & 0 \\ 0 & P_n \end{smallmatrix}\right)$, $P_n := \left(\begin{smallmatrix} I_q & \frac{1}{2} I_q \\ -\mu I_q & \frac{\mu}{2} I_q \end{smallmatrix}\right)$. By straightforward computation we find that the Cartan subalgebra of $\mathscr{L}$ is

$$\langle \{\left(\begin{smallmatrix} E_{ii} - E_{i'i'} & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & E_{jj} - E_{j^v, j^v} \end{smallmatrix}\right) \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle,$$

and the positive root vectors are
$\{\left(\begin{smallmatrix} E_{ij} - E_{j'i'} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for $1 \leqslant i < j \leqslant r$; $\left(\begin{smallmatrix} E_{ij'} + E_{ji'} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for $1 \leqslant i < j \leqslant r$;
$\left(\begin{smallmatrix} E_{ii'} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for $1 \leqslant i \leqslant r$; $\left(\begin{smallmatrix} 0 & E_{ij} \\ -E_{j^v i'} & 0 \end{smallmatrix}\right)$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q$;
$\left(\begin{smallmatrix} 0 & E_{ij^v} \\ -E_{ji'} & 0 \end{smallmatrix}\right)$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q$; $\left(\begin{smallmatrix} 0 & 0 \\ 0 & E_{ij} - E_{j^v i^v} \end{smallmatrix}\right)$, for $m+1 \leqslant i < j \leqslant m+q$;
$\left(\begin{smallmatrix} 0 & 0 \\ 0 & E_{ij^v} - E_{ji^v} \end{smallmatrix}\right)$, for $m+1 \leqslant i < j \leqslant m+q\}$

By the isomorphism φ, we can show that the Cartan subalgebra of H_0 is $A = \langle \{-D_H(x_i x_{i'}), \mu D_H(x_{j^v} x_j) \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle$ and the set of positive root vectors of H_0 are $\{D_H(-x_i x_{j'}), 1 \leqslant i < j \leqslant r; D_H(x_i x_j), 1 \leqslant i < j \leqslant r; D_H(x_i^2), 1 \leqslant i \leqslant r; D_H(-\frac{1}{2} x_i x_j - \frac{\mu}{2} x_i x_{j^v}), 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; D_H(-x_i x_j + \mu x_i x_{j^v}), 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; D_H(\frac{1}{2} x_j x_i + \frac{\mu}{2} x_{j^v} x_i + \frac{\mu}{2} x_{i^v} x_j + \frac{1}{2} x_{j^v} x_{i^v}), m+1 \leqslant i < j \leqslant m+q; D_H(x_j x_i + \mu x_{i^v} x_j + \mu x_i x_{j^v} + x_{i^v} x_{j^v}), m+1 \leqslant i < j \leqslant m+q\}$. $\square$

From the Definition 2.1 and Lemma 5.1, we can get $V_\lambda = \{y \in V \mid D_H(x_i x_{i'}) \cdot y = \lambda_i y, D_H(x_j x_{j^v}) \cdot y = \lambda_j y, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\}$, where y is a maximal vector

and satisfies the following statements:
$D_H(x_ix_j)\cdot y=0$, for $1\leqslant i,j\leqslant r$, or $1\leqslant i\leqslant r, i'<j\leqslant 2r$;
$D_H(x_ix_j)\cdot y=0$, for $1\leqslant i\leqslant r, m+1\leqslant j\leqslant s$;
$D_H(x_ix_j+\mu x_jx_{i^v})\cdot y=0$, for $m+1\leqslant i<j\leqslant m+q$;
$D_H(x_ix_j-\mu x_jx_{i^v})\cdot y=0$, for $m+q+1\leqslant i<j\leqslant s$.

5.2. Lemma. Suppose that $M:=<u(H)D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega>$, where $1\leqslant i\leqslant r$. If the following situations hold, respectively,

(1) $\lambda_i\neq 0, \beta_i=p-1$.
(2) $\lambda_i=0, \beta_i\neq p-1$.
(3) $\lambda_i=-1, \beta_i\neq 1$.
(4) There exists $j\in Y_0$, $1\leqslant i<j\leqslant r$, such that $\lambda_j\neq 0$ and $\beta_j=p-1$.
(5) There exists $j\in Y_0$, $1\leqslant j<i\leqslant r$, such that $\lambda_j\neq -1$ and $\beta_j=0$.

Then $D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

(6) If $\lambda_i\neq -1, \beta_i=p-2$, then $D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

Proof. By virtue of $[D_j,D_H(f)]=D_H(D_j(f))$, for $j\in Y$, we obtain

$$\begin{aligned}
&D_H(x^{(2\varepsilon_i)}x_{i'})\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&(D_{i'}\cdot D_H(x^{(2\varepsilon_i)}x_{i'})-D_H(x^{(2\varepsilon_i)}))\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{i'-1}^{p-1}D_{i'}^{p-2}D_{i'+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&D_{i'}^{p-1}\cdot D_H(x^{(2\varepsilon_i)}x_{i'})\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
&+D_{i'}^{p-2}\cdot D_H(x^{(2\varepsilon_i)})\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
=\;&D_{i'}^{p-1}(D_i\cdot D_H(x^{(2\varepsilon_i)}x_{i'})-D_H(x_ix_{i'}))\cdot D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
&+D_{i'}^{p-2}\cdot(D_i\cdot D_H(x^{(2\varepsilon_i)})-D_H(x_i))\cdot D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
=\;&D_{i'}^{p-1}D_i^2\cdot D_H(x^{(2\varepsilon_i)}x_{i'})\cdot D_i^{\beta_i-2}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
&-D_{i'}^{p-1}D_i\cdot D_H(x_ix_{i'})\cdot D_i^{\beta_i-2}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
&-D_{i'}^{p-1}\cdot D_H(x_ix_{i'})\cdot D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
&+D_{i'}^{p-2}D_i^{\beta_i}\cdot D_H(x^{(2\varepsilon_i)})\cdot D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
&-\beta_iD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega, \qquad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
&D_H(x_ix_{i'})\cdot D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
=\;&(D_i\cdot D_H(x_ix_{i'})-D_H(x_{i'}))\cdot D_i^{\beta_i-2}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
=\;&D_i\cdot D_H(x_ix_{i'})\cdot D_i^{\beta_i-2}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
&+D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
=\;&D_i^{\beta_i-1}\cdot D_H(x_ix_{i'})\cdot D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
&+(\beta_i-1)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega. \qquad (5.2)
\end{aligned}$$

By (5.2), we find that (5.1) equals

$$\begin{aligned}
&(-(\lambda_i+\beta_i-1)-(\lambda_i+\beta_i-2)\cdots-(\lambda_i+\beta_i-\beta_i)-\beta_i)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&-\frac{\beta_i}{2}(2\lambda_i+\beta_i+1)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega
\end{aligned}$$

With the conditions (1), (2) or (3), respectively, we have $D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

(4) In situation (4), we have

$$D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_{j'}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.$$

Hence,

$$\begin{aligned}
& D_H(x_{j'}x_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_{j'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & (D_{j'}\cdot D_H(x_{j'}x_i)-D_H(x_i))D_{j'}^{p-2}D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots\widehat{D_{j'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
= & D_{j'}^{p-1}\cdot D_H(x_{j'}x_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots\widehat{D_{j'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
= & D_{j'}^{p-1}(D_i\cdot D_H(x_{j'}x_i)-D_H(x_{j'}))D_i^{\beta_i-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots\widehat{D_{j'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
= & \beta_i D_i^{\beta_i-1}\cdots D_j^{p-1}\cdots D_{i'}^{p-1}\cdots D_{j'}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.
\end{aligned}$$

The assertion follows from the above equation.

(5) By a straightforward calculation, we obtain

$$\begin{aligned}
& D_H(x_{j'}x_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & (D_i\cdot D_H(x_{j'}x_i)-D_H(x_{j'}))D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j'-1}^{p-1}\cdot D_H(x_{j'}x_i)\cdot D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
& +\beta_i D_j D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j'-1}^{p-1}(D_{j'}\cdot D_H(x_{j'}x_i)-D_H(x_i))D_{j'}^{p-2}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
& +\beta_i D_j D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\cdot D_H(x_{j'}x_i)\otimes\omega\\
& +\beta_i D_j D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.
\end{aligned}\tag{5.3}$$

(5.3) multiplied by $D_H(x^{(2\varepsilon_j)}x_{j'})$, then we have $-(\lambda_j+1)\beta_i D_i^{\beta_i-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$. By the situation (5), we have the desired result.

(6) In the particular case, $\beta_i=p-2$, we obtain

$$\begin{aligned}
& D_H(x^{((p-1)\varepsilon_i)}x_{i'})\cdot D_i^{p-2}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & -(\lambda_i+1)D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega.
\end{aligned}$$

For $\lambda_i\neq-1$, we obtain the asserted result, i.e., $D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$. □

5.3. Lemma. Suppose that $M:=<u(H)D_{r+1}^{p-1}\cdots D_{i'-1}^{p-1}D_{i'}^{\beta_{i'}}\otimes\omega>$, for $1\leqslant i\leqslant r$. If the following situations hold, respectively,

(1) $\lambda_i\neq-1,\beta_{i'}=p-1$.

(2) $\lambda_i=-1,\beta_{i'}\neq p-1$.

(3) $\lambda_i=0,\beta_{i'}\neq 1$.

(4) There exists $j'\in Y_0$, $r+1\leqslant j'<i'\leqslant m$ such that $\lambda_j\neq-1$ and $\beta_{j'}=p-1$.

(5) There exists $j'\in Y_0$, $r+1\leqslant i'<j'\leqslant m$ such that $\lambda_j\neq 0$ and $\beta_{j'}=0$. Then $D_{r+1}^{p-1}\cdots D_{i'-1}^{p-1}D_{i'}^{\beta_{i'}-1}\otimes\omega\in M$.

(6) If $\lambda_i\neq 0,\beta_{i'}=p-2$, then $D_{r+1}^{p-1}\cdots D_{i'+1}^{p-1}\otimes\omega\in M$.

Proof. For $1 \leqslant i \leqslant r$,

$$\begin{aligned} & D_H(x_i x^{(2\varepsilon_{i'})}) \cdot D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}} \otimes \omega \\ = \;& (D_{i'} \cdot D_H(x_i x^{(2\varepsilon_{i'})}) - D_H(x_i x_{i'})) D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \\ = \;& \frac{\beta_{i'}}{2}(-2\lambda_i + \beta_{i'} - 1) D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega. \end{aligned} \tag{5.4}$$

From (5.4), with the situations (1), (2) or (3), respectively, we can conclude $D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M$.

(4) We see that $D_{r+1}^{p-1} \cdots D_{j'-1}^{p-1} D_{j'}^{p-2} D_{j'+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M$. Hence, we have

$$\begin{aligned} & D_H(x_j x_{i'}) \cdot D_{r+1}^{p-1} \cdots D_{j'-1}^{p-1} D_{j'}^{p-2} D_{j'+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}} \otimes \omega \\ = \;& D_{j'}^{p-2}(D_{i'} \cdot D_H(x_j x_{i'}) - D_H(x_j)) D_{r+1}^{p-1} \cdots D_{j'-1}^{p-1} \widehat{D_{j'}^{p-2}} D_{j'+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \\ = \;& -\beta_{i'} D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M. \end{aligned}$$

(5) For $i < j$, we have

$$\begin{aligned} & D_H(x_j x_{i'}) \cdot D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}} \otimes \omega \\ = \;& D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}} \cdot D_H(x_j x_{i'}) \otimes \omega - \beta_{i'} D_{j'} D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M. \end{aligned} \tag{5.5}$$

(5.5) multiplied by $D_H(x_j x^{(2\varepsilon_{j'})})$, we have $\beta_{i'} \lambda_j D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M$.

(6) In particular, $\beta_i = p-2$, we see that

$$\begin{aligned} & D_H(x^{((p-1)\varepsilon_{i'})} x_i) \cdot D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{p-2} \otimes \omega \\ = \;& -\lambda_i D_{r+1}^{p-1} \cdots D_{i'+1}^{p-1} \otimes \omega \in M. \end{aligned}$$

□

5.1. Theorem. $M_H(B(\lambda))$ is simple, if $(\lambda_1, \cdots, \lambda_r) \neq \sum_{t=1}^{k-1} -\varepsilon_t$, (the empty sum being zero) for all $1 \leqslant k \leqslant r+1$, where $\varepsilon_t := (0, \cdots, 1, \cdots, 0) \in \mathbb{N}_0^r$, 1 occurs at the tth place.

Proof. Let M' be a nonzero submodule of $M_H(B(\lambda))$. In the analogous proof for W described above, we can get $\prod_{t_1=1}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

First, without loss of generality, we assume $\prod_{t_1=i}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$, for $1 \leqslant i \leqslant r$.

(i) If $\lambda_i \neq 0$ and $\lambda_i \neq -1$. By (1) and (6) in Lemma 5.2, then we conclude that $\prod_{t_1=i+1}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(ii) If $\lambda_i = -1$, and $(\lambda_1, \cdots, \lambda_r) \neq \prod_{k=1}^{i} -\varepsilon_k$, then there exists $j \in Y_0$, such that $1 \leqslant j < i \leqslant r$ and $\lambda_j \neq -1$, or $i < j \leqslant r$, such that $\lambda_j \neq 0$. By (4) or (5) of Lemma 5.2, we get $\prod_{t_1=i+1}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(iii) If $\lambda_i = 0$, and $(\lambda_1, \cdots, \lambda_r) \neq \prod_{k=1}^{i-1} -\varepsilon_k$, then there exists $j \in Y_0$, such that $1 \leqslant j \leqslant i-1$, $\lambda_j \neq -1$, or $1 \leqslant i < j \leqslant r$, such that $\lambda_j \neq 0$. By (4) or (5) of Lemma

5.2, we get $\prod_{t_1=i+1}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$. Because i is arbitrary, we know $\prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

We assume $\prod_{t_2=r+1}^{i'} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega$, for $1 \leqslant i \leqslant r$.

(i) If $\lambda_i \neq 0$ and $\lambda_i \neq -1$, by (1) and (6) of Lemma 5.3, we conclude that $\prod_{t_2=r+1}^{i'+1} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(ii) If $\lambda_i = -1$, and $(\lambda_1, \cdots, \lambda_r) \neq \prod_{k=1}^{i} -\varepsilon_k$, then there exists j, $1 \leqslant j < i \leqslant r$ such that $\lambda_j \neq -1$, or $i < j \leqslant r$ such that $\lambda_j \neq 0$. By (4) or (5) of Lemma 5.3, we conclude that $\prod_{t_2=r+1}^{i'+1} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(iii) If $\lambda_i = 0$, and $\lambda \neq \prod_{k=1}^{i-1} -\varepsilon_k$, then there exists j, $1 \leqslant j < i \leqslant r$ such that $\lambda_j \neq -1$, or $i < j \leqslant r$, such that $\lambda_j \neq 0$. By (4) or (5) of Lemma 5.3, we conclude that $\prod_{t_2=r+1}^{i'+1} D_{t_2}^{p-1} \prod_{j=m+1}^{s} D_j \otimes \omega \in M'$. Because i is arbitrary, we know $\prod_{l=m+1}^{s} D_l \otimes \omega \in M'$. The condition (1) implies that there exists $i \in Y_0$, such that $\lambda_i \neq 0$. Then, for $j \in Y_1$,

$$
\begin{aligned}
& D_H(x_i x_{i'} x_j) \cdot D_j \cdots D_s \otimes \omega \\
= \;& (-D_j \cdot D_H(x_i x_{i'} x_j + D_H(x_i x_{i'}))D_{j+1} \cdots D_s \otimes \omega \\
= \;& \lambda_i D_{j+1} \cdots D_s \otimes \omega.
\end{aligned}
$$

Along with the arbitrary j, it yields $1 \otimes \omega \in M'$. Then $M_H(B(\lambda))$ is simple. $\square$

For the case of $n = 2q+1$, we substitute $P_n := \begin{pmatrix} I_q & \frac{1}{2} I_q \\ -\mu I_q & \frac{\mu}{2} I_q \end{pmatrix}$ in Lemma 5.1 with $P_n := \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_q & \frac{1}{2} I_q \\ 0 & -\mu I_q & \frac{\mu}{2} I_q \end{pmatrix}$. The remaining proof is treated similarly as above.

6. The simple module of Lie superalgebra $K(m, n, \underline{1})$

We first consider the case $n = 2q$.

6.1. Lemma. $A = \langle \{-x_i x_{i'}, x_m, \mu x_{j^v} x_j \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle$ is a Cartan subalgebra of K_0, where $\mu^2 = -1$, $j^v = \begin{cases} j+q, & m+1 \leq j \leq m+q \\ j-q, & m+q+1 \leq j \leq m+2q \end{cases}$. The positive root vectors of K_0 are $\{-x_i x_{j'}, 1 \leqslant i < j \leqslant r; x_i x_j, 1 \leqslant i < j \leqslant r; x_i^2, 1 \leqslant i \leqslant r; -\frac{1}{2} x_i x_j - \frac{\mu}{2} x_i x_{j^v}, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; -x_i x_j + \mu x_i x_{j^v}, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; \frac{1}{2} x_j x_i + \frac{\mu}{2} x_{j^v} x_i + \frac{\mu}{2} x_{i^v} x_j + \frac{1}{2} x_{j^v} x_{i^v}, m+1 \leqslant i < j \leqslant m+q; x_j x_i + \mu x_{i^v} x_j + \mu x_i x_{j^v} + x_{i^v} x_{j^v}, m+1 \leqslant i < j \leqslant m+q\}$.

Proof. Let $\varphi : K_0 \to \mathscr{L} = \{\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in pl(m-1, n) \mid A_1^t G + G A_1 = 0, B_1^t G + C_1 = 0, D_1^t + D_1 = 0\}$ be a mapping of vector spaces, given by

$$x_i x_j \mapsto \sigma(j)(-1)^{\tau(j)}(E_{ij'} + \sigma(i)\sigma(j)(-1)^{\tau(i)\tau(j)+\tau(i)+\tau(j)} E_{ji'}), (1 \leqslant i < j \leqslant s, i, j \neq m)$$

$$x_m \mapsto 1 \in \mathbb{F},$$

where $G = \begin{pmatrix} & I_r \\ -I_r & \end{pmatrix}$, $pl(m-1, n) := pl_{\bar{0}}(m-1, n) \oplus pl_{\bar{1}}(m-1, n)$, for $pl_{\bar{0}}(m-1, n) := \{\begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix} \mid A_1$ is the $(m-1) \times (m-1)$ matrix over $\mathbb{F}$, D_1 is the $n \times n$ matrix over $\mathbb{F}\}$, $pl_{\bar{1}}(m-1, n) := \{\begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix} \mid B_1$ is the $(m-1) \times n$ matrix over $\mathbb{F}$, C_1 is the $n \times (m-$

1) matrix over $\mathbb{F}\}$. It is obvious that $K_0 \cong \mathscr{L}\oplus\mathbb{F} \cong \mathscr{L}(P)\oplus\mathbb{F} := \{P^{-1}EP \mid E \in \mathscr{L}\}\oplus\mathbb{F}$, where $P := \begin{pmatrix} I_m & 0 \\ 0 & P_n \end{pmatrix}$, $P_n := \begin{pmatrix} I_q & \frac{1}{2}I_q \\ -\mu I_q & \frac{\mu}{2}I_q \end{pmatrix}$. By a straight computation, we get the Cartan subalgebra of $\mathscr{L}$ as

$$\langle\{\begin{pmatrix} E_{ii}-E_{i'i'} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & E_{jj}-E_{j^v,j^v} \end{pmatrix} \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\}\rangle,$$

and the positive root vectors are
$\{\begin{pmatrix} E_{ij}-E_{j'i'} & 0 \\ 0 & 0 \end{pmatrix}$, for $1 \leqslant i < j \leqslant r$; $\begin{pmatrix} E_{ij'}+E_{ji'} & 0 \\ 0 & 0 \end{pmatrix}$ for $1 \leqslant i < j \leqslant r$; $\begin{pmatrix} E_{ii'} & 0 \\ 0 & 0 \end{pmatrix}$, for $1 \leqslant i \leqslant r$; $\begin{pmatrix} 0 & E_{ij} \\ -E_{j^v i'} & 0 \end{pmatrix}$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q$; $\begin{pmatrix} 0 & E_{ij^v} \\ -E_{ji'} & 0 \end{pmatrix}$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q$; $\begin{pmatrix} 0 & 0 \\ 0 & E_{ij}-E_{j^v i^v} \end{pmatrix}$, for $m+1 \leqslant i < j \leqslant m+q$; $\begin{pmatrix} 0 & 0 \\ 0 & E_{ij^v}-E_{ji^v} \end{pmatrix}$, for $m+1 \leqslant i < j \leqslant m+q\}$

By the isomorphism φ, we can get the Cartan subalgebra of K_0 is $A = \langle\{-x_i x_{i'}, x_m, \mu x_{j^v} x_j \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\}\rangle$, where $\mu^2 = -1$, $j^v = \begin{cases} j+q, & m+1 \leq j \leq m+q \\ j-q, & m+q+1 \leq j \leq m+2q \end{cases}$.
Also, the positive root vectors of K_0 are $\{-x_i x_{j'}, 1 \leqslant i < j \leqslant r; x_i x_j, 1 \leqslant i < j \leqslant r); x_i^2, 1 \leqslant i \leqslant r; -\frac{1}{2}x_i x_j - \frac{\mu}{2}x_i x_{j^v}, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; -x_i x_j + \mu x_i x_{j^v}, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; \frac{1}{2}x_j x_i + \frac{\mu}{2}x_{j^v} x_i + \frac{\mu}{2}x_{i^v} x_j + \frac{1}{2}x_{j^v} x_{i^v}, m+1 \leqslant i < j \leqslant m+q; x_j x_i + \mu x_{i^v} x_j + \mu x_i x_{j^v} + x_{i^v} x_{j^v}, m+1 \leqslant i < j \leqslant m+q\}$. □

With respect to the Definition 2.1 and Lemma 6.1, we can get $V_\lambda = \{y \in V \mid (x_i x_{i'})\cdot y = \lambda_i y, x_m \cdot \omega = \lambda_m y, (x_j x_{j^v})\cdot y = \lambda_j y, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\}$, where y is a maximal vector and satisfies the following conditions,
$(x_i x_j)\cdot y = 0$, for $1 \leqslant i, j \leqslant r$, or $1 \leqslant i \leqslant r, i' < j \leqslant 2r$;
$(x_i x_j)\cdot y = 0$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant s$;
$(x_i x_j + \mu x_j x_{i^v})\cdot y = 0$, for $m+1 \leqslant i < j \leqslant m+q$;
$(x_i x_j - \mu x_j x_{i^v})\cdot y = 0$, for $m+q+1 \leqslant i < j \leqslant s$.

6.1. Theorem. $M_K(B(\lambda))$ is simple, if $(\lambda_1, \cdots, \lambda_r, \lambda_m) \neq \zeta_k + (\pm k - r - 1)\varepsilon_m$, for $1 \leqslant k \leqslant r+1$, where $\zeta_k = -\sum_{i=1}^{r-k+1} \varepsilon_i$ (the empty sum being zero), $\varepsilon_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{N}_0^{r+1}$, 1 occurs at the ith place, for $1 \leqslant i \leqslant r+1$.

Proof. Let M' be a nonzero submodule of $M_K(B(\lambda))$. Take $a \in M'$ and $a \neq 0$. We note

$$a = \Sigma_{\beta\in\mathscr{A}} c(\beta) i(x_1)^{\beta_1} \cdots \widehat{i(x_m)^{\beta_m}} \cdots i(x_s)^{\beta_s} x_0^{\beta_0} \otimes b_\beta,$$

where $\beta = (\beta_1, \cdots, \widehat{\beta_m}, \cdots, \beta_s, \beta_0)$, $c(\beta) \in \mathbb{F}$, $\mathscr{A} := \{a = \Sigma_k \beta_k \varepsilon_k \mid 0 \leqslant \beta_k \leqslant p-1$ for $1 \leqslant k \leqslant m-1$; $\beta_k = 0$ or 1 for $m+1 \leqslant k \leqslant s\} \subset \mathbb{Z}^{s-1}$. Write $i(x_j) = x_j, i(1) = x_0$ in $u(K)$.

Put $\alpha_0 = \min\{\beta_0 \mid a = \Sigma_{\beta\in\mathscr{A}} c(\beta) x_1^{\beta_1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{\beta_0} \otimes b_\beta, c(\beta) \neq 0\}$. Since $[1, x^\alpha] = x^{\alpha-\varepsilon_m}$, $x_0 x_\alpha = x_\alpha x_0 + x^{\alpha-\varepsilon_m}$ holds in $u(K)$. We can get
$x_0^{p-1-\alpha_0}\cdot v = \Sigma_{\beta'\in\mathscr{A}} c(\beta') x_1^{\beta_1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta'} \in M'$, where $\beta' = (\beta_1, \cdots, \widehat{\beta_m}, \cdots, \beta_s, \alpha_0)$.
Put $\alpha_1 := \min\{\beta_1 \mid a = \Sigma_{\beta'\in\mathscr{A}} c(\beta') x_1^{\beta_1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta'}, c(\beta') \neq 0\}$. Multiplying $\Sigma_{\beta'\in\mathscr{A}} c(\beta') x_1^{\beta_1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta'}$ by $x_1^{p-1-\alpha_1}$, we can obtain
$\Sigma_{\beta''\in\mathscr{A}} c(\beta'') x_1^{p-1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta''} \in M'$, where $\beta'' = (\alpha_1, \beta_2, \cdots, \widehat{\beta_m}, \cdots \beta_s, \alpha_0)$.
Eventually, we can conclude that

$$\Sigma_{\beta^{(m)}\in\mathscr{A}} c(\beta^{(m)}) x_1^{p-1} \cdots x_{m-1}^{p-1} \widehat{x_m^{\beta_m}} x_{m+1}^{\beta_{m+1}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta^{(m)}} \in M', \tag{6.1}$$

where $\beta^{(m)} = (\alpha_1, \alpha_2, \cdots, \alpha_{m-1}, \widehat{\beta_m}, \beta_{m+1}, \cdots, \beta_s, \alpha_0)$.

Put α_{m+1} :=min$\{\beta_{m+1} \mid a = \Sigma_{\beta^{(m)}\in\mathscr{A}} c(\beta^{(m)}) x_1^{p-1}\cdots x_{m-1}^{p-1}\widehat{x_m^{\beta_m}} x_{m+1}^{\beta_{m+1}}\cdots x_s^{\beta_s} x_0^{p-1}\otimes b_{\beta^{(m)}},\ c(\beta^{(m)})\neq 0\}$. Multiplying (6.1) by $x_{m+1}^{1-\alpha_{m+1}}$, we can obtain $\Sigma_{\beta^{(m+1)}\in\mathscr{A}} c(\beta^{(m+1)}) x_1^{p-1}\cdots x_{m-1}^{p-1}\widehat{x_m^{\beta_m}} x_{m+1}\cdots x_s^{\beta_s} x_0^{p-1}\otimes b_{\beta^{(m)}} \in M'$. Finally, we can conclude that there exists η such that $c(\eta)x_1^{p-1}\cdots x_{m-1}^{p-1}\widehat{x_m^{p-1}} x_{m+1}\cdots x_s x_0^{p-1}\otimes b_\eta \in M'$, where $\eta = (\alpha_1,\alpha_2,\cdots,\widehat{\alpha_m},\alpha_{m+1},\cdots,\alpha_s,\alpha_0)$, $c(\eta)\neq 0$. Similarly to the discussion for W, there exists a maximal vector ω such that $\prod\limits_{t=0}^{m-1} x_t^{p-1}\prod\limits_{l=m+1}^{s} x_l\otimes\omega\in M'$. By Lemma 2.12 and Lemma 2.13 of [9], we can get $\prod\limits_{l=m+1}^{s} x_l\otimes\omega\in M'$. By our assumption, we know that there exists i, for $1\leqslant i\leqslant r$ or $i=m$, such that $\lambda_i\neq 0$.

We assume $\prod\limits_{l=j}^{s} x_l\otimes\omega\in M'$, for $j\in Y_1$.

If $\lambda_i\neq 0$, for $1\leqslant i\leqslant r$, we have

$$\begin{aligned} &(x_i x_{i'} x_j)\cdot\prod_{l=j}^{s} x_l\otimes\omega \\ =\ & (-x_j\cdot(x_i x_{i'} x_j) - x_i x_{i'})\prod_{l=j+1}^{s} x_l\otimes\omega \\ =\ & -\lambda_i\prod_{l=j+1}^{s} x_l\otimes\omega. \end{aligned}$$

With the generality of $j\in Y_1$, we get $1\otimes\omega\in M'$.

If $\lambda_m\neq 0$, then

$$\begin{aligned} &(x_{m+1}\cdots x_s x_m)\cdot\prod_{l=m+1}^{s} x_l\otimes\omega \\ =\ & (x_{m+1}\cdot(x_{m+1}\cdots x_s x_m) + (x_{m+2}\cdots x_s x_m))\prod_{l=m+2}^{s} x_l\otimes\omega \\ =\ & (x_s x_m)\cdot x_s\otimes\omega \\ =\ & -x_m\otimes\omega \\ =\ & -\lambda_m\otimes\omega. \end{aligned}$$

We also can conclude that $1\otimes\omega\in M'$. In other words, $M_K(B(\lambda))$ is simple. □

For the case of $n=2q+1$, we substitute $P_n := \begin{pmatrix} I_q & \frac{1}{2}I_q \\ -\mu I_q & \frac{\mu}{2}I_q\end{pmatrix}$ in Lemma 6.1 with $P_n := \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_q & \frac{1}{2}I_q \\ 0 & -\mu I_q & \frac{\mu}{2}I_q\end{pmatrix}$. The remaining proof is treated similarly as above.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 153 – 163

Weak solutions of a hyperbolic-type partial dynamic equation in Banach spaces

Ahmet Yantir* and Duygu Soyoğlu†

Abstract

In this article, we prove an existence theorem regarding the weak solutions to the hyperbolic-type partial dynamic equation

$$\begin{matrix} z^{\Gamma\Delta}(x,y) = f(x,y,z(x,y)), \\ z(x,0) = 0, \quad z(0,y) = 0 \end{matrix} \quad , \quad x \in \mathbb{T}_1, \quad y \in \mathbb{T}_2$$

in Banach spaces. For this purpose, by generalizing the definitions and results of Cichoń *et.al.* we develop weak partial derivatives, double integrability and the mean value results for double integrals on time scales. DeBlasi measure of weak noncompactness and Kubiaczyk's fixed point theorem for the weakly sequentially continuous mappings are the essential tools to prove the main result.

2000 AMS Classification: 34G20, 34N05, 35L10, 35R20, 47N20, 46B50.

Keywords: Hyperbolic partial dynamic equation; Banach space; measure of weak noncompactness, time scale.

Received 29/07/2013 : Accepted 20/01/2014 Doi : 10.15672/HJMS.2015449412

1. Introduction

The *time scale* which unifies the discrete and continuous analysis was initiated by Hilger [24]. Henceforth, the equations which can be described by continuous and discrete models are unified as "dynamic equations". Nevertheless, the theory of dynamic equations does not provide only a unification of continuous and discrete models. It also gives an opportunity to study some difference schemes based on variable step-size such as q-difference (quantum) models under the frame of dynamic equations. The landmark studies in the theory of dynamic equations are collected in the books by Bohner and Peterson [5, 6].

*Department of Mathematics, Yaşar University, İzmir, Turkey.
Email: ahmet.yantir@yasar.edu.tr
†Department of Mathematics, İzmir University of Economics, İzmir, Turkey.
Email:duygusoyoglu@gmail.com

Since the difference and differential equations are also studied in infinite dimensional Banach spaces [1, 9, 10, 11, 12, 16, 18, 21, 29, 28, 30, 37], it is reasonable to study dynamic equations in Banach spaces. The pioneering work on dynamic equations in Banach spaces is by Hilger [25]. Nevertheless this area is not sufficiently developed. Recently, Cichoń et. al. [13] study the existence of weak solutions of Cauchy dynamic problem. After this work, there have been some research activities in the theory of dynamic equations in Banach spaces [14, 15, 31].

On the other hand, the bi-variety calculus on time scales dates back to the landmark articles of Bohner and Guseinov [7, 8]. Authors study the partial differentiation and multiple integration on time scales respectively. Jackson [26] and Ahlbrandt and Morian [2] employ these background for studying some specific kinds of partial dynamic equations on $\mathbb{R}$. However, there is no result for the partial dynamic equations in Banach spaces.

The hyperbolic Goursat problem

$$u_{xy} = f(x, y, u, u_x, u_y), \quad u(x, 0) = u(0, y) = 0, \quad (x, y) \in V$$

where V is a rectangle containing $(0, 0)$, has been studied by many authors for years. Picard proved that when $f(x, y, z_1, z_2, z_3)$ is Lipschitz continuous in the $z-$ variable, then the solution exists and unique [17, 27]. The existence of solutions when f is independent from z_2 and z_3 was proved by Montel [33]. Then the sharper results followed by weakening the conditions on f (see [32, 22, 3, 34, 35]). For an application of a hyperbolic partial differential equations in stochastic process, see [36].

Motivated by the above studies and the lack of the results for nonlinear partial dynamic equations, in this article, we concentrate on the hyperbolic type partial dynamic problem

$$\begin{array}{l} z^{\Gamma_w \Delta_w}(x, y) = f(x, y, z(x, y)), \\ z(x, 0) = 0, \quad z(0, y) = 0 \end{array} , \quad x \in \mathbb{T}_1, \quad y \in \mathbb{T}_2 \tag{1.1}$$

in Banach spaces. Here the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ both include 0 and the differential operators Γ_w and Δ_w are weak partial derivative operators with respect to the variables x and y respectively.

We assume that f is Banach space-valued, weakly-weakly sequentially continuous function. We also assume some regularity conditions expressed in terms of DeBlasi measure of weak noncompactness [19] on f. We define a weakly sequentially continuous integral operator associated to an integral equation which is equivalent to (1.1). The existence of a fixed point of such operator is verified by using the fixed point theorem for weakly sequentially continuous mappings given by Kubiaczyk [28].

2. Preliminaries and Notations

The time scale calculus (and weak calculus) for the Banach space valued functions is created by Cichoń *et.al.* [13, 15]. Authors generalize the definitions of Hilger [24]. On the other hand, the multi-variable time scale calculus is created by Ahlbrandt and Morian [2] and Jackson [26]. In this section, we construct the definitions of weak partial derivatives and the weak double integral of a Banach space valued function defined on $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$. Also the mean value result of Cichoń *et.al.* (see Thm 2.11 of [13]) is generalized for the multivariable case.

Before we state the preliminary definitions, we remark the readers about the notations. Throughout this article, if a function of two variables $f : \mathbb{T}_1 \times \mathbb{T}_2 \to E$ is considered, by $f^{\Gamma}(s, t)$, we mean the forward Γ-derivative with respect to the first variable $s \in \mathbb{T}_1$. Similarly $f^{\Delta}(s, t)$ stands for the forward Δ-derivative with respect to the second variable $t \in \mathbb{T}_2$. For a function of single variable $f : \mathbb{T} \to E$, the ordinary notation $f^{\Delta}(t)$ is used. The similar considerations are also valid for the integrals.

We refer to [13] for the weak calculus of functions of single variable defined on a time scale. We only state the core definitions to clarify the weak calculus of functions of several variables defined on product time scale.

Let $(E, ||\cdot||)$ be a Banach space with the supremum norm and E^* be its dual space.

2.1. Definition. [13] Let $f : \mathbb{T} \to E$. Then we say that f is Δ-weak differentiable at $t \in \mathbb{T}$ if there exists an element $F(t) \in E$ such that for each $x^* \in E^*$ the real valued function $x^* f$ is Δ-differentiable at t and $(x^* f)^{\Delta}(t) = (x^* F)(t)$. Such a function F is called Δ-weak derivative of f and denoted by f^{Δ_w}.

