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Some results of generalized k-fractional integral operator with k-Bessel function

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Abstract. In this research paper, we develop the generalized fractional k-integral operators (gFkIO) involving Appell k-function as its kernel, and investigate (gFkIO) with the composition of Bessel k-function of first kind (BkF-I). We shall obtain results by applying Sagio fractional integral k-operators (SFIkO) and Riemann Liouville fractional integral k-operators (RLFIkO) in which Gauss Hypergeometric k-function (GHkF) acting as a kernel in the left and right sense with product of power k-function and Bessel k-function of first kind (BkF-I) and results will be establish in the terms of generalized Wright Hypergeometric k-function (gWHkF).

1. Introduction

Fractional calculus is the field of mathematical analysis, which deals with the investigation and applications of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notions of integrals and derivatives. It has gained significance and recognition over the last four decades, specially because of its enormous capacity of tested programs in diverse seemingly expanded fields of science, applied mathematics and engineering [1–3]. We proposed a unified approach to the special functions of fractional calculus and our approach is based on the usage of generalized fractional calculus operators. Diaz and Pariguan [4, 5] paved the way for extensions of fractional calculus when they introduced the gamma k-function, beta k-function and hypergeometric k-functions based on Pochhammer's k-symbols [6, 7] and proved a number of their properties.

Different additions of numerous fractional integral operators and their properties have been investigated by many authors [8–10]. Many applications and special cases of generalized fractional integral operators are the recurring appearance of compositions of classical Riemann Liouville and Erdelyi Kober fractional operators in various problems of applied analysis and several properties of this operator can be located in [11, 12]. Many authors added a family of fractional integral operators with the Appell function F_3 in their kernel and extension of many acknowledged formulas given [13–15]. A distinct account of such operators along with their properties and applications had been considered [16–21].

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Definition 1.1. *The generalized fractional integral k-operator defined for α, α', β, β',$\gamma \in \mathbb{C}$ and $y > 0$, $\Re(\gamma) > 0$ and k is any real number respectively*

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(y) = \frac{y^{\frac{-\alpha}{k}}}{k\Gamma_k(\gamma)} \int_0^y (y-t)^{\frac{\gamma}{k}-1} t^{-\frac{\alpha'}{k}} F_{3,k}(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{y};1-\frac{y}{t}) f(t)dt \tag{1}$$

and

$$(I_{k,y^-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(y) = \frac{y^{\frac{-\alpha'}{k}}}{k\Gamma_k(\gamma)} \int_y^\infty (t-y)^{\frac{\gamma}{k}-1} t^{-\frac{\alpha}{k}} F_{3,k}(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{y}{t};1-\frac{t}{y}) f(t)dt. \tag{2}$$

Definition 1.2. *[22] The left and right sided Sagio fractional integral k-operator defined for $\alpha,\beta,\gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $y > 0$ and k is any real number respectively as*

$$(I_{k,0^+}^{\alpha,\beta,\gamma} f)(y) = \frac{y^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \int_0^y (y-t)^{\frac{\alpha}{k}-1} {}_2F_{1,k}(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{y}) f(t)dt \tag{3}$$

and

$$(I_{k,y^-}^{\alpha,\beta,\gamma} f)(y) = \frac{1}{k\Gamma_k(\alpha)} \int_y^\infty (t-y)^{\frac{\alpha}{k}-1} t^{\frac{-\alpha-\beta}{k}} {}_2F_{1,k}(\alpha+\beta,-\gamma;\alpha;1-\frac{y}{t}) f(t)dt. \tag{4}$$

Definition 1.3. *[22] The left and right sided Riemann Liouville fractional integral k-operator defined for $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, $y > 0$ and k is any positive real number respectively*

$$(I_{k,0^+}^{\alpha} f)(y) = \frac{1}{k\Gamma_k(\alpha)} \int_0^y (y-t)^{\frac{\alpha}{k}-1} f(t)dt \tag{5}$$

and

$$(I_{k,0^-}^{\alpha} f)(y) = \frac{1}{k\Gamma_k(\alpha)} \int_y^\infty (t-y)^{\frac{\alpha}{k}-1} f(t)dt. \tag{6}$$

Definition 1.4. *The k-beta function [24], defined for $\Re(l) > 0$, $\Re(h) > 0$, as*

$$\beta_k(l,h) = \frac{1}{k} \int_0^1 s^{\frac{l}{k}-1}(1-s)^{\frac{h}{k}-1} ds, \tag{7}$$

$$\text{so that} \qquad \beta_k(l,h) = \frac{1}{k}\beta\Big(\frac{l}{k},\frac{h}{k}\Big) \qquad \text{and} \qquad \beta_k(l,h) = \frac{\Gamma_k(l)\Gamma_k(h)}{\Gamma_k(l+h)}, \tag{8}$$

where $\Gamma_k(l)$, $\Gamma_k(h)$ and $\Gamma_k(l+h)$ are gamma k-functions.

Definition 1.5. *The gamma k-function [24], defined for $\Re(t) > 0$, $k > 0$, $t \in \mathbb{C}$ as*

$$\Gamma_k(t) = \int_0^\infty s^{t-1} e^{\frac{-s^k}{k}} ds, \tag{9}$$

$$\text{so that} \qquad \Gamma_k(z+k) = z\Gamma_k(z) \quad \text{and} \quad \Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1}\Gamma(\frac{\gamma}{k}). \tag{10}$$

Definition 1.6. *The Pochhammer's k-symbol for $k > 0$ [5], defined as*

$$(\alpha)_{n,k} = \begin{cases} \alpha(\alpha+k)(\alpha+2k)\cdots(\alpha+(n-1)k) & \text{for } n \geq 1 \\ 1 & \text{for } n = 0, \alpha \neq 0, \end{cases} \tag{11}$$

$$\text{So that} \qquad (\alpha)_{n,k} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)} \qquad \text{and} \qquad \frac{\Gamma_k(\alpha)}{\Gamma_k(\alpha-n)} = (-1)^n(k-\alpha)_{n,k} \tag{12}$$

where $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$.

Definition 1.7. *The Hypergeometric k-function defined for* $\forall\ \alpha', \beta', \eta' \in \mathbb{C},\ \eta' \neq 0, -1, -2, -3, \cdots,\ |t| < 1$, *as*

$$ {}_2F_{1,k}((\alpha', k), (\beta', k); (\eta', k); t) = \sum_{m=0}^{\infty} \frac{(\alpha')_{m,k}(\beta')_{m,k}}{(\eta')_{m,k}} \frac{t^m}{m!}, \qquad k > 0, \tag{13} $$

$$ {}_2F_{1,k}((a, k), (b, k); (c, k); 1) = \frac{\Gamma_k(c)\Gamma_k(c - a - b)}{\Gamma_k(c - a)\Gamma_k(c - b)}, \tag{14} $$

where $\Gamma_k(c)$, $\Gamma_k(c - a - b)$, $\Gamma_k(c - a)$ *and* $\Gamma_k(c - b)$ *are gamma k-functions.*

Definition 1.8. *The generalized Wright Hypergeometric k-function [25], defined by the series as*

$$ {}_l\psi_h^k(t) = {}_l\Psi_h^k\left[\begin{array}{c} (c_i, \alpha_i')_{1,l} \\ (d_j, \beta_j')_{1,h} \end{array} \middle| t \right] \equiv \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{l} \Gamma_k(c_i + \alpha_i' m) t^m}{\prod_{j=1}^{h} \Gamma_k(d_j + \beta_j' m) m!}, \tag{15} $$

where $k \in \Re^+$, $t \in \mathbb{C}$, $c_i, d_j \in \mathbb{C}$, *and* $\alpha_i', \beta_j' \in \Re$ $(i = 1, 2, \cdots, l;\ j = 1, 2, \cdots, h)$.

Definition 1.9. *The Bessel k-function of first kind* $W_{\upsilon,c}^k(t)$ *[12], defined for* $t \in \mathbb{C}$ *and* $\upsilon \in \mathbb{C}$ *by*

$$ W_{\upsilon,c}^k(t) = \sum_{p=0}^{\infty} \frac{(-c)^p (\frac{t}{2})^{\frac{\upsilon}{k}+2p}}{\Gamma_k(\upsilon + pk + k) p!}, \quad k > 0, c \in \Re. \tag{16} $$

We use the following notation in our results

$$ \mathcal{E}^{p,k} = \sum_{p=0}^{\infty} \frac{(-c)^p (\frac{1}{2})^{\frac{\upsilon}{k}+2p}}{\Gamma_k(\upsilon + pk + k) p!}, \qquad as \qquad W_{\upsilon,c}^k(t) = \mathcal{E}^{p,k}(t)^{\frac{\upsilon}{k}+2p}. \tag{17} $$

2. Left sided integral *k*-operators with Bessel *k*-function

In this section, we derive the fundamental results for left sided Sagio fractional integral *k*-operator in which Gauss hypergeometric *k*-function using as a kernel with the composition of power function and Bessel *k*-function, and also discuss the left sided Riemann Liouville fractional integral *k*-operator. The following theorems are needed to prove our main results.

Theorem 2.1. *Let* α, α', β, β', γ, υ, $\sigma \in \mathbb{C}$, $k > 0$, $c \in \Re$ *and* $x > 0$ *be such that* $\Re(\upsilon) > -1, \Re(\gamma) > 0$ *and* $\Re(\frac{\sigma+\upsilon}{k}) > max[0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')]$, *then there holds the following relation:*

$$ (I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1} W_{\upsilon,c}^k(t)])(x) = x^{\frac{1}{k}(\sigma+\upsilon+\gamma-\alpha-\alpha')-1} (2k)^{\frac{-\upsilon}{k}} $$

$$ {}_3\psi_4^k\left[\begin{array}{c} (\sigma + \upsilon, 2)(\sigma + \upsilon + \gamma - \alpha - \alpha' - \beta, 2)(\sigma + \upsilon + \beta' k - \alpha', 2) \\ (\upsilon + 1, 1)(\sigma + \upsilon + \gamma - \alpha - \alpha', 2)(\sigma + \upsilon + \gamma - \alpha' - \beta, 2)(\sigma + \upsilon + \beta' k, 2) \end{array} \middle| -\frac{cx^2}{4k} \right]. $$

Proof. Consider the generalized *k*-fractional integral (1) with the product of power function and Bessel

k-function of first kind (16), we have

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(t)])(x)$$

$$= \sum_{p=0}^{\infty} \frac{(-c)^p(\frac{1}{2})^{2p+\frac{\upsilon}{k}}}{\Gamma_k(pk+\upsilon+k)p!}\Big[\frac{x^{-\frac{\alpha}{k}}}{k\Gamma_k(\gamma)}\int_0^x (x-t)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha'}{k}}F_{3,k}(\alpha,\alpha',\beta,\beta',\gamma;1-\frac{t}{x};1-\frac{x}{t})t^{\frac{\sigma+\upsilon}{k}+2p-1}\Big]dt$$

$$= \mathcal{E}^{p,k}\Big[\frac{x^{-\frac{\alpha}{k}}}{\Gamma_k(\gamma)}\frac{1}{k}\int_0^x (x-t)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha'}{k}}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}(1-\frac{t}{x})^m(1-\frac{x}{t})^n t^{\frac{\sigma+\upsilon}{k}+2p-1}\Big]dt$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{\gamma-\alpha}{k}-1}}{k\Gamma_k(\gamma)}\int_0^x (1-\frac{t}{x})^{\frac{\gamma}{k}+m-1}(1-\frac{x}{t})^n t^{\frac{\sigma+\upsilon-\alpha'}{k}+2p-1}\Big]dt. \tag{18}$$

By putting $u=\frac{t}{x}\Rightarrow xdu=dt$, if $t=0\Rightarrow u=0$, if $t=x\Rightarrow u=1$ in equation (18), we get

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(t)])(x)$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{\gamma-\alpha}{k}-1}}{\Gamma_k(\gamma)}\frac{1}{k}\int_0^1 (1-u)^{\frac{\gamma}{k}+m-1}(1-\frac{1}{u})^n (xu)^{\frac{\sigma+\upsilon-\alpha'}{k}+2p-1}\Big]xdu$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\frac{x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}}{\Gamma_k(\gamma)}\Big[\frac{1}{k}\int_0^1 u^{\frac{\sigma+\upsilon-\alpha'}{k}+2p-n-1}(1-u)^{\frac{\gamma}{k}+m+n-1}\Big]du.$$

Using equations (7) and equation (13) in equation (19), we have

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(t)])(x)$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{\Gamma_k(\gamma)(\gamma)_{m+n,k}\ m!n!}\Big[x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}\beta_k(\sigma+\upsilon-\alpha'+2pk-nk,\gamma+mk+nk)\Big].$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\frac{x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}}{\Gamma_k(\gamma)}\Big[\frac{\Gamma_k(\sigma+\upsilon-\alpha'+2pk-nk)\Gamma_k(\gamma+mk+nk)}{\Gamma_k(\sigma+\upsilon-\alpha'+2pk-nk+\gamma+mk+nk)}\Big]. \tag{19}$$

By using equation (12) in equation (19), we obtain

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(t)])(x)$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\frac{x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}}{\Gamma_k(\gamma)}\Big[\frac{\Gamma_k(\sigma+\upsilon-\alpha'+2pk-nk)\Gamma_k(\gamma)(\gamma)_{m+n,k}}{\Gamma_k(\sigma+\upsilon-\alpha'+2pk+\gamma)(\sigma+\upsilon-\alpha'+2pk+\gamma)_{m,k}}\Big]$$

$$= x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}\sum_{m=0}^{\infty}\frac{(\alpha)_{m,k}(\beta)_{m,k}(1)^m}{(\sigma+\upsilon-\alpha'+2pk+\gamma)_{m,k}m!}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}}{n!}\frac{\Gamma_k(\sigma+\upsilon-\alpha'+2pk-nk)}{\Gamma_k(\sigma+\upsilon-\alpha'+2pk+\gamma)}. \tag{20}$$

By using equation (14) in equation (20), we get

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x)$$

$$= x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}}{n!}\frac{\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha-\beta)\Gamma_k(\sigma+v-\alpha'+2pk-nk)}{\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\beta)}. \quad (21)$$

Now we use equation (12) in equation (21), we have

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x)$$

$$= \sum_{n=0}^{\infty}\frac{x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}(\alpha')_{n,k}(\beta')_{n,k}(-1)^n}{(k-(\sigma+v-\alpha'+2pk))_{n,k}n!}\frac{\Gamma_k(\sigma+v-\alpha'+2pk)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha-\beta)}{\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\beta)}$$

$$= \frac{x^{\frac{\sigma+v+\gamma}{k}-1}\mathcal{E}^{p,k}\Gamma_k(k-\sigma-v+\alpha'-2pk)\Gamma_k(k-\sigma-v-2pk-\beta')\Gamma_k(\sigma+v-\alpha'+2pk)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha-\beta)}{x^{\frac{\alpha+\alpha'}{k}-2p}\Gamma_k(k-\sigma-v+\alpha'-2pk-\beta')\Gamma_k(k-\sigma-v-2pk)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\beta)}$$

$$= \frac{x^{\frac{\sigma+v+\gamma}{k}}\mathcal{E}^{p,k}(\sigma+v-\alpha'+2pk)_{\beta',k}}{x^{\frac{\alpha+\alpha'}{k}+1-2p}(\sigma+v+2pk)_{\beta',k}}\frac{\Gamma_k(\sigma+v-\alpha'+2pk)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha-\beta)}{\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\beta)}. \quad (22)$$

Using the equation (12) in equation (22), we obtain

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x)$$

$$= x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}\frac{\Gamma_k(\sigma+v-\alpha'+2pk+\beta'k)\Gamma_k(\sigma+v+2pk)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha-\beta)}{\Gamma_k(\sigma+v+2pk+\beta'k)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\beta)}$$

$$= \frac{x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}\Gamma_k(\sigma+v+2pk)}{\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha)}\frac{\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\alpha-\beta)\Gamma_k(\sigma+v-\alpha'+2pk+\beta'k)}{\Gamma_k(\sigma+v-\alpha'+\gamma+2pk-\beta)\Gamma_k(\sigma+v+2pk+\beta'k)}. \quad (23)$$

By using equations (17) and equation (10) in equation (23), we get

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x) = \frac{x^{\frac{\sigma+v-\alpha-\alpha'+\gamma}{k}-1}}{(2k)^{\frac{v}{k}}}\sum_{p=0}^{\infty}\Bigg[\frac{\Gamma_k(\sigma+v+2pk)}{\Gamma_k(v+p+1)\Gamma_k(\sigma+v+\beta'k+2pk)}$$

$$\times\frac{\Gamma_k(\sigma+v+\gamma-\alpha-\alpha'-\beta+2pk)\Gamma_k(\sigma+v+\beta'k-\alpha'+2pk)}{\Gamma_k(\sigma+v+\gamma-\alpha-\alpha'+2pk)\Gamma_k(\sigma+v+\gamma-\alpha'-\beta+2pk)}\Bigg]\frac{(\frac{-cx^2}{4k})^p}{p!}. \quad (24)$$

By using equation (15) in equation (24), and get the final result

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x) = x^{\frac{1}{k}(\sigma+v+\gamma-\alpha-\alpha')-1}(2k)^{\frac{-v}{k}}$$

$$ {}_3\psi_4^k\left[\begin{array}{c}(\sigma+v,2)(\sigma+v+\gamma-\alpha-\alpha'-\beta,2)(\sigma+v+\beta'k-\alpha',2)\\(v+1,1)(\sigma+v+\gamma-\alpha-\alpha',2)(\sigma+v+\gamma-\alpha'-\beta,2)(\sigma+v+\beta'k,2)\end{array}\Bigg|-\frac{cx^2}{4k}\right].$$

□

Corollary 2.2. *Taking $k=1, c=1$ in Theorem (2.1), we get*

$$(I_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\sigma-1}J_v(t)])(x) = x^{\sigma+v+\gamma-\alpha-\alpha'-1}(2)^{-v}$$

$$ {}_3\psi_4\left[\begin{array}{c} (\sigma+v,2)(\sigma+v+\gamma-\alpha-\alpha'-\beta,2)(\sigma+v+\beta'-\alpha',2) \\ (v+1,1)(\sigma+v+\gamma-\alpha-\alpha',2)(\sigma+v+\gamma-\alpha'-\beta,2)(\sigma+v+\beta',2)\end{array}\bigg| -\frac{x^2}{4}\right].$$

Theorem 2.3. *Let $\alpha,\beta,\gamma,v,\sigma \in \mathbb{C}$, $k>0$, $c \in \Re$ and $x>0$ be such that $\Re(v) > -1, \Re(\alpha) > 0$ and $\Re(\frac{\sigma+v}{k}) > max[0, \Re(\beta-\gamma)]$, then the following results holds true:*

$$(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x) = \frac{x^{\frac{\sigma+v-\beta}{k}-1}}{(2k)^{\frac{v}{k}}}\ {}_2\psi_3^k\left[\begin{array}{c}(\sigma+v,2),(\sigma+v-\beta+\gamma,2)\\(v+1,1),(\sigma+v-\beta,2),(\sigma+v+\alpha+\gamma,2)\end{array}\bigg| -\frac{cx^2}{4k}\right].$$

Proof. Consider the left sided Saigo fractional k-integral operator (3) with the product of power function and Bessel k-function of first kind (16), we have

$$\begin{aligned}
&(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x)\\
&= \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n+\frac{v}{k}}}{\Gamma_k(nk+v+k)n!}\Big[\frac{x^{\frac{-\alpha-\beta}{k}}}{\Gamma_k(\alpha)}\frac{1}{k}\int_0^x (x-t)^{\frac{\alpha}{k}-1}{}_2F_{1,k}(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x})t^{\frac{\sigma+v}{k}+2n-1}\Big]dt\\
&= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{-\alpha-\beta}{k}}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{1}{k}\int_0^x (x-t)^{\frac{\alpha}{k}-1}(1-\frac{t}{x})^m\ t^{\frac{\sigma+v}{k}+2n-1}\Big]dt\\
&= \mathcal{E}^{n,k}\frac{x^{\frac{-\alpha-\beta+\alpha}{k}-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\frac{1}{k}\int_0^x (1-\frac{t}{x})^{\frac{\alpha}{k}+m-1}t^{\frac{\sigma+v}{k}+2n-1}\Big]dt.
\end{aligned}\tag{25}$$

By putting $u=\frac{t}{x} \Rightarrow xdu = dt$ if $t=0 \Rightarrow u=0$ if $t=x \Rightarrow u=1$ in (25), we have

$$(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x) = \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+v-\beta}{k}+2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\frac{1}{k}\int_0^1 u^{\frac{\sigma+v}{k}+2n-1}(1-u)^{\frac{\alpha}{k}+m-1}\Big]du. \tag{26}$$

Using equation (7) in equation (26), we obtain

$$\begin{aligned}
(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x) &= \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+v-\beta}{k}+2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\beta_k(\sigma+v+2nk,\ \alpha+mk)\Big]\\
&= \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+v-\beta}{k}+2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\frac{\Gamma_k(\sigma+v+2nk)\Gamma_k(\alpha+mk)}{\Gamma_k(\sigma+v+2nk+\alpha+mk)}\Big].
\end{aligned}\tag{27}$$

Using equation (12) in equation (27), we have

$$\begin{aligned}
(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x) &= \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+v-\beta}{k}+2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\frac{\Gamma_k(\sigma+v+2nk)\Gamma_k(\alpha)(\alpha)_{m,k}}{\Gamma_k(\sigma+v+\alpha+2nk)(\sigma+v+\alpha+2nk)_{m,k}}\Big]\\
&= \mathcal{E}^{n,k}x^{\frac{\sigma+v-\beta}{k}+2n-1}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}(1)^m}{(\sigma+v+\alpha+2nk)_{m,k}m!}\Big[\frac{\Gamma_k(\sigma+v+2nk)}{\Gamma_k(\sigma+v+\alpha+2nk)}\Big].
\end{aligned}\tag{28}$$

By using equation (13) in equation (28), we have

$$(I^{\alpha,\beta,\gamma}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = \mathcal{E}^{n,k}x^{\frac{\sigma+\upsilon-\beta}{k}+2n-1}\frac{\Gamma_k(\sigma+\upsilon+2nk)\Gamma_k(\sigma+\upsilon+2nk-\beta+\gamma)}{\Gamma_k(\sigma+\upsilon+2nk-\beta)\Gamma_k(\sigma+\upsilon+\alpha+2nk+\gamma)}. \tag{29}$$

By using equations (17) and equation (10) in equation (29), we attain

$$(I^{\alpha,\beta,\gamma}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = \frac{x^{\frac{\sigma+\upsilon-\beta}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\Big[\frac{\Gamma_k(\sigma+\upsilon+2nk)}{\Gamma(\frac{\upsilon}{k}+1+n)}\frac{\Gamma_k(\sigma+\upsilon-\beta+\gamma+2nk)}{\Gamma(\sigma+\upsilon-\beta+2nk)\Gamma(\sigma+\upsilon+\alpha+\gamma+2nk)}\Big]\frac{(\frac{-cx^2}{4k})^n}{n!}. \tag{30}$$

By using equation (15) in equation (30), we get the final result

$$(I^{\alpha,\beta,\gamma}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = \frac{x^{\frac{\sigma+\upsilon-\beta}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\,{}_2\psi^k_3\left[\begin{array}{c|c} (\sigma+\upsilon,2),(\sigma+\upsilon-\beta+\gamma,2) & \\ (\frac{\upsilon}{k}+1,1),(\sigma+\upsilon-\beta,2),(\sigma+\upsilon+\alpha+\gamma,2) & -\frac{cx^2}{4k}\end{array}\right].$$

□

Theorem 2.4. *Let $\alpha,\upsilon,\sigma \in \mathbb{C}$, $k>0$, $c \in \Re$ and $x>0$ be such that $\Re(\upsilon) > -1 and \Re(\alpha) > 0$, then there holds following formula:*

$$(I^{\alpha}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = \frac{x^{\frac{\sigma+\upsilon+\alpha}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\,{}_1\psi^k_2\left[\begin{array}{c|c} (\sigma+\upsilon,2) & \\ (\frac{\upsilon}{k}+1,1),(\sigma+\upsilon+\alpha,2) & -\frac{cx^2}{4k}\end{array}\right].$$