2.2. Definition. [26](Partial Differentiability) Let $f : \mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be a function and let $(s,t) \in \mathbb{T}^k$. We define $f^{\Gamma}(s,t)$ to be the number (provided that it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood N of s, with $N = (s-\delta, s+\delta) \cap \mathbb{T}_1$ for $\delta > 0$ such that

$$\Big|[f(\sigma(s),t) - f(u,t)] - f^{\Gamma}(s,t)[\sigma(s) - u]\Big| \le \varepsilon\, |\sigma(s) - u|$$

for all $u \in N$. $f^{\Gamma}(s,t)$ is called the partial delta derivative of f with respect to the variable s.

Similarly we define $f^{\Delta}(s,t)$ to be the number (provided that it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood N of t, with $N = (t-\delta, t+\delta) \cap \mathbb{T}_2$ for $\delta > 0$ such that

$$\Big|[f(s,\sigma(t)) - f(s,u)] - f^{\Delta}(s,t)[\sigma(t) - u]\Big| \le \varepsilon\, |\sigma(t) - u|$$

for all $u \in N$. $f^{\Delta}(s,t)$ is called the partial delta derivative of f with respect to the variable t.

Since we have the definitions of weak Δ-derivative and the partial derivatives on time scales, it is reasonable to combine these definitions to construct the definition of weak partial derivative of a Banach space valued function.

2.3. Definition. Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to E$. Then we say that f is Γ-weak partial differentiable at $(s,t) \in \mathbb{T}$ if there exists an element $F(s,t) \in E$ such that for each $z^* \in E^*$ the real valued function $z^* f$ is Γ partial differentiable at (s,t) and $(z^* f)^{\Gamma}(s,t) = (z^* F)(s,t)$. Such a function F is called Γ-weak partial derivative of f and denoted by f^{Γ_w}.

Similarly, f is said to be Δ-weak partial differentiable at $(s,t) \in \mathbb{T}$ if there exists an element $F(s,t) \in E$ such that for each $z^* \in E^*$ the real valued function $z^* f$ is Δ partial differentiable at (s,t) and $(z^* f)^{\Delta}(s,t) = (z^* F)(s,t)$. Such a function F is called Δ-weak partial derivative of f and denoted by f^{Δ_w}.

2.4. Definition. If $F^{\Gamma_w}(s,t) = f(s,t)$ for all (s,t), then we define Γ-weak Cauchy integral by

$$(C_w)\int_a^s f(\tau,t)\Gamma\tau = F(s,t) - F(a,t).$$

The Riemann, Cauchy-Riemann, Borel and Lebesgue integrals on time scales for the Banach space-valued functions are defined by Aulbach *et. al.* [4]. Since the weak Cauchy integral is defined by means of weak anti-derivativatives, the space of weak integrable functions is too restricted. Therefore it is conceivable to define the weak Riemann integral.

2.5. Definition. [13] Let $P = \{a_0, a_1, \cdots, a_n\}$ be a partition of the interval $[a,b]$. P is called $finer$ than $\delta > 0$ either $\mu_{\mathbb{T}}([a_{i-1}, a_i]) \le \delta$ or $\mu_{\mathbb{T}}([a_{i-1}, a_i]) > \delta$ only if $a_i = \sigma(a_{i-1})$, where $\mu_{\mathbb{T}}$ denotes the time scale measure.

2.6. Definition. (Riemann Double Integrability) A Banach space valued-function $f : [a,b] \times [c,d] \to E$ is called weak Riemann double integrable if there exists $I \in E$ such that for any $\varepsilon > 0$ there exists a positive number δ with the following property: For any partition $P_1 = \{a_0, a_1, \cdots, a_n\}$ of $[a,b]$ and $P_2 = \{c_0, c_1, \cdots, c_n\}$ of $[c,d]$ which are finer than δ and the set of points $s_j \in [a_{j-1}, a_j)$ and $t_j \in [c_{j-1}, c_j)$ for $j = 1, 2, \cdots, n$ one has

$$\left| z^*(I) - \sum_{j=1}^{n} z^*(f(s_j, t_j))\mu_{\mathbb{T}}([a_{j-1}, a_j) \times [c_{j-1} c_j)) \right| \leq \ \varepsilon, \text{ for all } z^* \in E^*.$$

The uniquely determined function I is called weak Riemann double integral f and denoted by

$$I = (\mathcal{R}_w) \int\int_{[a,b]\times[c,d]} f(s,t)\Delta t\Gamma s.$$

Using Theorem 4.3 of Guseinov [23] and regarding the definition of weak Cauchy and Riemann integrals, it can be remarked that every Riemann weak integrable function is Cauchy weak integrable and therefore these two integrals coincide.

The measure of weak noncompactness which is developed by DeBlasi [19] is the fundamental tool in our main result. The regularity conditions on the nonlinear term f is expressed in terms of measure of weak noncompactness. Let A be a bounded nonempty subset of E. The measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{there exists } C \in K^{\omega} \text{ such that } A \subset C + tB_1\}$$

where K^{ω} is the set of weakly compact subsets of E and B_1 is the unit ball in E.

We make use of the following properties of the measure of weak noncompactness β. For bounded nonempty subsets A and B of E,

(1) If $A \subset B$ then $\beta(A) \leq \beta(B)$,
(2) $\beta(A) = \beta(\bar{A^w})$, where $\bar{A^w}$ denotes the weak closure of A,
(3) $\beta(A) = 0$ if and only if A is relatively weakly compact,
(4) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,
(5) $\beta(\lambda A) = |\lambda|\beta(A)$ $(\lambda \in R)$,
(6) $\beta(A + B) \leq \beta(A) + \beta(B)$,
(7) $\beta(\overline{\text{conv}}(A)) = \beta(\text{conv}(A)) = \beta(A)$, where $\text{conv}(A)$ denotes the convex hull of A.

If β is an arbitrary set function satisfying the above properties *i.e.*, if β is an axiomatic measure of weak noncompactness, then the following lemma is true.

2.7. Lemma. *If* $||E_1|| = \sup\{||x|| : x \in E_1\} < 1$ *then*

$$\beta(E_1 + E_2) \leq \beta(E_2) + ||E_1||\beta(K(E_2, 1)),$$

where $K(E_2, 1) = \{x : d(E_2, x) \leq 1\}$.

The generalization of Ambrosetti Lemma for $C(\mathbb{T}_1 \times \mathbb{T}_2, E)$ is as follows:

2.8. Lemma. *Let* $H \subset C(\mathbb{T}_1 \times \mathbb{T}_2, E)$ *be a family of strongly equicontinuous functions. Let* $H(x,y) = \{h(x,y) \in E, h \in H\}$, *for* $(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2$. *Then*

$$\beta(H(\mathbb{T}_1 \times \mathbb{T}_2)) = \sup_{(x,y)\in\mathbb{T}_1\times\mathbb{T}_2} \beta(H(x,y)),$$

and the function $(x,y) \mapsto \beta(H(x,y))$ *is continuous on* $\mathbb{T}_1 \times \mathbb{T}_2$.

Proof. The proof directly follows by generalizing the proof of Lemma 2.9 of [13]. □

2.9. Theorem. *(Mean Value Theorem for Double Integrals) If the function* $\phi : \mathbb{T}_1 \times \mathbb{T}_2 \to E$ *is* Δ- *and* Γ-*weak integrable, then*

$$\iint_{\Omega} \phi(s,t)\Delta t\Gamma s \in \mu_{\mathbb{T}}(\Omega) \cdot \overline{conv}\ \phi(\Omega)$$

where Ω is an arbitrary subset of $\mathbb{T}_1 \times \mathbb{T}_2$.

Proof. Let $\iint_\Omega \phi(s,t)\Delta t\Gamma s = w$ and $\mu_{\mathbb{T}}(R) \cdot \overline{conv}\ \phi(\Omega) = W$. Suppose to the contrary, that $w \notin W$. By separation theorem for the convex sets there exists $z^* \in E^*$ such that

$$\sup_{\varphi \in W} z^*(\varphi) = \alpha < z^*(w).$$

However

$$z^*(w) = z^*\left((C_w)\iint_\Omega \phi(s,t)\Delta t\Gamma s\right) = \iint_\Omega z^*(\phi(s,t))\Delta t\Gamma s.$$

Moreover, for $(s,t) \in \Omega$, we have $\phi(s,t) \in \phi(\Omega)$ and therefore

$$\mu_{\mathbb{T}}(\Omega) \cdot \phi(s,t) \subseteq \mu_{\mathbb{T}}(\Omega) \cdot \overline{conv}\ \phi(\Omega) = W, \quad \text{i.e. } \phi(s,t) \subseteq \frac{W}{\mu_{\mathbb{T}}(\Omega)}.$$

Hence

$$z^*(\phi(s,t)) \leq z^*\left(\frac{W}{\mu_{\mathbb{T}}(\Omega)}\right) < \frac{\alpha}{\mu_{\mathbb{T}}(\Omega)}.$$

Finally we obtain,

$$z^*(w) = \iint_\Omega z^*(\phi(s,t))\Delta t\Gamma s \leq \iint_\Omega \frac{\alpha}{\mu_{\mathbb{T}}(\Omega)}\Delta t\Gamma s = \frac{\alpha}{\mu_{\mathbb{T}}(\Omega)} \cdot \mu_{\mathbb{T}}(\Omega) = \alpha$$

which is a contradiction. □

In the proof of the main theorem, we make use of the following fixed point theorem of Kubiaczyk.

2.10. Theorem. [28] *Let X be a metrizable, locally convex topological vector space, D be a closed convex subset of X, and F be a weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication*

(2.1) $\quad \overline{V} = \overline{conv}(\{x\} \cup F(V)) \Rightarrow V$ *is relatively weakly compact,*

holds for every subset V of D, then F has a fixed point.

3. The Existence Result

We claim that in the case of weakly-weakly continuous f, finding a weak solution of (1.1) is equivalent to solving the integral equation

$$z(x,y) = (C_w)\int_0^x \int_0^y f(s,t,z(s,t))\Delta t\Gamma s, \quad (s,t) \in \mathbb{T}_1 \times \mathbb{T}_2. \tag{3.1}$$

To justify the equivalence, we first assume that a weakly continuous function $z : \mathbb{T}_1 \times \mathbb{T}_2 \to E$ is a weak solution of (1.1). We show that z solves the integral equation (3.1). By the definition of weak Cauchy integral (Definition 2.4), we have

$$(C_w)\int_0^y f(x,t,z(x,t))\Delta t = (C_w)\int_0^y z^{\Gamma\Delta}(x,t)\Delta t = z^\Gamma(x,y) - z^\Gamma(x,0) = z^\Gamma(x,y)$$

Note that $z^\Gamma(x,0) = 0$ since $z(x,0) = 0$. If we integrate the resulting equation on $[0,x]_{\mathbb{T}_1}$, we obtain

$$(C_w)\int_0^x \int_0^y f(s,t,z(s,t))\Delta t\Gamma s = (C_w)\int_0^x z^\Gamma(s,y)\Gamma s = z(x,y) - z(0,y) = z(x,y)$$

which points out that z solves the integral equation (3.1).

Conversely, we assume that $z(x,y)$ is a solution of the integral equation (3.1). For any $z^* \in E^*$, we have

$$(z^*z)(x,y) = z^*\left(\int_0^x \int_0^y f(s,t,z(s,t))\Delta t \Gamma s\right)$$

and therefore

$$\begin{aligned}(z^*z)^{\Gamma}(x,y) &= \left(\int_0^x \int_0^y z^*(f(s,t,z(s,t)))\Delta t \Gamma s\right)^{\Gamma} \\ &= \int_0^y z^*(f(x,t,z(x,t)))\Delta t.\end{aligned}$$

Differentiating the last expression we get

$$\begin{aligned}(z^*z)^{\Gamma\Delta}(x,y) &= \left(\int_0^y z^*(f(x,t,z(x,t)))\Delta t\right)^{\Delta} \\ &= z^*(f(x,y,z(x,y))).\end{aligned}$$

By the definition of weak partial derivatives (Definition 2.3), we obtain

$$z^{\Gamma_w \Delta_w}(x,y) = f(x,y,z(x,y)).$$

Clearly the boundary conditions of (1.1) hold. Hence $z(x,y)$ is the weak solution of (1.1).

We consider the space of continuous functions $\mathbb{T}_1 \times \mathbb{T}_2 \to E$ with its weak topology, *i.e.*,

$$(C(\mathbb{T}_1 \times \mathbb{T}_2, E), w) = (C(\mathbb{T}_1 \times \mathbb{T}_2, E), \tau(C(\mathbb{T}_1 \times \mathbb{T}_2, E), C^*(\mathbb{T}_1 \times \mathbb{T}_2, E))).$$

Let $G : \mathbb{T}_1 \times \mathbb{T}_2 \times [0,\infty) \to [0,\infty)$ be a continuous function and nondecreasing in the last variable. Assume that the scalar integral inequality

$$g(x,y) \geq \int_0^x \int_0^y G(s,t,g(s,t))\Delta t \Gamma s \tag{3.2}$$

has locally bounded solution $g_0(x,y)$ existing on $\mathbb{T}_1 \times \mathbb{T}_2$.

We define the ball B_{g_0} as follows:

$$\begin{aligned}B_{g_0} = \Big\{ &z \in (C(\mathbb{T}_1 \times \mathbb{T}_2, E), w) : ||z(x,y)|| \leq g_0(x,y) \text{ on } \mathbb{T}_1 \times \mathbb{T}_2, \\ &||z(x_1,y_1) - z(x_2,y_2)|| \leq \left|\int_0^{x_2} \int_{y_1}^{y_2} G(s,t,g_0(s,t))\Delta t \Gamma s\right| \\ &+ \left|\int_{x_1}^{x_2} \int_0^{y_1} G(s,t,g_0(s,t))\Delta t \Gamma s\right| \quad \text{for } x_1, x_2 \in \mathbb{T}_1 \text{ and } y_1, y_2 \in \mathbb{T}_2 \Big\}\end{aligned} \tag{3.3}$$

Clearly the set B_{g_0} is nonempty, closed, bounded, convex and equicontinuous.

Assume that a nonnegative, real-valued, continuous function $(x,y,r) \mapsto h(x,y,r)$ defined on $\mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R}^+$ satisfies the following conditions:

(H1) $h(x,y,0) = 0$,

(H2) $z(x,y) \equiv 0$ is the unique continuous solution of the integral inequality

$$u(x,y) \leq \int_0^x \int_0^y h(s,t,u(s,t))\Delta t \Gamma s$$

satisfying the condition $u(0,0)=0$.

We define the integral operator $F : (C(\mathbb{T}_1 \times \mathbb{T}_2, E), w) \to (C(\mathbb{T}_1 \times \mathbb{T}_2, E), w)$ associated to the integral equation (3.1) by

$$F(z)(x,y) = (\mathcal{R}_w) \int_0^x \int_0^y f(s,t,z(s,t))\Delta t \Gamma s, \quad x \in \mathbb{T}_1, \quad y \in \mathbb{T}_2. \tag{3.4}$$

By the considerations presented above, the fixed point of the integral operator F is the weak solution of (1.1). Our main result is as follows:

3.1. Theorem. *Assume that the function $f : \mathbb{T}_1 \times \mathbb{T}_2 \times B_{g_0} \to E$ satisfies the following conditions:*

- (C1) *$f(x, y, \cdot)$ is weakly-weakly sequentially continuous for each $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$,*
- (C2) *For each strongly absolutely continuous function $z : \mathbb{T}_1 \times \mathbb{T}_2 \to E$, $f(\cdot, \cdot, z(\cdot, \cdot))$is weakly continuous*
- (C3) *$||f(x, y, u)|| \leq G(x, y, ||u||)$ for all $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $u \in E$,*
- (C4) *For any function h satisfying the conditions (H1) and (H2)*

$$\beta(f(I_x \times I_y \times W)) \leq h(x, y, \beta(W))$$

for each $W \subset B_{g_0}$ and $I_x \subset \mathbb{T}_1,\ I_y \subset \mathbb{T}_2$.

Then there exists a weak solution of the partial dynamic problem (1.1).

Proof. By virtue of the condition (C2), the operator $F : B_{g_0} \to (C(\mathbb{T}_1 \times \mathbb{T}_2, E), w)$ is well-defined. Next we clarify that the operator F maps B_{g_0} into B_{g_0}. For this purpose first we verify $||F(z)(x, y)|| \leq g_0(x, y)$. For $z(x, y) \in B_{g_0}$, the condition (C3), the monotonicity of G in the last variable and the existence of locally bounded solution $g_0(x, y)$ of (3.2) guarantee that

$$\begin{aligned}
||F(z)(x, y)|| &= \left|\left|\int_0^x \int_0^y f(s, t, z(s, t))\Delta t \Gamma s\right|\right| \\
&\leq \int_0^x \int_0^y ||f(s, t, z(s, t))||\Delta t \Gamma s \\
&\leq \int_0^x \int_0^y G(s, t, ||z(s, t)||)\Delta t \Gamma s \\
&\leq \int_0^x \int_0^y G(s, t, ||g_0(x, y)||)\Delta t \Gamma s \leq g_0(x, y). \qquad (3.5)
\end{aligned}$$

Consequently, we claim that

$$\begin{aligned}
||F(z)(x_1, y_1) - F(z)(x_2, y_2)|| \leq & \left|\int_0^{x_2} \int_{y_1}^{y_2} G(s, t, g_0(s, t))\Delta t \Gamma s\right| \\
& + \left|\int_{x_1}^{x_2} \int_0^{y_1} G(s, t, g_0(s, t))\Delta t \Gamma s\right|.
\end{aligned}$$

For all $z^* \in E^*$ with $||z^*|| \leq 1$, we have

$$\begin{aligned}
|z^*(f(s, t, z(s, t)))| &\leq \sup_{z^* \in E^*, ||z^*|| \leq 1} |z^*(f(s, t, z(s, t)))| \\
&= ||(f(s, t, z(s, t)))|| \\
&\leq G(s, t, ||z(s, t)||),
\end{aligned}$$

where we use the condition (C3) for the last step. Hence

$$\begin{aligned}
|z^*[F(z)(x_1, y_1) - F(z)(x_2, y_2)]| &= \left|z^*\left(\int_0^{x_2} \int_{y_1}^{y_2} f(s, t, z)\Delta t \Gamma s - \int_{x_1}^{x_2} \int_0^{y_1} f(s, t, z)\Delta t \Gamma s\right)\right| \\
&\leq \int_0^{x_2} \int_{y_1}^{y_2} |z^*(f(s, t, z))|\Delta t \Gamma s + \int_{x_1}^{x_2} \int_0^{y_1} |z^*(f(s, t, z))|\Delta t \Gamma s
\end{aligned}$$

Utilizing the condition (C2) we acquire,

$$\begin{aligned} ||F(z)(x_1,y_1)-F(z)(x_2,y_2)|| \quad &\leq \quad \left|\int_0^{x_2}\int_{y_1}^{y_2} G(s,t,||z(s,t)||)\Delta t\Gamma s\right| \\ &\quad + \left|\int_{x_1}^{x_2}\int_0^{y_1} G(s,t,||z(s,t)||)\Delta t\Gamma s\right|. \end{aligned}$$

Since G is nondecreasing in the last variable, the desired result

$$\begin{aligned} ||F(z)(x_1,y_1)-F(z)(x_2,y_2)|| \quad &\leq \quad \left|\int_0^{x_2}\int_{y_1}^{y_2} G(s,t,g_0(s,t))\Delta t\Gamma s\right| \\ &\quad + \left|\int_{x_1}^{x_2}\int_0^{y_1} G(s,t,g_0(s,t))\Delta t\Gamma s\right| \end{aligned}$$

follows.

Next, we substantiate the weakly sequentially continuity of the integral operator F. Let $z_n \overset{w}{\to} z$ in B_{g_0}. Then for given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n > N$ and $(x,y) \in I_\alpha \times I_\beta \subset \mathbb{T}_1 \times \mathbb{T}_2$, we have $|z^*z_n(x,y) - z^*z(x,y)| < \epsilon$. Apparently, from condition (C1), one can obtain

$$|z^*f(x,y,z_n(x,y)) - z^*f(x,y,z(x,y))| \leq \frac{\epsilon}{\alpha\beta}.$$

Hence

$$\begin{aligned} |z^*(F(z_n)(x,y)-F(z)(x,y))| \;&= \left|z^*\left(\int_0^x\int_0^y f(s,t,z_n)\Delta t\Gamma s - \int_0^x\int_0^y f(s,t,z)\Delta t\Gamma s\right)\right| \\ &\leq \int_0^x\int_0^y \left|z^*\left(f(s,t,z_n(s,t)) - f(s,t,z(s,t))\right)\right|\Delta t\Gamma s \\ &\leq \int_0^\alpha\int_0^\beta \left|z^*f(s,t,z_n(s,t)) - z^*f(s,t,z(s,t))\right|\Delta t\Gamma s \\ &< \int_0^\alpha\int_0^\beta \frac{\epsilon}{\alpha\beta}\Delta t\Gamma s = \epsilon, \end{aligned}$$

(for the first integral inequality see [23, 5, 6]). Owing to the closedness of $\mathbb{T}_1 \times \mathbb{T}_2$, is it locally compact Hausdorff space. Thanks to the result of Dobrakov (see [20], Thm 9), $F(z_n)$ converges weakly to $F(z)$ in $(C(\mathbb{T}_1 \times \mathbb{T}_2, E), w)$. Therewith F is weakly sequentially continuous mapping.

As a result, F is well-defined, weakly sequentially continuous and maps B_{g_0} into B_{g_0}.

Now we prove that the fixed point of the integral operator (3.4) exists by employing Kubiaczyk's fixed point theorem (Theorem 2.10).

Let $W \subset B_{g_0}$ satisfying the condition

$$W = \overline{\text{conv}}\left(\{z\} \cup F(W)\right) \tag{3.6}$$

for some $z \in B_{g_0}$. We prove that W is relatively weakly compact. For $(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2$, we define $W(x,y) = \{w(x,y) \in E : w \in W\}$. Resulting from Ambrosetti's Lemma (Lemma 2.8), the function $(x,y) \mapsto w(x,y) = \beta(W(x,y))$ is continuous on $\mathbb{T}_1 \times \mathbb{T}_2$.

Since the integral is $\int_0^x\int_0^y G(s,t,g(s,t))\Delta t\Gamma s$ is bounded, there exist $\xi \in \mathbb{T}_1$ and $\eta \in \mathbb{T}_2$ such that

$$\iint\limits_R G(s,t,||z(s,t)||)\Delta t\Gamma s < \epsilon$$

where $R = \mathbb{T}_1 \times \mathbb{T}_2 - ([0,\xi]_{\mathbb{T}_1} \times [0,\eta]_{\mathbb{T}_2})$. We divide the interval $[0,\xi]_{\mathbb{T}_1}$ into m parts

$$0 < s_1 < s_2 < \ldots < s_m = \xi$$

and $[0,\eta]_{\mathbb{T}_2}$ into n parts

$$0 < t_1 < t_2 < \ldots < t_m = \eta$$

in a way that each partition is finer than $\delta > 0$. Also we define $\mathbb{T}_1^i = [s_i, s_{i+1}]_{\mathbb{T}_1}$ and $\mathbb{T}_2^j = [t_j, t_{j+1}]_{\mathbb{T}_2}$. By Abrosetti's Lemma there exists $(\sigma_i, \tau_j) \in \mathbb{T}_1^i \times \mathbb{T}_2^j = P_{ij}$ such that

$$\beta(W(P_{ij})) = \sup\{\beta(W(s,t)) : (s,t) \in P_{ij}\} = w(\sigma_i, \tau_j).$$

On the other hand, for $x > \xi$, $y > \eta$ and for any $w \in W$, we have

$$\begin{aligned} F(w)(x,y) &= \int_0^x \int_0^y f(s,t,w(s,t))\Delta t \Gamma s \\ &= \int_0^\xi \int_0^\eta f(s,t,w(s,t))\Delta t \Gamma s + \iint_{R_1} f(s,t,w(s,t))\Delta t \Gamma s. \end{aligned}$$

Therefore the mean value theorem (Theorem 2.9) entails

$$F(w(x,y)) \in \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) \overline{\text{conv}}(f(P_{ij} \times W(P_{ij}))) + \iint_{R_1} f(s,t,w(s,t))\Delta t \Gamma s,$$

which has the consequence

$$F(W(x,y)) \subset \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) \overline{\text{conv}}(f(P_{ij} \times W(P_{ij}))) + \iint_{R_1} f(s,t,W(s,t))\Delta t \Gamma s.$$

Using (C4), Lemma 2.7 and the properties of measure of weak noncompactness, we acquire

$$\begin{aligned} \beta(F(W(x,y))) &\leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) \beta(\overline{\text{conv}}(f(P_{ij} \times W(P_{ij})))) + \left\| \iint_{R_1} f(s,t,w(s,t))\Delta t \Gamma s \right\| \\ &\leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) \beta(f(P_{ij} \times W(P_{ij}))) + \sup_{w \in W} \iint_{R_1} f(s,t,w(s,t))\Delta t \Gamma s \\ &\leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) h((P_{ij} \times \beta(W(P_{ij})))) + \sup_{w \in W} \iint_{R_1} f(s,t,w(s,t))\Delta t \Gamma s \\ &\leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) h((P_{ij} \times \beta(W(P_{ij})))) + \sup_{w \in W} \iint_{R} f(s,t,w(s,t))\Delta t \Gamma s \\ &\leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) h((P_{ij} \times w(\sigma_i, \tau_j)) + \sup_{w \in W} \iint_{R} f(s,t,w(s,t))\Delta t \Gamma u \\ &\leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) h((P_{ij} \times w(\sigma_i, \tau_j)) + \sup_{w \in W} \iint_{R} G(s,t,||w(s,t)||)\Delta t \Gamma s \\ &\leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_{\mathbb{T}}(P_{ij}) h((P_{ij} \times w(\sigma_i, \tau_j)) + \epsilon \end{aligned}$$

Since ϵ is arbitrary,

$$\beta(F(W)(x,y)) \leq \int_0^x \int_0^y h(s,t,w(s,t))\Delta t \Gamma s. \tag{3.7}$$

By the condition (3.6), inequality (3.7) and the properties of measure of weak noncompactness

$$w(x,y) \leq \int_0^x \int_0^y h(s,t,w(s,t))\Delta t \Gamma s.$$

The condition (H2) implies that the integral inequality above has only trivial solution, i.e. $w(x,y) = \beta(W(x,y)) = 0$ which means that W ie relatively weakly compact. Thus the condition (2.1) of Theorem 2.10 is substantiated. So the integral operator F defined by (3.4) has a fixed point which is actually a weak solution of the hyperbolic partial dynamic equation (1.1). □

Acknowledgements. The authors wish to acknowledge the anonymous reviewer for his/her detailed and helpful comments to the manuscript.

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STATISTICS

Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 165 – 178

Exact distribution of Cook's distance and identification of influential observations

G.S. David Sam Jayakumar* and A.Sulthan†

Abstract

This paper proposed the exact distribution of Cook's distance used to evaluate the influential observations in multiple linear regression analysis. The authors adopted the relationship proposed by Weisberg (1980), Belsey et al. (1980) and showed the derived density function of the cook's distance in terms of the series expression form. Moreover, the first two moments of the distribution are derived and the authors computed the critical points of Cook's distance at 5% and 1% significance level for different sample sizes based on no.of predictors. Finally, the numerical example shows the identification of the influential observations and the results extracted from the proposed approach is more scientific, systematic and it's exactness outperforms the traditional rule of thumb approach.

2000 AMS Classification: 62H10

Keywords: Cook's distance, influential observations, Series expression form, moments, critical points

Received 16/08/2013 : Accepted 26/02/2014 Doi : 10.15672/HJMS.201487459

1. Introduction and Related work

Cook's Distance (Di) is used for assessing influential observations in regression models. The problem of outliers or influential data in the multiple or multivariate linear regression setting has been thoroughly discussed with reference to parametric regression models by the pioneers namely Cook (1977), Cook and Weisberg (1982), Belsey et al. (1980) and Chatterjee and Hadi (1988) respectively. In non-parametric regression models, diagnostic results are quite rare. Among them, Eubank (1985), Silverman (1985), Thomas (1991), and Kim (1996) studied residuals, leverages, and several types of Cook's distance in smoothing splines, and Kim and Kim (1998) proposed a type of Cook's distance in kernel density estimation. Later, Kim et al. (2001) suggested a type of Cook's distance

*Assistant Professor, Jamal Institute of Management, Tiruchirappalli - 620 020, India
Email: samjaya77@gmail.com

†Research scholar, Jamal Institute of Management, Tiruchirappalli - 620 020, India
Email:Sulthan90@gmail.com

in local polynomial regression. Recently, Diaz-Garcia and Gonzlez-Faras (2004) modified the classical cook's distance with generalized Mahalanobis distance in the context of multivariate elliptical linear regression models and they also establish the exact distribution for identification of outlier data points. Considering the above reviews, the authors proposed an alternative and scientific approach to identify the influential data points in multiple linear regression models and it is discussed in the subsequent sections.

2. Relationship between Cook's D and F-ratios

The multiple linear regression model with random error is given by

$Y = X\beta + e$ - (1)

where $\underset{(nX1)}{Y}$ is the matrix of the dependent variable, $\underset{(kX1)}{\beta}$ is the vector of beta co-efficients or partial regression co-efficients and $\underset{(nX1)}{e}$ is the residual followed normal distribution N $(0,\sigma_e^2 I_n)$. From (1), statisticians concentrate and give importance to the error diagnostics such as outlier detection, identification of leverage points and evaluation of influential observations. Several error diagnostics techniques exist in the literature proposed by statisticians, cook's distance is the most frequently and interesting technique used to identify the influential observations in the Y as well as in the X-space in a multiple linear regression model. The general form of the Cook's distance of the *ith* observation is given by

$$D_i = \frac{1}{(p+1)\,\hat{\sigma}_e^2}\left(\hat{\beta} - \underset{(-i)}{\hat{\beta}}\right)^T X^T X \left(\hat{\beta} - \underset{(-i)}{\hat{\beta}}\right) \text{ - (2)}$$

where $\underset{(-i)}{\hat{\beta}}$ is the vector of estimated regression co-efficient with the *ith* observation deleted, p is the no.of predictors and $\hat{\sigma}_e^2$ is the residual error variance for the full data set. Removing the *ith* observation should keep $\underset{(-i)}{\hat{\beta}}$ close to $\hat{\beta}$ unless the *ith* observation is an outlier. Cook and Weisberg (1982) indicate that D_i of about 1, corresponding to distances between $\hat{\beta}$ and $\underset{(-i)}{\hat{\beta}}$ beyond a 50% confidence region would generally be considered large. Similarly, Bollen et al (1990) suggested, Cook's distance for observations more than a cut-off of $4/n-p$ which is treated as the traditional approach of evaluating the influential observations. Cook's distance (Cook and Weisberg (1982) p.118) can also be written in an alternative form as

$$D_i = \frac{r_i^2}{(p+1)}\left(\frac{h_{ii}}{1-h_{ii}}\right) \text{ - (3)}$$

Where from (3), r_i is the studentized residual which is equal to $\hat{e}_i \,/\, \hat{\sigma}_e \sqrt{1-h_{ii}}$ and h_{ii} is the hat element. Thus Cook's distance measures the joint influence on the case being an outlier on Y-space and in the space of the predictors (X-space). An influential observation in a multiple linear regression model may or may not be an outlier. In order to overcome the rule of thumb approach of evaluating and identifying the influential observation, we utilize the relationship among the cook's distance(D_i), Studentized residual (r_i) and hat elements(h_{ii}).The terms (r_i) and (h_{ii}) are independent because the computation of (r_i) involves the error term (e_i) $\sim N(0,\sigma_e^2)$ and (h_{ii})values involves the set of predictors($H = X(X'X)^{-1}X'$).Therefore, from the property of least squares $E(eX) = 0$, so (r_i) and (h_{ii}) are also uncorrelated and independent. Using this assumption, we first determine the distribution of (r_i)based on the relationship given by Weisberg (1980) as

$$t_i = r_i\sqrt{\frac{n-p-2}{(n-p-1)-r_i^2}} \sim t_{(n-p-2)} \text{ - (4)}$$

From (4) it follows student's t- distribution with $(n-p-2)$ degrees of freedom and it can be written in terms of the F-ratio as

$$r_i^2 = \frac{(n-p-1)t_i^2}{(n-p-2)+t_i^2}$$

$r_i^2 = \frac{(n-p-1)F_{i(1,n-p-2)}}{(n-p-2)+F_{i(1,n-p-2)}}$ - (5)

From (5), if t_i follows student's t- distribution with $(n-p-2)$ degrees of freedom, then t_i^2 follows $F_{(1,n-p-2)}$ distribution with $(1, n-p-2)$ degrees of freedom. Similarly, we identify the distribution of (h_{ii})based on the relationship proposed by Belsey et al (1980) and they showed when the set of predictors is multivariate normal with(μ_X, Σ_X), then

$\frac{(n-p)(h_{ii}-1/n)}{(p-1)(1-h_{ii})} \sim F_{(p-1),(n-p)}$ - (6)

From (6) it follows F-distribution with $(p-1, n-p)$ degrees of freedom and it can be written in an alternative form as

$h_{ii} = \frac{\left((p-1)F_{i(p-1,n-p)}/(n-p)\right)+1/n}{1+(p-1)F_{i(p-1,n-p)}/(n-p)}$ - (7)

In order to derive the exact distribution of (D_i),substitute (5) and (6) in (2), we get the Cook's D in terms of the two independent F-ratios with $(1, n-p-2)$ and $(p-1, n-p)$ degrees of freedom respectively and the relationship is given by

$D_i = \frac{1}{(p+1)}\left(\frac{(n-p-1)F_{i(1,n-p-2)}}{(n-p-2)+F_{i(1,n-p-2)}}\right)\left(\frac{\left((p-1)F_{i(p-1,n-p)}/(n-p)\right)+1/n}{(n-1)/n}\right)$ - (8)

Based on the identified relationship from (8), the authors derived the distribution of the Cook's D-distance and it is discussed in the next section.