Proof. Consider the left sided Riemann Liouville k-fractional integral operator (5) with the product of power function and Bessel k-function of first kind (16), we have

$$\begin{aligned}(I^{\alpha}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) &= \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n+\frac{\upsilon}{k}}}{\Gamma_k(nk+\upsilon+k)n!}\Big[\frac{1}{k\Gamma_k(\alpha)}\int_0^x(x-t)^{\frac{\alpha}{k}-1}t^{\frac{\sigma+\upsilon}{k}+2n-1}\Big]dt\\ &= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha)}\frac{1}{k}\int_0^x(1-\frac{t}{x})^{\frac{\alpha}{k}-1}t^{\frac{\sigma+\upsilon}{k}+2n-1}\Big]dt. \end{aligned}\tag{31}$$

By putting $u=\frac{t}{x} \Rightarrow xdu = dt$, if $t=0 \Rightarrow u=0$, if $t=x \Rightarrow u=1$ in equation (31), we get

$$(I^{\alpha}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+\upsilon+\alpha}{k}+2n-1}}{\Gamma_k(\alpha)}\Big[\frac{1}{k}\int_0^1 u^{\frac{\sigma+\upsilon+2nk}{k}-1}(1-u)^{\frac{\alpha}{k}-1}\Big]du. \tag{32}$$

By using equation (7) in equation (32), we attain

$$\begin{aligned}(I^{\alpha}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) &= \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+\upsilon+\alpha}{k}+2n-1}}{\Gamma_k(\alpha)}\beta_k(\sigma+\upsilon+2nk,\alpha)\\ &= x^{\frac{\sigma+\upsilon+\alpha}{k}+2n-1}\mathcal{E}^{n,k}\frac{\Gamma_k(\sigma+\upsilon+2nk)}{\Gamma_k(\sigma+\upsilon+\alpha+2nk)}.\end{aligned}\tag{33}$$

By using the equations (10) and equation (17) in equation (33), we have

$$\begin{aligned}(I^{\alpha}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) &= x^{\frac{\sigma+\upsilon+\alpha}{k}+2n-1}\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n}(\frac{1}{2})^{\frac{\upsilon}{k}}}{k^{\frac{\upsilon}{k}+n+1-1}\Gamma(\frac{\upsilon}{k}+n+1)}\frac{\Gamma_k(\sigma+\upsilon+2nk)}{\Gamma_k(\sigma+\upsilon+\alpha+2nk)}\\ &= \frac{x^{\frac{\sigma+\upsilon}{k}+\alpha-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\frac{\Gamma(\sigma+\upsilon+2nk)}{\Gamma(\frac{\upsilon}{k}+1,1)\Gamma(\sigma+\upsilon+\alpha+2nk)}\frac{(\frac{-cx^2}{4k})^n}{n!}.\end{aligned}\tag{34}$$

By using equation (15) in equation (34), we get the final result

$$(I^{\alpha}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = \frac{x^{\frac{\sigma+\upsilon+\alpha}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\ {}_1\psi^k_2\left[\begin{array}{c}(\sigma+\upsilon,2)\\ (\frac{\upsilon}{k}+1,1),(\sigma+\upsilon+\alpha,2)\end{array}\middle|-\frac{cx^2}{4k}\right].$$

□

3. Right sided fractional k-operators with Bessel k-function

In this section, we elaborate the right sided Sagio fractional integral k-operator in which hypergeometric k-function using as a kernel with Bessel k-function, and also derived Riemann Liouville fractional k-operator in the form of theorems.

Theorem 3.1. *Let α, α', β, β', γ, υ, $\sigma \in \mathbb{C}$, $k > 0$, $c \in \Re$ and $x > 0$ be such that $\Re(\upsilon) > -1, \Re(\gamma) > 0$ and $\Re(\frac{\sigma+\upsilon}{k}) > max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')]$. then there holds the following relation:*

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma+\upsilon+\gamma-\alpha-\alpha')-1}(2k)^{\frac{-\upsilon}{k}}$$

$${}_3\psi^k_4\left[\begin{array}{c}(k-\sigma+\upsilon-\beta,2),(k-\sigma+\upsilon-\gamma+\alpha+\alpha'k,2),(k-\sigma+\upsilon+\alpha+\beta'-\gamma,2)\\ (\frac{\upsilon}{k}+1,1),(k-\sigma+\upsilon-\gamma+\alpha+\alpha'k+\beta',2),(k-\sigma+\upsilon-\gamma+\alpha-\beta,2),(k-\sigma+\upsilon,2)\end{array}\middle|-\frac{cx^2}{4k}\right].$$

Proof. Consider the right sided generalized fractional k-operator (2) with the composition of power function and Bessel k-function of first kind (16), we have

$$\begin{aligned}
&(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x)\\
&= \sum_{p=0}^{\infty}\frac{(-c)^p(\frac{1}{2})^{2p+\frac{\upsilon}{k}}}{\Gamma_k(pk+\upsilon+k)p!}\Big[\frac{x^{-\frac{\alpha'}{k}}}{\Gamma_k(\gamma)}\times\frac{1}{k}\int_x^{\infty}(t-x)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha}{k}}F_{3,k}(\alpha,\alpha',\beta,\beta',\gamma;1-\frac{x}{t};1-\frac{t}{x})t^{\frac{\sigma-\upsilon}{k}-2p-1}\Big]dt\\
&= \mathcal{E}^{p,k}\Big[\frac{x^{-\frac{\alpha'}{k}}}{k\Gamma_k(\gamma)}\int_x^{\infty}(t-x)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha}{k}}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}(1-\frac{x}{t})^m(1-\frac{t}{x})^n t^{\frac{\sigma-\upsilon}{k}-2p-1}\Big]dt\\
&= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{-\frac{\alpha'}{k}}}{\Gamma_k(\gamma)}\frac{1}{k}\int_x^{\infty}(1-\frac{x}{t})^{\frac{\gamma}{k}+m-1}(1-\frac{t}{x})^n t^{\frac{\sigma-\upsilon-\alpha+\gamma}{k}-2p-2}\Big]dt.
\end{aligned}\tag{35}$$

By putting $u = \frac{x}{t} \Rightarrow -xu^2du = dt$, if $t = \infty \Rightarrow u = 0$, if $t = x \Rightarrow u = 1$ in equation (35), we have

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) \tag{36}$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{-\alpha'}{k}-1}}{\Gamma_k(\gamma)}\frac{1}{k}\int_1^0(1-u)^{\frac{\gamma}{k}+m-1}(1-\frac{1}{u})^n(xu^{-1})^{\frac{\sigma-\upsilon-\alpha+\gamma}{k}-2p-2}\Big](-xu^{-2})du$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}}{\Gamma_k(\gamma)}\frac{1}{k}\int_0^1 u^{\frac{k-\sigma+\upsilon+\alpha-\gamma+2pk-nk}{k}-1}(1-u)^{\frac{\gamma+mk+nk}{k}-1}\Big]du. \tag{37}$$

By using equation (7) and equation (8) in equation (37), we obtain

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\left[\frac{x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}}{\Gamma_k(\gamma)}\beta_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk,\gamma+mk+nk)\right]$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\frac{x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}}{\Gamma_k(\gamma)}\left[\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk)\Gamma_k(\gamma+mk+nk)}{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk+\gamma+mk+nk)}\right]. \tag{38}$$

By using the equation (12) in equation (38), we have

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta')_{m,k}(\beta)_{n,k}}{\Gamma_k(\gamma)(\gamma)_{m+n,k}\ m!n!}\left[\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk)\Gamma_k(\gamma)(\gamma)_{m+n,k}}{\Gamma_k(k-\sigma+\upsilon+\alpha+2pk)(k-\sigma+\upsilon+\alpha+2pk)_{m,k}}\right].$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\sum_{m=0}^{\infty}\frac{(\alpha)_{m,k}(\beta)_{m,k}(1)^m}{(k-\sigma+\upsilon+\alpha+2pk)_{m,k}\ m!}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}}{n!}\left[\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk)}{\Gamma_k(k-\sigma+\upsilon+\alpha+2pk)}\right]. \tag{39}$$

By using the equation (14) in equation (39), we get

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}}{n!}\frac{\Gamma_k(k-\sigma+\upsilon+2pk-\beta)\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk)}{\Gamma_k(k-\sigma+\upsilon+\alpha+2pk-\beta)\Gamma_k(k-\sigma+\upsilon+2pk)}. \tag{40}$$

Now, we use the equation (12) in equation (40), we have

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}(-1)^n}{(\sigma-\upsilon-\alpha+\gamma-2pk)_{n,k}n!}\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk)\Gamma_k(k-\sigma+\upsilon+2pk-\beta)}{\Gamma_k(k-\sigma+\upsilon+\alpha+2pk-\beta)\Gamma_k(k-\sigma+\upsilon+2pk)}$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\frac{\Gamma_k(\sigma-\upsilon+\gamma-\alpha-2pk)\Gamma_k(\sigma-\upsilon+\gamma-\alpha-\alpha'-\beta'-2pk)}{\Gamma_k(\sigma-\upsilon+\gamma-\alpha-\alpha'-2pk)\Gamma_k(\sigma-\upsilon+\gamma-\alpha-\beta'-2pk)}$$
$$\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk)\Gamma_k(k-\sigma+\upsilon-\beta+2pk)}{\Gamma_k(k-\sigma+\upsilon+\alpha-\beta+2pk)\Gamma_k(k-\sigma+\upsilon+2pk)}$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}-2p-1}\frac{(k-\sigma+\upsilon-\gamma+\alpha+2pk)_{\alpha',k}}{(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk)_{\alpha',k}}\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk)\Gamma_k(k-\sigma+\upsilon-\beta+2pk)}{\Gamma_k(k-\sigma+\upsilon+\alpha-\beta+2pk)\Gamma_k(k-\sigma+\upsilon+2pk)}$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}-2p-1}\frac{\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+2pk+\alpha'k)\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk)\Gamma_k(k-\sigma+\upsilon-\beta+2pk)}{\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk+\alpha'k)\Gamma_k(k-\sigma+\upsilon+\alpha-\beta+2pk)\Gamma_k(k-\sigma+\upsilon+2pk)} \tag{41}$$

By using equations (17) and equation (10) in (41), we get

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(\frac{1}{t})])(x)$$

$$= \sum_{p=0}^{\infty} \frac{x^{\frac{\sigma+\gamma-\alpha-\alpha'}{k}-1}(-c)^p(\frac{x}{2})^{\frac{\upsilon}{k}-2p}\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+2pk+\alpha'k)\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk)\Gamma_k(k-\sigma+\upsilon-\beta+2pk)}{\Gamma_k(pk+\upsilon+k)\Gamma_k(k-\sigma+\upsilon+\alpha-\beta+2pk)\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk+\alpha'k)\Gamma_k(k-\sigma+\upsilon+2pk)}. \tag{42}$$

By using equation (17) in equation (42), we get the final result

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma+\upsilon+\gamma-\alpha-\alpha')-1}(2k)^{\frac{-\upsilon}{k}}$$

$${}_3\psi_4^k\left[\begin{array}{c} (k-\sigma+\upsilon-\beta,2),(k-\sigma+\upsilon-\gamma+\alpha+\alpha'k,2),(k-\sigma+\upsilon+\alpha+\beta'-\gamma,2) \\ (\frac{\upsilon}{k}+1,1),(k-\sigma+\upsilon-\gamma+\alpha+\alpha'k+\beta',2),(k-\sigma+\upsilon-\gamma+\alpha-\beta,2),(k-\sigma+\upsilon,2) \end{array}\middle| -\frac{c}{4kx^2}\right].$$

□

Corollary 3.2. *Taking $k = 1, c = 1$ in Theorem* (3.1)*, we get*

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\sigma-1}J_\upsilon(\frac{1}{t})])(x)) = x^{\sigma+\upsilon+\gamma-\alpha-\alpha'-1}2^{-\upsilon}$$

$$\times\, {}_3\psi_4\left[\begin{array}{c} (1-\sigma+\upsilon-\beta,2),(1-\sigma+\upsilon-\gamma+\alpha+\alpha',2),(1-\sigma+\upsilon+\alpha+\beta'-\gamma,2) \\ (\upsilon+1,1),(1-\sigma+\upsilon-\gamma+\alpha+\alpha'+\beta',2),(1-\sigma+\upsilon-\gamma+\alpha-\beta,2),(1-\sigma+\upsilon,2) \end{array}\middle| \frac{-1}{4x^2}\right].$$

Theorem 3.3. *Let α, β, γ, υ, $\sigma \in \mathbb{C}$, $k > 0$, $c \in \Re$ and $x > 0$ be such that $\Re(\upsilon) > -1, \Re(\alpha) > 0$ and $\Re(\frac{\sigma-\upsilon}{k}) < 1 + min[\Re(\beta), \Re(\gamma)]$. Then there holds the following relation:*

$$(I_{k,0^-}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma-\upsilon-\beta)-1}(2k)^{\frac{-\upsilon}{k}}\,{}_2\psi_3^k\left[\begin{array}{c} (k-\sigma+\upsilon+\beta,2),(k-\sigma+\upsilon+\gamma,2) \\ (\frac{\upsilon}{k}+1,1),(k-\sigma+\upsilon,2),(1-\sigma+\upsilon+\alpha+\beta+\gamma,2) \end{array}\middle| -\frac{c}{4kx^2}\right].$$

Proof. Consider the right sided Saigo fraction k-integral operator (4) with the product of power function with Bessel k-function (16), we have

$$\begin{aligned}
(I_{k,0^-}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(\frac{1}{t})])(x) &= \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n+\frac{\upsilon}{k}}}{\Gamma_k(nk+\upsilon+k)n!}\Big[\frac{1}{\Gamma_k(\alpha)}\frac{1}{k}\int_x^\infty(t-x)^{\frac{\alpha}{k}-1}t^{\frac{-\alpha-\beta}{k}}\,{}_2F_{1,k}(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t})t^{\frac{\sigma-\upsilon}{k}-2n-1}\Big]dt \\
&= \mathcal{E}^{n,k}\Big[\frac{1}{k\Gamma_k(\alpha)}\int_x^\infty t^{\frac{\alpha}{k}-1}(1-\frac{x}{t})^{\frac{\alpha}{k}-1}t^{\frac{-\alpha-\beta}{k}}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}(1-\frac{x}{t})^m t^{\frac{\sigma-\upsilon}{k}-2n-1}\Big]dt \\
&= \mathcal{E}^{n,k}\Big[\frac{1}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{1}{k}\int_x^\infty(1-\frac{x}{t})^{\frac{\alpha}{k}+m-1}t^{\frac{\sigma-\upsilon+\alpha-\alpha-\beta}{k}-2n-2}\Big]dt.
\end{aligned} \tag{43}$$

By putting $u = \frac{x}{t} \Rightarrow dt = -xu^{-2}du$ if $t = x \Rightarrow u = 1$ if $t = \infty \Rightarrow u = 0$ in (43), we obtain

$$
\begin{aligned}
(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) &= \mathcal{E}^{n,k}\Big[\frac{1}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{1}{k}\int_1^0(1-u)^{\frac{\alpha}{k}+m-1}(xu^{-1})^{\frac{\sigma-\upsilon-\beta}{k}-2n-2}\Big](-xu^{-2})du\\
&= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon-\beta}{k}-2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{1}{k}\int_0^1 u^{\frac{k-\sigma+\upsilon+\beta+2nk}{k}-1}(1-u)^{\frac{\alpha+mk}{k}-1}\Big]du. \qquad (44)
\end{aligned}
$$

By using equation (7) in equation (44), we obtain

$$
\begin{aligned}
(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) &= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon-\beta}{k}-2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\beta_k(k-\sigma+\upsilon+\beta+2nk,\alpha+mk)]\\
&= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon-\beta}{k}-2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{\Gamma_k(k-\sigma+\upsilon+\beta+2nk)\Gamma_k(\alpha+mk)}{\Gamma_k(k-\sigma+\upsilon+\beta+2nk+\alpha+mk)}\Big]. \qquad (45)
\end{aligned}
$$

By using equations (12) and equation (14) in equation (45), we have

$$
\begin{aligned}
(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) &= \mathcal{E}^{n,k}x^{\frac{\sigma-\upsilon-\beta}{k}-2n-1}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}(1)^m}{(k-\sigma+\upsilon+\beta+2nk+\alpha)_{m,k}m!}\frac{\Gamma_k(k-\sigma+\upsilon+\beta+2nk)}{\Gamma_k(k-\sigma+\upsilon+\beta+2nk+\alpha)}\\
&= \frac{\Gamma_k(k-\sigma+\upsilon+\gamma+2nk)\Gamma_k(k-\sigma+\upsilon+\beta+2nk)}{\Gamma_k(k-\sigma+\upsilon+2nk)\Gamma_k(k-\sigma+\upsilon+\alpha+\beta+2nk+\gamma)}. \qquad (46)
\end{aligned}
$$

By using equations (10) and equation (17) in (46), we get

$$
\begin{aligned}
&(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x)\\
&= x^{\frac{\sigma-\upsilon-\beta}{k}-2n-1}\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n}(\frac{1}{2})^{\frac{\upsilon}{k}}}{k^{\frac{\upsilon}{k}+1+2n-1}\Gamma(\frac{\upsilon}{k}+1+n)n!}\frac{\Gamma_k(k-\sigma+\upsilon+\gamma+2nk)\Gamma_k(k-\sigma+\upsilon+\beta+2nk)}{\Gamma_k(k-\sigma+\upsilon+2nk)\Gamma_k(k-\sigma+\upsilon+\alpha+\beta+\gamma+2nk)}\\
&= \frac{x^{\frac{\sigma-\upsilon-\beta}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\Big[\frac{\Gamma_k(k-\sigma+\upsilon+\gamma+2nk)}{\Gamma_k(\frac{\upsilon}{k}+1+n)\Gamma_k(k-\frac{\sigma}{+}\upsilon+2nk)}\frac{\Gamma_k(k-\sigma+\upsilon+\beta+2nk)}{\Gamma_k(k-\sigma+\upsilon+\alpha+\beta+\gamma+2nk)}\Big]\frac{(\frac{-c}{4kx^2})^n}{n!}. \qquad (47)
\end{aligned}
$$

By using equation (15) in equation (47), we get the final result

$$
(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma-\upsilon-\beta)-1}(2k)^{\frac{-\upsilon}{k}}\,{}_2\psi^k_3\left[\begin{array}{c}(k-\sigma+\upsilon+\beta,2),(k-\sigma+\upsilon+\gamma,2)\\(\frac{\upsilon}{k}+1,1),(k-\sigma+\upsilon,2),(1-\sigma+\upsilon+\alpha+\beta+\gamma,2)\end{array}\Big| -\frac{c}{4kx^2}\right].
$$

□

Theorem 3.4. *Let α, υ, $\sigma \in \mathbb{C}$, $k > 0$, $c \in \Re$ and $x > 0$ be such that $\Re(\upsilon) > -1, \Re(\alpha) > 0$, then there holds the following relation:*

$$
(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma-\upsilon+\alpha)-1}(2k)^{\frac{-\upsilon}{k}}\,{}_1\psi^k_2\left[\begin{array}{c}(k-\sigma+\upsilon-\alpha,2)\\(\frac{\upsilon}{k}+1,1),(k-\sigma-\upsilon,2)\end{array}\Big| -\frac{c}{4kx^2}\right].
$$

Proof. Consider the right sided Rieman Liuville fractional k-integral operator (6) with the product of power function and Bessel k-function (16), we have

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^{k}_{\upsilon,c}(\frac{1}{t})])(x) = \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n+\frac{\upsilon}{k}}}{\Gamma_k(nk+\upsilon+k)n!}\Big[\frac{1}{k\Gamma_k(\alpha)}\int_x^{\infty}(t-x)^{\frac{\alpha}{k}-1}t^{\frac{\sigma-\upsilon}{k}-2n-1}\Big]dt$$
$$= \mathcal{E}^{n,k}\Big[\frac{1}{\Gamma_k(\alpha)}\frac{1}{k}\int_x^{\infty}(1-\frac{x}{t})^{\frac{\alpha}{k}-1}t^{\frac{\sigma-\upsilon+\alpha}{k}-2n-2}\Big]dt. \quad (48)$$

By putting $u=\frac{x}{t} \Rightarrow dt=-xu^{-2}du$ if $t=x \Rightarrow u=1$ and $t=\infty \Rightarrow u=0$ in (48), we have

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^{k}_{\upsilon,c}(\frac{1}{t})])(x) = \mathcal{E}^{n,k}\Big[\frac{1}{k\Gamma_k(\alpha)}\int_1^{0}(1-u)^{\frac{\alpha}{k}-1}(xu^{-1})^{\frac{\sigma-\upsilon+\alpha}{k}-2n-2}(-xu)^{-2}\Big]du$$
$$= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{\Gamma_k(\alpha)}\frac{1}{k}\int_0^{1}u^{\frac{k-\sigma+\upsilon-\alpha+2nk}{k}-1}(1-u)^{\frac{\alpha}{k}-1}\Big]du. \quad (49)$$

By using equation (7) in equation (49), we get

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^{k}_{\upsilon,c}(\frac{1}{t})])(x) = \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{\Gamma_k(\alpha)}\beta_k(k-\sigma+\upsilon-\alpha+2nk,\alpha)\Big].$$
$$= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{\Gamma_k(\alpha)}\frac{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk)\Gamma_k(\alpha)}{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk+\alpha)}\Big]. \quad (50)$$

By using equations (10) and equation (17) in (50), we obtain

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^{k}_{\upsilon,c}(\frac{1}{t})])(x) = \frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{\Gamma_k(\alpha)}\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n}(\frac{1}{2})^{\frac{\upsilon}{k}}}{k^{2n+\frac{\upsilon}{k}\Gamma(n+\frac{\upsilon}{k}+1)n!}}\frac{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk)\Gamma_k(\alpha)}{\Gamma_k(k-\sigma+\upsilon+2nk)}$$
$$= x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n}(\frac{1}{2})^{\frac{\upsilon}{k}}}{k^{2n+\frac{\upsilon}{k}}n!}\frac{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk)}{\Gamma(n+\frac{\upsilon}{k}+1)\Gamma_k(k-\sigma+\upsilon+2nk)}$$
$$= \frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\Big[\frac{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk)}{\Gamma(n+\frac{\upsilon}{k}+1)\Gamma_k(k-\sigma+\upsilon+2nk)}\Big]\frac{(\frac{-c}{4kx^2})^n}{n!}. \quad (51)$$

By using equation (15) in equation (51), we get the final result

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^{k}_{\upsilon,c}(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma-\upsilon+\alpha)-1}(2k)^{\frac{-\upsilon}{k}}\ {}_1\psi^k_2\left[\begin{array}{c|c} (k-\sigma+\upsilon-\alpha,2) & \\ (\frac{\upsilon}{k}+1,1),(k-\sigma-\upsilon,2) & -\frac{c}{4kx^2}\end{array}\right].$$

□

Conclusion

In this paper, we have derived generalized k-fractional integral operators involving Appell k-function as its kernels with Bessel k-function. We have proved some composition formulae for Saigo, Riemann-Liouville k-fractional integral operators. The results have been established in terms of generalized k-Wright hypergeometric function. Furthermore if we take $k=1$, then we find out the results which are discussed in the form of corollaries (2.2) and (3.2).

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The alpha power Weibull Frechet distribution: properties and applications

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Abstract. Modeling everyday life processes play a great role in human existence. Thus, distribution theory has helped to understand how our everyday life processes are distributed. However, this depends on how researchers in distribution theory compound several distributions to derive a more flexible distribution. This study proposes the alpha power Weibull Frechet distribution for real-life datasets. However, some statistical structural properties of the model such as kurtosis, hazard rate and odd functions, cumulative, quantiles, reversed hazard, skewness, order statistics and survival function were derived. The parameters of the proposed model were obtained using the maximum likelihood method. The behavioural nature of the model was studied through simulation. Finally, a two real life data was used to investigate the performance of the proposed model. The results show that the new model performs better than some existing continuous models in statistical literature.

1. Introduction

Integral representations of solutions for differential equations and operators are used in many scientific fields [1, 2]. Several methods for generating family of univariate distributions were based on differential equation (Pearson 1895). Of most important, is the translation method proposed in [3] . This method is based on quantile function that was developed in [4]. Lifetime processes have received several attentions through modeling the way and manner in which they are distributed, thus developing a flexible distribution depending on how the researcher compounds one or more distribution(s) to form a better or a comparable distribution [5]. The Weibull distribution plays a very important role in modeling lifetime processes. The Weibull distribution was proposed by a famous statistician called Weibull [6]. This Weibull

distribution has a wide range of applications in modelling lifetime processes, failure time processes, survival time, mechanical and electrical systems and machine learning. More so, the Frechet distribution is used in modeling extreme value theory. Its applications ranging from horse racing accelerated life testing in earthquakes, floods, rainfall, queues in supermarkets, wind speed and sea waves. The Frechet distribution can also be used in modelling material properties in engineering materials.