3. Exact Distribution of Cook's Distance

Using the technique of two-dimensional Jacobian of transformation, the joint probability density function of the two F-ratios namely $F_{i(1,n-p-2)}$,$F_{i(p-1,n-p)}$ with $(1, n-p-2)$ and $(p-1, n-p)$ degrees of freedom was transformed into density function of Cook's distance (D_i) and it is given as

$f(D_i, u_i) = f(F_{i(1,n-p-2)}, F_{i(p-1,n-p)})\,|J|$ - (9)

From (8), we know $F_{i(1,n-p-2)}$and $F_{i(p-1,n-p)}$ are independent then rewrite (9) as

$f(D_i, u_i) = f(F_{i(1,n-p-2)})f(F_{i(p-1,n-p)})\,|J|$ - (10)

Using the change of variable technique, substitute $F_{i(1,n-p-2)} = u_i$ in (8) we get

$F_{i(p-1,n-p)} = \frac{n-p}{p-1}\left(\frac{D_i((n-p-2)+u_i)((n-1)/n)}{((n-p-1)/(p+1))u_i} - 1/n\right)$ - (10a)

Then partially differentiate (10a) and compute the Jacobian determinant in (10) as

$f(D_i, u_i) = f(F_{i(1,n-p-2)})f(F_{i(p-1,n-p)})\left|\frac{\partial(F_{i(1,n-p-2)}, F_{i(p-1,n-p)})}{\partial(D_i, u_i)}\right|$ - (11)

$f(D_i, u_i) = f(F_{i(1,n-p-2)})f(F_{i(p-1,n-p)})\begin{vmatrix} \frac{\partial F_{i(1,n-p-2)}}{\partial D_i} & \frac{\partial F_{i(1,n-p-2)}}{\partial u_i} \\ \frac{\partial F_{i(p-1,n-p)}}{\partial D_i} & \frac{\partial F_{i(p-1,n-p)}}{\partial u_i} \end{vmatrix}$ - (12)

From (12), we know the F-ratios are independent, then the density function of the joint distribution of $F_{i(1,n-p-2)}$ and $F_{i(p-1,n-p)}$ are given as

$$f(F_{i(1,n-p-2)}, F_{i(p-1,n-p)}) = f(F_{i(1,n-p-2)})f(F_{i(p-1,n-p)})$$

$$f(F_{i(1,n-p-2)}, F_{i(p-1,n-p)}) = \left(\frac{(1/n-p-2)^{1/2}}{B(\frac{1}{2}, \frac{n-p-2}{2})} \left(F_{i(1,n-p-2)}\right)^{(1/2)-1} \left(1 + \frac{F_{i(1,n-p-2)}}{n-p-2}\right)^{-(\frac{1}{2}+\frac{n-p-2}{2})} \right)$$
$$* \left(\frac{((p-1)/n-p)^{(p-1)/2}}{B(\frac{p-1}{2}, \frac{n-p}{2})} \left(F_{i(p-1,n-p)}\right)^{((p-1)/2)-1} (1 + \frac{p-1}{n-p} F_{i(p-1,n-p)})^{-(\frac{p-1}{2}+\frac{n-p}{2})} \right) \quad \text{- (13)}$$

where $0 \leq F_{i(1,n-p-2)}, F_{i(p-1,n-p)} \leq \infty, n, p > 0$

and
$$\begin{vmatrix} \frac{\partial F_{i(1,n-p-2)}}{\partial D_i} & \frac{\partial F_{i(1,n-p-2)}}{\partial u_i} \\ \frac{\partial F_{i(p-1,n-p)}}{\partial D_i} & \frac{\partial F_{i(p-1,n-p)}}{\partial u_i} \end{vmatrix} = \begin{vmatrix} 0 & \frac{n-p}{p-1} \left(\frac{((n-p-2)+u_i)((n-1)/n)}{((n-p-1)/(p+1))u_i} \right) \\ 1 & -\frac{n-p}{p-1} \left(\frac{D_i(n-p-2)((n-1)/n)}{((n-p-1)/(p+1))u_i^2} \right) \end{vmatrix} \quad \text{- (14)}$$
$$= \frac{n-p}{p-1} \left(\frac{((n-p-2)+u_i)((n-1)/n)}{((n-p-1)/(p+1))u_i} \right)$$

Then substitute (13) and (14) in (12) in terms of the substitution of u_i we get the joint distribution of Cook's D and u_i as

$$f(D_i, u_i) = \left(\frac{(1/n-p-2)^{1/2}}{B(\frac{1}{2}, \frac{n-p-2}{2})} u_i^{(1/2)-1} (1 + \frac{u_i}{n-p-2})^{-(\frac{1}{2}+\frac{n-p-2}{2})} \right)$$
$$* \left(\begin{array}{l} \frac{((p-1)/n-p)^{(p-1)/2}}{B(\frac{p-1}{2}, \frac{n-p}{2})} \left(\frac{n-p}{p-1} \left(\frac{D_i((n-p-2)+u_i)((n-1)/n)}{((n-p-1)/(p+1))u_i} - 1/n \right) \right)^{((p-1)/2)-1} \\ \left(1 + \left(\frac{D_i((n-p-2)+u_i)((n-1)/n)}{((n-p-1)/(p+1))u_i} - 1/n \right) \right)^{-(\frac{p-1}{2}+\frac{n-p}{2})} \end{array} \right) |J| \quad \text{- (15)}$$

where $0 \leq D_i \leq \infty, 0 \leq u_i \leq \infty$ and $|J| = \frac{n-p}{p-1} \left(\frac{((n-p-2)+u_i)((n-1)/n)}{((n-p-1)/(p+1))u_i} \right)$

Rearrange (15) and integrate with respect tou_i, we get the marginal distribution of D_i *as*

$$f(D_i, u_i) = \alpha(n,p) \sum_{q=0}^{(p-3)/2} \sum_{k=0}^{\infty} \binom{(p-3)/2}{q} \binom{-(n-1)/2}{k}$$
$$\gamma^{((p-3)/2)-q} \lambda^k \int_0^\infty u_i^{q-(\frac{p-1}{2}+k)-1} (1 + \frac{u_i}{n-p-2})^{-(q-(\frac{p-1}{2}+k)+\frac{n-p}{2})} du_i \quad \text{-(16)}$$

where $0 \leq D_i \leq \infty$,

$$\alpha(n,p) = \frac{1}{B(\frac{1}{2}, \frac{n-p-2}{2}) B(\frac{p-1}{2}, \frac{n-p}{2})} \left(\frac{(n-p-2)^{1/2}(-1/n)^{(p-3)/2}}{((n-p-1)/(p+1))((n-1)/n))^{(\frac{n-3}{2})}} \right)$$

$$\lambda = \left(\frac{D_i(n-p-2)}{(n-p-1)/(p+1)} \right)$$

and $\gamma = -\left(\frac{D_i(n-p-2)(n-1)}{(n-p-1)/(p+1)} \right)$

Finally from (16), after the integration arranging the terms, we get the density of Cook's D distance as the form of series expression as

$$f(D_i; n, p) = \alpha(n, p) \sum_{q=0}^{(p-3)/2} \sum_{k=0}^{\infty} \begin{pmatrix} (p-3)/2 \\ q \end{pmatrix} \begin{pmatrix} -(n-1)/2 \\ k \end{pmatrix}$$
$$\beta(n, p, q, k) D_i^{((p-3)/2)-q+k}$$

\- (17)

where, $0 \leq D_i \leq \infty$,$n, p > 0$, $n > p$

$$\alpha(n, p) = \frac{1}{B(\frac{1}{2}, \frac{n-p-2}{2})B(\frac{p-1}{2}, \frac{n-p}{2})} \left(\frac{(n-p-2)^{-1/2}(-1/n)^{(p-3)/2}}{((n-p-1)/(p+1))((n-1)/n))^{(n-3)/2}} \right)$$

$$\beta(n, p, q, k) = \left(\frac{1}{((n-p-1)/(p+1))} \right)^k \left(-\frac{(n-1)}{(n-p-1)/(p+1)} \right)^{((p-3)/2)-q}$$
$$B(q - (\frac{p-1}{2} + k), \frac{n-p-1}{2})$$

From (17), it is the density function of Cook's D distance which involves the normalizing constants such as$\alpha(n, p)$,$\beta(n, p, q, k)$ and$B(\frac{1}{2}, \frac{n-p-2}{2})$,$B(\frac{p-1}{2}, \frac{n-p}{2})$,$B(q - (\frac{p-1}{2} + k), \frac{n-p-1}{2})$ are the Beta functions respectively with two parameters (n, p), where n is the sample size and p is the no. of predictors used in the multiple linear regression model. In order to know the location and dispersion of Cook's D, the authors derived the first two moments in terms of mean, variance from (8) and it is given as follows.

$$D_i =$$
$$\frac{(n-p-1)}{(p+1)(n-1)} \left(\left(F_{i(1,n-p-2)}/(n-p-2)\right) \sum_{k=0}^{\infty} (-1)^k \left(F_{i(1,n-p-2)}/(n-p-2)\right)^k \right)$$
$$* \left(1 + n(p-1)F_{i(p-1),(n-p)}/(n-p)\right)$$

$$D_i = \frac{(n-p-1)}{(p+1)(n-1)} \left(\sum_{k=0}^{\infty} (-1)^k \left(1/(n-p-2)\right)^{k+1} F_{i(1,n-p-2)}^{k+1} \right)$$
$$* \left(1 + n(p-1)F_{i(p-1),(n-p)}/(n-p)\right)$$

\- (18)

Therefore,

$$E(D) = \frac{(n-p-1)}{(p+1)(n-1)} \left(\sum_{k=0}^{\infty} (-1)^k \left(1/(n-p-2)\right)^{k+1} E(F_{(1,n-p-2)}^{k+1}) \right)$$
$$* \left(1 + n(p-1)E(F_{(p-1),(n-p)})/(n-p)\right)$$

$E(D) = \frac{(p(n-1)-2)(n-p-1)}{2(n-1)(p+1)(n-p+1)}$ - (19)

From (18), Squaring on both sides and take expectation, we get the second moment of the cook's D as

$$D_i^2 = \left(\frac{(n-p-1)}{(p+1)(n-1)}\right)^2 \left(\sum_{k=0}^{\infty}(-1)^k(k+1)\left(1/(n-p-2)\right)^{k+2}\left(F_{i(1,n-p-2)}\right)^{k+2}\right)$$
$$*\left(1+(n(p-1)/(n-p))^2 F^2_{i(p-1),(n-p)} + 2n(p-1)F_{i(p-1),(n-p)}/(n-p)\right)$$

$$E(D^2) = \left(\frac{(n-p-1)}{(p+1)(n-1)}\right)^2 \left(\sum_{k=0}^{\infty}(-1)^k(k+1)\left(1/(n-p-2)\right)^{k+2} E(F^{k+2}_{(1,n-p-2)})\right)$$
$$*\left(1+(n(p-1)/(n-p))^2 E(F^2_{(p-1),(n-p)}) + 2n(p-1)E(F_{(p-1),(n-p)})/(n-p)\right)$$

$$E(D^2) = \frac{((n-p-1)/(p+1)(n-1))^2}{B(\frac{1}{2},\frac{n-p-2}{2})}$$
$$*\left(\sum_{k=0}^{\infty}(-1)^k(k+1)B(\frac{1}{2}+k+2,\frac{n-p-2}{2}-(k+2))\right)$$
$$*\left(1+\left(n^2(p^2-1)/(n-p-2)(n-p-4)\right)+(2n(p-1)/(n-p-2))\right)$$

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Therefore, we know

$V(D) = E(D^2) - (E(D))^2$ - (21)

Substitute (19) and (20) in (21), we get

$V(D) = \frac{(n-p-1)}{(n-p+1)}\left(\frac{1}{(p+1)(n-1)}\right)^2 *$

$$\left(3\left(\frac{p^2+6p+n^2(p^2+2p-2)+n(2-2p^2-8p)+8}{(n-p-2)(n-p-4)}\right)-\frac{(n-p-1)}{(n-p+1)}\left(\frac{(p(n-1)-2)}{2}\right)^2\right)$$

Moreover, the authors adopted test of significance approach of evaluating and identifying the influential observations in a sample. The approach is to derive the critical points of the Cook's distance by using (8) for different values of (n,p) and the significance probability is given by by $p\left(D_i > D_{i(n,p)}(\alpha)\right) = \alpha$. Using the critical points, we can test the significance of the influential observation computed from a multiple linear regression model. The following tables 1 and 2 show the significance points of the distribution of Cook's D for varying sample size*(n)* and no.of predictors *(p)* at 5% and 1% significance (α).

Table 1 Significant two-tail values of the Distribution of Cook's D at 5% level of Significance
$(P\left(D_i > D_{i(0.05)}\right) = 0.05)$

n	***p***													
	2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*
3	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4	.33	-	-	-	-	-	-	-	-	-	-	-	-	-
5	.34	2.96	-	-	-	-	-	-	-	-	-	-	-	-
6	.31	2.27	3.90	-	-	-	-	-	-	-	-	-	-	-
7	.27	1.76	2.85	4.37	-	-	-	-	-	-	-	-	-	-
8	.24	1.41	2.15	3.14	4.65	-	-	-	-	-	-	-	-	-
9	.22	1.17	1.69	2.33	3.30	4.83	-	-	-	-	-	-	-	-
10	.20	.99	1.38	1.81	2.43	3.40	4.96	-	-	-	-	-	-	-
11	.18	.86	1.16	1.47	1.88	2.49	3.48	5.06	-	-	-	-	-	-
12	.16	.76	1.00	1.23	1.51	1.92	2.53	3.53	5.13	-	-	-	-	-
13	.15	.68	.87	1.05	1.26	1.54	1.94	2.56	3.57	5.19	-	-	-	-
14	.14	.61	.78	.92	1.08	1.28	1.56	1.96	2.59	3.60	5.24	-	-	-
15	.13	.56	.70	.81	.94	1.09	1.29	1.57	1.97	2.60	3.62	5.27	-	-
16	.12	.51	.63	.73	.83	.94	1.09	1.30	1.57	1.98	2.61	3.64	5.31	-
17	.12	.47	.58	.66	.74	.83	.95	1.10	1.30	1.58	1.99	2.62	3.66	5.33
18	.11	.44	.54	.60	.67	.75	.84	.95	1.10	1.30	1.58	1.99	2.63	3.67
19	.10	.41	.50	.55	.61	.67	.75	.84	.95	1.10	1.30	1.58	2.00	2.64
20	.10	.39	.46	.51	.56	.61	.67	.75	.84	.95	1.10	1.30	1.59	2.00
21	.09	.36	.43	.48	.52	.56	.61	.67	.75	.84	.95	1.10	1.30	1.59
22	.09	.34	.41	.45	.48	.52	.56	.61	.67	.74	.84	.95	1.10	1.30
23	.09	.33	.38	.42	.45	.48	.52	.56	.61	.67	.74	.83	.95	1.10
24	.08	.31	.36	.40	.42	.45	.48	.52	.56	.61	.67	.74	.83	.95
25	.08	.29	.34	.37	.40	.42	.45	.48	.52	.56	.61	.67	.74	.83
26	.08	.28	.33	.35	.38	.40	.42	.45	.48	.51	.56	.61	.67	.74
27	.07	.27	.31	.34	.36	.38	.40	.42	.45	.48	.51	.55	.60	.66
28	.07	.26	.30	.32	.34	.36	.37	.39	.42	.44	.47	.51	.55	.60
29	.07	.25	.29	.31	.32	.34	.35	.37	.39	.42	.44	.47	.51	.55
30	.07	.24	.27	.29	.31	.32	.34	.35	.37	.39	.41	.44	.47	.51
40	.05	.17	.20	.21	.21	.22	.22	.23	.24	.24	.25	.26	.27	.28
50	.04	.13	.15	.16	.16	.16	.17	.17	.17	.18	.18	.18	.19	.19
60	.03	.11	.12	.13	.13	.13	.13	.13	.14	.14	.14	.14	.14	.15
70	.03	.09	.10	.11	.11	.11	.11	.11	.11	.11	.11	.11	.12	.12
80	.02	.08	.09	.09	.09	.09	.09	.09	.10	.10	.10	.10	.10	.10
90	.02	.07	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08
100	.02	.06	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07
∞	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 2 Significant two-tail values of the Distribution of Cook's D at 1% level of Significance
$(P\left(D_i > D_{i(0.05)}\right) = 0.01)$

n	*p*													
	2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*
3	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4	.33	-	-	-	-	-	-	-	-	-	-	-	-	-
5	.37	9.64	-	-	-	-	-	-	-	-	-	-	-	-
6	.37	6.43	12.42	-	-	-	-	-	-	-	-	-	-	-
7	.35	4.85	7.84	13.81	-	-	-	-	-	-	-	-	-	-
8	.33	3.87	5.71	8.50	14.63	-	-	-	-	-	-	-	-	-
9	.30	3.20	4.44	6.08	8.87	15.16	-	-	-	-	-	-	-	-
10	.28	2.72	3.60	4.66	6.27	9.10	15.53	-	-	-	-	-	-	-
11	.26	2.36	3.02	3.75	4.77	6.39	9.26	15.80	-	-	-	-	-	-
12	.25	2.08	2.59	3.12	3.81	4.83	6.46	9.37	16.01	-	-	-	-	-
13	.23	1.86	2.26	2.66	3.15	3.84	4.87	6.51	9.45	16.17	-	-	-	-
14	.22	1.68	2.01	2.31	2.67	3.16	3.85	4.89	6.55	9.52	16.30	-	-	-
15	.20	1.53	1.80	2.04	2.32	2.67	3.16	3.86	4.90	6.57	9.56	16.41	-	-
16	.19	1.41	1.63	1.82	2.04	2.31	2.67	3.16	3.86	4.91	6.59	9.60	16.49	-
17	.18	1.30	1.49	1.65	1.82	2.03	2.30	2.66	3.15	3.85	4.91	6.60	9.64	16.57
18	.17	1.21	1.38	1.50	1.64	1.81	2.02	2.29	2.65	3.14	3.85	4.91	6.61	9.66
19	.16	1.13	1.27	1.38	1.49	1.63	1.79	2.00	2.28	2.64	3.14	3.85	4.91	6.62
20	.16	1.06	1.19	1.28	1.37	1.48	1.61	1.78	1.99	2.26	2.63	3.13	3.84	4.91
21	.15	1.00	1.11	1.19	1.26	1.35	1.46	1.60	1.76	1.98	2.25	2.62	3.12	3.84
22	.14	.94	1.04	1.11	1.17	1.25	1.34	1.45	1.58	1.75	1.97	2.24	2.61	3.12
23	.14	.89	.98	1.04	1.09	1.16	1.23	1.32	1.43	1.57	1.74	1.96	2.24	2.61
24	.13	.85	.93	.98	1.02	1.08	1.14	1.22	1.31	1.42	1.56	1.73	1.95	2.23
25	.13	.81	.88	.92	.96	1.01	1.06	1.12	1.20	1.30	1.41	1.55	1.72	1.94
26	.12	.77	.84	.88	.91	.95	.99	1.05	1.11	1.19	1.28	1.40	1.54	1.71
27	.12	.74	.80	.83	.86	.89	.93	.98	1.03	1.10	1.18	1.27	1.39	1.53
28	.11	.71	.76	.79	.82	.84	.88	.92	.97	1.02	1.09	1.17	1.27	1.38
29	.11	.68	.73	.76	.78	.80	.83	.86	.91	.95	1.01	1.08	1.16	1.26
30	.11	.65	.70	.72	.74	.76	.79	.82	.85	.89	.94	1.00	1.07	1.15
40	.08	.47	.50	.50	.51	.51	.52	.52	.54	.55	.56	.58	.60	.62
50	.07	.37	.38	.39	.38	.38	.38	.39	.39	.39	.40	.40	.41	.42
60	.05	.30	.31	.31	.31	.31	.30	.30	.30	.31	.31	.31	.31	.32
70	.05	.26	.26	.26	.26	.26	.25	.25	.25	.25	.25	.25	.25	.25
80	.04	.22	.23	.23	.22	.22	.22	.21	.21	.21	.21	.21	.21	.21
90	.04	.20	.20	.20	.19	.19	.19	.19	.18	.18	.18	.18	.18	.18
100	.03	.18	.18	.18	.17	.17	.17	.16	.16	.16	.16	.16	.16	.16
∞	0	0	0	0	0	0	0	0	0	0	0	0	0	0

4. Numerical Results and Discussion

In this section, the authors show a numerical study of evaluating the influential observation based on cook's distance of the *ith* observation in a regression model. For this, the authors fitted Step-wise linear regression models with different set of predictors in a Brand equity study. The data in the study comprised of 18 different attributes about a car brand and the data was collected from 275 car users. A well-structured questionnaire was prepared and distributed to 300 customers and the questions were anchored at five point Likert scale from 1 to 5. After the data collection is over, only 275 completed questionnaires were used for analysis. The Step-wise regression results reveals 4 models were extracted from the regression procedure by using IBM SPSS version 22. For each model, the cook's distance were computed and the identification of influential observations, comparison of proposed approaches with the traditional approach of identifying influential observations are visualized in the following table.3

Table 3

Model	p	Traditional approach		
		Cut-off $4/(n-p)$	No.of Influential observation (n)	Mean Cook's D of Influential observations
1	1	.014599	22	.0797472
2	2	.014652	20	0.074233
3	3	.014706	19	0.084601
4	4	.014760	24	0.062829
Proposed approach				
Model	p	5% Significance level		
		Critical Cook's D	No.of Influential observation (n)	Mean Cook's D of Influential observations
1	1	.00700	31	.0586684
2	2	.02288	15	0.093052
3	3	.02493	13	0.113835
4	4	.02528	15	0.088706
Model	p	1% Significance level		
		Critical Cook's D	No.of Influential observation (n)	Mean Cook's D of Influential observations
		.01203	22	.0797470
1	1	.06236	9	0.129777
2	2	.06297	10	0.13628
3	3	.06126	9	0.125272

p-no.of predictors n=275

Table-3 visualizes the results of the identification and evaluation process of the influential observation based on the cook's D distance in a multiple linear regression model. As far as the traditional approach is concern, the cut-off cook's D distances are 0.014599 for model-1, 0.014652 for model-2, 0.014706 for model -3 and 0.014760 for model-4 respectively. From model-1, we identified 22 observations are more than the prescribed cut-off followed by 20, 19, 24 observations from model-2 model-3 and model-4 respectively. This approach is traditional and if the analyst may change the cut-off then, it will give different results. As far as the proposed approach is concern, the authors identified the influential observations at 5% and 1% test of significance. As far as model 1 is concern 31 observations are said to be influential because the cook's D for these observations are more than the critical cook's D distance. Similarly 15 observations from model 2, 13 observations from model-3 and 15 observations from model-4 are also influential at 5% significance level. In the same manner, 22 observations are influential in model-1 at 1% level of significance followed by 9 observations from model-2 and 10 observations from model-3 and 9 observations from model-4 are more than the critical cook's D at 1% level of significance respectively. Another good evidence was also provided by the authors that is the mean cook's distance of the influential observations are higher than the critical cook's D for all the models at 5% and 1% significance level. This shows the identification of influential observation based on the test of significance gives different results when compared to the traditional approach we recommend the proposed approach is more scientific and it over rides the use of traditional approach in identifying influential observation in multiple regression model. The following control charts exhibits the results of Table 3 graphically.

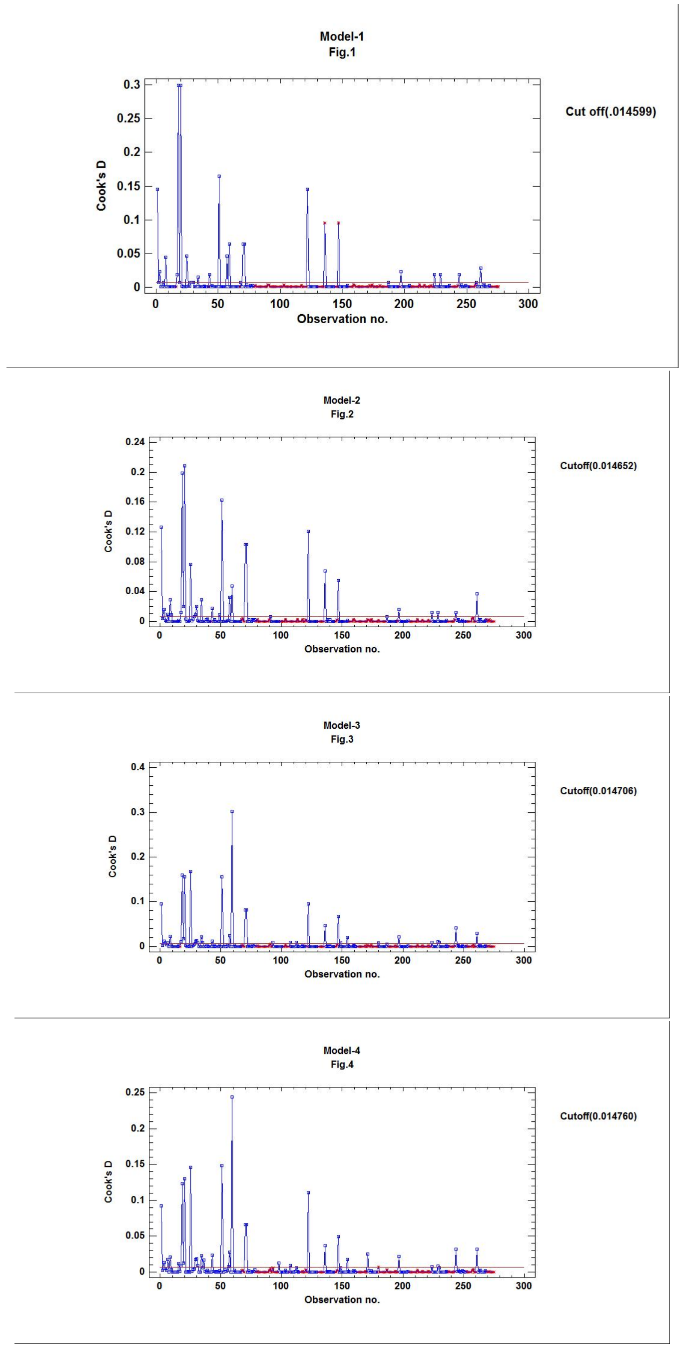

Figure 1. Control charts for each fitted model shows the identification of influential observations based on Traditional approach

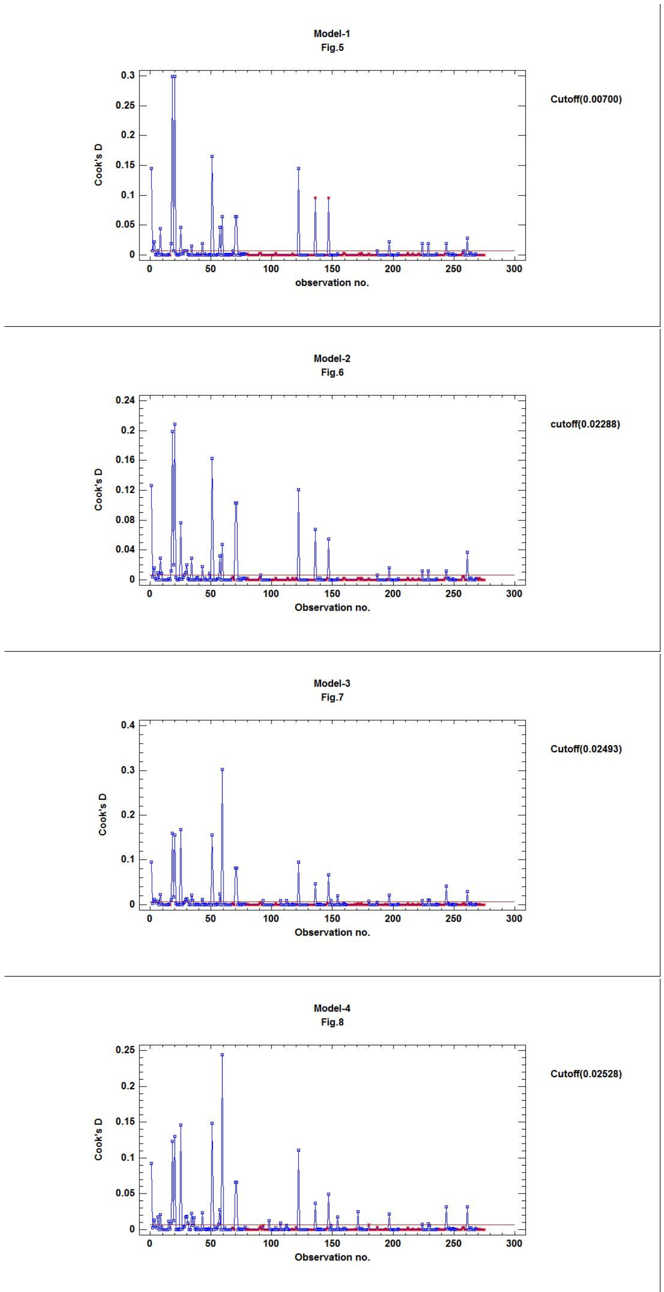

Figure 2. Control charts for each fitted model shows the identification of influential observations at 5% significance level proposed approach

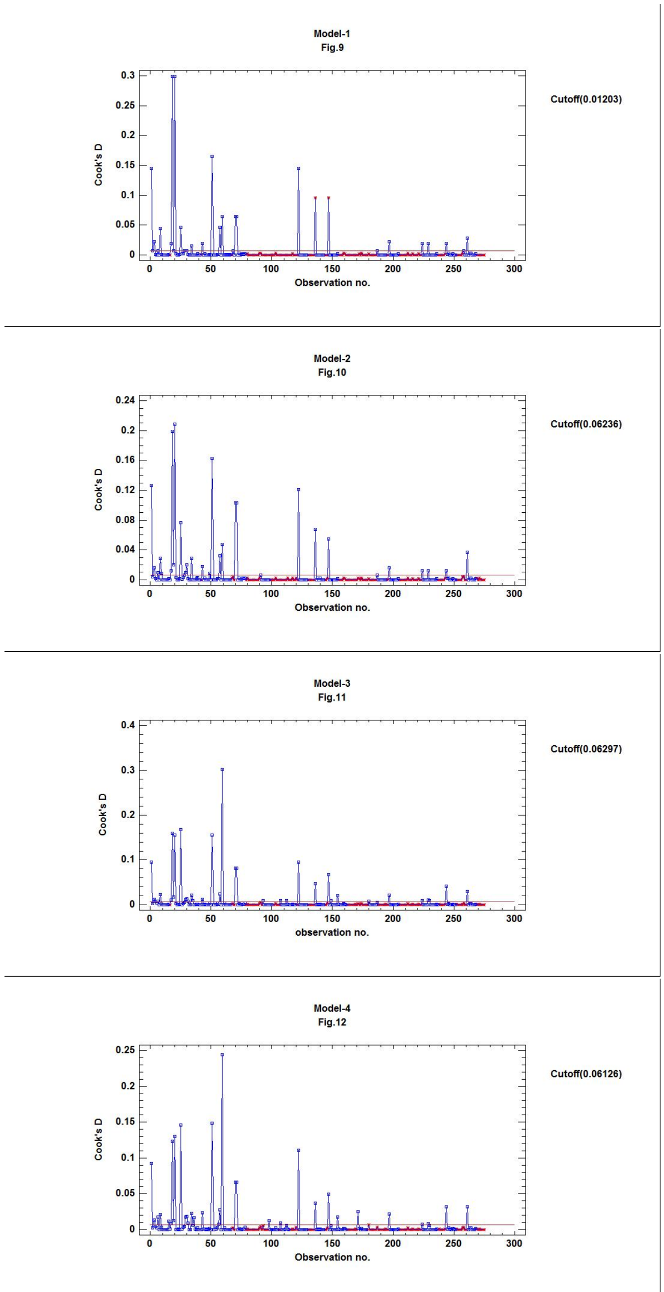

Figure 3. Control charts for each fitted model shows the identification of influential observations at 1% significance level based on proposed approach

5. Conclusion

From the previous sections, the authors proposed a scientific approach which is based on test of significance for identifying and evaluating the influential observation in a multiple linear regression model. At first, the exact distribution of the Cook's D distribution was derived and the authors proved, it followed a beta distribution with 2 shape parameters n and p and we expressed the density function of Cook's D in series expression form. Moreover, the authors computed the Critical points of Cook's D and it is utilized to evaluate the influential observations. Finally, the proposed approach which is more systematic and scientific method of identifying the influential observation because it is based on the test of significance and the results are different when compared it with traditional approach. So the authors found that the proposed approach over rides the use of traditional approach in identifying influential observation in multiple regression models.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 179 – 188

Some results on dynamic discrimination measures of order (α, β)

Suchandan Kayal*

Abstract

In this paper we propose two measures of discrimination of order (α, β) for residual and past lifetimes. Lower and upper bounds of the proposed measures are derived. Some bounds are obtained by considering weighted distributions and subsequently, examples are presented. Finally, characterization results of the proportional hazards and proportional reversed hazards models are given.

2000 AMS Classification: 62N05, 90B25

Keywords: Residual lifetimes, past lifetimes, hazard rate function, reversed hazard rate function, weighted distribution, proportional hazards model, proportional reversed hazards model.

Received 12/09/2013 : Accepted 14/02/2014 Doi : 10.15672/HJMS.201467457

1. Introduction

Discrimination measures are often useful in many applications of probability theory in comparing two probability distributions. They have great importance in information theory, reliability theory, genetics, economics, approximations of probability distributions, signal processing and pattern recognition. Several divergence measures have been proposed for this purpose. Of these the most fundamental one is Kullback-Leibler [13]. Let X and Y be two absolutely continuous random variables (rv's) representing lifetimes of two units. Let $f(x)$, $F(x)$ and $\bar{F}(x)$, respectively be the probability density function (pdf), cumulative distribution function (cdf) and survival function (sf) of X; and the corresponding functions for Y be $g(x)$, $G(x)$ and $\bar{G}(x)$. Let us to take into account that the pdf's are differentiable in their common support. Denote $\eta_X(x) = f(x)/\bar{F}(x)$ and $\eta_Y(x) = g(x)/\bar{G}(x)$ as the hazard rate functions of X and Y, respectively; and $\xi_X(x) = f(x)/F(x)$ and $\xi_Y(x) = g(x)/G(x)$, as their reversed hazard rate functions.

*Department of Mathematics, National Institute of Technology Rourkela, Rourkela-769008, India.
Email: suchandan.kayal@gmail.com, kayals@nitrkl.ac.in

Kullback and Leibler's (KL) discrimination measure, known as relative entropy, between two probability distributions with pdf's $f(x)$ and $g(x)$ is given by

$$I^{KL}_{X,Y} = \int_0^\infty f(x) \ln \frac{f(x)}{g(x)} dx. \tag{1.1}$$

The discrimination measure (1.1) is not appropriate in reliability and life-testing studies as the current age of a system needs to be included. Ebrahimi and Kirmani [11] proposed KL discrimination measure between X and Y at time t (> 0) as

$$I^{KL}_{X,Y}(t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} \ln \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx. \tag{1.2}$$

The measure (1.2) is also known as relative entropy of residual lifetimes $X^+_t = [X - t|X > t]$ and $Y^+_t = [Y - t|Y > t]$. Residual lifetime is an important concept in biology. It is defined as the remaining time to an event given that the survival time X of a patient is at least t. In several clinical studies, particularly when the associated diseases are chronic or/and incurable, it is great concern to patients to know residual lifetime. However, it is reasonable to presume that in many realistic situations, the random lifetime variable is not necessarily related to the future but can also refer to the past. For example, consider a system which is working during a specified time interval and its state is observed only at certain pre-specified inspection times. Suppose the system is inspected for the first time and it is found to be down, then the uncertainty relies in the interval $(0, t)$, it has stopped working. Let X be the failure time of the system, then the variable of interest is $X^-_t = [t - X|X < t]$. It indeed measures the time elapsed from the failure of the component given that its lifetime is less than t. The random variable X^-_t is known as past lifetime of a system. Di Crescenzo and Longobardi [6] proposed a discrimination measure between past lifetimes $X^-_t = [t - X|X < t]$ and $Y^-_t = [t - Y|Y < t]$, which is given by

$$\bar{I}^{KL}_{X,Y}(t) = \int_0^t \frac{f(x)}{F(t)} \ln \frac{f(x)/F(t)}{g(x)/G(t)} dx. \tag{1.3}$$

It is clear that $I^{KL}_{X,Y}(t) = I^{KL}_{X^+_t, Y^+_t}$ and $\bar{I}^{KL}_{X,Y}(t) = I^{KL}_{X^-_t, Y^-_t}$. Discrimination measures are used to measure mutual information concerning two variables. The measures given in (1.2) and (1.3) are respectively useful to compare the residual and past lifetimes of two biological systems, say left or right kidneys. Several researchers have studied KL discrimination measure by including the current age. In this direction we refer to Asadi *et al.* [2], Di Crescenzo and Longobardi [7] and Ebrahimi and Kirmani [10, 11]. Later the discrimination measure (1.1) was generalized, called discrimination measure of order α, as

$$I^R_{X,Y} = \frac{1}{\alpha - 1} \ln \int_0^\infty f^\alpha(x) g^{1-\alpha}(x) dx, \tag{1.4}$$

where $\alpha > 0$ but $\neq 1$. Note that as α tends to 1, $I^R_{X,Y}$ reduces to $I^{KL}_{X,Y}$. As similar measure to (1.2), discrimination measure of order α between two rv's X and Y at time t can be defined by (see Asadi *et al.* [3])

$$I^R_{X,Y}(t) = \frac{1}{\alpha - 1} \ln \int_t^\infty \frac{f^\alpha(x)}{\bar{F}^\alpha(t)} \frac{g^{1-\alpha}(x)}{\bar{G}^{1-\alpha}(t)} dx. \tag{1.5}$$

In literature, it is also dubbed as the relative entropy of order α between X^+_t and Y^+_t. Note that $I^R_{X,Y}(t) = I^R_{X^+_t, Y^+_t}$. Discrimination measure of order α between past lifetimes

X_t^- and Y_t^- is given by (see Asadi *et al.* [4])

$$\bar{I}^R_{X,Y}(t) = \frac{1}{\alpha-1}\ln\int_0^t \frac{f^\alpha(x)}{F^\alpha(t)}\frac{g^{1-\alpha}(x)}{G^{1-\alpha}(t)}dx. \tag{1.6}$$

Note that $\bar{I}^R_{X,Y}(t) = I^R_{X_t^-,Y_t^-}$. For more details we refer to Asadi *et al.* [3], Asadi *et al.* [4], Maya and Sunoj [14], Sunoj and Linu [18] and Sunoj and Sreejith [19]. Based on Varma's entropy (see Varma [20]) the discrimination measure of order α given in (1.4) can be further generalized as

$$I^V_{X,Y} = \frac{1}{\alpha-\beta}\ln\int_0^\infty f^\gamma(x)g^{1-\gamma}(x)dx, \tag{1.7}$$

where $\alpha \neq \beta,\ \beta \geq 1,\ \beta-1 < \alpha < \beta$ and $\gamma = \alpha+\beta-1 > 0$. We shall call it generalized discrimination measure of order (α,β), or discrimination measure of order (α,β). It is worthwhile noting that as β tends to 1, $I^V_{X,Y}$ reduces to $I^R_{X,Y}$, whereas $I^V_{X,Y}$ reduces to $I^{KL}_{X,Y}$, when both α and β tend to 1. In this paper we propose two new dynamic (time dependent) discrimination measures of order (α,β) similar to (1.5) and (1.6) with the following forms:

$$I^V_{X,Y}(t) = \frac{1}{\alpha-\beta}\ln\int_t^\infty \frac{f^\gamma(x)}{\bar{F}^\gamma(t)}\frac{g^{1-\gamma}(x)}{\bar{G}^{1-\gamma}(t)}dx \tag{1.8}$$

and

$$\bar{I}^V_{X,Y}(t) = \frac{1}{\alpha-\beta}\ln\int_0^t \frac{f^\gamma(x)}{F^\gamma(t)}\frac{g^{1-\gamma}(x)}{G^{1-\gamma}(t)}dx. \tag{1.9}$$

It is clear that $I^V_{X,Y}(t) = I^V_{X_t^+,Y_t^+}$ and $\bar{I}^V_{X,Y}(t) = \bar{I}^V_{X_t^-,Y_t^-}$. When β tends to 1, dynamic discrimination measures (1.8) and (1.9) reduce to (1.5) and (1.6), respectively. The dynamic discrimination measures (1.8) and (1.9), respectively reduces to (1.2) and (1.3) when both α and β tend to 1.