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Let S be a random variable, say $s > 0$. Then, the Frechet distribution is defined as

$$g(s, \alpha, \beta) = \beta\alpha^{\beta} s^{-beta-1} exp\Big[-(\frac{\alpha}{s})^{\beta}\Big] \quad \alpha, \beta > 0. \tag{1}$$

The corresponding cdf is expressed as

$$G(s, \alpha, \beta) = exp\Big[-(\frac{\alpha}{s})^{\beta}\Big] \quad \alpha, \beta > 0, \tag{2}$$

where α and β are the scale and shape parameters respectively.
More so, the Weibull pdf with the parameters $\alpha > 0$ and $\beta > 0$ is defined as

$$f(s, \lambda, \beta) = \lambda\gamma s^{\gamma-1} exp(-\gamma s^{\gamma-1}); \quad \lambda\ \gamma > 0. \tag{3}$$

The cdf that corresponds to the Weibull pdf is given as

$$F(s, \lambda, \beta) = 1 - exp(-\gamma s^{\gamma}); \quad \lambda\ \gamma > 0 \tag{4}$$

where λ and γ are the shaped and scale parameters respectively.
[7] Proposed the Weibull Frechet (WFr) distribution and obtained the its pdf as

$$f(s) = \psi b\beta\tau^{\beta} s^{-\beta-1} exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]\Big\{1 - exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big\}^{-b-1} exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big] - 1\Big]^{-b}\Big] \tag{5}$$

The corresponds cdf is expressed as

$$F(s) = 1 - exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big] - 1\Big]^{-b}\Big], \tag{6}$$

whereτ is the scale parameter,β, ψ and b are the shape parameters.
The alpha power transformation (AP) was proposed in [8]. The pdf of the alpha power transformed family of distribution is given as

$$f_{AP}(s) = \begin{cases} g(s)\frac{\log\alpha}{(\alpha-1)}\alpha^{G(s)}, & \text{if } \alpha \in (\Re^{+} - (1)) \\ \\ g(s), & \text{otherwise}\alpha = 1. \end{cases} \tag{7}$$

The corresponding cdf is defined as

$$F_{AP}(s) = \frac{\alpha^{G(s)} - 1}{\alpha - 1} \quad \alpha \in (\Re^{+} - (1)) \,. \tag{8}$$

Otherwise, $F(s)$, for $\alpha = 1$ where $g(s)$ is the baseline pdf and $G(s)$ is the baseline cdf.
Several research works have been done in literature researched. [9] Proposed the Weibull-G family of distribution. The alpha power inverted exponential distribution was proposed in [10]. Gompertz-G distribution was proposed in [11]. Gompertz alpha power inverted exponential distribution was proposed in [12]. The extended new generalized exponential distribution was proposed in [13]. The Weibull alpha power inverted exponential distribution was proposed in [14]. Alpha power Weibull distribution was proposed in [15].
However, many distributions have been proposed in literature to extend distributions that are significant to the progress of distribution frontiers and to make life more meaningful. Thus, this study set up a model called alpha power Weibull Frechet (APWF) distribution to push back the frontiers of knowledge in data science, data analysis and distribution theory.

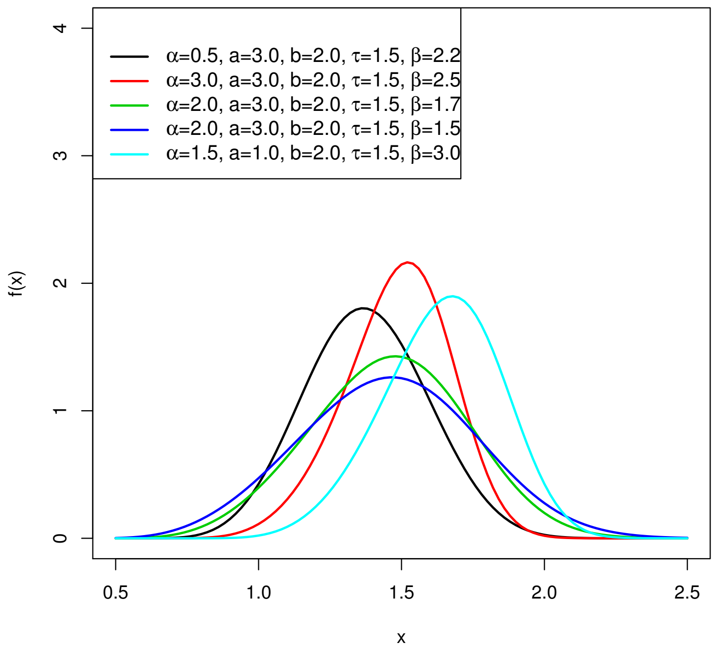

Figure 1: The APWF density for different parameter values cases

This study was motivated by studies and events obtained from some literature research in probability and distribution theories. However, the APWF model was proposed to push back the frontiers of knowledge in data science, data analysis and distribution theory by addition of a parameter to improve the existing models using the AP characterization.

The aim of this study was to introduce a class of Frechet distribution in distribution theory together with its mathematical properties. It worthy to note that this study was proposed to address APWF model, since, say , we obtained the usual WFr model.

2. The APWF Distribution

This section proposed a class of the Frechet family of distribution called APWF model. Let $s_1, s_2, s_3, \cdots s_n$ be a random sample of the APWF distribution. Then, the pdf of the APWF is given as

$$\begin{aligned} f_{APWF}(s) =&\psi b\beta\tau^{\beta}s^{-\beta-1}exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]\Big\{1-exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big\}^{-b-1}exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big]-1\Big]^{-b}\Big] \\ &\times\frac{\log\alpha}{(\alpha-1)}\alpha^{\Big[1-exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big]-1\Big]^{-b}\Big]\Big]}, \qquad \alpha\in(\Re^{+}-(1)). \end{aligned} \tag{9}$$

Figure 1 shows the plot of the pdf for different parameter values cases. In Figure 1, the shape of the pdf could be increasing, decreasing, unimodal and symmetrical depending on the parameter values.

The cdf that corresponds to Equation (9) is given as

$$F_{APWF}(s) = \Big\{\alpha^{\Big[1-exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big]-1\Big]^{-b}\Big]\Big]}-1\Big\}(\alpha-1)^{-1}, \qquad \alpha\in(\Re^{+}-(1)). \tag{10}$$

3. Mathematical Mixture Representation

In this section, we expressed the APWF distribution in power series. First and foremost, we expressed the Weibull Frechet distribution before the proposed distribution is addressed. Thus, the Equation (5) can be defined as

$$f(s) = \psi b\beta\tau^{\beta}s^{-(\beta+1)}exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]exp\Big[-\psi\Big[\frac{exp\Big[-(\frac{\tau}{s})^{\beta}\Big]}{1-exp\Big[-(\frac{\tau}{s})^{\beta}\Big]}\Big]^{b}\Big]\Big\{1-exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big\}^{-(b+1)} \tag{11}$$

Let the middle quantity in Equation (11) be A. Then, expanding the exponential function in A, we expressed

$$A = \sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha^{k}}{k!}\frac{exp\Big[-bk(\frac{\tau}{s})^{\beta}\Big]}{\Big[1-exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]\Big]^{kb}}. \tag{12}$$

Inserting the Equation (12) into Equation (11), we have

$$f(s) = b\beta\tau^{\beta}s^{-(\beta+1)}\sum_{\xi=0}^{\infty}\frac{(-1)^{\xi}\alpha^{\xi+1}}{\xi!}exp\Big[-(\xi+1)b(\frac{\tau}{s})^{\beta}\Big]\Big[1-exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big]^{-(\xi b+b+1)} \tag{13}$$

Further expansion of the last quantity in power series gives

$$f(s) = b\beta\tau^{\beta}s^{-(\beta+1)}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{(-1)^{\xi}\Psi^{\xi+1}\Big[(\xi+1)b+1\Big]^{j}}{j!\xi!}exp\Big[-[(\xi+1)b+j](\frac{\tau}{s})^{\beta}\Big], \tag{14}$$

where $\Psi^{j} = \frac{\Gamma(\Psi+j)}{\Gamma(\Psi)}$ is the rising factional for any real Ψ.

However, the Equation (14) can be expressed as

$$f(s) = \beta[(\xi+1)b+j]\tau^{\beta}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\upsilon_{j,\xi}s^{-(\beta+1)}exp\Big[-[(\xi+1)b+j](\frac{\tau}{s})^{\beta}\Big], \tag{15}$$

where

$$\upsilon_{j,\xi} = \frac{(-1)^{\xi}\Psi^{\xi+1}\Big[(\xi+1)b+1\Big]^{j}}{j!\xi![(\xi+1)b+j]}. \tag{16}$$

Thus the Equation (11) reduces to

$$f(s) = \sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\upsilon_{j,\xi}h_{(\xi+1)b+j}(s), \tag{17}$$

where is the scale parameter $\alpha[(\xi+1)b+j]^{\frac{1}{\beta}}$ of the Frechet distribution $h_{(\xi+1)b+j}(s)$ and shape parameter β .

Integrating Equation (17), the cdf of can be expressed as

$$F(s) = \sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\upsilon_{j,\xi}H_{(\xi+1)b+j}(s), \tag{18}$$

where

$$h(s) = \psi b\beta\tau^{\beta}s^{-(\beta+1)}exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]\Big\{1 - exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big\}^{-(b+1)} \tag{19}$$

and

$$H(s) = \Psi\Big\{exp\Big[(\frac{\tau}{s})^{\beta}\Big] - 1\Big\}^{-b}. \tag{20}$$

Also, $\alpha^{G(s)}$ can be written as

$$\alpha^{G(s)} = \sum_{i=0}^{\infty}\frac{(\log\alpha)^{i}G(s)^{i}}{i}, \tag{21}$$

where $G(s)$ is the baseline pdf. Hence, $F(s)^i$ in Equation (18) can be expressed as

$$F(s)^{i} = \sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty} \upsilon^{i}_{j,\xi}H^{i}_{(\xi+1)b+j}(s). \tag{22}$$

Hence, Equation (21) becomes

$$\alpha^{G(s)} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{(\log\alpha)^{i}}{i}\upsilon^{i}_{j,\xi}H^{i}_{(\xi+1)b+j}(s). \tag{23}$$

However, the pdf of the APWF distribution is given in mixture representation as

$$f_{APWF}(s) = \frac{\log\alpha}{(\alpha-1)}g(s)\alpha^{G(s)} = \frac{1}{\alpha-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i}\upsilon^{i+1}_{j,\xi}h_{(\xi+1)b+j}H^{i}_{(\xi+1)b+j}(s) \tag{24}$$

The corresponding cdf is defined as

$$F_{APWF}(s) = \frac{1}{\alpha-1}\Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i}\upsilon^{i}_{j,\xi}H^{i}_{(\xi+1)b+j}(s) - 1\Big) \tag{25}$$

where $H_{(\xi+1)b+j}(s)$ is the Frechet cdf with scale parameter $\alpha[(\xi+1)b+j]^{\frac{1}{\beta}}$ and shape parameter β .

4. Mathematical Properties

This section investigates the properties of the APWF density. The structural properties of the APWF density was computed efficiently by using programming software like R, Maple, Matlab and Mathematical.