To overcome the difficulty of modeling non-experimental, non-replicated and non-random data set which usually occur in environmental and ecological studies, Rao [17] introduced the concept of weighted distributions. Let $f(x)$ be the pdf of X and $w(x)$ be a non-negative function with $\mu_w = E(w(X)) < \infty$. Also let $f_w(x),\ F_w(x)$ and $\bar{F}_w(x)$, respectively be the pdf, cdf and sf of a weighted rv X_w, where $f_w(x) = w(x)f(x)/\mu_w$, $F_w(x) = E(w(X)|X < t)F(x)/\mu_w$ and $\bar{F}_w(x) = E(w(X)|X > t)\bar{F}(x)/\mu_w$. We refer to Di Crescenzo and Longobardi [8], Gupta and Kirmani [12], Maya and Sunoj [14], Navarro *et al.* [15] and Navarro *et al.* [16] for various results and applications on weighted distributions.

Throughout this paper, the terms decreasing and increasing are used for non-increasing and non-decreasing, respectively.

1.1. Definition *Let X and Y be two rv's with pdf's $f(x)$ and $g(x)$, respectively. Then X is said to be less than or equal to Y in likelihood ratio ordering, denoted by $X \overset{lr}{\leq} Y$, if $f(t)/g(t)$ is decreasing in t.*

The rest of the paper is arranged as follows. In Section 2, we obtain some bounds of dynamic discrimination measure of order (α,β) between residual lifetimes. Furthermore a characterization result is stated for the proportional hazard rate models through this discrimination measure. Afterward, analogous results are given for the dynamic discrimination measure of order (α,β) between past lifetimes in Section 3.

2. Residual Lifetimes

In this section we consider dynamic discrimination measure of order (α,β) between two residual lifetimes given in (1.8) and obtain some bounds which are functions of the hazard rates and/or residual entropy of order (α,β). The residual entropy of order (α,β) of a rv X at time t is defined by

$$I_X^V(t) = \frac{1}{\beta-\alpha}\ln\int_t^\infty \frac{f^\gamma(x)}{\bar{F}^\gamma(t)}dx. \tag{2.1}$$

Note that as $\beta \to 1$, $I_X^V(t)$ reduces to residual entropy of order α (see Abraham and Sankaran [1]) and it reduces to residual entropy (see Ebrahimi [9]) when both α and β tend to 1. In the following theorem we obtain lower and upper bounds of $I_{X,Y}^V(t)$ which are functions of hazard rates.

2.1. Theorem *Let* $X \overset{lr}{\leq} Y$. *Then*

$$\begin{aligned}(i)\ I_{X,Y}^V(t) &\geq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_X(t)}{\eta_Y(t)}\Big)\ if\ \gamma>1,\ and\\ (ii)\ I_{X,Y}^V(t) &\leq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_X(t)}{\eta_Y(t)}\Big)\ if\ \gamma<1.\end{aligned}$$

Proof. (i) As $X \overset{lr}{\leq} Y$ and $x>t$, we have $f^{\gamma-1}(t)g^{1-\gamma}(t) \geq f^{\gamma-1}(x)g^{1-\gamma}(x)$ for $\gamma>1$. Thus, from (1.8) we immediately observe that,

$$I_{X,Y}^V(t) \geq \frac{1}{\alpha-\beta}\ln\Big(\frac{f^{\gamma-1}(t)}{\bar{F}^{\gamma-1}(t)}\frac{\bar{G}^{\gamma-1}(t)}{g^{\gamma-1}(t)}\Big) = \frac{1}{\alpha-\beta}\ln\Big(\frac{\eta_X^{\gamma-1}(t)}{\eta_Y^{\gamma-1}(t)}\Big) = \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_X(t)}{\eta_Y(t)}\Big).$$

Moreover, the inequality in (ii) can be yielded similarly by using $f^{\gamma-1}(t)g^{1-\gamma}(t) \leq f^{\gamma-1}(x)g^{1-\gamma}(x)$ when $\gamma<1$.
This completes the proof of the theorem. □

Again since $\eta_X(t)/\eta_{X_w}(t) = E(w(X)|X>t)/w(t)$, Theorem 2.1. leads to the following corollary.

2.1. Corollary *Let* $X \overset{lr}{\leq} X_w$. *Then*

$$\begin{aligned}(i)\ I_{X,X_w}^V(t) &\geq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{E(w(X)|X>t)}{w(t)}\Big)\ if\ \gamma>1,\ and\\ (ii)\ I_{X,X_w}^V(t) &\leq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{E(w(X)|X>t)}{w(t)}\Big)\ if\ \gamma<1.\end{aligned}$$

We consider the following example as an application of the Corollary 2.1.

2.1. Example *Let X be a rv following Pareto distribution with pdf*

$$f(x|a,b) = \frac{ab^a}{x^{a+1}},\ x>b>0,\ a>1.$$

Consider the weight function $w(x)=x$. Here $X \overset{lr}{\leq} X_w$, because the expression $f_w(x)/f(x) = ((a-1)/ab)x$ is an increasing function in x for $a>1$. The dynamic discrimination measure of order (α,β) between X and X_w can be obtained by

$$\begin{aligned}I_{X,X_w}^V(t) &= \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{a}{a-1}\Big)+\frac{1}{\alpha-\beta}\ln\Big(\frac{a}{\gamma+a-1}\Big)\\ &= \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{E(w(X)|X>t)}{w(t)}\Big)+\frac{1}{\alpha-\beta}\ln\Big(\frac{a}{\gamma+a-1}\Big). \qquad (2.2)\end{aligned}$$

Therefore from (2.2), Corollary 2.1. can be verified.

In the following theorem we present upper and lower bounds for $I^V_{X,Y}(t)$, which are the functions of the hazard rate and residual entropy of order (α, β) given in (2.1).

2.2. Theorem *Let $g(x)$ be a decreasing function in x. Then*

$$(i)\ I^V_{X,Y}(t) \leq -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln(\eta_Y(t))\ if\ \gamma > 1,\ and$$
$$(ii)\ I^V_{X,Y}(t) \geq -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln(\eta_Y(t))\ if\ \gamma < 1.$$

Proof: The proof is straightforward. Hence omitted. □

With reference to this fact that the hazard rate function can be written as $\eta_{X_w}(t) = (w(t)\eta_X(t))/E(w(X)|X > t)$, the next corollary follows as a direct consequence of the Theorem 2.2.

2.2. Corollary *Let $f_w(x)$ be a decreasing function in x. Then*

$$(i)\ I^V_{X,X_w}(t) \leq -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\eta_X(t)}{E(w(X)|X>t)}\Big)\ if\ \gamma > 1,\ and$$
$$(ii)\ I^V_{X,X_w}(t) \geq -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\eta_X(t)}{E(w(X)|X>t)}\Big)\ if\ \gamma < 1.$$

The following example illustrates the Corollary 2.2.

2.2. Example *Consider the rv X and the weighted rv X_w as described in Example* 2.1. *Also $f_w(x)$ is decreasing in x. The dynamic discrimination measure of order (α, β), obtained in Example* 2.1. *can be written as*

$$I^V_{X,X_w}(t) = -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\eta_X(t)}{E(w(X)|X>t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{\gamma-1+a\gamma}{\gamma-1+a}\Big), \tag{2.3}$$

provided $\gamma - 1 + a\gamma > 0$. From (2.3) *we easily obtain the inequalities given in the Corollary 2.2.*

In the next result, we consider three rv's X_1, X_2 and X_3, and obtain a lower bound of $I^V_{X_1,X_3}(t) - I^V_{X_2,X_3}(t)$.

2.3. Theorem *Let $X_1,\ X_2,\ X_3$ be three rv's with pdf's $f_1(x),\ f_2(x),\ f_3(x)$; sf's $\bar{F}_1(x),\ \bar{F}_2(x), \bar{F}_3(x)$ and hazard rate functions $\eta_{X_1}(x),\ \eta_{X_2}(x),\ \eta_{X_3}(x)$, respectively. Also let $X_1 \stackrel{lr}{\leq} X_2$. Then the inequality*

$$I^V_{X_1,X_3}(t) - I^V_{X_2,X_3}(t) \geq \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{\eta_{X_1}(t)}{\eta_{X_2}(t)}\Big)$$

holds for $\gamma > 0$.

Proof. Given $X_1 \stackrel{lr}{\leq} X_2$. Therefore, $f_2(x)/f_1(x)$ is an increasing function in x. Thus from (1.8), we get

$$I^V_{X_1,X_3}(t) \geq \frac{1}{\alpha-\beta}\ln\int_t^\infty \frac{f_2^\gamma(x)f_1^\gamma(t)}{f_2^\gamma(t)\bar{F}_1^\gamma(t)}\frac{f_3^{1-\gamma}(x)}{\bar{F}_3^{1-\gamma}(t)}dx,$$

which leads to the required inequality. □

2.1. Remark *Let X_1, X_2 and X_3 be three rv's as described in the Theorem 2.3. with $X_2 \overset{lr}{\leq} X_3$. Then*

$$\begin{aligned}(i)\ I^V_{X_1,X_2}(t) - I^V_{X_1,X_3}(t) &\leq -\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_{X_2}(t)}{\eta_{X_3}(t)}\Big)\ if\ \gamma>1,\ and\\ (ii)\ I^V_{X_1,X_2}(t) - I^V_{X_1,X_3}(t) &\geq -\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_{X_2}(t)}{\eta_{X_3}(t)}\Big)\ if\ \gamma<1.\end{aligned}$$

In the following we shall here derive examples to verify the inequalities stated in the Theorem 2.3. and Remark 2.1.

2.3. Example *Let X_1 and X_2 be two independent rv's following exponential distributions with means $1/\sigma_1$ and $1/\sigma_2$, respectively, where $\sigma_1, \sigma_2 > 0$ and $\sigma_1 > \sigma_2$. It is easy to verify that $X_1 \overset{lr}{\leq} X_2$. With further assumption, $X_3 = \min(X_1, X_2)$, it can be written*

$$(2.4)\ I^V_{X_1,X_3}(t) - I^V_{X_2,X_3}(t) = \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{\eta_{X_1}(t)}{\eta_{X_2}(t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{\sigma_1+\sigma_2-\sigma_1\gamma}{\sigma_1+\sigma_2-\sigma_2\gamma}\Big),$$

provided $\sigma_1 + \sigma_2 - \sigma_1\gamma > 0$ and $\sigma_1 + \sigma_2 - \sigma_2\gamma > 0$. From (2.4) we get

$$I^V_{X_1,X_3}(t) - I^V_{X_2,X_3}(t) \geq \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{\eta_{X_1}(t)}{\eta_{X_2}(t)}\Big).$$

Hence, the Theorem 2.3. is verified.

2.4. Example *Let X_2 and X_3 be two independent rv's with pdf's $f_2(x|a_2, b_2) = a_2 b_2^{a_2}/x^{a_2+1}, x > b_2 > 0, a_2 > 0$ and $f_3(x|a_3, b_3) = a_3 b_3^{a_3}/x^{a_3+1},\ x > b_3 > 0,\ a_3 > 0$, respectively, where $b_2 > b_3$. It can be shown that $X_2 \overset{lr}{\leq} X_3$. Moreover, consider another rv $X_1 = \min(X_2, X_3)$. Then*

$$I^V_{X_1,X_2}(t) - I^V_{X_1,X_3}(t) = -\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_{X_2}(t)}{\eta_{X_3}(t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{a_2\gamma+a_3}{a_3\gamma+a_2}\Big).$$

Hence the inequalities given in Remark 2.1. follow.

Proportional hazards rate model was introduced by Cox in 1972 in order to estimate the effects of different covariates influencing the times to the failures of a system. Since then this model is extensively used in biomedical applications and reliability engineering. We refer to Cox and Oakes [5] for various applications of this model. In the following we obtain a characterization result of the proportional hazard rates models through the dynamic discrimination measure of order (α, β) given in (1.8). Assume that the survival functions of the rv's X and Y are related by

$$(2.5)\qquad \bar{F}(t) = (\bar{G}(t))^{\theta},\ t > 0,$$

where $\theta > 0$ is called proportionality constant.

2.4. Theorem *The dynamic discrimination measure $I^V_{X,Y}(t)$ is independent of t, for $\gamma\theta - \gamma + 1 > 0$, if and only if $F(x)$ and $G(x)$ have proportional hazard rate models.*

Proof. Assume that $F(x)$ and $G(x)$ have proportional hazard rate models, that is, (2.5) holds. Thus using (2.5) in (1.8) we obtain

$$(2.6)\qquad I^V_{X,Y}(t) = \frac{1}{\alpha-\beta}\ln\Big(\frac{\theta^{\gamma}}{\theta\gamma-\gamma+1}\Big),$$

provided $\theta\gamma-\gamma+1>0$. Note that (2.6) is free from t. Next we assume that $I^V_{X,Y}(t)=c_1$, where c_1 is a non-zero constant free from t. Therefore, we have

$$\int_t^\infty \frac{f^\gamma(x)}{\bar F^\gamma(t)}\frac{g^{1-\gamma}(x)}{\bar G^{1-\gamma}(t)}dx = exp\{(\alpha-\beta)c_1\} = c_2(\neq 1),\ say. \tag{2.7}$$

Differentiating (2.7) with respect to t, we get

$$\gamma\phi^{\gamma-1}(t)+(1-\gamma)\phi^\gamma(t)=c_2^{-1}, \tag{2.8}$$

where $\phi(t)=\eta_Y(t)/\eta_X(t)$. We also assume that $\phi(t)$ is a differentiable function. By differentiating from (2.8) with respect to t, we compute

$$\gamma(\gamma-1)\phi'(t)\phi^{\gamma-2}(t)[1-\phi(t)]=0, \tag{2.9}$$

where $\phi'(t)=\frac{d\phi}{dt}$. Therefore, from (2.9), either $\phi'(t)=0$, or $\phi(t)=1$, since $\gamma\neq 1$ and $\phi(t)\neq 0$. Note that $\phi(t)=1$ implies $f(x)=g(x)$, which leads to $c_1=0$. But it is assumed that $c_1\neq 0$. Hence, $\phi(t)=1$ is not a feasible choice. Thus we have $\phi'(t)=0$, that is, there exists a constant $\theta(>0)$ such that $\eta_F(t)=\theta\eta_G(t)$.
This completes the proof of the theorem. □

2.5. Example *We consider a series system of n components with lifetimes X_i, $i=1,\dots,n$, which are identically, independently distributed having exponential distribution with mean lifetime $1/\sigma$. The lifetime of the system is $Z=\min(X_1,\dots,X_n)$. It is easy to see that $\bar F_Z(x)=(\bar F_{X_i}(x))^n$, that is, Z and X_i satisfy the proportional hazard rates models. Here by using (2.6), $I^V_{Z,X_i}(t)$ can be obtained as*

$$I^V_{Z,X_i}(t)=\frac{1}{\alpha-\beta}\ln\Big(\frac{n^\gamma}{n\gamma-\gamma+1}\Big),$$

which is independent of t. Conversely, assuming

$$I^V_{Z,X_i}(t)=\frac{1}{\alpha-\beta}\ln\int_t^\infty \frac{f_Z^\gamma(x)}{\bar F_Z^\gamma(t)}\frac{f_{X_i}^{1-\gamma}(x)}{\bar F_{X_i}^{1-\gamma}(t)}dx = constant$$

and along the lines (Equation 2.7. onwards) of the proof of the Theorem 2.4. it can be shown that $\bar F_Z(x)=(\bar F_{X_i}(x))^n$.

3. Past Lifetimes

Due to duality it is natural to study the dynamic discrimination measure of order (α,β) between past lifetimes given in (1.9). In this section we derive some of its bounds which are functions of the reversed hazard rates and/or past entropy of order (α,β). Note that proofs of the theorems stated for past lifetime case have analogous methodology with the residual lifetime case, hence they are omitted. The past entropy of order (α,β) of a rv X at time t is given by

$$\bar I^V_X(t)=\frac{1}{\beta-\alpha}\ln\int_0^t \frac{f^\gamma(x)}{F^\gamma(t)}dx. \tag{3.1}$$

We have the following theorem regarding upper and lower bounds of $\bar I^V_{X,Y}(t)$, which are functions of reversed hazard rates.

3.1. Theorem *Let $X\overset{lr}{\leq}Y$. Then*

$$(i)\ \bar I^V_{X,Y}(t) \leq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\xi_X(t)}{\xi_Y(t)}\Big)\ if\ \gamma>1,\ and$$

$$(ii)\ \bar I^V_{X,Y}(t) \geq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\xi_X(t)}{\xi_Y(t)}\Big)\ if\ \gamma<1.$$

Note that $\xi_X(t)/\xi_{X_w}(t) = E(w(X)|X < t)/w(t)$. An immediate corollary of this theorem is the following, which, in the weighted rv case can be useful result.

3.1. Corollary *Let* $X \overset{lr}{\leq} X_w$. *Then*

$$(i)\ \bar{I}^V_{X,X_w}(t) \leq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{E(w(X)|X<t)}{w(t)}\Big)\ if\ \gamma>1,\ and$$

$$(ii)\ \bar{I}^V_{X,X_w}(t) \geq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{E(w(X)|X<t)}{w(t)}\Big)\ if\ \gamma<1.$$

The next example describes the results stated in the Corollary 3.1.

3.1. Example *For a rv* X *with pdf*

$$f(x|a) = ax^{a-1}, \quad 0<x<1, \quad a>0. \tag{3.2}$$

Consider the weight function $w(x) = x^b,\ b>0$. *The pdf of* X_w *can be obtained as*

$$f_w(x) = (b+a)x^{b+a-1}, \quad 0<x<1.$$

Therefore, it can be checked that $X \overset{lr}{\leq} X_w$. *Now the expression of* $\bar{I}^V_{X,X_w}(t)$ *is computed by*

$$\bar{I}^V_{X,X_w}(t) = \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{a}{b+a}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{a}{a-b\gamma+b}\Big), \tag{3.3}$$

where $a - b\gamma + b > 0$. *Thus, from* (3.3) *we can easily obtain the inequalities given in the Corollary 3.1.*

In the following result we obtain upper and lower bounds of $\bar{I}^V_{X,Y}(t)$, which are functions of the reversed hazard rate as well as past entropy of order (α, β).

3.2. Theorem *Let* $g(x)$ *be an increasing function in* x. *Then*

$$(i)\ \bar{I}^V_{X,Y}(t) \leq -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln(\xi_Y(t))\ if\ \gamma>1,\ and$$

$$(ii)\ \bar{I}^V_{X,Y}(t) \geq -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln(\xi_Y(t))\ if\ \gamma<1.$$

The Theorem 3.2. leads to the following corollary as, $\xi_{X_w}(t) = w(t)\xi_X(t)/E(w(X)|X < t)$.

3.2. Corollary *Let* $f_w(x)$ *be increasing in* x. *Then*

$$(i)\ \bar{I}^V_{X,X_w}(t) \leq -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\xi_X(t)}{E(w(X)|X<t)}\Big)\ if\ \gamma>1,\ and$$

$$(ii)\ \bar{I}^V_{X,X_w}(t) \geq -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\xi_X(t)}{E(w(X)|X<t)}\Big)\ if\ \gamma<1.$$

In this part of paper we state the following example to illustrate the Corollary 3.2.

3.2. Example *Let* X *be a rv with pdf given by* (3.2). *Consider weight function* $w(x) = x$. *Then*

$$\bar{I}^V_{X,X_w}(t) = -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\xi_X(t)}{E(w(X)|X<t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{a\gamma-\gamma+1}{a-\gamma+1}\Big),$$

provided $a\gamma - \gamma + 1 > 0$ *and* $a - \gamma + 1 > 0$. *Hence, the results in Corollary* 3.2. *follow.*

Furthermore, we consider three rv's $X_1,\ X_2$ and X_3 in the following theorem and obtain an upper bound of $\bar{I}^V_{X_1,X_3}(t) - \bar{I}^V_{X_2,X_3}(t)$.

3.3. Theorem *Let there be three rv's $X_1,\ X_2,\ X_3$ with pdf's $f_1(x),\ f_2(x),\ f_3(x)$; cdf's $F_1(x),\ F_2(x),\ F_3(x)$ and reversed hazard rate functions $\xi_{X_1}(x),\ \xi_{X_2}(x),\ \xi_{X_3}(x)$, respectively. Also let $X_1 \overset{lr}{\leq} X_2$. Then for $\gamma > 0$,*

$$\bar{I}^V_{X_1,X_3}(t) - \bar{I}^V_{X_2,X_3}(t) \leq \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{\xi_{X_1}(t)}{\xi_{X_2}(t)}\Big).$$

3.1. Remark *Consider three rv's $X_1,\ X_2$ and X_3 as described in Theorem 3.3. and $X_2 \overset{lr}{\leq} X_3$. Then*

$$(i)\ \bar{I}^V_{X_1,X_2}(t) - \bar{I}^V_{X_1,X_3}(t) \geq -\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\xi_{X_2}(t)}{\xi_{X_3}(t)}\Big) \text{ if } \gamma > 1, \text{ and}$$

$$(ii)\ \bar{I}^V_{X_1,X_2}(t) - \bar{I}^V_{X_1,X_3}(t) \leq -\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\xi_{X_2}(t)}{\xi_{X_3}(t)}\Big) \text{ if } \gamma < 1.$$

As an application of the Theorem 3.3 and Remark 3.1, the upcoming example is presented

3.3. Example *Let X_1 and X_2 be two independent rv's with pdf's*

$$f_1(x|a_1) = a_1 x^{a_1-1},\quad 0<x<1,\quad a_1>0$$

and

$$f_2(x|a_2) = a_2 x^{a_2-1},\quad 0<x<1,\quad a_2>0,$$

where $a_1 < a_2$. It can be shown that $X_1 \overset{lr}{\leq} X_2$. Consider another rv $X_3 = \max(X_1, X_2)$. Then the inequality of the Theorem 3.3. is provided as,

$$\bar{I}^V_{X_1,X_3}(t) - \bar{I}^V_{X_2,X_3}(t) = \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{\xi_{X_1}(t)}{\xi_{X_2}(t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{a_1+a_2-a_1\gamma}{a_1+a_2-a_2\gamma}\Big),$$

where $a_1+a_2-a_1\gamma > 0$ and $a_1+a_2-a_2\gamma > 0$.

3.4. Example *Let X_2 and X_3 be two independent rv's with pdf's*

$$f_2(x|a_2) = a_2 x^{a_2-1},\quad 0<x<1,\quad a_2>0$$

and

$$f_3(x|a_3) = a_3 x^{a_3-1},\quad 0<x<1,\quad a_3>0,$$

where $a_2 < a_3$. It is easy to see that $X_2 \overset{lr}{\leq} X_3$. Consider another rv $X_1 = \max(X_2, X_3)$. Then

$$\bar{I}^V_{X_1,X_2}(t) = \bar{I}^V_{X_1,X_3}(t) + \frac{1-\gamma}{\alpha-\beta}\ln\Big(\frac{\xi_{X_2}(t)}{\xi_{X_3}(t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{a_2\gamma+a_3}{a_3\gamma+a_2}\Big). \tag{3.4}$$

From (3.4), Remark 3.1. can be verified.

We now conclude this article by presenting a characterization result of proportional reversed hazard rates models through the dynamic discrimination measure of order (α, β) given in (1.9). Suppose the cdf's of two rv's X and Y satisfy the following relation:

$$F(t) = (G(t))^{\theta},\ t > 0, \tag{3.5}$$

where $\theta > 0$.

3.5. Theorem *The dynamic past discrimination measure of order (α, β) $\bar{I}^V_{X,Y}(t)$ is independent of t, for $\gamma\theta - \gamma + 1 > 0$, if and only if $F(x)$ and $G(x)$ have proportional reversed hazard rates models.*

It is worthwhile to mention that if we consider a parallel system of n components instead of series system in Example 2.5 the result in the theorem can be verified.

Acknowledgements

The author sincerely wish to thank Professor Somesh Kumar, Department of Mathematics, Indian Institute of Technology Kharagpur, India for helpful discussions. The author also thanks the reviewers for their suggestions which have improved the content and the presentation of the paper.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 189 – 202

A class of estimators for population median in two occasion rotation sampling

Kumari Priyanka* and Richa Mittal †

Abstract

The present work deals with the problem of estimation of finite population median at current occasion, in two occasion successive (rotation) sampling. A class of estimators has been proposed for the estimation of population median at current occasion, which includes many existing estimators as a particular case. Asymptotic properties including the asymptotic convergence of proposed class of estimators are elaborated. Optimum replacement strategies are also discussed. The proposed class of estimators at optimum condition is compared with the sample median estimator when there is no matching from the previous occasion as well as with some other members of the class. Theoretical results have been justified through empirical interpretation with the help of some natural populations.

2000 AMS Classification: 62D05

Keywords: Population median, Skewed distribution, Successive sampling, Class of estimators, Optimum replacement policy.

Received 28/10/2013 : Accepted 11/03/2014 Doi : 10.15672/HJMS.2014107461

1. Introduction

When both, the characteristic and the composition of the population change over time, then the cross-sectional surveys at a particular point of time become important. The survey estimates are therefore time specific, a feature that is particularly important in some context. For example, the unemployment rate is a key economic indicator that varies over time, the rate may change from one month to the next because of a change in the economy (with business laying off or recruiting new employees). To deal with such kind of circumstances, sampling is done on successive occasions with partial replacement of the units.

*Department of Mathematics, Shivaji College (University of Delhi), New Delhi 110 027, India Email: priyanka.ism@gmail.com

†

Email:sovereignricha@gmail.com

The problem of sampling on two successive occasions was first considered by Jessen (1942), and latter this idea was extended by Patterson (1950), Narain (1953), Eckler (1955), Gordon (1983), Arnab and Okafor (1992), Feng and Zou (1977), Singh and Singh (2001), Singh and Priyanka (2008), Singh et al. (2012) and many others. All the above efforts were devoted to the estimation of population mean or variance on two or more occasion successive sampling.

Often, there are many practical situations where variables involved, consists of extreme values and resulting strong influence on the value of mean. In such cases the study variable is having a highly skewed distribution. For example, the study of environmental issues, the study of income as well as expenditure, the study of social evils such as abortions etc.. In these situations, the mean as a measure of central tendency may not be representative of the population because it moves with the direction of asymmetry leaving the median as a better measure since it is not affected by extreme values.

Most of the studies related to median have been developed by assuming simple random sampling or its ramification in stratified random sampling (Gross (1980), Sedransk and Meyer (1978), Smith and Sedransk (1983)).

As noted earlier, a large number of estimators for estimating the population mean at current occasion have been proposed by various authors, but only a few efforts (namely Martinez-Miranda et al. (2005), Singh et al. (2007) and Rueda and Munoz (2008)) have been made to estimate the population median on current occasion in two occasion successive sampling.

The present work develops a one-parameter class of estimators that estimate the population median on the current occasion in two-occasion successive sampling. The proposed class of estimators includes some of the estimators proposed by Singh et al. (2007) for second quantile as particular cases.

Asymptotic expressions for bias and mean square error including the asymptotic convergence of the proposed class of estimators are derived. The optimum replacement strategies are discussed. The proposed class of estimators at optimum conditions is compared with sample median estimator when there is no matching from the previous occasion as well as with some of the estimators due to Singh et al. (2007) and few other members of its class. Theoretical results are justified by empirical interpretation with the help of some natural populations.

2. Sample Structure and Notations

Let $U = (U_1, U_2, \ldots, U_N)$ be the finite population of N units, which has been sampled over two occasions. It is assumed that size of the population remains unchanged but values of units change over two occasions. The character under study be denoted by x (y) on the first (second) occasions respectively. Simple random sample (without replacement) of n units is taken on the first occasion. A random subsample of $m = n\lambda$ units is retained for use on the second occasion. Now at the current occasion a simple random sample (without replacement) of $u = (n - m) = n\mu$ units is drawn afresh from the remaining $(N - n)$ units of the population so that the sample size on the second occasion is also n. μ and λ, $(\mu + \lambda = 1)$ are the fractions of fresh and matched samples respectively at the second (current) occasion. The following notations are considered for the further use:

M_x, M_y : Population median of the variables x and y, respectively.

$\widehat{M}_x(n), \widehat{M}_x(m), \widehat{M}_y(m), \widehat{M}_y(u)$: Sample medians of the respective variables shown in suffices and based on the sample sizes given in braces.

$f_x(M_x), f_y(M_y)$: The marginal densities of variables x and y, respectively.

3. Proposed Class of Estimators

To estimate the population median M_y on the current (second) occasion, two independent estimators are suggested. One is based on sample of the size $u = n\mu$ drawn afresh on the current (second) occasion and which is given by

$$T_u = \widehat{M}_y(u)\,. \tag{3.1}$$

Second estimator is a one-parameter class of estimators based on the sample of size $m = n\lambda$ common to the both occasions and is defined as

$$T_m(d) = \widehat{M}_y(m)\left[\frac{(A+C)\widehat{M}_x(n) + fB\widehat{M}_x(m)}{(A+fB)\widehat{M}_x(n) + C\widehat{M}_x(m)}\right], \tag{3.2}$$
$$A = (d-1)(d-2), \quad B = (d-1)(d-4),$$
$$C = (d-2)(d-3)(d-4) \text{ and } f = \frac{n}{N},$$

where d is a non-negative constant, identified to minimize the mean square error of the estimator $T_m(d)$.

Now considering the convex linear combination of the estimators T_u and $T_m(d)$, a class of estimators for M_y is proposed as

$$\widehat{T}_d = \varphi T_u + (1-\varphi)T_m(d), \tag{3.3}$$

where φ is an unknown constant to be determined so as to minimize the mean square error of the class of the estimators $\widehat{T}_d$.

3.1. Remark. For estimating the median on each occasion, the estimator T_u is suitable, which implies that more belief on T_u could be shown by choosing φ as 1 (or close to 1), while for estimating the change from occasion to occasion, the estimator $T_m(d)$ could be more useful so φ might be chosen 0 (or close to 0). For asserting both problems simultaneously, the suitable (optimum) choice of φ is desired.

3.2. Remark. The following estimators can be identified as a particular case of the suggested class of estimators $\widehat{T}_d$ to estimate population median on the current occasion in two occasion successive (rotation) sampling for different values of the unknown parameter 'd':

(i) $\widehat{T}_1 = \varphi_1 T_u + (1-\varphi_1)T_m(1)$; (Ratio type estimator)
(ii) $\widehat{T}_2 = \varphi_2 T_u + (1-\varphi_2)T_m(2)$; (Product type estimator)
(iii) $\widehat{T}_3 = \varphi_3 T_u + (1-\varphi_3)T_m(3)$; (Dual to Ratio type estimator)

where

$$T_m(1) = \widehat{M}_y(m)\left[\frac{\widehat{M}_x(n)}{\widehat{M}_x(m)}\right],$$
$$T_m(2) = \widehat{M}_y(m)\left[\frac{\widehat{M}_x(m)}{\widehat{M}_x(n)}\right],$$
$$T_m(3) = \widehat{M}_y(m)\left[\frac{n\widehat{M}_x(n) - m\widehat{M}_x(m)}{(n-m)\widehat{M}_x(n)}\right]$$

and φ_i $(i = 1,2,3)$ are unknown constants to be determined so as to minimize the mean square error of the estimators $\widehat{T}_i$ $(i = 1,2,3)$.

3.3. Remark. The Ratio and Product type estimators, proposed by Singh et al. (2007) for second quantile become particular cases of the proposed family of the estimators $\widehat{T}_d$ for $d = 1$ and 2, respectively.

4. Properties of the Proposed Class of Estimators

The properties of the proposed class of estimators $\widehat{T}_d$ are derived under the following assumptions:

(i) Population size is sufficiently large (i.e. $N \to \infty$), therefore finite population corrections are ignored.
(ii) As $N \to \infty$, the distribution of the bivariate variable (x, y) approaches a continuous distribution,which depend on population under consideration with marginal densities $f_x(\cdot)$ and $f_y(\cdot)$, respectively (see Kuk and Mak (1989)).
(iii) The marginal densities $f_x(\cdot)$ and $f_y(\cdot)$ are positive.
(iv) The sample medians $\widehat{M}_y(u)$, $\widehat{M}_y(m)$, $\widehat{M}_x(m)$ and $\widehat{M}_x(n)$ are consistent and asymptotically normal (see Gross (1980)).
(v) Following Kuk and Mak (1989), P_{yx} is assumed to be the proportion of elements in the population such that $x \le \widehat{M}_x$ and $y \le \widehat{M}_y$.
(vi) The following large sample approximations are assumed:

$$\widehat{M}_y(u) = M_y(1 + e_0), \quad \widehat{M}_y(m) = M_y(1 + e_1), \quad \widehat{M}_x(m) = M_x(1 + e_2),$$
$$\widehat{M}_x(n) = M_x(1 + e_3) \;\text{ such that } |e_i| < 1 \;\; \forall\, i = 0, 1, 2 \text{ and } 3.$$

The values of various related expectations can be seen in Allen et al. (2002) and Singh (2003). Under the above transformations, the estimators T_u and $T_m(d)$ takes the following forms:

$$T_u = M_y(1 + e_0), \tag{4.1}$$

$$\begin{aligned} T_m(d) = M_y[1 &+ e_1 + d_1e_3 + d_2e_2 - d_3e_3 - d_4e_2 - d_1d_3e_3^2 \\ &- d_1d_4e_2e_3 - d_2d_3e_2e_3 - d_2d_4e_2^2 + d_3^2e_3^2 + d_4^2e_2^2 \\ &+ 2d_3d_4e_2e_3 + (d_1 - d_3)e_1e_3 + (d_2 - d_4)e_1e_2] \end{aligned} \tag{4.2}$$

where $d_1 = \dfrac{A + C}{A + fB + C}$, $d_2 = \dfrac{fB}{A + fB + C}$, $d_3 = \dfrac{A + fB}{A + fB + C}$ and $d_4 = \dfrac{C}{A + fB + C}$.