4.1. The Quantile and Random Number Generation of the APWF Distribution

Let S be a random variable such that $S \sim APWF(\psi, b, \beta, \tau, \alpha)$. Then, the quantile function of the variable S for $\mu \in (0.1)$ is given as

$$s_{\mu} = \tau\Big[\log\Big[\Big[-\psi^{-1}\log\Big[1-\Big(\log\alpha\Big)^{-1}\log\Big[\mu\Big(\alpha-1\Big)+1\Big]\Big]\Big]^{\frac{1}{b}}+1\Big]^{\frac{1}{\beta}}\Big]. \tag{26}$$

By setting $\mu = 0.5$ in Equation (26), we obtain the median of the random variable S is obtained as

$$s_{0.5} = \tau\Big[\log\Big[\Big[-\psi^{-1}\log\Big[1-\Big(\log\alpha\Big)^{-1}\log\Big[0.5\Big(\alpha-1\Big)+1\Big]\Big]\Big]^{\frac{1}{b}}+1\Big]^{\frac{1}{\beta}}\Big]. \tag{27}$$

However, the 25th and 75th percentile for the random variable of the APWF distribution are obtained as

$$s_{0.25} = \tau\Big[\log\Big[\Big[-\psi^{-1}\log\Big[1-\Big(\log\alpha\Big)^{-1}\log\Big[0.25\Big(\alpha-1\Big)+1\Big]\Big]\Big]^{\frac{1}{b}}+1\Big]^{\frac{1}{\beta}}\Big], \tag{28}$$

$$s_{0.75} = \tau\Big[\log\Big[\Big[-\psi^{-1}\log\Big[1-\Big(\log\alpha\Big)^{-1}\log\Big[0.75\Big(\alpha-1\Big)+1\Big]\Big]\Big]^{\frac{1}{b}}+1\Big]^{\frac{1}{\beta}}\Big]. \tag{29}$$

Simulating the APWF random variable deviate from a uniform variates on the interval $(0, 1)$. The Bowley's formula for finding the coefficient of skewness is given as

$$S_k(s) = \frac{x_{0.75} + x_{0.25} - 2x_{0.5}}{x_{0.75} - x_{0.25}}. \tag{30}$$

The corresponding Moor's formula for coefficient of Kurtosis is given as

$$K_k(s) = \frac{x_{0.875} - x_{0.625} + x_{0.125} - x_{0.375}}{x_{0.75} - x_{0.25}}. \tag{31}$$

4.2. Survival and Reliability Function

The reliability function of the APWF random variable X is given as

$$R_{APWF}(s) = \frac{1}{(\alpha-1)}\Big(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\Big(\frac{\log\alpha}{i!}v_{j,\xi}H_{(\xi+1)b+j}(s)\Big)^{i}\Big). \tag{32}$$

4.3. Hazard Rate Function of the APWF Distribution

The failure rate function of the APWF random variable is given as

$$h_{APWF}(s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\Bigg(\frac{(\log\alpha)^{i+1}v_{j,\xi}^{i+1}h_{(\xi+1)b+j}(s)H^{i}_{(\xi+1)b+j}(s)\Big)}{i!\Big(\alpha-\Big(\frac{\log\alpha}{i!}v_{j,\xi}H_{(\xi+1)b+j}(s)\Big)^{i}\Big)}\Bigg) \tag{33}$$

Figure 2 shows the plot for the hazard rate function of the APWF distribution.

4.4. APWF Cumulative Hazard Function

The Cumulative hazard function of the APWF distribution is given as

$$H_{APWF}(s) = \log(\alpha-1) - \log\Big[\alpha - \sum_{i,j,\xi=0}^{\infty}\Big(\log\alpha v_{j,\xi} + H_{(\xi+1)b+j}(s)\Big)^{i}\Big] \tag{34}$$

4.5. APWF Reversed Hazard Function

The Reversed Hazard Function of the APWF distribution is the ratio of the pdf of the APWF distribution to the cdf of the APWF distribution. Thus,

$$r_{APWF}(s) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\Big[\frac{[\log(\alpha)]^{i+1}}{i!}v_{j,\xi}^{i+1}h_{(\xi+1)b+1}(s)H^{i}_{(\xi+1)b+j}(s)\Big]\Big[\frac{[\log(\alpha]^{i}}{i!}v^{i}_{j,\xi}H^{i}_{(\xi+1)b+j}(s)-1\Big]^{-1}. \tag{35}$$

4.6. APWF Odds Function

The Odds function of the APWF distribution is given as

$$O_{APWF}(s) = F_{APWF}(s)R_{APWF}(s)^{-1}, \tag{36}$$

where $R_{APWF}(s)$ is the APWF reliability function.

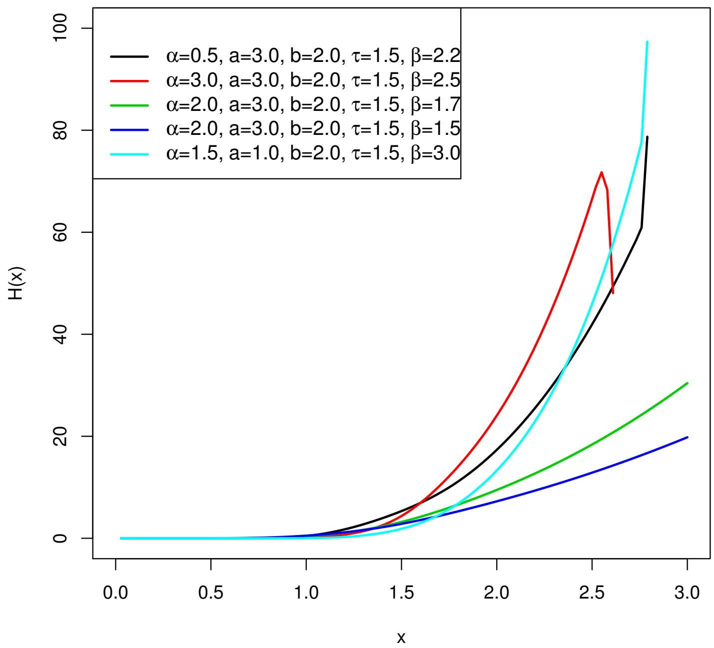

Figure 2: The hazard rate function of the APWF distribution for different parameter values

4.7. The APWF Order Statistics

Let $s_1, s_2, s_3, \cdots, s_n$ be a APWF random variable from a finite population which has the value $f(s)$ at s, then the pdf of the p^{th} order statistics is given as

$$\begin{aligned} g_p(s) =& \frac{n!}{(p-1)!(n-p)!}\left[\frac{1}{(\alpha-1)}\right]^n \left[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^i}{i!} v_{j,\xi}^i H_{(\xi+1)b+j}^i(s) - 1\right]^{p-1} \\ &\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^{i+1}}{i!} v_{j,\xi}^{i+1} h_{(\xi+1)b+j(s)} H_{(\xi+1)b+j}^i(s) \\ &\left(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\left(\frac{\log \alpha v_{j,\xi} H_{(\xi+1)b+j}(s)}{i!}\right)^i\right)^{n-p}. \end{aligned} \tag{37}$$

The following is observed for $p = 1$, we obtained the minimum order statistics distribution as

$$\begin{aligned} g_1(s) =& \frac{n!}{(n-p)!}\left[\frac{1}{(\alpha-1)}\right]^n \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^{i+1}}{i!} v_{j,\xi}^{i+1} h_{(\xi+1)b+j(s)} H_{(\xi+1)b+j}^i(s) \\ &\left(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\left(\frac{\log \alpha v_{j,\xi} H_{(\xi+1)b+j}(s)}{i!}\right)^i\right)^{n-1}. \end{aligned} \tag{38}$$

$p = n$ we obtained the maximum order statistics distribution as

$$g_n(s) = \frac{n!}{(n-1)!}\Big[\frac{1}{(\alpha-1)}\Big]^n \Big[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^i}{i!}v^i_{j,\xi}H^i_{(\xi+1)b+j}(s) - 1\Big]^{n-1}$$
$$\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^{i+1}}{i!}v^{i+1}_{j,\xi}h_{(\xi+1)b+j(s)}H^i_{(\xi+1)b+j}(s). \tag{39}$$

When n is odd. $n = 2m + 1$,and setting $p = m + 1$, then the distribution of median is given as

$$g_p(s) = \frac{(2m+)!}{m!m!}\frac{1}{(\alpha-1)^{2m+1}}\Big[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^i}{i!}v^i_{j,\xi}H^i_{(\xi+1)b+j}(s) - 1\Big]^{m}$$
$$\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^{i+1}}{i!}v^{i+1}_{j,\xi}h_{(\xi+1)b+j(s)}H^i_{(\xi+1)b+j}(s)$$
$$\Big(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\Big(\frac{\log\alpha v_{j,\xi}H_{(\xi+1)b+j}(s)}{i!}\Big)^i\Big)^m. \tag{40}$$

when n is even, $n = m2m$ and $p = m + 1$

$$g_{m+1}(s) = \frac{2m!}{m!m!}\Big[\frac{1}{(\alpha-1)}\Big]^{2m} \Big[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^i}{i!}v^i_{j,\xi}H^i_{(\xi+1)b+j}(s) - 1\Big]^{m}$$
$$\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^{i+1}}{i!}v^{i+1}_{j,\xi}h_{(\xi+1)b+j(s)}H^i_{(\xi+1)b+j}(s)$$
$$\Big(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\Big(\frac{\log\alpha v_{j,\xi}H_{(\xi+1)b+j}(s)}{i!}\Big)^i\Big)^{m-1}. \tag{41}$$

4.8. Probability Weighted Moments (PWM)

The PWM is a function can be used to obtain the parameter and quantiles function of a particular distribution that may not be obtained in a closed form. The $(\mu, v)^{th}$ of PWM of random variable S is defined as

$$\rho(\mu, v) = \int_0^{\infty} s^{\mu} f(s)F^{v}(s)ds = \sum_{i,m=0}^{\infty}\sum_{j,\xi=0}^{\infty}\Gamma\Big(1-\frac{\mu}{\beta}\Big)t_{i,j,\xi,m}\tau^{\mu}\Big[(\xi+1)b+j\Big]^{\frac{\mu}{\beta}}\frac{(\log\alpha)^{i+1}}{(\alpha-1)i!},$$

where

$$t_{i,j,\xi,m} = \Big[(\xi+1)b+j+1\Big]^{j}\frac{(-1)^{\xi+m+1}b\psi^{\xi+1}(j+1)^{\xi}}{j!\xi!((\xi+1)b+j)}\Big[(\xi+1)b+1\Big]^{\frac{\mu}{\beta-1}}\binom{v}{i}\binom{i}{m}.$$

4.9. Parameter Estimation of the APWF Distribution

The parameter of the APWF distribution are obtained by maximum likelihood (MLE) method as follows: Let $s_1, s_2, s_3, \cdots, s_n$ be a APWF random sample from an infinite population with a pdf $f(s)$ at the point s with distribution of the vector APWF of parameter $\theta(\psi, b, \beta, \tau, \alpha)^T$, then the likelihood function is given as

$$\begin{aligned}\prod_{i=1}^{n} f(s,\psi,b,\beta,\tau,\alpha) =&\psi^n b^n \beta^n \tau^{n\beta} (\log\alpha)^n \frac{1}{(\alpha-1)^n}\\ &\times\prod_{i=1}^{n} s_i^{-(\beta+1)} exp\Big[\sum_{i=1}^{n}\Big[-b(\frac{\tau}{s_i})^{\beta}\Big]\Big]\prod_{i=1}^{n}\Big\{1-exp\Big[-(\frac{\tau}{s_i})^{\beta}\Big]\Big\}^{-b-1}\\ &\times exp\Big[\sum_{i=1}^{n}-\psi\Big\{exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]-1\Big\}^{-b}\Big]\alpha^{\sum_{i=1}^{n}\Big[1-exp\Big[-\psi\Big\{exp\Big[(\frac{\tau}{s_i})^{\beta}\Big]-1\Big\}^{b}\Big]\Big]}\end{aligned} \tag{42}$$

Let ℓ denotes the log-likelihood function, then

$$\begin{aligned}\ell =& n\log\psi + n\log b + n\log\beta + n\beta\log\tau - n\log(\alpha-1) + n\log(\log\alpha) - (\beta+1)\sum_{i=1}^{n}\log s_i\\ &\sum_{i=1}^{n}\Big[-b\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] + (1-b)\sum_{i=1}^{n}\log\Big[1-exp\Big[-\Big(\frac{\tau}{s_i}\Big)^{b}\Big]\Big] - \sum_{i=1}^{n}\psi\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]-1\Big]^{-b}\\ &+\sum_{i=1}^{n}\Big[1-exp\Big[-\psi\Big\{exp\Big[(\frac{\tau}{s_i})^{\beta}\Big]-1\Big\}^{b}\Big]\Big]\log\alpha\end{aligned} \tag{43}$$