Thus we have the following theorems:

4.1. Theorem. *The bias of the estimator $\widehat{T}_d$ to the first order of approximation is obtained as*

$$B(\widehat{T}_d) = (1 - \varphi)B\{T_m(d)\} \tag{4.3}$$

where

$$B\{T_m(d)\} = \frac{1}{n}Q_1 + \frac{1}{m}Q_2\,, \tag{4.4}$$

$$\begin{aligned} Q_1 = (-d_1d_3 - d_1d_4 - d_2d_3 + d_3^2 + 2d_3d_4)&\frac{\{f_x(M_x)\}^{-2}}{4M_x^2} \\ + (d_1 - d_3)(P_{yx} - 0.25)&\frac{\{f_y(M_y)\}^{-1}\{f_x(M_x)\}^{-1}}{M_yM_x} \end{aligned}$$

and

$$Q_2 = (-d_2 d_4 + d_4^2)\frac{\{f_x(M_x)\}^{-2}}{4M_x^2} + (d_2 - d_4)(P_{yx} - 0.25)\frac{\{f_y(M_y)\}^{-1}\{f_x(M_x)\}^{-1}}{M_y M_x}.$$

Proof. The bias of the estimator $\widehat{T}_d$ is given by

$$B\{\widehat{T}_d\} = E\{\widehat{T}_d - M_y\}$$

$$= \varphi B\{T_u\} + (1-\varphi)B\{T_m(d)\}\,. \tag{4.5}$$

Since, the estimator T_u is unbiased for M_y and $T_m(d)$ is biased for M_y, so the bias of the estimator $T_m(d)$is given by

$$B\{T_m(d)\} = E\{T_m(d) - M_y\}\,.$$

Now, substituting the value of $T_m(d)$ from equation (4.2) in the above equation we get the expression for bias of $T_m(d)$ as in equation (4.4).

Finally substituting the value of $B\{T_m(d)\}$ in equation (4.5), we get the expression for the $B\{\widehat{T}_d\}$as in equation (4.3). □

4.2. Theorem. *The mean square error of the estimator $\widehat{T}_d$ is given by*

$$M(\widehat{T}_d) = \varphi^2 V(T_u) + (1-\varphi)^2 M(T_m(d))_{opt.} \tag{4.6}$$

where

$$V(T_u) = \frac{1}{u}\frac{\{f_y(M_y)\}^{-2}}{4} \tag{4.7}$$

and

$$M(T_m(d))_{opt.} = \frac{1}{m}A_1 + \left(\frac{1}{m} - \frac{1}{n}\right)\{\alpha^{*2}A_2 + 2\alpha^* A_3\} \tag{4.8}$$

where

$$A_1 = \frac{\{f_y(M_y)\}^{-2}}{4}, \qquad A_2 = \frac{\{f_x(M_x)\}^{-2}}{4}\left[\frac{M_y^2}{M_x^2}\right],$$

$$A_3 = (P_{yx} - 0.25)\{f_y(M_y)\}^{-1}\{f_x(M_x)\}^{-1}\left[\frac{M_y}{M_x}\right],$$

$$\alpha^* = [\alpha]_{d=d_0},$$

$\alpha = (d_2 - d_4) = (d_3 - d_1) = \dfrac{fB - C}{A + fB + C}$ *and d_0 is the optimum value of d.*

Proof. The mean square error of the estimator $\widehat{T}_d$ is given by

$$\widehat{M}(T_d) = E[\widehat{T}_d - M_y]^2$$

$$= E[\varphi(T_u - M_y) + (1-\varphi)\{T_m(d) - M_y\}]^2$$

$$= \varphi^2 V(T_u) + (1-\varphi)^2 M[T_m(d)] + 2\varphi(1-\varphi)\,\mathrm{cov}(T_u, T_m(d)) \tag{4.9}$$

where

$$V(T_u) = E[T_u - M_y]^2 \tag{4.10}$$

and

$$M[T_m(d)] = E[T_m(d) - M_y]^2\,. \tag{4.11}$$

As T_u and $T_m(d)$ are based on two independent samples of sizes u and m respectively, hence $\mathrm{cov}(T_u, T_m(d)) = 0$. Now, substituting the values of T_u and $T_m(d)$ from equations

(4.1) and (4.2) in equation (4.10) and (4.11) respectively, taking expectations and ignoring finite population corrections we get the expression for $V(T_u)$ as in equation (4.7) and mean square error of $T_m(d)$is obtained as

$$M[T_m(d)] = \left[\frac{1}{m}A_1 + \left(\frac{1}{m} - \frac{1}{n}\right)\{\alpha^2 A_2 + 2\alpha A_3\}\right]$$

where

$$A_1 = \frac{\{f_y(M_y)\}^{-2}}{4}, \quad A_2 = \frac{\{f_x(M_x)\}^{-2}}{4}\left[\frac{M_y^2}{M_x^2}\right],$$
$$A_3 = (P_{yx} - 0.25)\{f_y(M_y)\}^{-1}\{f_x(M_x)\}^{-1}\left[\frac{M_y}{M_x}\right]$$

and

$$\alpha = (d_2 - d_4) = (d_3 - d_1) = \frac{fB - C}{A + fB + C}.$$

The mean square error of the $T_m(d)$ is a function of α, which in turns is a function of d, hence it can be minimized for d, and therefore we have

$$\frac{\partial\{M[T_m(d)]\}}{\partial d} = 0.$$

This gives $\alpha = \frac{-A_3}{A_2}$, assuming $\frac{\partial\alpha}{\partial d} \neq 0$ which in turns yields a cubic equation in 'd' given by

$$z_1 d^3 + z_2 d^2 + z_3 d + z_4 = 0 \tag{4.12}$$

where

$$z_1 = \left(\frac{A_3}{A_2} - 1\right), \qquad z_2 = (f+9) + \frac{A_3}{A_2}(f-8),$$
$$z_3 = (-5f - 26) + \frac{A_3}{A_2}(23 - 5f)$$

and

$$z_4 = (4f + 24) + \frac{A_3}{A_2}(4f - 22).$$

Now for given values of M_x, M_y, $f_x(M_x)$ and $f_y(M_y)$ one will get the three optimum values of d for which $M[T_m(d)]$ attains the minimum value. The possibility of getting negative or imaginary roots cannot be ruled out. However, Singh and Shukla (1987) has pointed out that for any choice of f, M_x, M_y, $f_x(M_x)$ and $f_y(M_y)$, there exists at least one positive real root of the equation (4.12) ensuring that $M[T_m(d)]$ attaints its minimum within the parameter space $(0, \infty)$. Since, there may exist at most three optimum values of d, a criterion for suitable value of optimum d may be set as follows: "Out of all possible values of optimum d, choose $d = d_0$ as an adequate choice, which makes $|B[T_m(d)]|$ smallest".

Hence, the minimum mean square error of $T_m(d)$ is given by

$$M[T_m(d)]_{\text{opt.}} = \frac{1}{m}A_1 + \left(\frac{1}{m} - \frac{1}{n}\right)A_4 \tag{4.13}$$

where $A_1 = \frac{\{f_y(M_y)\}^{-2}}{4}$, $A_4 = \alpha^{*2}A_2 + 2\alpha^* A_3$, and $\alpha^* = [\alpha]_{d=d_0}$.

Further, substituting the expression for $V(T_u)$ and $M[T_m(d)]_{\text{opt.}}$ in equation (4.9) we get the expression for $M(\widehat{T}_d)$ as in equation (4.6). □

4.3. Remark. The cubic equation (4.12) depends on the population parameters P_{yx}, $f_y(M_y)$ and $f_x(M_x)$. If these parameters are known, the proposed estimator can be easily applied. Otherwise, which is the most often situation in practice, the unknown population parameters are replaced by their sample estimates. The population proportion P_{yx} can be replaced by the sample estimate $\widehat{P}_{yx}$ and the marginal densities $f_y(M_y)$ and $f_x(M_x)$ can be substituted by their kernel estimator or nearest neighbour density estimator or generalized nearest neighbour density estimator related to the kernel estimator (Silverman (1986)). Here, the marginal densities $f_y(M_y)$ and $f_x(M_x)$ are replaced by $\widehat{f}_y(\widehat{M}_y(m))$ and $\widehat{f}_x(\widehat{M}_x(n))$ respectively, which are obtained by method of generalized nearest neighbour density estimation related to kernel estimator.

To estimate $f_y(M_y)$ and $f_x(M_x)$, by generalized nearest neighbour density estimator related to the kernel estimator, following procedure has been adopted:

Choose an integer $h \approx n^{\frac{1}{2}}$ and define the distance $\delta(x_1, x_2)$ between two points on the line to be $|x_1 - x_2|$.

For $\widehat{M}_x(n)$, define $\delta_1(\widehat{M}_x(n)) \leq \delta_2(\widehat{M}_x(n)) \leq \cdots \leq \delta_n(\widehat{M}_x(n))$ to be the distances, arranged in ascending order, from $\widehat{M}_x(n)$ to the points of the sample.

The generalized nearest neighbour density estimate is defined by

$$\widehat{f}(\widehat{M}_x(n)) = \frac{1}{n\delta_h(\widehat{M}_x(n))} \sum_{i=1}^{n} K\left[\frac{\widehat{M}_x(n) - x_i}{\delta_h(\widehat{M}_x(n))}\right]$$

where the kernel function K, satisfies the condition $\int_{-\infty}^{\infty} K(x)dx = 1$.

Here, the kernel function is chosen as Gaussian Kernel given by $K(x) = \dfrac{1}{2\pi} e^{-(\frac{1}{2}x^2)}$.

The estimate of $f_y(M_y)$ can be obtained by the above explained procedure in similar manner.

4.4. Theorem. *The estimator $\widehat{T}_d$, its bias and mean square error are asymptotically convergent to the estimator $\widehat{T}_1$, its bias and mean square error respectively for large d.*

Proof. Taking limit as $d \to \infty$ in equation (3.3) we get

$$\lim_{d\to\infty} \widehat{T}_d = \varphi T_u + (1-\varphi) \lim_{d\to\infty} T_m(d)\,.$$

Since, $d \neq 0$, dividing numerator and denominator of the second term in right hand side of above equation by $d^3$and taking limit as $d \to \infty$, we have

$$\lim_{d\to\infty} \widehat{T}_d = \varphi T_u + (1-\varphi) T_m(1) = \widehat{T}_1\,.$$

This is the ratio type estimator to estimate population median in two occasion rotation sampling as given in Remark 3.2. Similarly, using the expressions of bias and mean square error of the estimator $\widehat{T}_d$, it is easy to see that

$$\lim_{d\to\infty} B\{\widehat{T}_d\} = B\{\widehat{T}_1\}$$

and

$$\lim_{d\to\infty} M\{\widehat{T}_d\} = M\{\widehat{T}_1\}\,.$$

Thus the proposed class of estimators converges to a well-defined estimator even if one chooses arbitrary, a larger value of the unknown parameter d. The bias and mean square error also tends asymptotically to that of ratio type estimator to estimate finite population median. There is no need to bother about the existence of the estimator while choosing a larger value of d. □

5. Minimum Mean Square Error of the Proposed Class of Estimators $\widehat{T}_d$

Since, mean square error of $\widehat{T}_d$ in equation (4.6) is function of unknown constant φ, therefore, it is minimized with respect to φ and subsequently the optimum value of φ is obtained as

$$\varphi_{\text{opt.}} = \frac{M\{T_m(d)\}_{\text{opt.}}}{V(T_u) + M\{T_m(d)\}_{\text{opt.}}} \tag{5.1}$$

and substituting the value of $\varphi_{\text{opt.}}$ from equation (5.1) in equation (4.6), we get the optimum mean square error of the estimator $\widehat{T}_d$ as

$$M(\widehat{T}_d)_{\text{opt.}} = \frac{V(T_u) \cdot M\{T_m(d)\}_{\text{opt.}}}{V(T_u) + M\{T_m(d)\}_{\text{opt.}}}\,. \tag{5.2}$$

Further, by substituting the values from equation (4.7) and equation (4.8) in equation (5.2), we get the simplified value of $M(\widehat{T}_d)_{\text{opt.}}$ as

$$M(\widehat{T}_d)_{\text{opt.}} = \frac{A_1[A_1 + \mu A_4]}{n[A_1 + \mu^2 A_4]}, \tag{5.3}$$

where $\mu\,(= u/n)$ is the fraction of fresh sample drawn on the current (second) occasion. Again $M(\widehat{T}_d)_{\text{opt.}}$ derived in equation (5.3) is the function of μ. To estimate the population median on each occasion the better choice of μ is 1 (case of no matching); however, to estimate the change in median from one occasion to the other, μ should be 0 (case of complete matching). But intuition suggests that an optimum choice of μ is desired to devise the amicable strategy for both the problems simultaneously.

6. Optimum Replacement Policy

The key design parameter affecting the estimates of change is the overlap between successive samples. Maintaining high overlap between repeats of a survey is operationally convenient, since many sampled units have been located and have some experience in the survey. Hence to decide about the optimum value of μ (fraction of sample to be drawn afresh on current occasion) so that M_y may be estimated with maximum precision, we minimize $M(\widehat{T}_d)_{\text{opt.}}$ in equation (5.3) with respect to μ.

The optimum value of μ so obtained is one of the two roots given by

$$\widehat{\mu} = \frac{-A_1 \pm \sqrt{A_1(A_1 + A_4)}}{A_4}\,. \tag{6.1}$$

The real value of $\widehat{\mu}$ exists, iff $A_1(A_1 + A_4) \geq 0$. For any situation, which satisfies this condition, two real values of $\widehat{\mu}$ may be possible, hence in choosing a value of $\widehat{\mu}$, care should be taken to ensure that $0 \leq \widehat{\mu} \leq 1$, all other values of $\widehat{\mu}$ are inadmissible. If both the real values of $\widehat{\mu}$ are admissible, the lowest one will be the best choice as it reduces the total cost of the survey. Substituting the admissible value of $\widehat{\mu}$ say μ_0 from equation (6.1) in equation (5.3), we get the optimum value of the mean square error of the estimator $\widehat{T}_d$ with respect to φ and μ both as

$$M(\widehat{T}_d)_{\text{opt.}^*} = \frac{A_1[A_1 + \mu_0 A_4]}{n[A_1 + \mu_0^2 A_4]}\,.$$

7. Efficiency Comparison

To evaluate the performance of the estimator $\widehat{T}_d$, the estimator $\widehat{T}_d$ at optimum conditions is compared with respect to the estimator $\widehat{M}_y(n)$(the sample median), when there is no matching from previous occasion. Since, $\widehat{M}_y(n)$ is unbiased for population median, its variance for large N is given by

$$(7.1) \qquad V[\widehat{M}_y(n)] = \frac{1}{n}\frac{\{f_y(M_y)\}^{-2}}{4}.$$

The percent relative efficiency of the estimator $\widehat{T}_d$ (under optimal condition) with respect to $\widehat{M}_y(n)$ is given by

$$(7.2) \qquad \text{P.R.E.}(\widehat{T}_d, \widehat{M}_y(n)) = \frac{V[\widehat{M}_y(n)]}{M(\widehat{T}_d)_{\text{opt.}^*}} \times 100\,.$$

The estimator $\widehat{T}_d$(at optimal conditions) is also compared with respect to the estimators $\widehat{T}_1$, $\widehat{T}_2$ and $\widehat{T}_3$, respectively. Hence for large N, the expressions for optimum mean square errors of $\widehat{T}_1$, $\widehat{T}_2$ and $\widehat{T}_3$ are given by

$$M(\widehat{T}_1)_{\text{opt.}^*} = \frac{A_1[A_1+\mu_1 A_5]}{n[A_1+\mu_1^2 A_5]},$$
$$M(\widehat{T}_2)_{\text{opt.}^*} = \frac{A_1[A_1+\mu_2 A_6]}{n[A_1+\mu_2^2 A_6]}$$

and

$$M(\widehat{T}_3)_{\text{opt.}^*} = \frac{A_1[A_1+\mu_3 A_7]}{n[A_1+\mu_3^2 A_7]}$$

where

$$\mu_1 = \frac{-A_1 \pm \sqrt{A_1^2 + A_1A_5}}{A_5}, \qquad \mu_2 = \frac{-A_1 \pm \sqrt{A_1^2 + A_1A_6}}{A_6},$$

$$\mu_3 = \frac{-A_1 \pm \sqrt{A_1^2 + A_1A_7}}{A_7}, \qquad A_1 = \frac{\{f_y(M_y)\}^{-2}}{4},$$

$$A_5 = A_2 - 2A_3, \quad A_6 = A_2 + 2A_3 \text{ and } A_7 = \left(\frac{f}{1+f}\right)^2 A_2 + 2\left(\frac{f}{1+f}\right)A_3\,,$$

where $A_2 = \dfrac{\{f_x(M_x)\}^{-2}}{4}\left[\dfrac{M_y^2}{M_x^2}\right]$ and $A_3 = (P_{yx} - 0.25)\{f_y(M_y)\}^{-1}\{f_x(M_x)\}^{-1}\left[\dfrac{M_y}{M_x}\right]$.

The percent relative efficiencies of $\widehat{T}_d$ at optimum conditions with respect to the estimators $\widehat{T}_i$ for $i = 1, 2$ and 3 at optimum conditions are given by

$$\text{P.R.E.}(\widehat{T}_d, \widehat{T}_i) = \frac{M(\widehat{T}_i)_{\text{opt.}^*}}{M(\widehat{T}_d)_{\text{opt.}^*}} \times 100 \text{ for } i = 1, 2 \text{ and } 3\,.$$

8. Numerical Illustrations

The various results obtained in previous sections are now illustrated using two natural populations.

Population Source. (Free access to the data by Statistical Abstracts of the United States) In the first case, a real life situation consisting $N = 51$ states of United States has been considered. Let y_i represent the number of abortions during 2007 in the ith state of U.S. and x_i be the number of abortions during 2005 in the ith state of U.S. The data are presented pictorially in Figure 8.1 as under:

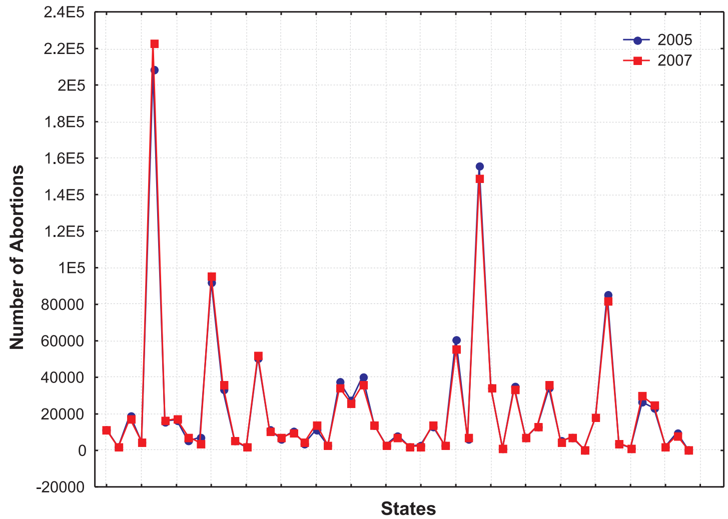

Figure 8.1. Number of Abortions during 2005 and 2007 versus different states of U.S.

Similarly in the second case, the study population consist of $N = 51$ states of United States for year 2004. Let y_i (study variable) be the percent of bachelor degree holders or more in the year 2004 in the ith state of U.S. and x_i be the percent of bachelor degree holders or more in the year 2000 in the ith state of U.S. The data are represented pictorially in Figure 8.2 as under:

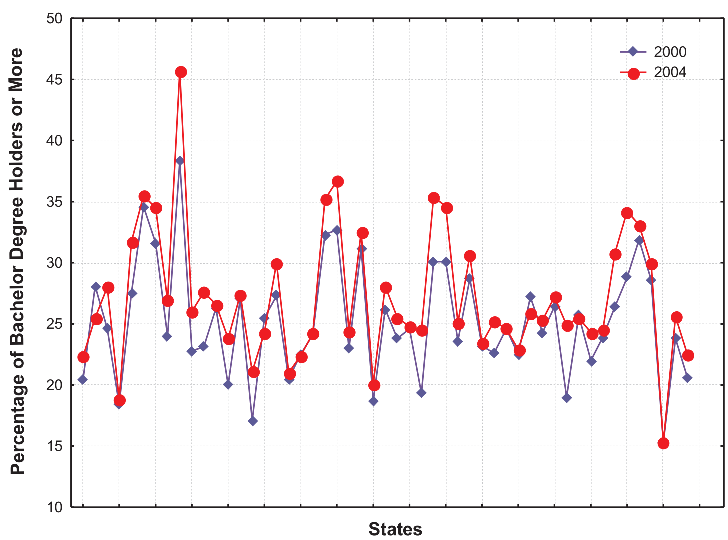

Figure 8.2. Percent of Bachelor Degree Holders or More during 2000 and 2004 versus Different States of U.S.

Table 1. Descriptive Statistics for Population-I and Population-II

	Population-I		**Population-II**	
	Number of Abortions in 2005	**Number of Abortions in 2007**	**% of Bachelor Degree Holders or More in 2000**	**% of Bachelor Degree Holder or More in 2004**
Mean	23651.76	23697.65	27.19	27.17
Standard Error	5389.35	5510.75	0.65	0.75
Median	10410.00	9600.00	24.60	25.50
Standard Deviation	38487.71	39354.65	4.66	5.40
Kurtosis	12.39	14.42	0.29	1.67
Skewness	3.31	3.52	0.40	0.89
Minimum	70.00	90.00	15.30	15.30
Maximum	208430.00	223180.00	30.30	45.70

The graph in Figure 8.1 shows that the distribution of number of abortions in different states is skewed towards right. Similar graph is obtained for Population-II as indicated in Figure 8.2. One reason of skewness may be the distribution of population in different states, that is, the states having larger populations are expected to have larger number of abortion cases and the larger percent of bachelor degree holders or more for the second case as well. Thus skewness of the data indicates that the use of median may be a good measure of central location than mean in such a situation.

Based on the above description, the descriptive statistics for both populations have been computed and are presented in Table 1.

For the two populations under consideration, the cubic equation (4.12) is solved for "d" for some choices of "f". The optimum mean square errors of the proposed class of estimators are found to be same for all the three values of "d" obtained. So, using the criteria set in the proof of Theorem 4.1, Table 2 shows the best choice of the optimum value of "d" for different choices of "f" for both, Population-I and Population-II.

Table 2. Best choice of d for Population-I and Population-II, for different choices of f

f	**Population-I**			**Population-II**		
	d	|Bias|	d_0	d	|Bias|	d_0
0.9800	10.0002	3.6526	**2.4170**	22.8356	0.1419	**2.3553**
	2.4170	0.3097		2.3533	0.1089	
	1.4705	4.1206		1.2030	0.1467	
0.1960	10.7520	1.8948	**2.6449**	2.5878	1.3940	**25.5834**
	2.6449	1.2919		25.5834	0.0702	
	1.3740	2.1515		1.1537	0.0748	
0.2941	11.5280	1.3005	**11.5280**	28.3715	0.0486	**28.3715**
	2.8115	1.5131		2.7621	0.1526	
	1.3146	1.4675		1.1244	0.0504	
0.3922	12.3230	0.9984	**12.3230**	31.1885	0.0367	**31.1885**
	2.9414	1.5271		2.8979	0.1562	
	1.2729	1.1168		1.1047	0.0381	
0.4902	13.1327	0.8141	**13.1327**	34.0268	0.0296	**34.0268**
	3.0462	1.4584		3.0070	0.1532	
	1.2417	0.9026		1.0905	0.0306	

Table 3. Optimum value of μ and percent relative efficiencies of $\widehat{T}_d$ at optimum conditions with respect to $\widehat{M}_y(n)$ and $\widehat{T}_i$ for $i = 1, 2$ and 3 at optimum conditions

	Population-I	**Population-II**
f	0.9800	0.9800
d_0	2.4170	2.3553
μ_0	0.6800	0.6271
P.R.E.$(\widehat{T}_d, \widehat{M}_y(n))$	136.00	125.41
P.R.E.$(\widehat{T}_d, \widehat{T}_1)$	103.33	100.16
P.R.E.$(\widehat{T}_d, \widehat{T}_2)$	206.73	173.48
P.R.E.$(\widehat{T}_d, \widehat{T}_3)$	128.93	120.81

9. Interpretation of Results and Conclusion

(1) From Table 2, it can clearly be seen that the real optimum value of 'd' always exists for both the considered populations. This justifies the feasibility of the proposed class of estimators $\widehat{T}_d$.

(2) From Table 3, it can be seen that the optimum value of μ also exist for both the considered populations. Hence, it indicates that the proposed class of estimators $\widehat{T}_d$ is quite feasible under optimal conditions.

(3) Table 3 indicates that the proposed class of estimators $\widehat{T}_d$ at optimum conditions is highly preferable over sample median estimator $\widehat{M}_y(n)$. It also performs better than the estimators $\widehat{T}_1$ and $\widehat{T}_2$ which are the estimators proposed by Singh et al. (2007) for second quantile. It also proves to be highly efficient than the estimator $\widehat{T}_3$ which is a Dual to Ratio type estimator, a member of its own class.

Hence, it can be concluded that the estimation of median at current occasion is certainly feasible in two occasion successive sampling. The enchanting convergence property of proposed class of estimators $\widehat{T}_d$ justifies the incorporation of unknown parameter in the structure of proposed class of estimators, since the optimum value of the parameter always exists. Hence the proposed class of estimators $\widehat{T}_d$ can be recommended for its further use by survey practitioners.

Acknowledgement. Authors are thankful to the honourable referees for their valuable and inspiring suggestions. Authors are also thankful to UGC, New Delhi, India and Shivaji College (University of Delhi), India for providing the financial assistance to carry out the present work. Authors also acknowledge the free access of the data from Statistical Abstracts of the United States that is available on internet.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 203 – 214

Non-dominated sorting genetic algorithm (NSGA-II) approach to the multi-objective economic statistical design of variable sampling interval T^2 control charts

Asghar Seif* and Majide Sadeghifar†

Abstract

T^2 control charts are used to primarily monitor the mean vector of quality characteristics of a process. Recent studies have shown that using variable sampling interval (VSI) schemes results in charts with more statistical power for detecting small to moderate shifts in the process mean vector. In this study, we have presented a multiple-objective economic statistical design of VSI T^2 control chart when the in-control process mean vector and process covariance matrix are unknown. Then we exert to find the Pareto-optimal designs in which the two objectives are minimized simultaneously by using the Non-dominated sorting genetic algorithm. Through an illustrative example, the advantages of the proposed approach is shown by providing a list of viable optimal solutions and graphical representations, thereby bolding the advantage of flexibility and adaptability.

2000 AMS Classification:

Keywords: Hotelling's T^2 control chart; Economic Statistical Design; NSGA-II Algorithm; Multiple-Objective Optimization; variable sampling interval (VSI) scheme

Received 24/10/2013 : Accepted 04/03/2014 Doi : 10.15672/HJMS.201497460

*Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran
Email: erfan.seif@gmail.com
†Department of Statistics, Bu-Ali Sina University, Hamedan, Iran.

1. Introduction

Control charts have been used widely to monitor industrial processes. Nowadays, in industry, there are many situations in which the simultaneous monitoring or control of two or more related quality process characteristics is necessary. Unfortunately, the current practice in industry toward these multivariate and highly correlated variables is usually to have one set of univariate control charts for each variable. This approach creates many control charts that could easily overwhelm the operator. Also, this approach produces misleading results.

Important literature on multivariate process control include Jackson [1, 2], Alt [3] and Mason, Tracy and Young [4]. Lowry and Montgomery [5] wrote an excellent literature review on multivariate control charts. Extensive discussions on multivariate statistical process control can be found in Mason and Young [6], as well as in Fuchs and Kenett [7].

A common statistical method to monitor multivariate processes is to use the Hotelling T^2 control chart. The Hotelling T^2 control chart, an extension of the univariate Shewhart control chart, was developed by Hotelling [8]. However, because computing the T^2 statistic requires a lot of computations and requires some knowledge of matrix algebra, acceptance of multivariate control charts by industry was slow and hesitant.

Nowadays, with rapid progress in sensor technology and computing power, we are getting more and more data in production, manufacturing, and business operation. Most of these data are correlated multivariate data. The need to implement multivariate process control is growing. Also, with the increasing capability of modern computers, most of the laborious computational work can be accomplished in a split second, and it is getting easier and easier to implement multivariate process control.

The reduction of defective products and non-conformities is a fundamental principle of any quality improvement program and control charts are a powerful statistical tool to reach this goal. Duncan [9] was the first who evaluates the economic consequences of control charts which are affected by the choice of the control chart parameters such as the selection of the sample size (n), the control limits (k), and the time interval between samples (h). Consequently, Duncan [9] showed that statistical control charts may not be cost-effective and may increase the cost of production. Therefore, a wise attention should be given to economic objectives while designing control charts, i.e. selecting the control chart parameters.

Woodall [10] criticized economic designs by their poor statistical performance or their high Type I error rates. Saniga [11] developed a new approach named Economic Statistical Design (ESD) by adding statistical constraints on an economic model to combine the benefits of both pure statistical and economic designs. The ESD approach is very popular in the academic literature; in fact, Montgomery and Woodall [12] mentioned that the trend in economic modeling and design for control charts is to incorporate statistical constraints.

The traditional implementation of control charts is to apply a fixed ratio sampling (FRS) scheme in which samples of fixed size n_0 are obtained at constant intervals h_0 to monitor a process. Taylor [13] noted that economic control charts using the FRS scheme are non-optimal.

Accordingly, some researchers studied the ESDs of control charts with adaptive sampling schemes such as: Variable Sampling Intervals (VSI) (e.g. Chen [14] and Chao et al. [15]), Variable sample sizes (VSS) (e.g. Burr [16], Daudin [17] and Prabhu et al. [18]), Variable Sample Sizes and Sampling Intervals $(VSSI)$ (e.g. Chen [19]), Variable Sampling Intervals and control limits ($VSIC$) (e.g. Torabian et al. [20]), Variable Sample sizes and Control limits ($VSSC$) (e.g. Seif et al. [21, 22]) and Variable Parameters (VP) (e.g. Costa et al. [23]).

One major problem with any of the above mentioned designs is that they may not be flexible and adaptive. Faraz and Saniga [24] addressed the control chart design problem in a way that users are provided with a set of optimal designs which can be tailored to the temporal imperatives of the specific industrial situation. They showed that the proposed approach has the advantages of flexibility and thus adaptability when compared to the traditional economic statistical designs and yet preserve the statistical strengths and economic optimality of traditional designs.

Different solution algorithms are developed to obtain the optimal solution of the multi-objective optimization models. However, the quality of a Pareto optimal set can be evaluated based on three desirable properties, namely, diversity (a wide range of non-dominated solutions), uniformity (a uniform distribution of non-dominated solutions), and cardinality (a large number of non-dominated solutions) ([25, 26]).

The Pareto optimal solutions with the abovementioned properties can be obtained through the evolutionary algorithms such as multi-objective tabu search [27], vector evaluated genetic algorithm [25], multi-objective genetic algorithm [28], and non-dominated sorting genetic algorithm (NSGA and NSGA II) [27]. Unlike most of aforesaid methods that use one elite preservation strategy, NSGA II finds much spread solutions over the Pareto optimal set. It is one of the most popular multi-objective evolutionary algorithms known for its capacity to promote the quality of solutions [27].

Hence, NSGA-II that is an efficient method to identify the Pareto optimal set has been utilized in this research. The proposed Pareto optimization method searches for non- dominated solutions; optimization through the Pareto dominance compares each objective only with itself which remove the need for standardization of objectives.

In this paper, we develop the double objective ESD design of the VSI T^2 control chart, a study that hasn't been found in the literature yet. First we apply the Non-dominated Sorting Genetic Algorithm (NSGA-II) as a solution method. It's been proven that NSGA-II has a better capability in multi-objective optimization problems (see, Deb et al. [29]). Second, we theoretically develop an adaptive sampling intervals scheme with two sampling intervals. We also compare the results with the classical economic statistical designs through an illustrative example.

This paper is organized as follows: In Section 2, the VSI T^2 control scheme and Markov chain approach are briefly reviewed. In Section 3, the cost model proposed by Costa and Rahim [23] is described for our situation then double-objective optimization problem of the ESD VSI T^2 are presented in Section 3.3. Section 4 provides a brief introduction to the principle of the Non-dominated Sorting Genetic Algorithm (NSGAII). Numerical illustrations and comparisons are made in Section 5. Finally, concluding remarks make up the last section.

2. VSI T^2 Control Scheme and Markov Chain Approach

In order to control a process with p correlated characteristics using the T^2 scheme, it is first assumed that the joint probability distribution of the quality characteristics is a p-variate normal distribution with in-control mean vector $\mu_0' = (\mu_{01}, ..., \mu_{0p})$ and variance-covariance matrix $\sum$. Then the subgroups (each of size n) statistics $T_i^2 = n(\bar{X}_i - \mu_0)' \sum^{-1}(\bar{X}_i - \mu_0)$ are plotted in sequential order to form the T^2 control chart. The chart signals as soon as $T_i^2 \geqslant k$.

In statistical design methodology, If the process parameters(μ_0, Σ) are known, k is given by the upper α percentage point of chi-square variable with p degrees of freedom. However μ_0 and Σ are generally unknown and have to be estimated through m initial samples when the process is in control. In this case, the parameter k is obtained upon the $1-\alpha$ percentage point F distribution with p and ν degrees of freedom as follows:

$$(2.1) \qquad k = c(m, n, p)F_\alpha(p, \nu)$$

$c(m, n, p) = \frac{p(m+1)(n-1)}{m(n-1)-p+1}$ and $\nu = m(n-1) - p + 1$. Note that if $n = 1$ then we have $c(m, n, p) = \frac{p(m+1)(n-1)}{m(m-p)}$ and $\nu = m(m-p)$.

In this paper, it is assumed that the process starts in a state of statistical control with mean vector μ_0 and covariance matrix Σ and then after a while assignable causes occur resulting in a shift in the process mean (μ_1). The magnitude of the shift is measured by $d = n(\mu_1 - \mu_0)' \sum^{-1}(\mu_1 - \mu_0)$. Further it is assumed that the time before the assignable cause occurs has an exponential distribution with parameter λ.Thus, the mean time that the process remains in state of statistical control is λ^{-1}.

When an $FRST^2$ chart is used to monitor a multivariate process, a sample of size n_0 is drawn every h_0 hour, and the value of the T^2 statistic (sample point) is plotted on a control chart with $k_0 = c(m, n_0, p)F_\alpha(p, \nu_0)$ as the control limit or action limit. One procedure to improve the statistical performance of the FRS control schemes is Variable Sampling Interval (VSI) scheme that varies the sampling interval between successive samples as a function of prior sample results. In this procedure, the area between the control limits and the origin has been divided into two zones by a warning line w for the use of two different sampling intervals$(h_1 > h_2)$. If the current sample value falls in a particular zone, then the next sample is to be drawn from the process after according to corresponding sampling interval. The use of the VSI control schemes requires the user to select five design parameters: the long and short sampling intervals h_1 and h_2, the fixed sample size n, the warning limit w and the control limit k.

In the literature, the most commonly used measure for comparing control schemes with different sampling strategies is the adjusted average time to signal $(AATS)$. This is also the average time from a process mean shift until the chart produces a signal and is defined as follows::

$$(2.2) \qquad AATS = ATC - \lambda^{-1}$$

where ATC (the average time of the cycle) is the average time from the beginning of the process until the first signal after the process shift. One method of calculating ATC is using Markov chains. Readers are referred to Cinlar [30] for the fundamental ideas behind the Markov chain approach we use. Now, upon the VSI scheme, each sampling stage can be considered as one of the following five transient states:
State 1: $0 \le T^2 < w$ and the process is in control;
State 2: $w \le T^2 < k$ and the process is in control;
State 3: $T^2 \ge k$ and the process is in control (false alarm);
State 4: $0 \le T^2 < w$ and the process is out of control;
State 5: $w \le T^2 < k$ and the process is out of control;

The control chart produces a signal when $T^2 \ge k$. If the current state is 3, the signal is a false alarm; the absorbing state (state 6) is reached when the true alarm occurs. The transition probability matrix is given by

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & p_{46} \\ 0 & 0 & 0 & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

Where p_{ij} denotes the probability of transitioning from state i to j state . In what follows, $F(x, p, \nu, \eta)$ will denote the cumulative probability distribution function of a non-central F distribution with p and ν degrees of freedom and non-centrality parameter $\eta = nd^2$.

$$p_{11} = p(T^2 < w) \times e^{-\lambda h_1} = F(\frac{w}{c(m,n,p)}, p, \eta = 0) \times e^{-\lambda h_1} \tag{2.4}$$

$$p_{12} = p(w \leq T^2 < k) \times e^{-\lambda h_1} = [F(\frac{k}{c(m,n,p)}, p, \eta = 0) - F(\frac{w}{c(m,n,p)}, p, \eta = 0)] \times e^{-\lambda h_1} \tag{2.5}$$

$$p_{13} = p(T^2 \geq k) \times e^{-\lambda h_1} = [1 - F(\frac{k}{c(m,n,p)}, p, \eta = 0)] \times e^{-\lambda h_1} \tag{2.6}$$

$$p_{14} = p(T^2 < w) \times (1 - e^{-\lambda h_1}) = F(\frac{w}{c(m,n,p)}, p, \eta = nd^2) \times (1 - e^{-\lambda h_1}) \tag{2.7}$$

$$p_{15} = p(w \leq T^2 < k) \times (1 - e^{-\lambda h_1}) = [F(\frac{k}{c(m,n,p)}, p, \eta = nd^2) - F(\frac{w}{c(m,n,p)}, p, \eta = nd^2)] \times (1 - e^{-\lambda h_1}) \tag{2.8}$$

$$p_{16} = p(T^2 \geq k) \times (1 - e^{-\lambda h_1}) = [1 - F(\frac{k}{c(m,n,p)}, p, \eta = nd^2)] \times (1 - e^{-\lambda h_1}) \tag{2.9}$$

$$p_{21} = p_{31} = p(T^2 < w) \times e^{-\lambda h_2} = F(\frac{w}{c(m,n,p)}, p, \eta = 0) \times e^{-\lambda h_2} \tag{2.10}$$

$$p_{22} = p_{32} = p(w \leq T^2 < k) \times e^{-\lambda h_2} = [F(\frac{k}{c(m,n,p)}, p, \eta = 0) - F(\frac{w}{c(m,n,p)}, p, \eta = 0)] \times e^{-\lambda h_2} \tag{2.11}$$

$$p_{23} = p_{33} = p(T^2 \geq k) \times e^{-\lambda h_2} = [1 - F(\frac{k}{c(m,n,p)}, p, \eta = 0)] \times e^{-\lambda h_2} \tag{2.12}$$

$$p_{24} = p_{34} = p(T^2 < w) \times (1 - e^{-\lambda h_2}) = F(\frac{w}{c(m,n,p)}, p, \eta = nd^2) \times (1 - e^{-\lambda h_2}) \tag{2.13}$$

$$p_{25} = p_{35} = p(w \leq T^2 < k) \times (1 - e^{-\lambda h_2}) = [F(\frac{k}{c(m,n,p)}, p, \eta = nd^2) - F(\frac{w}{c(m,n,p)}, p, \eta = nd^2)] \times (1 - e^{-\lambda h_2}) \tag{2.14}$$

$$p_{26} = p_{36} = p(T^2 \geq k) \times (1 - e^{-\lambda h_2}) = [1 - F(\frac{k}{c(m,n,p)}, p, \eta = nd^2)] \times (1 - e^{-\lambda h_2}) \tag{2.15}$$

(2.16) $p_{44} = p_{54} = p(T^2 < w) = F(\frac{w}{c(m,n,p)}, p, \eta = nd^2)$

(2.17) $p_{45} = p_{55} = p(w \leq T^2 < k) = F(\frac{k}{c(m,n,p)}, p, \eta = nd^2) - F(\frac{w}{c(m,n,p)}, p, \eta = nd^2)$

(2.18) $p_{46} = p_{56} = p(T^2 \geq k) = 1 - F(\frac{k}{c(m,n,p)}, p, \eta = nd^2)$

Now, ATC is calculated as follows:

(2.19) $ATC = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{h}$

where $\mathbf{h}' = (h_1, h_2, h_2, h_1, h_2)$ is the vector of sampling time intervals, **Q** is the 5×5 matrix obtained from **P** by deleting the elements corresponding to the absorbing state, **I** is the identity matrix of order 5 and $\mathbf{b}' = (p_1, p_2, p_3, p_4, p_5)$ is a vector of initial probabilities, with $\sum_{i=1}^{5} p_i = 1$. In this paper, the vector $\mathbf{b}'$ is set to $(0, 1, 0, 0, 0)$ to provide extra protection and prevent problems that are encountered during start-up.

3. The cost model

3.1. Assumptions. In building our model of a process controlled by a $VSIT^2$ control chart we make the usual assumptions about the process, namely:

1. The p quality characteristics follow a multivariate normal distribution with mean vector μ and covariance matrix Σ.
2. The process is characterized by an in-control state $\mu = \mu_0$.
3. A single assignable cause produces "step changes" in the process mean from $\mu = \mu_0$ to a known $\mu = \mu_1$. This results in a known value of the Mahalanobis distance.
4. "Drifting processes" are not a subject of this research. That is, assignable causes that affect process variability are not addressed, and hence it is assumed that the covariance matrix Σ is constant over time.
5. Before the shift, the process is considered to be in a state of statistical control.
6. The assignable cause is assumed to occur according to a Poisson distribution with intensity λ occurrences per hour.
7. The process is not self-correcting.
8. The quality cycle starts with the in-control state and continues until the process is repaired after an out-of-control signal. It is assumed that the quality cycle follows a renewal reward process.
9. During the search for an assignable cause, the process is shut down.

3.2. The loss function. The process cycle consists of the following four phases: in control, out of control, assignable cause detection, and repair. Therefore, the expected length of a production cycle is given by

(3.1) $E(T) = ATC + T_0 ANF + T_1$

where T_0 is the average amount of time wasted searching for the assignable cause when the process is in control, T_1 is the average time to find and remove the assignable cause, and ANF is the expected number of false alarms per cycle. The expected number of false alarms per cycle is given by

(3.2) $ANF = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-1}(0, 0, 1, 0, 0)$

The expected net profit from a production cycle is given by

$$E(C) = V_0 \times (\frac{1}{\lambda}) + V_1 \times (ATC - \frac{1}{\lambda}) - C_0 \times ANF - C_1 - S \times ANI \tag{3.3}$$

where V_0 is the average profit per hour earned when the process is operating in control, V_1 is the average profit per hour earned when the process is operating out of control, C_0 is the average cost of a false alarm, C_1 is the average cost for detecting and removing the assignable cause, S the cost per inspected item, and ANI is the average number of inspected items per cycle. The average number of inspected items per cycle is given by

$$ANI = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-1}(n, n, n, n, n) \tag{3.4}$$

and the loss function $E(L)$ is given by

$$E(L) = V_0 - \frac{E(C)}{E(T)} \tag{3.5}$$

3.3. Double-objective ESD of the VSIT^2 chart. Equations (2), (21) and (24) give the three important objectives for designing a control chart. By minimizing ANF, a practitioner can reduce false alarm rates. In a similar fashion minimizing $AATS$ guarantees detecting assignable causes as quickly as possible and minimizing the quality cycle cost, or $E(L)$, satisfies the firm's economic objectives. Saniga's [11] ESD approach considers all of the above mentioned criteria but it lacks flexibility and adaptability. This approach provides the practitioners with solutions that consider the trade offs between the statistical and economic objectives.

Let $\overrightarrow{x} = (k, w, h_1, h_2, n)$ be the VSI design vector comprising control limit k, warning line w, and sampling frequencies h_1 and h_2 and sample size n. The most plausible approach to determine the optimal values of the design vector $\overrightarrow{x}$ is that proposed by Saniga [11], called the ESD approach. This approach considers the design problem as an economic single-objective problem with several statistical constraints which has a major focus on reducing the cost of applying control charts. However, in designing control charts, there are three objectives: the expected loss per hour $E(L)$ and the two statistical objectives Type II and Type I error rates, or equivalently $AATS$ and ANF, which should be traded off in some way.

Usually the Type I error rate is somewhat fixed by the practitioners but there is no clear relative preference of the other two objectives. Hence, in this paper, we consider two objectives $E(L)$ and $AATS$ which are of the minimization type and tackle the Type I error issue in constraints. The goal of the double-objective ESD of the $VSIT^2$ scheme is to find $\overrightarrow{x}$ to simultaneously minimize both $E(L)$ and $AATS$ objectives subject to some constraints. Therefore, the double-objective problem is defined as follows:

$$\begin{aligned} & Min \ \ (E(L); AATS) \\ & \quad s.t.: \\ & ANF \leq ANF_0 \\ & \quad k < w \\ & \quad 1 \leq n \in Z^+ \\ & h_2 \leq h_1 \leq h_{max} \end{aligned} \tag{3.6}$$

In the above double-objective model, the constraint $ANF \leq ANF_0$ is added to form the best protection against false alarms; in this paper, without loss of generality, the value of $ANF_0 = 0.05$

shall be used. The parameter h_{max} is added to keep the chart more practical; in particular, we use the values of $h_{max} = 15h$ to eliminate other solutions that may prove problematic in a work shift. The goal of Double-objective ESD of the $VSIT^2$ control chart is to find the seven chart parameters (k, w, h_1, h_2, n) which optimization problem (25), given the five process parameters $(p, \lambda, d, T_0, T_1)$ and the five cost parameters (V_0, V_1, C_0, C_1, S).

4. Elitist non-dominated sorting genetic algorithm (NSGA-II)

A solution to the optimization problem (25) can be described by a decision vector $\overrightarrow{x} = (x_1, x_2, \ldots, x_5)$ in the design space X. The objective functions (2) and (24) define the function f which assigns an objective vector $\overrightarrow{y} = (y_1, y_2)$ in the objective space Y to each solution vector $\overrightarrow{x}$, i.e. f is a vector map of the form $f : X \to Y$. In the multi-objective optimization the optimal solutions form a dominant boundary which is defined as follows:
Suppose $(\overrightarrow{x}_1)$ and $(\overrightarrow{x}_2)$ are two arbitrary and viable solutions in X. we say:
- $(\overrightarrow{x}_1)$ dominates $(\overrightarrow{x}_2)$ $(\overrightarrow{x}_1 < \overrightarrow{x}_2)$ if the two components $\overrightarrow{y}_1 = f(\overrightarrow{x}_1)$ are less than or equal to their corresponding components in $\overrightarrow{y}_2 = f(\overrightarrow{x}_2)$.
- A solution $\overrightarrow{x}$ in X belongs to the dominant boundary if there is no other solution in X that dominates $\overrightarrow{x}$.

Dominant boundary includes all non-dominated optimal solutions to the problem. The set of these solutions is named Pareto set while its image in objective space is named Pareto front. A generic multi-objective optimization solver searches for non-dominated solutions that correspond to trade-offs between all the objectives. The genetic algorithms (GA) are semi-stochastic methods, based on an analogy with Darwin's laws of natural selection. The first multi-objective genetic algorithm (MOGA), called vector-evaluated GA (or VEGA), was proposed by Schaffer [31]. Recently, more advanced MOGA approaches are proposed, for example: the Niched Genetic Algorithm (NPGA) [32], the Non-dominated Sorting Genetic Algorithm (NSGA). Through a comparative case study, Zitzler and Thiele [33] showed that the NSGA has a better capability in multi-objective optimization problems than the VEGA and NPGA. Deb et al. [29] presented a fast and elitist NSGA algorithm called NSGA-II which is proven to have a better capability than the NSGA algorithm. Its main features are as follows:
- A sorting non-dominated procedure where all the individuals are sorted according to the level of non-dominance.
- It implements elitism which stores all non-dominated solutions and enhances convergence properties.
- It adapts a suitable automatic mechanism based on the crowding distance in order to guarantee the diversity of solutions.
- Constraints are implemented using a modified definition of dominance without the use of penalty functions.

In the NSGA-II procedure we have used the following settings of the control parameters: population size (N_{pop}) is set to 100; crossover percentage (p_c) is set to 0.2; mutation rate (r_m) is set to 0.1; mutation percentage (r_p) is set to 0.9; and the maximum number of iterations is set to 500.

5. Numerical analysis

In this section, the model application is illustrated through an industrial example. Consider a product with two important quality characteristics that should be monitored jointly $(p = 2)$. The estimated fixed and variable cost of sampling is \$5 $(S = 5)$ per item. The process is subject to several different types of assignable causes. However, on the average, when the process goes out of control, the magnitude of the mean shift is approximately 0.5 $(d = 0.5)$ and the process mean shift occurs every 100 hours of operation which reasonably can be modeled with an exponential distribution with parameter $\lambda = 0.01$. The average time to investigate an out-of-control signal and repairing the process is 60 minute ($T_1 = 1$), while the time spent to investigate a false alarm is 5 hours ($T_0 = 5$). The cost of detecting and removing the assignable cause is \$500 ($C_1 = 500$), while the cost of investigating a false alarm is \$500 ($C_0 = 500$). The average profit per hour earned when the process is operating in-control is \$500 per hour ($V_0 = 500$), while the average profit per hour earned when the process is operating out-of-control is \$50 per hour ($V_1 = 50$).

In the following, Hotelling's T^2 control charts with the VSI scheme (Table2), and the FRS scheme (Table1), for $d = 0.5$, are compared with respect to the loss function. For example, approximately 16% more savings per hour can be achieved by applying the VSI scheme than the FRS scheme and better statistical properties are also obtained. Consider the process working 8 h a day, 5 days a week and 22 days a month; here, the VSI scheme results in more than \$111724 savings annually. The VSI scheme is also able to detect the process shift $d = 0.5$ after 367-399 min with $AATS$ close to 6.5, but if someone is interested in detecting that shift sooner (around 294-304 min, say) the bolded designs with $AATS$ close to 5 are the good choices, costing 7.73-8.77 dollars per hour more than the economic ones.

Table 1

The optimal design of FRS scheme.

k	h	n	ANF	$AATS$	$Loss$
11.07	9.57	50	0.05	9.70	75.53

In Table 2, we list 20 designs on the Pareto optimal contour or Pareto front. Note that the first design is the least costly, and we see a consistent increase in cost as the $AATS$ becomes smaller, an expected result because Pareto optimal designs, unlike pure statistical design, are cost optimal for these prescribed constraints on $AATS$ and ANF. As illustrated in Figure 1, the multiple-objective economic statistical design(MOESD), using NSGA-II approach, gives a visual indication of how the $AATS$ and $E(L)$ trade off; this easily allow users to consider the costs of improved quality monitoring; that is, tighter control costs more. The advantage of the MOESD using NSGA-II approach is apparent in this example; by providing a set of designs, including graphical representations, each with its own cost, $AATS$, and ANF, the user can tailor the design to the temporal imperative of the industrial process, thereby having the advantage of flexibility and adaptability. Several findings from Tables (1-2) are spelled out as follows.

- The Loss values of the VSI control schemes are consistently smaller than that of the FRS control scheme.
- Compared with the FRS schemes, the corresponding VSI scheme requires more often sampling with a wider upper control limit and a smaller sample size.
- All the cases from the tables indicate that the optimal value of h_2 is close to zero, which means the process should be sampling immediately if T^2 falls into the warning region.
- Smaller $AATS$ implies the VSI control schemes offering a quicker speed for detecting a mean shift.
- The multi-objective solution has the added advantage of demonstrating the tradeoffs between the statistical and economic objectives.

Finally, we point out some more advantages of the proposed multi-objective model in a comparison with the traditional ESD designs introduced by Saniga [11]. Table 3 gives the classical ESDs for the example with two constraints on $AATS$, i.e, $AATS \leq 7$ and $AATS \leq 6$. First, setting these constraints is subjective. Second, in the classical ESDs practitioners have no clear idea about the trade-off between the cost function and statistical constraints. Third, a good guess can be setting the statistical constraints close to the Pareto optimal contour ($AATS \leq 6.65$) of the MOESD approach, i.e., $AATS \leq 7$ or $AATS \leq 6$. Please note that this would not be true in general because setting proper statistical constraints as one does in classical ESDs is different than optimizing the statistical constraints (such as that on $AATS$) as one does in the MOESD approach. Note also the lack of flexibility of the classical ESDs versus the MOESDs. In the latter case, the user has a choice of 20 designs each of which is Pareto optimal, whereas in the former case, only a single design is provided. Reducing control, as the second ESD example shows (larger $AATS$ constraint), results in a decrease in cost.

Table 2
The MOESD $VSIT^2$ chart.

$No.$	k	w	h_1	h_2	n	ANF	$AATS$	$Loss$
1	**11.58**	**3.46**	**10.32**	**0.0001**	**39**	**0.05**	**6.65**	**64.95**
2	**11.70**	**3.48**	**9.91**	**0.0001**	**39**	**0.05**	**6.42**	**64.99**
3	**11.83**	**3.45**	**9.90**	**0.0001**	**40**	**0.05**	**6.28**	**65.14**
4	**11.85**	**3.52**	**9.36**	**0.0001**	**39**	**0.05**	**6.11**	**65.16**
5	11.85	3.21	9.36	0.0001	39	0.05	5.90	65.36
6	11.98	3.47	9.03	0.0001	39	0.05	5.88	65.38
7	11.98	3.47	9.03	0.0001	40	0.05	5.75	65.53
8	12.15	3.60	8.70	0.0001	40	0.04	5.65	65.75
9	12.16	3.50	8.50	0.0001	40	0.05	5.45	66.01
10	12.08	3.47	8.27	0.0001	40	0.05	5.28	66.26
11	12.17	3.57	7.92	0.0001	40	0.05	5.12	66.64
12	**12.18**	**3.47**	**7.93**	**0.0001**	**40**	**0.05**	**5.07**	**66.76**
13	**12.18**	**3.47**	**7.72**	**0.0001**	**40**	**0.05**	**4.93**	**67.10**
14	**12.46**	**3.36**	**8.17**	**0.0001**	**43**	**0.04**	**4.90**	**67.80**
15	12.44	3.46	7.04	0.0001	38	0.05	4.75	67.89
16	12.43	3.48	7.02	0.0001	40	0.05	4.52	68.66
17	13.39	3.27	7.00	0.0001	40	0.03	4.51	69.73
18	13.17	3.46	6.64	0.0001	40	0.04	4.35	70.16
19	12.98	3.41	6.65	0.0001	41	0.04	4.21	70.63
20	12.67	3.51	6.49	0.0001	41	0.05	4.12	70.78

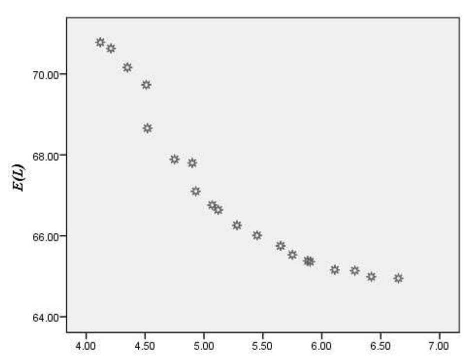

Figure1. Pareto front graph of Table 2.

6. Concluding remark

In this study we have presented a multiple-objective economic statistical design of $VSIT^2$ control chart when the in-control process mean vector and process covariance matrix are unknown. Therefore, a cost model was derived by the Markov Chain approach, and NSGA-II approach was applied to find the optimal design parameters. These solutions define a Pareto optimal set of solutions which greatly increase the flexibility and adaptability of control chart design in practical applications. Using the VSI scheme has been shown to give substantially faster detection of most process shifts than the conventional FRS scheme.

Table 3
The ESD $VSIT^2$ chart.

$Design$	$Constraints$	k	w	h_1	h_2	n	ANF	$AATS$	$Loss$
ESD	$ANF \leq 0.05 \& AATS \leq 7$	11.58	3.52	10.23	0.0001	39	0.05	6.6	64.95
ESD	$ANF \leq 0.05 \& AATS \leq 6$	11.56	4.02	9.81	0.0001	45	0.05	6	65.6

Acknowledgment:
We'd like to thank and express our best gratitude to Islamic Azad University, Hamedan Branch, Hamedan, Iran that financially supported this research project. We also would like to acknowledge and extend our heartfelt gratitude to the anonymous reviewers, the Editor and Editor in chief, all of whom helped us to improve this paper.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 215 – 228

An optimal controlled selection procedure for sample coordination problem using linear programming and distance function

Neeraj Tiwari* and Akhil Chilwal†

Abstract

A number of procedures have been developed for maximizing and minimizing the overlap of sampling units in different/repeated surveys. The concepts of controlled selection, transportation theory and controlled rounding have been used to solve the sample co-ordination problem. In this article, we proposed a procedure for sample co-ordination problem using linear programming with the concept of distance function that facilitates variance estimation using the Horvitz-Thompson estimator. The proposed procedure can be applied to any two-sample surveys having identical universe and stratification. Some examples have been considered to demonstrate the utility of the proposed procedure in comparison to the existing procedures.

2000 AMS Classification: 62D05

Keywords: Controlled Sampling, Distance Function, Sample Coordination, Variance Estimation.

Received 21/06/2013 : Accepted 16/02/2014 Doi : 10.15672/HJMS.201477458

1. Introduction

In practical life, we face situations where the same population is sampled in various surveys so as to obtain information on variety of characters or to obtain current estimates of a characteristic of the population. There are certain applications for which samples are selected at the same time point, for two or more surveys for the same population. For example, a sample can be designed for collecting information about the education status

*Department of Statistics, Kumaun University, S.S.J. Campus, Almora-263601,Uttarakhand (INDIA)
Email: kumarn_amo@yahoo.com
†Department of Statistics, Kumaun University, S.S.J. Campus, Almora-263601,Uttarakhand (INDIA)
Email:akhil.stat@gmail.com

of the families and another sample can be designed to collect information about the income of the families for the same population. On the other hand, if we have improved data after conducting a survey, then it would be desirable to improve the stratification and measures of size. In each survey, it is possible that both the stratification and the measure of size (i.e. the selection probabilities) of the sampling units are different. With the help of updated data a redesign is attempted in which the old units remain the same but the stratification and the selection probabilities are changed. In the redesigning of a survey for the same population, the two samples must be selected sequentially since the designs pertain to different time points. Moreover, it may be considered desirable to contain as many old units as possible in the new sample so that the costs associated with hiring of new enumerators, providing training to the enumerators, etc., can be reduced. Mostly in all the surveys, the cost of sampling is roughly proportional to the total number of units sampled in surveys. Thus, if we select the same units twice instead of selecting two different units, it will reduce the cost of the survey. When the cost of the survey is limited, it is usually desirable to select the units which can be taken as a sample for both the surveys (in case of simultaneous as well as in sequential selection). It can be achieved by minimizing the number of different units in the union of the samples. This is known as the problem of maximization of overlap between the sampling units or the positive sample co-ordination problem. On the other hand, a situation also exists, where it is desirable to withdraw or minimize the likelihood of selecting the same unit in more than one survey. This kind of problem is known as minimization of overlap of sampling units or the problem of negative sample co-ordination.

The problem of co-ordination of sampling units has been a topic of interest for more than fifty years. Various procedures have been proposed by different researchers in order to solve the sample co-ordination problem. Early developments on this topic were due to Patterson (1950) and Keyfitz (1951). Raj (1956) introduced the sample co-ordination problem as a transportation problem in linear programming by considering one unit per stratum. Kish and Hess (1959), Fellegi (1963, 1966), Gray and Platek (1963) and Kish (1963) also proposed some procedures for sample co-ordination problem but these procedures were in general restricted to either two successive samples or to small sample size. To solve the problem in context of a large sample size, Kish and Scott (1971) proposed a procedure for sample co-ordination problem. Brewer et al. (1972) proposed the concept of permanent random number (PRN) for solving the sample co-ordination problem. Causey et al. (1985) proposed an optimum linear programming procedure for maximizing the expected number of sampling units which are common to the two designs, when the two sets of sample units are chosen sequentially. Ernst and Ikeda (1995) also proposed a linear programming procedure for overlap maximization under very general conditions. Ernst (1996) introduced a procedure for sample co-ordination problem, with one unit per stratum designs where the two designs may have different stratifications. Ernst (1998) also proposed a procedure for sample co-ordination problem with no restriction on the number of sample units per stratum, but the stratification must be identical. Both of these procedures proposed by Ernst used the controlled selection algorithm of Causey et al. (1985) and can be used for simultaneous as well as sequential sample surveys. Based on the procedures of Ernst (1996, 1998), Ernst and Paben (2002) introduced a new procedure for sample co-ordination problem, which has no restriction on the number of sample units selected per stratum and also does not require that the two designs have identical stratification. Deville and Tille (2000) used random partition of population to solve the sample co-ordination problem in repeated sample surveys. Matei and Tille (2006) introduced a methodology for sample co-ordination problem based on iterative proportional fitting (IPF), to compute the probability distribution of a bi-design. Their procedure can be applied to any type of sampling design for which it is possible to compute the probability distribution for the two samples. Matei and Skinner (2009) developed optimal sampling design for given unit inclusion probabilities in order to realize maximum co-ordination. Their method is based on the concept of controlled selection and some theoretical conditions on joint selection probability of two samples. Tiwari and Sud (2012) introduced a procedure for solving sample co-ordination problem using the multiple objective linear programming. There procedure is efficient but quite cumbersome in the sense that before applying the idea of nearest proportional to size design to obtain the desired controlled IPPS design they have to first obtain an appropriate uncontrolled IPPS design and then define a non-IPPS design which totally avoids the non-preferred samples to make their probabilities zero. The procedure of Tiwari and Sud (2012) can be used for situations where two surveys are conducted for the same population with identical stratification and the sample units are selected simultaneously for the two designs.

In this article, using the linear programming approach with distance function, we propose an improved method for sample co-ordination problem which maximize (or minimize) the overlap of sampling units between two designs. The proposed procedure got its inspiration from the sample co-ordination procedure of Ernst (1998). The basic concept of the proposed procedure is adopted from Ernst (1998), however the way of solving the controlled selection problem is different and made quite simple in the proposed procedure. The proposed procedure also facilitates variance estimation using Yates–Grundy (1953) form of Horvitz-Thompson (1952) variance estimator, a feature not available with the procedure of Ernst (1998). SAS 9.3 and MATLAB 10.0 windows version packages have been used to solve this problem.

In Section 2, we describe the preliminaries and notations adopted in this article. In Section 3, we discuss the proposed methodology for positive and negative sample co-ordination problem. In Section 4, some numerical examples have been considered to demonstrate the utility of the proposed procedure. Finally in Section 5, the findings of this article are summarized.

2. The Basic Notations and Preliminaries

Let us consider a two-dimensional population array A of N units, consisting of cells that have real numbers, a_{ij}, $(i = 1, \dots, R, j = 1, \dots, C)$. Suppose a sample of size n is to be obtained from this population. Let y be the characteristic under study, Y_{ij} the y-value for the i^{th} & j^{th} unit in the population $(i = 1, \dots, R, j = 1, \dots, C)$ and y_l the y-value for the l^{th} unit in the sample ($l = 1, \dots, n$). Let s_{ij} be each internal entry of a sample (s). Then s_{ij} equals either $[a_{ij}]$ or $[a_{ij}]+1$, where $[a_{ij}]$ is the integer part of a_{ij}. We have to consider a set of samples with selection probabilities that satisfy the constraints:

$$E(s_{ij} \mid i, j) = \sum_{i,j \in s, s \in S} \mathrm{s}_{ij} p(s) = n a_{ij} \tag{2.1}$$

and

$$\sum_{s \in S} p(s) = 1 \tag{2.2}$$

where S is the set of all possible samples $\{s\}$, and $p(s)$ is the selection probability of each sample s.

There can be many sets of probability distributions $p(s)$ satisfying Eq. (2.1) and Eq. (2.2), although only one set of probabilities can be used to obtain a solution to the sample co-ordination problem. We may consider an algorithm based on an appropriate and objective principle to find the solution that reflects the closeness of each sample s to A. For this purpose we consider the several following measures of closeness between A and s.

The ordinary distance, which is often called the Euclidean distance, given by

$$\xi_1(A,s) = [\sum_{i=1}^{R}\sum_{j=1}^{C}(a_{ij} - s_{ij})^2]^{1/2} \tag{2.3}$$

is the most common measure to define the distance between A and s, as it is easy to calculate.

Two other distance measures can also be used to define the distance between A and s. These are:

(i) Cosine Distance Function:

$$\xi_2(A,s) = 1 - \sum_{i=1}^{R}\sum_{j=1}^{C}\frac{a_{ij}s_{ij}}{\|A\|_2\|s\|_2} \tag{2.4}$$

(ii) Bray-Curtis Distance Function:

$$\xi_3(A,s) = \frac{\sum_{i=1}^{R}\sum_{j=1}^{C}(a_{ij}-s_{ij})}{\sum_{i=1}^{R}\sum_{j=1}^{C}(a_{ij}+s_{ij})} \tag{2.5}$$

We have applied these three distance functions given in Eq.(2.3), Eq.(2.4) and Eq.(2.5), to all co-ordination problems considered by us and found that the distance function ξ_2 given in Eq.(2.4) provides best result in terms of the value of the objective function. Therefore, we shall be using ξ_2 as the distance measure in this article.

Following the notations of Ernst (1998), we consider two sampling designs D_1 and D_2, with identical population and stratification, consisting N units, with S denoting one of the strata. We have to select the given number of sample units from the two designs. The selection probability of each unit in S is different for the two designs. First of all we consider the problem of maximizing the overlap of sampling units in D_1 and D_2 designs. For this purpose, the sample units are selected subject to the following conditions originally introduced by Ernst (1998):

(i) There are a predetermined number of units, n_l, selected from S for the D_l sample, $l = 1, 2$. That is, the sample size for each stratum and design combination is fixed.
(ii) The i^{th} unit in S is selected for the D_l sample with its assigned probability, denoted by $(\pi_i)_l$, $l = 1, 2$.
(iii) The expected value of the number of sample units common to the two designs is maximized.
(iv) The number of sample units common to any D_1 and D_2 samples is always within one of the maximum expected value.

In each stratum S, we applied the described procedure separately. First of all, we construct a real valued tabular array $W = (w_{ij})$, which is a two–dimensional array. Here (w_{ij}) represents the internal units of W. A tabular array is one in which the final row and column contain the marginal values (marginal values are the sum of internal values for each row and column). Array W is known as the controlled selection problem, as it specifies the probability and expected value conditions to be satisfied for the problem under consideration.

Ernst (1998) suggested that the problem of maximizing the overlap of sampling units for the two designs can be converted into the "Controlled Selection" problem $W = (w_{ij})$, where W is an $(N+1) \times 5$ array with N internal rows and 4 internal columns, here N is the number of units in the stratum universe. The internal units of W are computed for each internal row $i = 1, \ldots, N$ as follows:

$$w_{i3} = min[(\pi_i)_1, (\pi_i)_2] \tag{2.6}$$

$$w_{il} = (\pi_i)_l - w_{i3}, \qquad l = 1, 2 \tag{2.7}$$

$$w_{i4} = 1 - \sum_{l=1}^{3} \mathrm{w}_{il} \tag{2.8}$$

$$w_{i5} = \sum_{l=1}^{4} \mathrm{w}_{il} \tag{2.9}$$

Array W can be considered as controlled selection problem. The first unit w_{i1} in the i^{th} internal row of this array denotes the probability that the i^{th} unit is in the sample D_1 but not in D_2; the second unit w_{i2} is the probability that the i^{th} unit is in the sample D_2 but not in D_1; the third unit w_{i3} is the probability that the i^{th} unit is in both the samples D_1 and D_2; and the fourth element w_{i4} is the probability that it is in neither sample. The marginals in the first four columns of the last row represents the expected number of units in the corresponding category.

The controlled selection problem W can be solved by constructing a sequence of integer valued tabular array, $M_1 = (m_{ij1})$, $M_2 = (m_{ij2})$,...., $M_u = (m_{iju})$, with the same number of rows and columns as W and associated probabilities p_1, p_2,..., p_u, which specify certain conditions. At last, a random array $M = (m_{ij})$, is then chosen among these u arrays using the indicated probability. Now we discuss the conditions which must be satisfied by this sequence of integer valued arrays. In each internal row of these arrays, one of the four internal columns has the value 1 and the other three have value 0. The value 1 in the first column indicates that the unit is only in the D_1 sample; value 1 in the second column indicates that the unit is only in the D_2 sample. Similarly, value 1 in the third column indicates that the unit is in both samples; and the value 1 in the forth column indicates that the unit is in neither sample.

Ernst (1998) has derived a set of conditions which, if met by the random array M, are sufficient to satisfy the conditions (i)-(iv). These conditions are as follows:

Condition (ii) will be satisfied if

$$p(m_{il} = 1) + p(m_{i3} = 1) = w_{il} + w_{i3} = (\pi_i)_l, \quad i = 1, ..., N, l = 1, 2 \tag{2.10}$$

Similarly, condition (*iii*) will be satisfied if

$$p(m_{i3} = 1) = w_{i3}, \quad i = 1, ..., N, \tag{2.11}$$

If it can be established that if

$$E(m_{il}) = \sum_{v=1}^{u} \mathrm{p}_v m_{ijv} = w_{ij}, \;\; i = 1, ..., N, j = 1, ..., 4 \tag{2.12}$$

then conditions (*ii*) and (*iii*) will hold, since (2.12) implies (2.10) and (2.11).

To establish condition (*i*), one only needs to show that

$$m_{(N+1)lv} + m_{(N+1)3v} = n_l, \quad l = 1, 2 \;\; v = 1, ..., u \tag{2.13}$$

Finally, to establish (*iv*), it is sufficient to show that

$$\mid m_{(N+1)3v} - w_{(N+1)3} \mid < 1, \qquad v = 1, ..., u \tag{2.14}$$

here $w_{(N+1)3}$ is the maximum expected number of units which are common to the two samples and $m_{(N+1)3v}$ is the number of units common to the v^{th} possible sample.

Now the problem reduces to the solution of the controlled selection problem *W* in such a way as to satisfy the conditions (2.12)-(2.14). The solution of the controlled selection problem *W*, will then maximize the overlap of sampling units in the design D_1 and D_2. To find the solution of the controlled selection problem *W*, Ernst (1998) used the procedure of Causey et al. (1985) and showed that the solution obtained through the procedure of Causey et al. (1985) satisfied all the conditions of maximization of overlap. The procedure of Causey et al. (1985) is based on the theory of controlled rounding, developed by Cox and Ernst (1982). In general, a controlled rounding of an $(N+1) \times (L+1)$ tabular array $W = (w_{ij})$ to a positive integer base b is an $(N+1) \times (L+1)$ tabular array $M = (m_{ij})$ for which $m_{ij} = \lfloor w_{ij}/b \rfloor \; or (\lfloor w_{ij}/b \rfloor + 1) b$ for all i,j where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

One drawback of the procedure of Ernst (1998) is that it is quite tedious in implementation. At each step of the procedure one has to obtain the zero-restricted controlled rounding of the adjusted array. Only after this the procedure of controlled selection can be achieved. The other drawback of this procedure is that the variance estimation is not possible in most of the cases due to non-fulfilment of the non-negativity condition $\pi_{ij} \leq \pi_i \pi_j$ of Y-G form of the H-T variance estimator, where π_i and π_j denote the first order inclusion probabilities and π_{ij} is the second order inclusion probability of the units i and j.

Recently, Tiwari and Sud (2012) proposed a procedure for solving sample coordination problem using the multiple objective linear programming. First of all, they constructed a two dimensional real valued array *W*, with internal units w_{ij}, as defined in (2.6)-(2.9). Using FORTRAN 77 program, they obtained a set *A* of all possible combinations of units according to the probabilities of the array *W*. The all possible combinations of units were nothing but the sequence of arrays, M_1, M_2,..., M_t. The probabilities p_1, p_2,..., p_t, satisfying the conditions (2.12)-(2.14), associated with these arrays, were also calculated. After that, they excluded all those arrays from the set A,which do not satisfy the condition (2.13). This set, termed as the set of non-preferred samples, was denoted by A_1. Next, they obtained an IPPS design $p(s)$ for each sample (s) in the set of all possible samples. This IPPS design is known as uncontrolled IPPS design as no restrictions were imposed on the selection probabilities. Tiwari and Sud (2012) used Sampford (1967) IPPS design to obtained the initial uncontrolled IPPS design p(s), as this design imposed only one restriction on selection probabilities. Using the initial IPPS design $p(s)$, they obtained a design $p_0(s)$ given by:

$$p_0(s) = \begin{cases} \frac{p(s)}{1-\sum_{s\in A_1} p(s)} & for s \in (A - A_1) \\ 0 & (otherwise) \end{cases}$$

The design p_0(s) assigned zero probability to the non-preferred samples and was termed as 'controlled design'. This design $p_0(s)$ is no longer an IPPS design. So, Tiwari and Sud (2012) proposed a new IPPS design $p^*(s)$, which is as near as possible to the design $p_0(s)$. To achieve the design $p^*(s)$, they minimized the directed distance D from the sampling design $p^*(s)$ to $p_0(s)$, given as:

$$D(p_0, p^*) = \sum_{s\in(A-A_1)} \frac{p^{*2}(s)}{p_0(s)} - 1$$

The maximization of overlap of units between the two designs was obtained through the solution of the controlled selection problem W. To find the solution of the controlled selection problem W, Tiwari and Sud (2012) applied the theory of multiple objective linear programming. With the help of this multiple objective linear programming they obtained an IPPS design that assigns zero probability to non-preferred samples. Tiwari and Sud (2012) also modified their procedure for the situations where minimization of overlap of sampling units was desirable. To minimize the overlap of sampling units, they redefine the internal units of W and made some changes in the objective function of the linear programming. Their procedure also provided the facility of variance estimation using the HT variance estimator.