However, taking the partial derivation of the Equation (43) with respect to the parameter ψ, b, β, τ and α and equation to zero, we have

$$\frac{\partial\ell}{\partial\psi} = \frac{n}{\psi} - \sum_{i=1}^{n}\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]-1\Big]^{-b} = 0, \tag{44}$$

$$\frac{\partial\ell}{\partial b} = \frac{n}{b} - \sum_{i=0}^{n}\Big(\frac{\tau}{s_i}\Big)^{\beta} - \sum_{i=0}^{n}\log\Big[1-exp\Big[-\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]\Big] + \sum_{i=1}^{n}\psi\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]-1\Big]^{-b}\log\Big[\sum_{i=1}^{n}\psi\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]-1\Big]\Big], \tag{45}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + n\log\tau - \sum_{i=1}^{n}\log s_i + \sum_{i=1}^{n}\Big[-b\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]\log\Big[-b\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] + (1-b)\sum_{i=1}^{n}\frac{S'_{i\beta}}{S_i} - \sum_{i=0}^{n}p'_{i\beta} + \sum_{i=1}^{n}z'_{i\beta}, \tag{46}$$

$$\frac{\partial\ell}{\partial\tau} = \frac{n\beta}{\tau} - \sum_{i=1}^{n}\Big(\frac{b}{s_i}\Big)^{\beta}\tau^{\beta-1} + (1-b)\sum_{i=1}^{n}\frac{S'_{i\tau}}{S_i} - \sum_{i=0}^{n}p'_{i\tau} + \sum_{i=1}^{n}z'_{i\tau}, \tag{47}$$

$$\frac{\partial\ell}{\partial\alpha} = -\frac{n}{\alpha-1} + \psi'_{\alpha} + \alpha^{-1}\sum_{i=1}^{n}\frac{z_i}{log\alpha} = 0, \tag{48}$$

where

$$\psi = n\log(\log\alpha),$$

$$S_i = 1 - exp\Big[-\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big],$$

$$p_i = \psi\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]-1\Big]^{\beta},$$

$$z_i = \Big[1-exp\Big[-\psi\Big\{exp\Big[(\frac{\tau}{s_i})^{\beta}\Big]-1\Big\}^{b}\Big]\Big]\log\alpha.$$

5. Simulation Study and Real Life Applications

A simulation was carried out to test the flexibility and efficiency of the APWF distribution. Table 1 shows the simulation for different values of parameters for the APWF distribution. The simulation is performed as follows:

- Data are generated using
- $x_\mu = \tau\left[\log\left[\left[-\psi^{-1}\log\left[1-\left(\log\alpha\right)^{-1}\log\left[\mu\left(\alpha-1\right)+1\right]\right]\right]^{\frac{1}{b}}+1\right]^{\frac{1}{\beta}}\right] \qquad 0 < u < 1$
- The values of the parameters are set as $\alpha = 0.5, \tau = 2.0, \psi = 1.5, b = 0.5$, and $\beta = 3.0$.
- The APWF random sample sizes were taken as $n = 50, 100, 150$, and 350.
- Each APWF random sample is replicated 5000 times.

In this simulation study, we investigated the mean estimates (MEs), variance, biases and means squared errors (RMSEs) of the maximum likelihood estimate (MLEs).

The bias is calculated by for $(S = \alpha, \tau, \psi, b, \beta)$

$$\hat{B}ias = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{S}_i - S\right).$$

Also, the MSE is obtained as

$$\hat{M}SE = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{S}_i - S\right)^2.$$

Table 1 shows the simulation results for the Mean, Biases, Variances and MSE of the MLEs of APWF model for some fixed parameter values. The results of the APWF Monte Carlo study in Table 1 shows the MSEs and the biases decrease as the sample size increases and approach zero that corresponds to the first-order asymptotic theory. The mean estimates of the parameters approach the true parameter values as the sample size increases. The variance decreases in all the cases as the sample size increases.

5.1. Real life applications

The performance of the APWF model was examined with other competing distributions using the gas fiber and carbon data real-life datasets. We considered the Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hannan-Quinn Information Criteria (HQIC), The Anderson Darling (A) statistic, Cramer-von Mises statistic (W), Kolmogorov Smirnov (KS) statistic, Log-likelihood and the P value to compare the fits of the APWF model to other competing models such as the Gompertz Weibull (GOW), Weibull Frechet (WFr), Kumaraswamy Lomax (KL), Gompertz (GL), Beta Lomax (BL), and the Alpha Power Inverted Exponential (APIE) distributions.

5.1.1. First set of data is glass fiber data

Datasets were collected for 1.5 cm strengths of glass fibres data at the UK National Physical Laboratory and was used to test the performance of the APWF distribution as used in [16- 20] .

Table 2 is the measure of comparison for the various distributions under consideration with APIE as alpha power inverted exponential.

Table 1: Simulation results: mean estimates (AE), biases, Variance and mean squared errors (MSE) of $\hat{\alpha}, \hat{\psi}, \hat{b}, \hat{\tau}$ and $\hat{\beta}$

Sample size	Parameter	AE	Bias	Variance	MSE
50	$\alpha = 0.5$	0.3788	-0.1212	0.0484	0.0631
	$\tau = 2.0$	1.8534	-0.1466	0.3809	0.4024
	b= 0.5	0.5646	-2.4354	0.2211	6.1521
	$\psi = 1.5$	1.2534	-0.2466	0.2564	0.3172
	$\beta = 3.0$	1.6367	1.1367	0.3772	1.6692
100	$\alpha = 0.5$	0.3866	-0.1134	0.0408	0.0537
	$\tau = 2.0$	1.9041	-0.0959	0.2558	0.2650
	b= 0.5	0.4993	-2.5007	0.1585	6.4120
	$\psi = 1.5$	1.2571	-0.2429	0.1206	0.1795
	$\beta = 3.0$	1.5858	1.0858	0.1951	1.3741
150	$\alpha = 0.5$	0.4062	-0.0938	0.0433	0.0521
	$\tau = 2.0$	1.9177	-0.0823	0.1878	0.1945
	b= 0.5	0.5215	-2.4785	0.1457	6.2888
	$\psi = 1.5$	1.2847	-0.2153	0.0692	0.1155
	$\beta = 3.0$	1.5570	1.0570	0.1239	1.2412
350	$\alpha = 0.5$	0.4575	-0.0425	0.0439	0.0457
	$\tau = 2.0$	1.9665	-0.0335	0.0858	0.0869
	b= 0.5	0.5285	-2.4715	0.0992	6.2074
	$\psi = 1.5$	1.3219	-0.1781	0.0255	0.0572
	$\beta = 3.0$	1.4698	0.9698	0.0325	0.9731
500	$\alpha = 0.5$	0.4841	-0.0159	0.0393	0.0396
	$\tau = 2.0$	1.9681	-0.0319	0.0671	0.0681
	b= 0.5	0.5089	-2.4911	0.0993	6.3051
	$\psi = 1.5$	1.3464	-0.1536	0.0149	0.0385
	$\beta = 3.0$	1.4609	0.9609	0.0238	0.9472

Table 2: The performace rating of the APWF distribution with glass fibres dataset

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
APWF	$\hat{\psi} = 11.049$						
	$\hat{b} = 0.1156$						
	$\hat{\beta} = 0.3353$	37.3734	38.4260	48.0891	41.5880	0.1808	0.9911
	$\hat{\tau} = 10.098$						
	$\hat{\alpha} = 0.3012$						
Gompertz Weibull	$\hat{\alpha} = 0.2245$						
	$\hat{\beta} = 0.0092$						
		38.3769	39.0666	46.9495	41.7486	0.2330	1.2832
	$\hat{\psi} = 0.7973$						
	$\hat{b} = 5.6176$						
Gompertz Lomax	$\hat{\alpha} = 0.0046$						
	$\hat{\beta} = 8.1791$						
		39.0055	37.6951	45.5780	40.3771	0.1685	0.9462
	$\hat{a} = 0.5070$						
	$\hat{b} = 1.5158$						
Weibull Frechet	$\hat{\alpha} = 3.61218$						

Table 2 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
	$\hat{m}$ = 25.1859						
		39.0276	39.7812	47.3686	42.1676	0.2472	1.3566
	$\hat{\beta}$ = 0.1623						
	$\hat{a}$ = 0.2131						
Kumaraswamy Lomax	$\hat{\alpha}$ = 9.8352						
	$\hat{\beta}$ = 45.3107						
		44.2055	44.8951	52.7779	47.5771	1.6446	1.9915
	$\hat{a}$ = 15.1182						
	$\hat{b}$ = 0.0483						
Beta Lomax	$\hat{\alpha}$ = 18.1737						
	$\hat{\beta}$ = 26.7645						
		56.8068	57.4964	65.3793	60.1784	2.5426	3.1986
	$\hat{a}$ = 10.8769						
	$\hat{b}$ = 0.0329						
APIE	$\hat{\alpha}$ = 53.5634						
	$\hat{\lambda}$ = 0.3509	196.3253	196.5253	200.611	198.0111	0.7775	4.2384

Table 3: Test statistic for the APWF distribution with glass fibres dataset

Distribution	KS	p-Value	Log-likelihood
APWF	0.1236	0.2910	13.6867
Gompertz Weibull	0.1521	0.1087	15.1887
Gompertz Lomax	0.1542	0.0998	14.5027
Weibull Frechet	0.1552	0.0960	14.8177
Kumaraswamy Lomax	0.1854	0.0263	18.1027
Beta Lomax	0.2182	0.0049	24.4034
Alpha power inverted exponential	0.4646	3.0e-12	96.1627

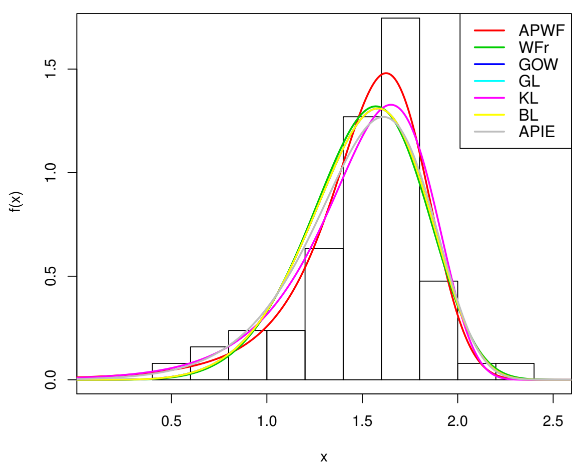

Figure 3: A plot of APWF distributions with the empirical histogram of the glass fibres data

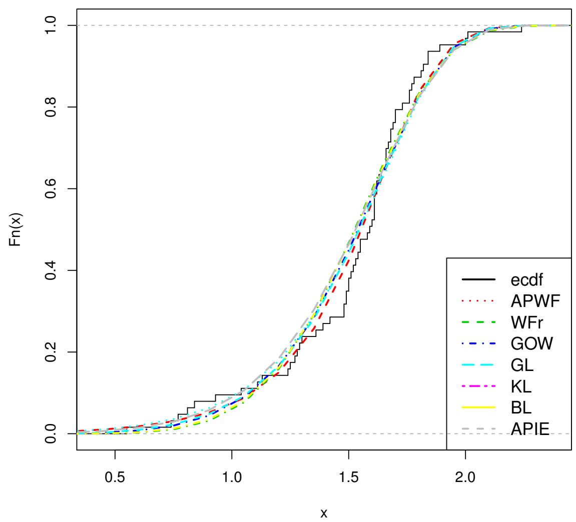

Figure 4: The fitted cdf of the APWF model for the glass data set

5.1.2. Second set of data carbon data

Our second set of data is from [21]. It consists of 100 observations taken on breaking stress of carbon fibers (in Gba). Table 4 and Table 5 are the goodness-of-fit and the performance rating of the APWF distribution using several test statistics for the carbon fibers dataset.