3. Optimal Controlled Procedure

We propose a procedure for positive co-ordination problem which also provides the advantage of variance estimation using HT variance estimator. The proposed procedure is compared with the procedure of Tiwari and Sud (2012) and it was found that it provides better results than the procedure of Tiwari and Sud (2012). The linear programming approach in conjunction with a distance function was used in the proposed procedure for maximizing the probability of those sample combinations which consist of maximum number of overlapped sampling units. The non-negativity condition of HT estimator is also achieved in the proposed procedure to facilitate variance estimation through Horvitz-Thompson estimator. The proposed procedure is described as follows:

First of all, we obtain the $(N+1) \times 5$ array W with internal units as discussed in (2.6)-(2.9). Using 'combcoms' command in MATLAB 10.0, we obtain all possible combinations of units according to the probabilities of the array W. Let the set of all possible pairs of D_1 and D_2 samples be denoted by B. The set of all possible samples 'B' satisfy the conditions (2.12)-(2.14). In order to satisfy the condition (2.13), we neglect all those arrays from the set of all possible arrays which do not satisfy the condition (2.13). Let this set of arrays be denoted by B_1. The set B_1 shows the set of non-preferred samples. Now, we obtain an appropriate controlled inclusion probability proportional to size (IPPS) design $p(s)$, for each sample (s) in the set of all possible samples (B), using the values of the internal units (w_{ij}'s) of the array W. The design $p(s)$ assigns zero probability to the non preferred samples and is termed as a controlled IPPS design.

The maximization of overlap of units between the two designs D_1 and D_2 is obtained through the solution of the controlled selection problem W, which satisfies the conditions (2.12)-(2.14). This is achieved through the solution of the following linear programming problem:

Maximize $$\phi = \sum_{s \in B} \xi_2(A, s)p(s) \tag{3.1}$$

Subject to the constraints:

$$\begin{aligned}
&i)\ \sum_{s \in B-B_1} p(s) = 1 \\
&ii)\ \sum_{s \in B-B_1} p(s)m_{ij} = w_{ij}, \qquad i = 1, ..., N, j = 1, ..., 4 \\
&iii)\ \lfloor w_{ijv} \rfloor \leq m_{ijv} \leq \lfloor w_{ijv} \rfloor + 1, \qquad i = 1, ..., N; j = 1, ..., 4; v = 1, ..., u \\
&iv)\ p(s) \geq 0 \\
&v)\ \sum_{s \ni i,j} p(s) \leq (\pi_i)_l(\pi_j)_l, \qquad i < j = 1,...,\text{N};\ l = 1,\ 2 \\
&vi)\ \sum_{s \ni i,j} p(s) > 0, \qquad i < j = 1, .., N
\end{aligned} \tag{3.2}$$

where B-B_1 shows the set of sample combinations consisting of maximum number of overlapped sampling units, s represents a sample in the set of all possible samples generated through the $(N+1) \times 5$ array W and $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

In the proposed procedure, constraints (i) and (iv) in (3.2) are necessary for any sampling design. Constraints (ii) and (iii) are required to satisfy (2.11) and (2.13), respectively. Constraint (ii) also ensures that the resultant design is an IPPS design. Constraint (v) in (3.2) is desirable as it ensures the sufficient condition for non-negativity of the Y-G form of the HT variance estimator and constraint (vi) is desirable as it ensures unbiased variance estimation using HT estimator.

In many situations, it is often desirable to withdraw the selection of same unit for two or more surveys. In these situations, we have to minimize the overlap of sampling units for two or more surveys. The proposed procedure can be easily modified to minimize the overlap of sampling units. In order to minimize the overlap of sampling units, we have to redefine the internal units of the array W. For negative co-ordination, condition (2.6) is replaced by

$$w_{i3} = max(\pi_{i1} + \pi_{i2} - 1, 0) \tag{3.3}$$

Conditions (2.7), (2.8) and (2.9) will remain the same as for the case of maximization of overlap of sampling units. The objective function, in the case of minimization of overlap of sampling units is redefined as:

$$Maximize \qquad \phi = \sum_{s \in C} \xi_2(A,s)p(s) \tag{3.4}$$

where C denotes the set of all sample combinations, which consists of minimum number of overlapped sampling units.

One limitation of the proposed linear programming approach is that it becomes cumbersome when the population size is large, as the process of enumeration of all possible samples and formation of the objective function and constraints becomes quite tedious. With the help of faster computing techniques and modern statistical tools, there may not be much difficulty in using the proposed procedure for large populations. The proposed plan takes lesser computing time in comparison to the procedures of Ernst (1998) and Tiwari & Sud (2012).

The proposed procedure can be used for the situations when the two surveys are conducted for the same population with identical stratification. These two surveys can be conducted sequentially or simultaneously. There is no restriction on the number of units selected per stratum. The proposed procedure is superior to the procedures of Ernst (1998) and Tiwari and Sud (2012) as the proposed procedure maximizes the probability of those sample combinations which consists of maximum number of overlapped sampling units (in case of positive co-ordination) or minimize the probability of those sample combinations which consists of maximum number of overlapped sampling units (in case of negative co-ordination). The proposed procedure also ensures variance estimation using H-T variance estimator and in the situations, where the conditions of H-T estimator could not be satisfied, some alternative variance estimator can be used.

4. Empirical Evaluation

In this section, we shall present some empirical examples to demonstrate the utility of the proposed procedure. We also compare the proposed procedure with the procedures of Ernst (1998) and Tiwari and Sud (2012).

Example 1.1 (Maximization Case): Let consider the following example taken from Ernst (1998), with inclusion probabilities and values of characteristic Y (given in Table 4.1) for two sampling designs with 5 ($N = 5$) different units in each stratum.

Table 4.1

Inclusion probabilities of units

i	1	2	3	4	5
π_{i1}	0.6	0.4	0.8	0.6	0.6
π_{i2}	0.8	0.4	0.2	0.4	0.2

Consider that a sample of size 3 is to be selected for sampling design D_1 and a sample of size 2 for the sampling design D_2, then find the values of internal units of W. Using (2.6)-(2.9), the array W is obtained as:

	0.0	0.2	0.6	0.2	1
	0.0	0.0	0.4	0.6	1
W =	0.6	0.0	0.2	0.2	1
	0.2	0.0	0.4	0.4	1
	0.4	0.0	0.2	0.4	1
	1.2	0.2	1.8	1.8	5

Now we have to solve the above controlled selection problem with $4N$ $(= 20)$ internal units in W and $n = 5$, where n denotes the total number of sample units to be selected from the two designs. The set of all possible samples (B) consists of 15,504 samples. Out of these 15,504 samples, only 24 samples satisfy the condition (2.13). Therefore, all arrays M_l that belongs to the set $(B\text{-}B_1)$ consists of 24 samples given as:

Sample 1 0.0 0.2 0.0 0.0 0.0 0.0 0.4 0.0 0.6 0.0 0.0 0.0 0.2 0.0 0.0 0.0 0.0 0.0 0.0 0.4	Sample 2 0.0 0.2 0.0 0.0 0.0 0.0 0.0 0.6 0.6 0.0 0.0 0.0 0.2 0.0 0.0 0.0 0.0 0.0 0.2 0.0	Sample 3 0.0 0.2 0.0 0.0 0.0 0.0 0.4 0.0 0.6 0.0 0.0 0.0 0.0 0.0 0.0 0.4 0.4 0.0 0.0 0.0	Sample 4 0.0 0.2 0.0 0.0 0.0 0.0 0.0 0.6 0.6 0.0 0.0 0.0 0.0 0.0 0.4 0.0 0.4 0.0 0.0 0.0
Sample 5 0.0 0.0 0.6 0.0 0.0 0.0 0.4 0.0 0.6 0.0 0.0 0.0 0.0 0.0 0.0 0.4 0.0 0.0 0.0 0.4	Sample 6 0.0 0.0 0.6 0.0 0.0 0.0 0.0 0.6 0.6 0.0 0.0 0.0 0.0 0.0 0.4 0.0 0.0 0.0 0.0 0.4	Sample 7 0.0 0.0 0.6 0.0 0.0 0.0 0.0 0.6 0.6 0.0 0.0 0.0 0.0 0.0 0.0 0.4 0.0 0.0 0.2 0.0	Sample 8 0.0 0.0 0.0 0.2 0.0 0.0 0.4 0.0 0.6 0.0 0.0 0.0 0.0 0.0 0.4 0.0 0.0 0.0 0.0 0.4
Sample 9 0.0 0.0 0.0 0.2 0.0 0.0 0.4 0.0 0.6 0.0 0.0 0.0 0.0 0.0 0.0 0.4 0.0 0.0 0.2 0.0	Sample 10 0.0 0.0 0.0 0.2 0.0 0.0 0.0 0.6 0.6 0.0 0.0 0.0 0.0 0.0 0.4 0.0 0.0 0.0 0.2 0.0	Sample 11 0.0 0.2 0.0 0.0 0.0 0.0 0.4 0.0 0.0 0.0 0.0 0.2 0.2 0.0 0.0 0.0 0.4 0.0 0.0 0.0	Sample 12 0.0 0.2 0.0 0.0 0.0 0.0 0.0 0.6 0.0 0.0 0.2 0.0 0.2 0.0 0.0 0.0 0.4 0.0 0.0 0.0
Sample 13 0.0 0.0 0.6 0.0 0.0 0.0 0.4 0.0 0.0 0.0 0.0 0.2 0.2 0.0 0.0 0.0 0.0 0.0 0.0 0.4	Sample 14 0.0 0.0 0.6 0.0 0.0 0.0 0.0 0.6 0.0 0.0 0.2 0.0 0.2 0.0 0.0 0.0 0.0 0.0 0.0 0.4	Sample 15 0.0 0.0 0.6 0.0 0.0 0.0 0.0 0.6 0.0 0.0 0.0 0.2 0.2 0.0 0.0 0.0 0.0 0.0 0.2 0.0	Sample 16 0.0 0.0 0.0 0.2 0.0 0.0 0.4 0.0 0.0 0.0 0.2 0.0 0.2 0.0 0.0 0.0 0.0 0.0 0.0 0.4
Sample 17 0.0 0.0 0.0 0.2 0.0 0.0 0.4 0.0 0.0 0.0 0.0 0.2 0.2 0.0 0.0 0.0 0.0 0.0 0.2 0.0	Sample 18 0.0 0.0 0.0 0.2 0.0 0.0 0.0 0.6 0.0 0.0 0.2 0.0 0.2 0.0 0.0 0.0 0.0 0.0 0.2 0.0	Sample 19 0.0 0.0 0.6 0.0 0.0 0.0 0.4 0.0 0.0 0.0 0.0 0.2 0.0 0.0 0.0 0.4 0.4 0.0 0.0 0.0	Sample 20 0.0 0.0 0.6 0.0 0.0 0.0 0.0 0.6 0.0 0.0 0.2 0.0 0.0 0.0 0.0 0.4 0.4 0.0 0.0 0.0

Sample 21	Sample 22	Sample 23	Sample 24
0.0 0.0 0.6 0.0 0.0 0.0 0.0 0.6 0.0 0.0 0.0 0.2 0.0 0.0 0.4 0.0 0.4 0.0 0.0 0.0	0.0 0.0 0.0 0.2 0.0 0.0 0.4 0.0 0.0 0.0 0.2 0.0 0.0 0.0 0.0 0.4 0.4 0.0 0.0 0.0	0.0 0.0 0.0 0.2 0.0 0.0 0.4 0.0 0.0 0.0 0.0 0.2 0.0 0.0 0.4 0.0 0.4 0.0 0.0 0.0	0.0 0.0 0.0 0.2 0.0 0.0 0.0 0.6 0.0 0.0 0.2 0.0 0.0 0.0 0.4 0.0 0.4 0.0 0.0 0.0

Here B-B_1 consists of the set of sample combinations which have two overlapped sampling units, that is, sample numbers 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24. Thus, the objective function (3.1) becomes:

Maximize $\phi = 0.2476*x[5] + 0.1874*x[6] + 0.223*x[7] + 0.3557*x[8] + 0.4013*x[9] + 0.3271*x[10] + 0.4013*x[13] + 0.3271*x[14] + 0.3705*x[15] + 0.5444*x[16] + 0.6115*x[17] + 0.5047*x[18] + 0.3557*x[19] + 0.2863*x[20] + 0.2863*x[21] + 0.4860*x[22] + 0.4860*x[23] + 0.4013*x[24]$

Applying the method discussed in Section 3 and solving the resultant linear programming problem through the SAS 9.3 and MATLAB 10.0 windows version packages, we obtain the controlled IPPS plan given in Table 4.2.

The objective function has the value: $\phi = 0.260624$

Table 4.2

Optimal controlled IPPS plan corresponding to proposed procedure

s	$p(s)$	s	$p(s)$	s	$p(s)$	s	$p(s)$
1	0.0	7	0.12	13	0.08	19	0.04
2	0.0	8	0.04	14	0.0	20	0.12
3	0.0	9	0.0	15	0.0	21	0.0
4	0.20	10	0.0	16	0.04	22	0.0
5	0.12	11	0.0	17	0.08	23	0.0
6	0.12	12	0.0	18	0.0	24	0.0

Ernst (1998) obtained the following solution for this problem:

$p_3 = 0.2$, $p_6 = 0.4$, $p_{18} = 0.2$, $p_{19} = 0.2$

Tiwari and Sud (2012) obtained the following solution for this problem:

$p_1 = 0.08$, $p_3 = 0.04$, $p_4 = 0.08$, $p_5 = 0.08$, $p_6 = 0.24$, $p_9 = 0.08$,
$p_{15} = 0.12$, $p_{20} = 0.16$, $p_{22} = 0.04$, $p_{23} = 0.08$

Using the procedures of Ernst (1998) and Tiwari and Sud (2012), we find the value of ϕ is 0.8 and 0.8, respectively. Thus, we observe that the value of ϕ for the proposed procedure is very small in comparison to the procedures of Ernst (1998) and Tiwari and Sud (2012). With the help of proposed procedure we can also estimate the value of variance using the Horvitz-Thompson variance estimator.

Example 1.2 (Minimization Case): Let us suppose the inclusion probabilities of Example 1.1 for the two sampling designs for 5 different units. For the sampling design

D_1, we have to select a sample size of 3, and a sample of size 2 for the sampling design D_2, in such a way that the overlap between the two designs is minimized. First of all we find the values of internal units of W. Using (2.7)-(2.9) and (3.3), W is obtained as:

$$W = \begin{array}{cccc|c} 0.2 & 0.4 & 0.4 & 0.0 & 1 \\ 0.4 & 0.4 & 0.0 & 0.2 & 1 \\ 0.8 & 0.2 & 0.0 & 0.0 & 1 \\ 0.6 & 0.4 & 0.0 & 0.0 & 1 \\ 0.6 & 0.2 & 0.0 & 0.2 & 1 \\ \hline 2.6 & 1.6 & 0.4 & 0.4 & 5 \end{array}$$

Solving the controlled selection problem with $N = 20$ and $n = 5$ the possible combinations satisfying condition (2.13) are given in Appendix. After solving this example with the help of proposed scheme, we obtain the controlled IPPS sampling plan given in Table 4.3.

Table 4.3

Optimal controlled IPPS plan corresponding to proposed scheme

s	$p(s)$	s	$p(s)$	s	$p(s)$	s	$p(s)$
1	0.12	5	0.0	9	0.04	13	0.0
2	0.0	6	0.08	10	0.0	14	0.20
3	0.08	7	0.16	11	0.0	15	0.0
4	0.12	8	0.0	12	0.2	16	0.0

The value of the objective function is: $\phi\ = 0.165016$

We also solve this example by the procedure of Ernst (1998), we get the following result:

$$p_1 = 0.4,\ p_6 = 0.2,\ p_{13} = 0.2,\ p_{16} = 0.2$$

Tiwari and Sud (2012) obtained the following solution for this problem:

$$p_2 = 0.08,\ p_3 = 0.04,\ p_4 = 0.16,\ p_5 = 0.12,\ p_7 = 0.16,\ p_8 = 0.04,$$
$$p_{11} = 0.08,\ p_{12} = 0.12,\ p_{14} = 0.20;$$

Following the procedures of Ernst (1998) and Tiwari and Sud (2012), the value of ϕ is same for both the procedures is 0.6.

5. Conclusion

In this article, we have proposed a linear programming approach with distance function as a weight for each sample, to obtain an optimum solution for the sample co-ordination problem. The proposed procedure is superior to the procedures of Ernst (1998) and Tiwari and Sud (2012) as it maximizes the probability of sample combinations having maximum number of overlapped samplin units (in case of positive co-ordination) or minimize the probability of sample combinations having maximum number of overlapped sampling units (in case of negative co-ordination). The proposed procedure also ensures variance estimation using Y-G (1953) form of H-T (1952) variance estimator as it satisfies the non-negativity condition of Horvitz-Thompson variance estimator through constraint (v) in Eq. (3.2). The proposed procedure takes lesser computing time in comparison to the procedures of Ernst (1998) and Tiwari and Sud (2012) and is found to be more advantageous than these procedures.

ACKNOWLEDGEMENT

The authors are thankful to the two referees and the Editor for their constructive comments, which led to considerable improvement in presentation of this manuscript.

Appendix

Example 1.2 (Minimization Case). For this example the all possible combinations are as follows:

Sample 1	Sample 2	Sample 3	Sample 4
0.0 0.4 0.0 0.0	0.0 0.4 0.0 0.0	0.2 0.0 0.0 0.0	0.0 0.4 0.0 0.0
0.0 0.4 0.0 0.0	0.4 0.0 0.0 0.0	0.0 0.4 0.0 0.0	0.4 0.0 0.0 0.0
0.8 0.0 0.0 0.0	0.0 0.2 0.0 0.0	0.0 0.2 0.0 0.0	0.8 0.0 0.0 0.0
0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0	0.0 0.4 0.0 0.0
0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0
Sample 5	Sample 6	Sample 7	Sample 8
0.2 0.0 0.0 0.0	0.2 0.0 0.0 0.0	0.0 0.4 0.0 0.0	0.2 0.0 0.0 0.0
0.0 0.4 0.0 0.0	0.4 0.0 0.0 0.0	0.4 0.0 0.0 0.0	0.0 0.4 0.0 0.0
0.8 0.0 0.0 0.0	0.0 0.2 0.0 0.0	0.8 0.0 0.0 0.0	0.8 0.0 0.0 0.0
0.0 0.4 0.0 0.0	0.0 0.4 0.0 0.0	0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0
0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0	0.0 0.2 0.0 0.0	0.0 0.2 0.0 0.0
Sample 9	Sample 10	Sample 11	Sample 12
0.2 0.0 0.0 0.0	0.2 0.0 0.0 0.0	0.0 0.0 0.4 0.0	0.0 0.0 0.4 0.0
0.4 0.0 0.0 0.0	0.4 0.0 0.0 0.0	0.0 0.0 0.0 0.2	0.0 0.0 0.0 0.2
0.0 0.2 0.0 0.0	0.8 0.0 0.0 0.0	0.0 0.2 0.0 0.0	0.8 0.0 0.0 0.0
0.6 0.0 0.0 0.0	0.0 0.4 0.0 0.0	0.6 0.0 0.0 0.0	0.0 0.4 0.0 0.0
0.0 0.2 0.0 0.0	0.0 0.2 0.0 0.0	0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0
Sample 13	Sample 14	Sample 15	Sample 16
0.0 0.0 0.4 0.0	0.0 0.0 0.4 0.0	0.0 0.0 0.4 0.0	0.0 0.0 0.4 0.0
0.0 0.0 0.0 0.2	0.0 0.4 0.0 0.0	0.4 0.0 0.0 0.0	0.4 0.0 0.0 0.0
0.8 0.0 0.0 0.0	0.8 0.0 0.0 0.0	0.0 0.2 0.0 0.0	0.8 0.0 0.0 0.0
0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0	0.6 0.0 0.0 0.0	0.0 0.4 0.0 0.0
0.0 0.2 0.0 0.0	0.0 0.0 0.0 0.2	0.0 0.0 0.0 0.2	0.0 0.0 0.0 0.2

Now we apply the proposed model as follows:

Max. $\phi = 0.1563*x[1] + 0.3235*x[2] + 0.3622*x[3] + 0.2081*x[4] + 0.2409*x[5] + 0.4325*x[6] + 0.2409*x[7] + 0.2751*x[8] + 0.4792*x[9] + 0.3362*x[10]$;

After solving the above model, we find the desired results shown in example 1.2.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 229 – 238

Wavelet decomposition for time series: Determining input model by using mRMR criterion

Budi Warsito* , Subanar† and Abdurakhman‡

Abstract

Determining the level of decomposition and coefficients used as input in the wavelet modeling for time series has become an interesting problem in recent years. In this paper, the detail and scaling coefficients that would be candidates of input determined based on the value of Mutual Information. Coefficients generated through decomposition with Maximal Overlap Discrete Wavelet Transform (MODWT) were sorted by Minimal Redundancy Maximal Relevance (mRMR) criteria, then they were performed using an input modeling that had the largest value of Mutual Information in order to obtain the predicted value and the residual of the initial (unrestricted) model. Input was then added one based on the ranking of mRMR. If additional input no longer produced a significant decrease of the residual, then process was stopped and the optimal model was obtained. This technique proposed was applied in both generated random and financial time series data.

2000 AMS Classification: 62M10, 65T60

Keywords: time series, MODWT, Mutual Information, mRMR

Received 20/01/2014 : Accepted 22/04/2014 Doi : 10.15672/HJMS.2014117462

*Department of Statistics, Diponegoro University, Semarang, Indonesia
Email: budiwrst2@gmail.com

†Department of Mathematics, Gadjah Mada University, Yogyakarta, Indonesia
Email:subanar@yahoo.com

‡Department of Mathematics, Gadjah Mada University, Yogyakarta, Indonesia
Email:rachmanstat@ugm.ac.id

1. Introduction

Wavelet transform for time series analysis has been proposed in many papers in recent years. Previous researches that deserve to be references are in [6] and [10]. Several approaches have been proposed for time series prediction by wavelet transform, as in [12] that used undecimated Haar transform. The choice of Haar transform was motivated by the fact that wavelet coefficients are calculated only from data obtained previously. One of the issues raised in this modeling is the determination of lagged value as an input so that it needs a technique to obtain the optimal input. Input selection aims to select the most relevant input set for a given task. [11] proposed the input selection uses sparse modeling based on a small number of coefficients on each of the signal in autoregressive case, and it is called Multiscale Autoregressive. Wavelet transform used in the method is redundant "*â trous*" wavelet transform which is similar with Maximal Overlap Discrete Wavelet Transform (MODWT) introduced by [10], which has the advantage of being shift-invariant. In this paper, we will utilize Minimal Redundancy Maximal Relevance (mRMR) feature selection technique proposed in [8] to select the scaling and detail coefficients of wavelet decomposition MODWT up to a certain level. Selection criteria used is the Mutual Information that measures the relationship between input variables and output.

Some researches on Mutual Information have been conducted mainly deal with the feature selection as in [4], [13] and [14], while [5] used it for detection of the input time series data and [7] applied for input selection on Wavelet Neural Network Model. On wavelet modeling for time series with mRMR, the initial model is a model formed with only one input, i.e the coefficient of detail or scale generated by MODWT, which has the largest value of Mutual information criterion. Input is then added one by one based on mRMR criteria until the desired amount achieved. Restrictions on the number of coefficients based on the difference of residual are obtained from the addition of the input with the previous model. If there are no significant differences, then the addition is stopped and optimal model is obtained. This paper is organized as follows; Section 2 discusses the wavelet decomposition, especially MODWT; Section 3 discusses the Mutual Information and input selection algorithm with mRMR; and a set of experiments illustrating the method is discussed in Section 4, covers random generate and the real data in financial field.

2. Wavelet Decomposition

Wavelet is a mathematical function that contains certain properties such as oscillating around zero (such as sine and cosine functions) and is localized in time domain, meaning that when the domain value is relatively large, wavelet function will be worth zero. Wavelet is divided into two types, namely father wavelet (ϕ) and mother wavelet (ψ) which has the properties:

$$\int_{-\infty}^{\infty} \phi(x)dx = 1 \quad dan \int_{-\infty}^{\infty} \psi(x)dx = 0 \tag{2.1}$$

Father and mother wavelet will give birth wavelet family by dyadic dilation and integer translation, those are:

$$\phi_{j,k}(x) = \left(2^j\right)^{1/2} \phi\left(2^j x - k\right) \tag{2.2}$$

$$\psi_{j,k}(x) = \left(2^j\right)^{1/2} \psi\left(2^j x - k\right) \tag{2.3}$$

In this case, j is the dilation parameter and k is the translation parameter.

Base wavelet can be seen as a form of dilation and translation with j = 0 and k = 0. Dilation index j and translation index k influence the change of support and range of base wavelet. Dilation index j influences the change of support and range in reverse, i.e if the support is narrow, the range will be widened. The translation index k affects the shift in position on the horizontal axis without changing the width of the support. In this case, the support is closure of the set of points which gives the value of function domain that is not equal to zero. Suppose a mapping belongs to $f : x\epsilon\mathbb{R} \to y = f(x)\epsilon\mathbb{R}$ then $support(f) = \overline{\{x\,|f(x) \neq 0\}}$.

Wavelet function can build a base for $L^2\mathbb{R}$ space, or in other words every function $f\epsilon L^2\mathbb{R}$ can be expressed as a linear combination of a base built by wavelet, and can be written in the following equation.

$$f(x) = \sum_{k\in Z} c_{J,k}\phi_{J,k}(x) + \sum_{j<J}\sum_{k\in Z} d_{j,k}\psi_{j,k}(x) \tag{2.4}$$

where

$$c_{J,k} = \int f(x)\phi_{J,k}(x)dx$$

$$d_{j,k} = \int f(x)\psi_{j,k}(x)dx$$

The transformation in equation (2.4) is *Continue Wavelet Transform* (CWT) in which the wavelet coefficients are obtained through the integration process, so that the value of wavelet must be defined at each $x\epsilon\mathbb{R}$. Another form of transformation is *Discrete Wavelet Transform* (DWT) where the wavelet values are defined only at finite points. Vector containing the values of wavelet is called *wavelet filter* or *detail filter* $\{h_l : l = 0, ..., L-1\}$, where L is the length of the filter and must be an even integer. A detail filter must meet the need of the following three basic properties [10]:

(1) $\sum_{l=0}^{L-1} h_l = 0$ and $\sum_{i=0}^{L-1} g_i^2 = 1$ where L is the length of filter

(2) $\sum_{l=0}^{L-1} h_l^2 = 1$

(3) $\sum_{l=0}^{L-1} h_l h_{l+2n} = \sum_{l=-\infty}^{\infty} h_l h_{l+2n} = 0$

for all nonzero integers n. To fulfill these properties, it is required that the filter length L is even. The required second filter is the *scaling filter* $\{g_l\}$ that corresponds to $\{h_l\}$:

$$g_l \equiv (-1)^{l+1} h_{L-1-l}$$

or in the inverse relationship:

$$h_l = (-1)^l g_{L-1-l}$$

Suppose given wavelet filter $\boldsymbol{h} = (h_0, h_1, ..., h_L - 1)$ while the $\boldsymbol{f} = (f_1, f_2, ..., f_n)$ is a realization of function $\boldsymbol{f}$ on $x_1, x_2, ..., x_n$. In this case, n = 2^J for some positive integer J. DWT can be written as:

$$\mathbf{W} = \mathcal{W}\mathbf{f} \tag{2.5}$$

where $\mathbf{W}$ = result of DWT and $\mathcal{W}$ = transformation matrix with the size nxn. DWT will map the vector $\mathbf{f} = (f_1, f_2, ..., f_n)$ to the coefficient vector $\mathbf{W} = (W_1, W_2, ..., W_J)$ where $\mathbf{W}$ contains wavelet coefficients $d_{j,k}$ and scaling coefficients $c_{J,k}$, for j = 1,2, ..., J. These are an approximation of the coefficients in equation (2.4). DWT can be used to reduce or eliminate the random disturbances in a data (*de-noising* process) by the absence of wavelet coefficients which are quite small. Wavelet coefficients are great values, they have a major contribution in reconstruction of a function, while the small coefficients contribute negligibly small (essentially zero).

Filtering by DWT as in equation (2.5) cannot be done on any sample size, which cannot be expressed in the form 2^J where J is a positive integer. As an alternative, the calculation of coefficients $d_{j,k}$ and $c_{J,k}$ can be done with *Maximal Overlap Discrete Transform* (MODWT). The advantage of MODWT is that it can eliminate data reduction by half (*down-sampling*), so that in each level there will be wavelet and scaling coefficients as much as length of the data [10]. Suppose a time series data of length N, MODWT transformation will give the column vector $w_1, w_2, ..., w_{J_0}$ and v_{J_0} each of length N.

In order to easily make relations between DWT and MODWT, it is convenient to define an MODWT wavelet filter $\{\tilde{h}_l\}$ through $\tilde{h}_l \equiv h_l/\sqrt{2}$ and scaling filter $\{\tilde{g}_l\}$ through $\tilde{g}_l \equiv h_l/\sqrt{2}$. Wavelet filter and scaling filter from MODWT must fulfill the following conditions:

$$\sum_{l=0}^{L-1} \tilde{h}_l = 0, \sum_{l=0}^{L-1} \tilde{h}_l^2 = \frac{1}{2}, \quad and \quad \sum_{l=-\infty}^{\infty} \tilde{h}_l \tilde{h}_{l+2n} = 0$$

$$\sum_{l=0}^{L-1} \tilde{g}_l = 1, \sum_{l=0}^{L-1} \tilde{g}_l^2 = \frac{1}{2}, \quad and \quad \sum_{l=-\infty}^{\infty} \tilde{g}_l \tilde{g}_{l+2n} = 0$$

In MODWT, the number of wavelet coefficients at each level is always the same, so it is more suitable for time series modeling compared with DWT. Prediction one step forward of time series data **X** is modeled linearly, based on coefficients of wavelet decomposition at previous times. Lag of coefficients that will be candidate of input to predict t are detail and scaling coefficients resulted from MODWT transformation in the form $d_{j,t-k}$ and $c_{j,t-k}$ or can be written in the following equation:

$$\hat{X}_t = \sum_{j=1}^{J} \sum_{k=1}^{A_j} (\hat{a}_{j,k} c_{j,t-k} + \hat{b}_{j,k} d_{j,t-k}) \tag{2.6}$$

the J symbol states the level of decomposition, while A_j describes the number of lag coefficients on the level of decomposition. If the number of lag coefficients at each level is the same, $A_j = A$, for each level j, then the number of variables that become candidates of input is 2AJ. Lag of coefficients which serves as inputs of the model will be determined by Minimal Redundancy Maximal Relevance criteria based on Mutual Information.

3. Maximal Relevance Minimal Redundancy

3.1. Entropy and Mutual Information. The entropy of a random variable, denoted by H(X), quantifies an uncertainty present in the distribution of X [2]. It is defined as,

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x) \tag{3.1}$$

where the lower case x denotes a possible value that the variable X can adopt from the alphabet $\mathcal{X}$. If the distribution is highly biased toward one particular event $x \in \mathcal{X}$, that is little uncertainty over the outcome, then the entropy is low. If all events are equally likely, that is maximum uncertainty over the outcome, then H(X) is maximum [2]. Following the standard rules of probability theory, entropy can be conditioned on other events. The conditional entropy of X given Y is denoted as follows.

$$H(X|Y) = -\sum_{y \in \mathcal{Y}} p(y) \sum_{x \in \mathcal{X}} p(x|y) \log p(x|y) \tag{3.2}$$

This can be thought as the amount of uncertainty remaining in X after we learn the outcome of Y. The Mutual Information (MI) between X and Y is the amount of information shared by X and Y.

$$MI(X;Y) = H(X) - H(X|Y) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(xy)\log\frac{p(xy)}{p(x)p(y)} \tag{3.3}$$

This is the difference of two entropies, i.e. the uncertainty before Y is known, H(X), and the uncertainty after Y is known, $H(X|Y)$. This can also be interpreted as the amount of uncertainty in X which is removed by knowing Y. Thus it follows the intuitive meaning of mutual information as the amount of information that one variable provides about another [2]. On the other words, the mutual information is the amount by which the knowledge provided by the feature vector decreases the uncertainty about the class [1]. The Mutual Information is symmetric, MI(X;Y) = MI(Y;X). If the variables are statistically independent, p(xy) = p(x)p(y), the Mutual Information will be zero.

To compute (3.1), we need an estimate of the distribution p(X). When X is discrete this can be estimated by frequency counts from data, $\hat{p}(x) = \frac{\#x}{N}$, the fraction of observations takes on value x from the total N [2]. When at least one of variables X and Y is continuous we need to incorporate data discretization as a preprocessing step. An alternative solution is to use density estimation method [9]. Given N samples of a variable X, the approximate density function $\hat{p}(x)$ has the following form:

$$\hat{p}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - x^{(i)}, h\right) \tag{3.4}$$

where $x^{(i)}$ is the i^{th} sample, h is the window width and $\delta(z,h)$ is the Parzen Window, for example the Gaussian window:

$$\delta(z,h) = \exp\left(-\frac{z^T \sum^{-1} z}{2h^2}\right) \Big/ \left\{(2\pi)^{d/2} h^d \left|\sum\right|^{1/2}\right\} \tag{3.5}$$

where $z = x - x^{(i)}$; d is the dimension of the sample x and Σ is the covariance of z.

3.2. mRMR Selection. The feature selection's goal in terms of mutual information is to find a feature set S with k features x_i which jointly have the largest dependency to the target y called maximum dependency (Max-Dependency).

$$\max D(S,y), \quad D = MI(\{x_i, i = 1, ..., k\}; y) \tag{3.6}$$

To simplify the implementation of (3.6), [8] proposed an alternative way to select features based on maximal relevance (Max-Relevance) and minimal redundancy (Min-Redundancy) criterion. Max-Relevance is to search features satisfying (3.7), which approximates $D(S,y)$ in (3.6) with the mean value of all mutual information values between individual feature x_i and the output y:

$$\max D(S,y), \quad D = \frac{1}{|S|}\sum_{x_i\in S} MI(x_i; y) \tag{3.7}$$

According to Max-Relevance, the features selected could have rich redundancy, i.e., the dependency among these features could be large. If two features highly depend on each other, the respective class-discriminative power would not change much if one of them were removed. Therefore, minimal redundancy (Min-Redundancy) condition is added to select mutually exclusive features:

$$\min D(S), \quad R = \frac{1}{|S|^2}\sum_{x_i,x_j\in S} MI(x_i, x_j) \tag{3.8}$$

The criterion combining the eq. (3.7) and (3.8) is called "minimal-redundancy-maximal-relevance" (mRMR). The operator to combine D and R is defined as a simple form to optimize D and R simultaneously:

$$\max \Phi\left(D,R\right), \Phi = D - R \tag{3.9}$$

In order to get the near optimal features defined by $\Phi(.)$, incremental search methods can be used. If we have n candidates of input, the first input X is included into the model that has the highest $MI(X,y)$. The remaining input consists of n-1 feature. To determine the next inputs, suppose we already have S_{m-1}, the feature set with m - 1 features. The task is to select the mth feature from the set $X - S_{m-1}$. This is undertaken by selecting the feature that optimizes the following condition:

$$\max_{x_j \in X - S_{m-1}} \left[MI\left(x_j;y\right) - \frac{1}{m-1} \sum_{x_i \in S_{m-1}} MI\left(x_j, x_i\right) \right] \tag{3.10}$$

The main goal of this algorithm is to select a subset of features S from inputs X, which has minimum redundancy and has maximum relevance with the target y (output). Determination of the value of m is based on the difference between prediction accuracy of the model with m inputs and prediction accuracy of the model with m + 1 input. If the difference is smaller than the desired value, then the input is selected as m. This algorithm computes MI(x_i, x_j) and MI(x_i, y), where y is the target (output) and (x_i, x_j) are individual inputs, i.e. all of scaling and detail coefficients from MODWT until level J. Systematically, algorithm for determining the input of wavelet model uses MI as similarity measure are:

(1) Use MODWT to decompose the data up to a certain level in order to obtain detail and scaling coefficients of each level
(2) At each level of decomposition, specify the detail and scaling coefficients that would be candidates of input until a certain lag
(3) Compute the Mutual Information between candidate of input x_i and target y, MI(x_i,y)
(4) Select the initial input x_i so that MI(x_i,y) is the highest MI and $x_i \in S$ then compute MSE of the initial (unrestricted) model
(5) Sort by ascending the remaining input based on mRMR:
$$\text{mRMR} = \left[MI\left(x_i, y\right) - \tfrac{1}{|S|} \sum_{x_j \in S} MI\left(x_i, x_j\right) \right]$$
(6) (a) Add selected input to S based on greatest value of mRMR then calculate the MSE
(b) Calculate the difference in MSE from the previous model and model with the addition of input
(c) If the difference is greater than the desired number then back to 6(a)
(7) Process of adding suspended and optimal model is obtained

The desired number in step 6(c) was chosen, the one that was small enough to the initial MSE. The addition of input was stopped when errors no longer decrease significantly. In this paper, the desired number chosen was equal to 1/100000 of MSE of the initial model.

4. Experimental Results

The using of mRMR in wavelet for time series would be applied in three types of data, they are randomly generated data from Autoregressive models, randomly generated data from GARCH model and real data in the financial fields.

4.1. Autoregressive Simulation Data. The data used is randomly generated by AR (2), AR (3) and ARMA (2,1) model of 500 respectively, the following equations are the description:

$$X_t = 1.5X_{t-1} - 0.7X_{t-2} + \varepsilon_t \quad (4.1)$$

$$X_t = 0.8X_{t-1} + 0.4X_{t-2} - 0.7X_{t-3} + \varepsilon_t \quad (4.2)$$

$$X_t = 1.5X_{t-1} - 0.7X_{t-2} + 0.5\varepsilon_{t-1} + \varepsilon_t \quad (4.3)$$

After getting the data generated from random generation, the first step taken is to decompose the data with MODWT up to 4^{th} level to obtain the detail and scaling coefficients of each level. In each level of decomposition, the lags of detail and scaling coefficients are determined as potential inputs. In this case, we choose the coefficients up to lag 16, so there will be 2x4x16 = 128 candidates of input. This value chosen on the ground can accommodate different types of past data that affects the present data. The next stage is to calculate the value of Mutual Information of each candidate and determine the highest MI used as the initial of input. From this result, modeling is executed by ordinary least squares method to obtain prediction values and the residuals.

The next stage is to sort mRMR value of each candidate without lagged value selected as the initial. One by one of the candidates is added into the model sequentially based on mRMR and then calculate the MSE. If additional input does not reduce the previous MSE by at least $1/100000$ of the initial MSE, then the process of adding is suspended, and the optimal model is obtained. This stopping criteria is chosen based on the thinking that the decreasing does not affect the difference of MSE significantly. The obtained results are compared with autoregressive models. To calculate the MODWT decomposition, we use the wmtsa toolkit for Matlab while to calculate Mutual Information and mRMR we use MIToolbox package. In each model generated we repeat it five times. The calculation results are presented in table 1.

Based on table 1, it appears that for data generated from linear autoregressive models, wavelet model with MODWT decomposition combined with mRMR procedure to obtain the input always provides a more predictive results than the original models, characterized by the value of both MSE and R square. For the data generated from AR(2), there are two coefficients that are always involved as inputs in wavelet model building, i.e 1^{st} lag of 1^{st} level and 1^{st} lag of 4^{th} level from the scaling coefficients. On the data generated from the AR(3), coefficients that always come up are 1^{st} and 5^{th} lags of 1^{st} level from the scaling coefficients, as well as 1^{st} lag from 4^{th} level. While the data generated from the ARMA(2,1), the 1^{st} lag of the 1^{st}, 2^{nd}, and 4^{th} level from scaling coefficients, respectively, have always become inputs of the model. Meanwhile, for the three data types, the detail coefficients are never entered as input irrespective of levels or lags.

By considering the selected input, the resulting model yields only a few parameter from a lot of candidates. The proposed procedure has succeeded in selecting candidates which have great contribution and dismiss a lot of candidates which are not considered giving significant contribution. This gives a wavelet model for time series with a few coefficients as input and still gives good results.

Table 1. Comparison of MODWT-mRMR with autoregressive models

		autoregressive		mRMR-MODWT			
model	exp	MSE	Rsq	Input	scaling coefficients (level;lags)	MSE	Rsq
AR(2)	1	0.1670	0.6539	3	(1;1,5)(4;1)	0.1309	0.7293
	2	0.2274	0.6759	9	(1;1,6,8)(2;1,11)(3;15)(4;1,9,16)	0.0975	0.8630
	3	0.2046	0.6364	3	(1;1,5)(4;1)	0.1681	0.7019
	4	0.1786	0.6174	3	(1;1,6)(4;1)	0.1604	0.6571
	5	0.1855	0.6439	6	(1;1,12,16)(2;1,6)(4;1)	0.1400	0.7334
AR(3)	1	0.1022	0.6105	4	(1;1,5)(3;3)(4;1)	0.0866	0.6705
	2	0.1017	0.7081	6	(1;1,5)(2;5)(3;6)(4;1,2)	0.0824	0.7649
	3	0.1033	0.6540	10	(1;1,5,7)(2;1,5)(3;4)(4;1,2,7,12)	0.0710	0.7655
	4	0.1118	0.7030	4	(1;1,4,5)(4;1)	0.0852	0.7740
	5	0.1124	0.6402	8	(1;1,5,6)(2;10)(3;3)(4;1,2,15)	0.0667	0.7887
ARMA(2,1)	1	0.0349	0.6918	4	(1;1,5)(3;3)(4;1)	0.0306	0.7311
	2	0.1017	0.7081	6	(1;1,5)(2;5)(3;6)(4;1,2)	0.0824	0.7649
	3	0.0419	0.7362	6	(1;1,6)(2;1)(3;11,16)(4;1)	0.0362	0.7738
	4	0.0421	0.6866	7	(1;1)(2;1,5)(3;6,16)(4;1,5)	0.0333	0.7545
	5	0.0365	0.7424	10	(1;1)(2;1,12)(3;5,6)(4;1,6,8,11,15)	0.0278	0.8070

4.2. GARCH Simulation Data. In this section, randomly generated data with a length of 500 will be discussed, following the ARIMA (0,0,0) as a mean model and GARCH (1,1) as a variant model with the following equation:

$$(4.4) \qquad y_t = 0.00001 + \varepsilon_t \qquad\qquad \sigma_t^2 = 0.00005 + 0.8\sigma_{t-1}^2 + 0.1\varepsilon_{t-1}^2$$

Further studies will be conducted with the application of the MODWT using mRMR input selection, for which the data are generated and then carried out a comparative study of the accuracy with GARCH model. We also repeat the experiments for five times and the results obtained are as table (2).

Table 2. MODWT-mRMR Model Comparison with GARCH

	GARCH(1,1)		mRMR-MODWT				
exp	MSE ($\times 10^{-4}$)	Rsq	Input	scaling (level;lags)	detail	MSE ($\times 10^{-4}$)	Rsq
1	4.7645	0.9744	4	(1;3,4)(2;2)(3;1)	-	3.0886	0.9926
2	4.3848	0.8739	8	(1;1)(2;1,2,3)(3;1,16)(4;1,5)	-	3.7462	0.9715
3	5.0352	0.9491	4	(1;3)(2;3)(3;1)(4;13)	-	2.9922	0.9717
4	5.4082	0.9383	4	(1;1,2)(2;1)(4;16)	-	2.5345	0.9918
5	4.9960	0.9167	8	(1;1,2,7)(2;1,16)(3;2)(4;2,16)	-	2.7526	0.9847

Calculation results in table (2) indicate that the MODWT with mRMR input selection yields better predictions compared to GARCH model. It is characterized by a smaller value of MSE and R square is greater. Although the lag of selected scaling coefficients are not consistent at a certain value, but it appears that the initial lagged of each level of decomposition dominates the coefficients entrance to the model. As in the random data from AR and ARMA models, in the randomly generated data from GARCH model the coefficients entered into the model are only the scaling coefficients, none of the lag of detail coefficients is chosen. As mentioned earlier, this procedure was successful for selecting a few coefficients included into the model.

4.3. Applications on the Financial Data. In this section we apply the method proposed in two financial time series data. The first is Quarterly Real Gross Private Domestic Investment from Bureau of Economic Analysis, U.S. Department of Commerce, January 1947 to January 2013 and the second is Monthly Price of the Indonesian Liquified Natural Gas data, from May 2003 to March 2013. The first data can be downloaded from http://research.stlouisfed.org/fred2/, while the second is http://www.indexmundi.com/commodities/. We have investigated that the first data were not stationer and after first order differencing, it would be stationer. The best linear model from the differenced data is AR(1) without constant, and by LM test we found that the residuals have an ARCH effect. The best model for the variance of residuals is GARCH(1,0) by BHHH optimization method.

In the second case, we focused on the monthly change price of the data. Investigation to the type of the data got the best linear model is ARMA(2,2) with constant, and by LM test we found that the residuals have an ARCH effect. The best model for the variance of residuals is GARCH(0,3) by BHHH optimization method. To show the efficiency of the proposed method we analyzed the both data and compared them with the appropriate models. R square value shows the influence of the price data instead of the change. The result is shown in the table (3).

Table 3. MODWT-mRMR Model of the Financial Time Series Data

Real Gross Private Domestic Investment data						
GARCH(1,0)		mRMR-MODWT				
MSE	Rsq	Input	scaling (level;lags)	detail	MSE	Rsq
1517.0429	0.9986	12	(1;1,2,8,10),(2;2,4),(3;3,4,5,7,9),(3;2)	-	1517.5143	0.9978
Indonesian Liquified Natural Gas data						
GARCH(0,3)		mRMR-MODWT				
MSE	Rsq	Input	scaling (level;lags)	detail	MSE	Rsq
40.8585	0.9995	3	(1;2)(4;1,3)	-	42.9082	0.9960

MSE value resulted from the calculation as shown in table (3) explains that in the first case, the proposed method gives result as good as the GARCH model, while in the second, the GARCH model is still superior. In both examined data, as in random generated data, only the scaling coefficients that are included into the model, but none of the detail coefficients is chosen. We can also make a conclusion that the scaling coefficients have dominant influence to the output, while the detail coefficients have almost no significant role. Overall, mRMR technique can be used to determine input of wavelet model for time series efficiently. It can be seen that the number of input selected with mRMR criteria was a few. This procedure has successfully resulted a model which was more parsimonious in the number of parameters and still gave a good description of the observed data.

5. Closing

A technique combining MODWT decomposition and mRMR criterion was proposed for constructing forecasting model for time series. In MODWT for time series we use a linear prediction based on some coefficients of decomposition of the past values. The mRMR criteria was used as a tool to determine the input. Coefficients which have high values of mRMR were chosen as the input. By this procedure, model resulted just contained coefficients that were considered important enough to gave influence to the present value. The advantage of this technique is opening up the possibility of development by utilizing more sophisticated processing such as Neural Network that results hybrid model, which is called Wavelet Neural Network.

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Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 239 – 245

A new nonparametric estimation method of the variance in a heteroscedastic model

Xiaoqin Zhang* † ‡, Hongxia Hao§ and Jiye Liang¶

Abstract

In most economic phenomena, the assumption of homoscedasticity in the classic linear regression model is not necessarily true, which leads to heteroscedasticity. The heteroscedastic estimate is an important aspect for the problem of heteroscedasticity. For this hot issue, this paper proposes a nonparametric estimation method with simple calculation for the estimation of heteroscedasticity through orthogonal arrays, which does not rely on the distribution of data. The performance of the proposed method is investigated by prediction error in real data sets and simulations. The results suggest that this method offers substantial improvements over the existing tests.

2000 AMS Classification: 37L15, 37M05, 37M25

Keywords: Orthogonal array; heteroscedasticity; non-parametric estimation

Received 17/07/2013 : Accepted 08/01/2014 Doi: 10.15672/HJMS.201417452

1. Introduction

In model analysis in many areas such as sociology, economics and technology, homoscedastic assumption in classic linear regression model is not necessarily true. That is to say, the variance of random error term changes with the observed values. This model is called a heteroscedasticity model[1]. What leads to the heteroscedasticity? One reason is because the random error term includes the measurement error and the impact of some factors omitted in the model on the dependent variable, on the other hand, the value of the dependent variable in different sampling unit may be very different. If we use ordinary least squares (OLS) to estimate the parameters under heteroscedasticity model,

*School of Mathematical Science, Shanxi University, Taiyuan 030006, Shanxi, P.R. China.
†Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Taiyuan, 030006, Shanxi, P.R. China.
‡Corresponding Author. Email: zhangxiaoqin@sxu.edu.cn
§Department of Mathematics, Southeast University, Nanjing, 210096, Jiangsu, P.R. China. Email: hong_xia_mm@163.com
¶Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Taiyuan, 030006, Shanxi, P.R. China. Email: ljy@sxu.edu.cn

it will have serious consequences, for example, the estimated variance of the parameter does not have the validity (i.e. minimum variance), although with no bias, significant test and interval estimation will draw the wrong statistical inference, which directly leads to the decline of the accuracy and the prediction accuracy[1][2]. Therefore, the research for heteroscedastic linear regression models is of great significance.

Currently, there are many ways to solve the problem of heteroscedasticity. Early, the weighted least squares method can be used in the situation that the variance is not constant[3], but it often requires the mean and covariances of the dependent variable satisfy the linear relationship except for known variance. In general, it is very difficult to meet these two requirements. Thus literature [4] studied the model $y_i = m(x_i) + \sigma(x_i)\varepsilon_i$ and estimated the unknown function $m(\cdot)$ and $\sigma(\cdot)$, but their method can only handle the situation where the covariate x_i is one-dimensional. When the covariate is high-dimensional, the paper [5] discussed the heteroscedastic model $Y_i = m(X_i\beta)+\sigma(X_i\beta, \theta)\varepsilon_i$ where $m(\cdot)$ and $\sigma(\cdot)$ are known. A high-dimensional X_i is projected to a direction of where $X_i\beta$ is and the function $m(\cdot)$, $\sigma(\cdot)$ are changed to unary function. Thereby, the paper solved the problem of dimensionality reduction. The model requires a known contact function $m(\cdot)$ and a variance affected by the mean, however, these two points often can not be satisfied in practice. The article [6] studied two estimators, namely: the HC3 estimator and the weighted bootstrap estimator. Furthermore, it evaluated the finite sample behavior of two bootstrap tests and proposed a new estimator. The literature [7] employed the maximum likelihood method to study parameter estimation based on Lognormal distribution jointly logarithmic mean and logarithmic variance model. $y_i \sim LN(\mu_i, \sigma_i^2), \mu_i = x_i'\beta, \ln(\sigma_i^2) = z_i'\gamma, i = 1, 2, \cdots, n$. The model asked y_i to obey a Lognormal distribution. The literature [8] proposed a method to estimate the coefficient in heteroscedastic model, but it still has some disadvantages.

So, with aid of orthogonal arrays, this paper proposes a nonparametric estimation method with simple calculation for the estimation of heteroscedasticity, which has improved the method in literature [8]. Most importantly, this method does not rely on the specific distribution type for y_i. As a consequence, compared with the result in the literature [7], the proposed method in this paper has a a wider range in use.

The paper is structured as follows: Section 2 gives the steps for the estimation of heteroscedasticity by orthogonal table. In section 3, in conjunction with simulated and real data sets, we illustrate the validity of proposed method. Section 4 does a brief summary and points out the direction of future research.

2. Estimation for heteroscedasticity

2.1. Assumptions of the model. Assume that data $(x_{i1}, x_{i2}, \cdots, x_{ip}, y_i)$, $i = 1, 2, \cdots, n$, has the following linear relationship:

$$
\begin{cases} Y = X\beta + \varepsilon, \\ E(\varepsilon) = 0_n, D(\varepsilon) = \Sigma = diag(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2), \\ \sigma_i^2 \text{ is not the same}, \ \ i = 1, 2, \cdots, n. \end{cases} \tag{2.1}
$$

where

$$
Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}.
$$

β is parameter to be estimated. The model is called a heteroscedastic linear model[2].

When heteroscedasticity occurs in the model, if covariance matrix of random item ε is known, we can use generalized least squares estimation (GLSE) $\hat{\beta}$ as the estimation of model coefficients β, that is:

$$\hat{\beta} = (X^{'}\Sigma^{-1}X)^{-1}X^{'}\Sigma^{-1}Y. \tag{2.2}$$

However, in most practical problems, Σ is unknown. To solve this problem, it is necessary to estimate Σ. Therefore, combining with the nature of the orthogonal array, this paper gives a reasonable estimate of covariance matrix Σ of random error term ε.

2.2. Estimation of variance. This subsection gives a method to estimate the covariance matrix Σ of random errors ε in the formula of (2.2) by using orthogonal array. For the convenience of description, we consider the case $p = 3$, i.e, there are three independent variables x_1, x_2, x_3 in the model, and other situations can be promoted similarly.

For example, orthogonal array $L_9(3^4)$, which is generated with the help of the knowledge of combinatorial mathematics and probability[9].

$$L_9(3^4) = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \end{pmatrix}^{'}. \tag{2.3}$$

The detailed process of heteroscedastic estimation for the case $p = 3$ is as follows:

(1) For each set of observation $(x_{i1}, x_{i2}, x_{i3}), i = 1, 2, \cdots, n$, taking each independent variable as a factor, we make the following treatment for it: using (x_{i1}, x_{i2}, x_{i3}) as central value and $(\frac{x_{i1}}{\Delta}, \frac{x_{i2}}{\Delta}, \frac{x_{i3}}{\Delta})$ as tolerance, we can get three levels of each factor:

$$(x_{i1}, x_{i2}, x_{i3}) \to \begin{pmatrix} x_{i1} - \frac{x_{i1}}{\Delta} & x_{i2} - \frac{x_{i2}}{\Delta} & x_{i3} - \frac{x_{i3}}{\Delta} \\ x_{i1} & x_{i2} & x_{i3} \\ x_{i1} + \frac{x_{i1}}{\Delta} & x_{i2} + \frac{x_{i2}}{\Delta} & x_{i3} + \frac{x_{i3}}{\Delta} \end{pmatrix}$$

Where $\frac{1}{\Delta}$ is usually called tolerance and its value often depends on magnitude of data.

(2) Regarding the data produced in the first step as three levels of each factor, we can obtain following data with the help of orthogonal array $L_9(3^4)$:

$$\begin{pmatrix} x_{i1} - \frac{x_{i1}}{\Delta} & x_{i2} - \frac{x_{i2}}{\Delta} & x_{i3} - \frac{x_{i3}}{\Delta} \\ x_{i1} - \frac{x_{i1}}{\Delta} & x_{i2} & x_{i3} \\ x_{i1} - \frac{x_{i1}}{\Delta} & x_{i2} + \frac{x_{i2}}{\Delta} & x_{i3} + \frac{x_{i3}}{\Delta} \\ x_{i1} & x_{i2} - \frac{x_{i2}}{\Delta} & x_{i3} \\ x_{i1} & x_{i2} & x_{i3} + \frac{x_{i3}}{\Delta} \\ x_{i1} & x_{i2} + \frac{x_{i2}}{\Delta} & x_{i3} - \frac{x_{i3}}{\Delta} \\ x_{i1} + \frac{x_{i1}}{\Delta} & x_{i2} - \frac{x_{i2}}{\Delta} & x_{i3} + \frac{x_{i3}}{\Delta} \\ x_{i1} + \frac{x_{i1}}{\Delta} & x_{i2} & x_{i3} - \frac{x_{i3}}{\Delta} \\ x_{i1} + \frac{x_{i1}}{\Delta} & x_{i2} + \frac{x_{i2}}{\Delta} & x_{i3} \end{pmatrix} \doteq \begin{pmatrix} x_{i11} & x_{i21} & x_{i31} \\ x_{i12} & x_{i22} & x_{i32} \\ x_{i13} & x_{i23} & x_{i33} \\ x_{i14} & x_{i24} & x_{i34} \\ x_{i15} & x_{i25} & x_{i35} \\ x_{i16} & x_{i26} & x_{i36} \\ x_{i17} & x_{i27} & x_{i37} \\ x_{i18} & x_{i28} & x_{i38} \\ x_{i19} & x_{i29} & x_{i39} \end{pmatrix}$$

(3) For each set of observation $y_i, i = 1, 2, \cdots, n$, we can take 9 independent random numbers $y_{ik}, (k = 1, 2, \cdots, 9)$ from normal distribution $N(y_i, \theta^2)$ or uniform distribution $U[y_i - h, y_i + h]$, where θ^2 and h often take a relatively small value to satisfy the need that produced data y_{ik} have little deviation.

(4) According to the source of data in previous step, we know that 9 random numbers are produced from one distribution independently, i.e. they have same variance. Further, from the regression model we can see the variance of random error term and the variance of dependent variable are the same. So we can consider that the regression of y_{ik} and x_{ijk}(fix $i, j = 1, 2, 3, k = 1, 2, \cdots, 9$) is homoscedastic. So, using the OLS for this

regression is reasonable. For each i, according to the regression of y_{ik} and x_{ijk},

$$(2.4)\qquad \begin{cases} y_{ik} = \gamma_0 + \gamma_1 x_{i1k} + \gamma_2 x_{i2k} + \gamma_3 x_{i3k} + \varepsilon_{ik}, \\ E(\varepsilon_{ik}) = 0, D(\varepsilon_{ik}) = \sigma_{ik}^2, k = 1, 2, \cdots, 9. \end{cases}$$

we can obtain residual squares:

$$(2.5)\qquad \begin{cases} e_{ik}^2 = (y_{ik} - \hat{y_{ik}})^2, \hat{y_{ik}} = \hat{\gamma_0} + \hat{\gamma_1} x_{i1k} + \hat{\gamma_2} x_{i2k} + \hat{\gamma_1} x_{i3k}, \\ i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, 9. \end{cases}$$

(5) Note the variance of ε_i in the model (2.1) as σ_i^2. According to the calculation formula of variance $\sigma_i^2 = E(\varepsilon_i^2) - [E(\varepsilon_i)]^2$ and the basic assumptions for ε_i, i.e. $E(\varepsilon_i) = 0$, there is a conclusion that $\sigma_i^2 = E(\varepsilon_i^2)$. So this paper uses $\sum_{k=1}^{9} e_{ik}^2/9$ to estimate $E(\varepsilon_i^2)$, i.e. $\hat{\sigma_i^2} = \sum_{k=1}^{9} e_{ik}^2/9$, $i = 1, 2, \cdots, n$, Finally we get the covariance matrix of ε as $\hat{\Sigma}$=diag$(\hat{\sigma_1^2}, \hat{\sigma_2^2}, \cdots, \hat{\sigma_n^2})$.

3. Example

Simulation Study: In this part, we want to confirm our results by simulation experiments. Here we consider a simple heteroscedastic variance problem where the variance is the square of first variable x_1 corresponding to it.

Let us consider a simple three variable linear model:

$$(3.1)\qquad \begin{cases} y_i = 0.2 + 2x_{i1} + 3x_{i2} + 4x_{i3} + \varepsilon_i, \\ \varepsilon_i \sim N(0, x_{i1}^2), i = 1, 2, \cdots, n. \end{cases}$$

Above all, in our simulation study, all the values of independent variables are being taken equally from the uniform distribution $U[0, 1]$. From the model we know ε_i are generated from $N(0, x_{i1}^2)$. Further, y_i is easily obtained.

Then, using the simulation data, we take advantage of the proposed method in subsection 2.2 to estimate the variance and get the simulation equation with the aid of formula (2.2). Further, we obtain the absolute value of prediction error. We run this simulation experiment under the following situation: normal distribution $N(y_i, \theta^2)$ or uniform distribution $U[y_i - h, y_i + h]$, three different sample sizes $n = 30, 60, 90$, the number of experiment $m = 100, 1000$, $\frac{1}{\Delta} = 0.01$ or 0.001 and θ^2=0.01 or 0.001. The absolute value of prediction error of this simulation experiment is arranged in table 1(using SAS macro).

From the table 1, we can find that the absolute value of prediction error has little differences by the proposed method (method 1) in this paper, which is unrelated to the the choice of distribution and parameters. Therefore the choice of distribution and parameters has little effect on the proposed method (method 1) in this paper and the proposed method is stable.

To confirm the performance of our method, we adopt the method (method 2) in paper [10] and the method (method 3) in paper [6] and also get the absolute value of prediction error with the help of weighted least square estimation (WLSE). See Table 1. The process of method 2 can be described as follows: Sort the explanatory variables x_1 from small to large and other variables y_i, x_2, x_3 maintain the original correspondence. Divide the x_1 into k groups and j-th group contains n_j numbers. Let the mean of numbers in the j-th group as x'_{1j} and use x'_{1j} in place of the original data in j-th group. So the data becomes $(x'_{1j}, x_{i2}, x_{i3}, y_i), j = 1, 2, \cdots, k.\ i = 1, 2, \cdots, n$. We divide the sample variance of the i-th group s_i^2 on the both sides of the classic linear regression model and use OLS to estimate the parameter. Meanwhile, the estimator proposed in paper [6], called HC4, is as formula (3.2).

Table 1: The absolute value of prediction error in simulation

		m=100				m=1000			
		$\frac{1}{\Delta}=0.01$ θ^2=0.01	$\frac{1}{\Delta}=0.01$ θ^2=0.001	$\frac{1}{\Delta}=0.001$ θ^2=0.01	$\frac{1}{\Delta}=0.001$ θ^2=0.001	$\frac{1}{\Delta}=0.01$ θ^2=0.01	$\frac{1}{\Delta}=0.01$ θ^2=0.001	$\frac{1}{\Delta}=0.001$ θ^2=0.01	$\frac{1}{\Delta}=0.001$ θ^2=0.001
normal distribution (by method 1)	n=30	0.4094	0.4092	0.4160	0.4133	0.4050	0.4038	0.4032	0.4046
	n=60	0.4152	0.4122	0.4104	0.4165	0.4108	0.4101	0.4108	0.4110
	n=90	0.4072	0.4096	0.4128	0.4106	0.4105	0.4099	0.4108	0.4088
		$\frac{1}{\Delta}=0.01$ h=0.02	$\frac{1}{\Delta}=0.01$ h=0.04	$\frac{1}{\Delta}=0.001$ h=0.02	$\frac{1}{\Delta}=0.001$ h=0.04	$\frac{1}{\Delta}=0.01$ h=0.02	$\frac{1}{\Delta}=0.01$ h=0.04	$\frac{1}{\Delta}=0.001$ h=0.02	$\frac{1}{\Delta}=0.001$ h=0.04
uniform distribution (by method 1)	n=30	0.4089	0.4064	0.4121	0.4110	0.4002	0.3998	0.4024	0.4012
	n=60	0.4077	0.4109	0.4115	0.4084	0.4071	0.4094	0.4079	0.4085
	n=90	0.4097	0.4089	0.4086	0.4102	0.4081	0.4072	0.4080	0.4083
		m=100				m=1000			
by method 2	n=30	0.6934				0.6711			
	n=60	0.5743				0.5759			
	n=90	0.5196				0.5222			
		m=100				m=1000			
by method 3	n=30	0.3974				0.3893			
	n=60	0.40387				0.40249			
	n=90	0.40650				0.40470			

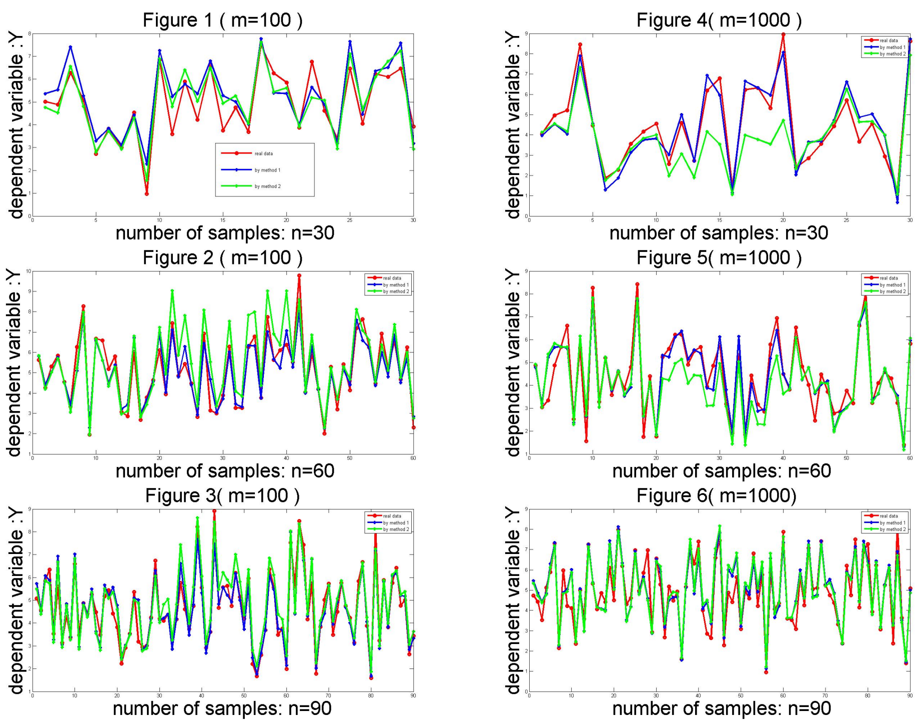

Figure 1-6: Simulation study

(3.2) $\hat{\Omega} = diag\{\hat{u_1}^2/(1-h_1)^{\delta_1}, \cdots, \hat{u_n}^2/(1-h_n)^{\delta_n}\}$

Where $\delta_i = min\{4, \frac{nh_i}{\sum_{j=1}^{n} h_i}\}$, h_i is the ith diagonal element of the "hat matrix" $H = X(X'X)^{-1}X'$ and $\hat{u_i}^2$ is i-th diagonal element of the diagonal matrix formed out of the vector of squared least-squares residuals.

From the table 1, we can see, on the one hand, the results using method 1 are smaller than them using method 2 on the whole. On the other hand, with the increase of the number of samples, the error by method 1 is changing little compared with those by method 2, which illustrates its stability and also shows that the newly proposed method in this paper is fitted with the data including 90 samples. Also, we can see that the results in table 1 by method 1 are almost the same with them by method 3. But the most important point we should not neglect is that the method 1(proposed method in this paper) does not rely on the distribution of data. However, according to formula of estimator proposed in paper [6], we can find that it depends on the normal distribution. So, the method we proposed in this paper has a wider range in practise.

Meanwhile, we use Figure 1-6(using MATLAB) to demonstrate the advantage of the method proposed (method 1) in this paper. In Figure 1-6, the horizontal axis represents the number of sample and the longitudinal axis notes the predicted value of the dependent variable obtained by different method. Red, blue and green lines respectively denote the value of independent variable using different methods. Red notes real values of dependent variable, blue indicates the predicted value of dependent variable using the proposed method (method 1, using $N(y_i, \theta^2)$, $\frac{1}{\Delta} = 0.01$, $\theta^2 = 0.01$) in this paper and green represents the values of dependent variable using the method proposed in article [10](method 2). As is shown in Figure 1-3(fix m), with the increasing of n, the value of dependent variable Y is closer to the real value using method 1. On the other hand, from the Figure 1 and 4, we can see that the effect of method 1(blue line) is better than method 2(green line) with the increasing of m.

Real Example: This example uses the proposed method to estimate the heteroscedasticity of data in example 2.6.2 in literature [7] and gives the regression equation in the presence of heteroscedasticity.

Let y, x_1, x_2, x_3 expresses total GDP and its components in the three industry respectively, namely primary industry, secondary industry and tertiary industry. We take the data from 31 provinces (autonomous regions and municipalities) of China in 2009 for example.

According to the way proposed in 2.2(N(y_i, θ^2), $\frac{1}{\Delta} = 0.01$ and $\theta^2 = 0.01$), calculate the variance of random term. By the formula (2.2), we get the regression equation and the prediction of dependent variable $\hat{y}$. See Table 2 (using SAS macro).

Table 2: data about real example

No.	y	x_1	x_2	x_3	$\hat{y}$	No.	y	x_1	x_2	x_3	$\hat{y}$
1	12153.03	118.29	2855.55	9179.19	12153.03	17	12961.10	1795.90	6038.08	5127.12	12961.10
2	7521.85	128.85	3987.84	3405.16	7521.85	18	13059.69	1969.69	5687.19	5402.81	13059.69
3	17235.48	2207.34	8959.83	6068.31	17235.48	19	39482.56	2010.27	19419.70	18052.59	39482.56
4	7358.31	477.59	3993.80	2886.92	7358.31	20	7759.16	1458.49	3381.54	2919.13	7759.16
5	9740.25	929.60	5114.00	3696.65	9740.25	21	1654.21	462.19	443.43	748.59	1654.21
6	15212.49	1414.90	7906.34	5891.25	15212.49	22	6530.01	606.80	3448.77	2474.44	6530.01
7	7278.75	980.57	3541.92	2756.26	7278.75	23	14151.28	2240.61	6711.87	5198.80	14151.28
8	8587.00	1154.33	4060.72	3371.95	8587.00	24	3912.68	550.27	1476.62	1885.79	3912.68
9	15046.45	113.82	6001.78	8930.85	15046.45	25	6169.75	1067.60	2582.53	2519.62	6169.753
10	34457.30	2261.86	18566.37	13629.07	34457.30	26	441.36	63.88	136.63	240.85	441.36
11	22990.35	1163.08	11908.49	9918.78	22990.35	27	8169.80	789.64	4236.42	3143.74	8169.80
12	10062.82	1495.45	4905.22	3662.15	10062.82	28	3387.56	497.05	1527.24	1363.27	3387.56
13	12236.53	1182.74	6005.30	5048.49	12236.53	29	1081.27	107.40	575.33	398.54	1081.27
14	7655.18	1098.66	3919.45	2637.07	7655.18	30	1353.31	127.25	662.32	563.74	1353.31
15	33896.65	3226.64	18901.83	11768.18	33896.65	31	4277.05	759.74	1929.59	1587.72	4277.05
16	19480.46	2769.05	11010.50	5700.91	19480.46						

Obtain the relation between y and x_1, x_2, x_3 by using σ_i^2 and formula: $y = x_1 + x_2 + x_3$. Compare and analyze the above regression equation with results of article from the following aspects:

(1) According to the meaning of independent variable and dependent variable, we can find the equation above is closer to the actual situation than the literature [7]. This can also be confirmed from the differences between the actual value of dependent variable and its prediction in Table 1.

(2) The literature [7] requires the specific distribution type for y_i, however the reported method in this paper does not rely on the limitation. As a consequence, the proposed method in this paper has a wider range in use.

4. Conclusions

When the covariance matrix of the random error term in the heteroscedastic regression model is unknown, this paper proposes a nonparametric method for the estimation of heteroscedasticity by orthogonal arrays. Most of all, this method does not rely on the distribution of data. Based on the fact that orthogonal arrays have good statistical properties, from the regression equation and the results in the simulation we can find that the proposed method is better than some other methods, which presents the validity and the stability of the proposed method in the paper. In most of the cases, people only focus on the test on the existence of heteroscedasticity and estimation of the heteroscedasticity, however, few people study the degree of impact of heteroscedasticity and variable that causes heteroscedasticity. These two aspects can be discussed further.

Acknowledgements

The authors are highly grateful to the anonymous reviewers for their helpful comments and suggestions for improving the paper. This research is partially supported by a grant of Project of National Natural Science Foundation of Nature(71031006), Youth Science and Technology Fund of Shanxi Province(201202105-1), National Natural Science Foundation of China(41101440).

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