Table 4: Test statistic for the APWF distribution with glass fibres dataset

Distribution	KS	p-Value	Log-likelihood
APWF	0.06082131	0.8687617	141.3111
Gompertz Weibull	0.0632502	0.8185524	141.2822
Gompertz Lomax	0.06365319	0.8125448	142.4323
Weibull Frechet	0.06251348	0.8293575	141.3857
Kumaraswamy Lomax	0.07543761	0.6198049	141.484
Beta Lomax	0.17654926	0.00459718	156.7625
Alpha power inverted exponential	0.3503104	4.384659e-11	209.1656

Table 5: The performace rating of the APWF distribution with glass fibres dataset

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
APWF	$\hat{\psi} = 0.4603$ $\hat{b} = 2.7010$ $\hat{\beta} = 0.6398$ $\hat{\tau} = 0.9554$ $\hat{\alpha} = 6.1598$	282.3754	283.0137	295.4013	287.6472	0.0609	0.3719
Gompertz Weibull	$\hat{\alpha} = 2.2594$ $\hat{\beta} = -0.2017$ $\hat{\psi} = 0.2650$ $\hat{b} = 2.9808$	290.6544	290.9854	300.985	294.7818	0.0648	0.3834
Gompertz Lomax	$\hat{\alpha} = 0.0091$ $\hat{\beta} = 5.0656$ $\hat{a} = 1.9848$ $\hat{b} = 0.6471$	292.8646	293.2857	303.2853	297.0821	0.0611	0.4763
Weibull Frechet	$\hat{\alpha} = 0.6942$ $\hat{m} = 3.5178$ $\hat{\beta} = 0.6178$ $\hat{a} = 0.0947$	294.6000	295.0000	305.0000	298.8000	0.06892	0.4169
Kumaraswamy Lomax	$\hat{\alpha} = 3.7970$ $\hat{\beta} = 24.367$ $\hat{a} = 0.0334$ $\hat{b} = 6.0885$	295.9681	291.3891	301.3888	295.1855	0.0842	0.4532
Beta Lomax	$\hat{\alpha} = 18.1737$ $\hat{\beta} = 26.7645$ $\hat{a} = 10.8769$ $\hat{b} = 0.0329$	315.0974	317.4653	320.1753	317.4653	1.0896	2.0088
APIE	$\hat{\alpha} = 11.0025$ $\hat{\lambda} = 0.8694$	422.3312	422.455	427.5416	424.44	0.3726	2.0427

6. Discussion

The performance of a model is determined by the value that corresponds to the highest Log-likelihood or the lowest Akaike Information Criteria (AIC) value is considered as the best model. In the two real life cases considered, the APWF distribution has the lowest AIC value with 37.37339 in glass fibres data and 282.3754 in carbon data respectively. Also, the APWF has the value of log-likelihood as 13.68669 and 136.1877 for glass fibres and carbon data respectively. Hence, it competes favourably with other existing model for the data used.

7. Conclusion

The concept of the APWF distribution has been defined, introduced and studied. The mathematical expression for the pdf and cdf were examined. The statistical properties which include the order statistics

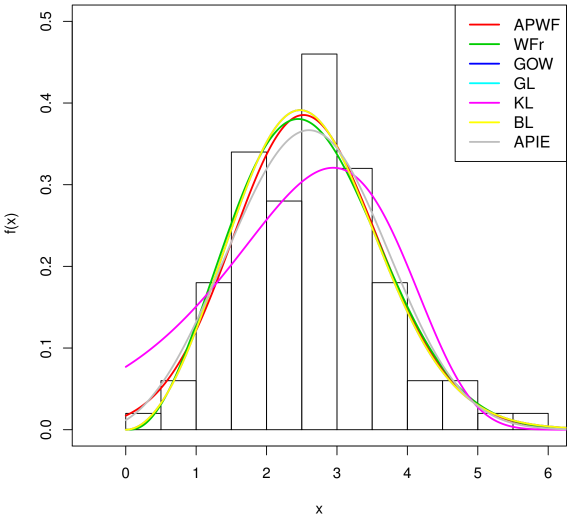

Figure 5: A plot of APWF distributions with the empirical histogram for the carbon data

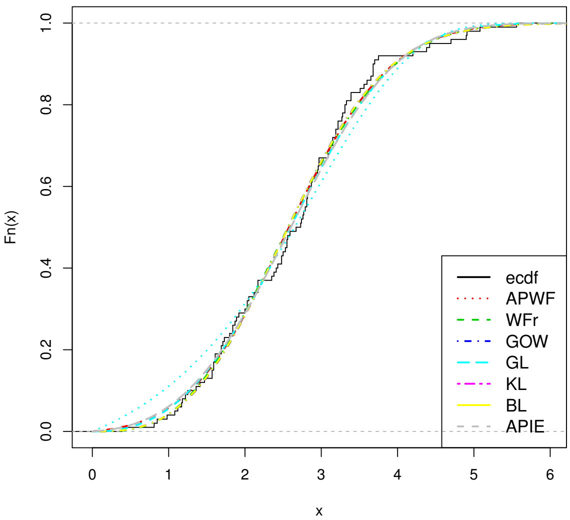

Figure 6: The fitted cdf of the APWF model for the carbon data set

distribution, cumulative hazard function, quantile, reversed hazard function, median, hazard rate function and odds function have been derived. The shape of the distribution could be inverted bathtub or decreasing. An application of the APWF model on a two real life data shows that the APWF distribution competes favourably with the Gompertz Weibull and Exponential, and better than the Kumaraswamy Lomax distribution, Beta Lomax distribution and some other families of distributions.

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Half inverse problems for the impulsive singular diffusion operator

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Abstract. In this paper, we consider the inverse spectral problem for the impulsive Sturm-Liouville differential pencils on $[0,\pi]$ with the Robin boundary conditions and the jump conditions at the point $\frac{\pi}{2}$. We prove that two potentials functious on the whole interval and the parameters in the boundary and jump conditions can be determined from a set of eigenvalues for two cases: (i) The potentials is given on $\left(0,\frac{\pi}{4}(\alpha+\beta)\right)$. (ii) The potentials is given on $\left(\alpha+\beta,\frac{\alpha+\beta}{2}\right)$, where $0<\alpha+\beta<1$, $\alpha+\beta>1$ respectively. Finally, was given interior inverse problem for same boundary problem.

1. Introduction

We consider the impulsive quadratic pencils of Sturm-Liouville operator of the form

$$ly := -y'' + [q(x) + 2\lambda p(x)]\,y = \lambda^2 \rho(x)\,y\ ,\ \ x \in \left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right] \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0 \tag{2}$$
$$V(y) := y'(\pi) + Hy(\pi) = 0 \tag{3}$$

and the jump conditions

$$\begin{aligned} y\left(\frac{\pi}{2}+0\right) &= ay\left(\frac{\pi}{2}-0\right) \\ y'\left(\frac{\pi}{2}+0\right) &= a^{-1}y'\left(\frac{\pi}{2}-0\right) + \gamma y\left(\frac{\pi}{2}\right) \end{aligned} \tag{4}$$

Where λ is the spectral parameter, $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real valued functions, $h, H \in$ R, a, γ, α, β are real numbers, $0<\alpha<\beta<1$, $\alpha+\beta>1$, $a>0$, $|a-1|^2+\gamma^2 \neq 0$ and

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$$\rho(x)=\begin{cases}\alpha^2, & 0<x<\dfrac{\pi}{2}\\ \beta^2, & \dfrac{\pi}{2}<x<\pi,\end{cases}$$

Here we denote by $W_2^m[0,\pi]$ the space of functions $f(x), x\in[0,\pi]$ such that the derivatives $f^{(m)}(x)$ $\left(m=\overline{0,n-1}\right)$ are absolute continuous and $f^{(n)}(x)\in L_2[0,\pi]$.

We can get $p(0)=0$without general exposure, otherwise, if $c_0=p(0)\neq 0$by direct calculation we note that equations (1) is equivalent to

$$ly:=-y''+\left[q(x)+2p(x)c_0-c_0^2+2(\lambda-c_0)(p(x)-c_0)\right]y=(\lambda-c_0)^2\rho(x)y \tag{5}$$

Let

$$\hat{q}(x)=q(x)+2p(x)c_0-c_0^2, \hat{p}(x)=p(x)-c_0, \hat{\lambda}=\lambda-c_0$$

then for the problem with the form (5) we have $\hat{p}(0)=0$.

Inverse spectral problems consist in recovering the coefficients of an operator from their spectral characteristics. The first results on inverse problems theory of classical Sturm-Liouville operator where given by Ambarzumyan and Borg (*see*[13, 24]). Inverse Sturm-Liouville problems which appear in mathematical physics, mechanics, electronics, geophysics an other branches of natural sciences have been studied for about ninety years (*see*[8, 9, 12]).

The half inverse Sturm-Liouville problem which is one of the important subjects of the inverse spectral theory has been studied firstly by Hochstadt and Lieberman in 1978 [*see*[20]]. They proved that spectrum of the problem

$$-y''+q(x)y=\lambda y, \quad x\in(0,1)$$

$$y'(0)-hy(0)=0=y'(1)+Hy(1)$$

and potential $q(x)$ on the $\left(\frac{1}{2},1\right)$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$ almost everywhere. Since then, this result has been generalized to various versions. In 1984, Hald [15] proved similar results in the case when there exist a impulse conditions inside the interval. He also gave some applications of this kinds of problem to geophysics. Recently, some new uniqueness results in inverse spectral analysis with partial information on the potential for some classes of differential equations have been given (see for example [18, 25, 32]). These kinds of results are known as Hochstadt and Lieberman type theorems. In particular, in the work [6] studied the inverse spectral problem for the impulsive Sturm-Liouville problem on $(0,\pi)$ with the Robin boundary conditions and the jump conditions at the point $\frac{\pi}{2}$. They proved that the potential $q(x)$ on the whole interval and the paremeters in the boundary conditions and jump conditions can be determined from a set of eigenvalues for two cases:

i) The potential $q(x)$ is given on $\left(0,\frac{1+\alpha}{4}\pi\right)$,

ii) The potential $q(x)$ is given on $\left(\frac{1+\alpha}{4}\pi,\pi\right)$, where $0<\alpha<1$,

and also shown that the potential and all the parameters can be uniquely recovered by one spectrum and some information on the eigenfunctions at some interior point. Similary problem studied in [25]. In particulary, they discuss Gesztesy-Simon theorem and show that if the potential function $q(x)$ is preseribed on the interval $\left[\frac{\pi}{2(1-\alpha)},\pi\right]$ for some $\alpha\in(0,1)$, then parts of a finite number of spectra suffice to determine $q(x)$ on $[0,\pi]$.

2. Preliminaries

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of the equation (1), satisfying the initial conditions $\varphi(0,\lambda)=1, \varphi'(0,\lambda)=h, \psi(\pi,\lambda)=1, \psi'(\pi,\lambda)=-H$ and the jump condition (4). Denote

$\sigma(x)=\int_0^x \sqrt{\rho(t)}dt, \tau=Im\lambda$, for every $\lambda\in C$

It is shown in [2] if $q(x)\in L_2[0,\pi]$ and $p(x)\in W_2^1[0,1]$ for every $\lambda\in C$,that there exist funvtions $A(x,t)$and $B(x,t)$ whose first order partial derivatives are summable on $[0,\pi]$ for each $x\in[0,\pi]$ such that

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^{\sigma(x)} A(x,t)\cos\lambda t dt+\int_0^{\sigma(x)} B(x,t)\sin\lambda t dt \tag{6}$$

Where

$$\varphi_0(x,\lambda)=\begin{cases} \cos\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+\frac{h}{\lambda\alpha}\sin\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right], & 0\le x<\frac{\pi}{2} \\ a^+\cos\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+a^-\cos\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right] & \\ +\frac{h}{\lambda\alpha}\left\{a^+\sin\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+a^-\sin\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\right\}, & \frac{\pi}{2}<x\le\pi \end{cases} \tag{7}$$

and $a^\pm=\frac{1}{2}\left(a\pm\frac{\alpha}{a\beta}\right), w^+(x)=\int_0^x p(t)\,dt, w^-(x)=\int_{\frac{\pi}{2}}^x p(t)\,dt$
It easy to verify from the integral representation (6) above that the solution $\varphi(x,\lambda)$ following asimptotic relation is valid as $|\lambda|\to\infty$. For $\frac{\pi}{2}<x\le\pi$

$$\begin{aligned}\varphi(x,\lambda)&=a^+\cos\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+a^-\cos\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\\ &+\frac{h}{\lambda\alpha}\left\{a^+\sin\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+a^-\sin\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\right\}\\ &+O\left(\lambda^{-2}\exp(|\tau|\sigma(x))\right)\end{aligned} \tag{8}$$

$$\begin{aligned}\varphi'(x,\lambda)&=-a^+\left(\lambda\beta-\frac{1}{\beta}p(x)\right)\sin\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]\\ &+a^-\left(\lambda\beta-\frac{1}{\beta}p(x)\right)\sin\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\\ &+\frac{h}{\lambda\alpha}a^+\left(\lambda\beta-\frac{1}{\beta}p(x)\right)\cos\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]\\ &-\frac{h}{\lambda\alpha}a^-\left(\lambda\beta-\frac{1}{\beta}p(x)\right)\cos\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]+O\left(\lambda^{-1}\exp(|\tau|\sigma(x))\right)\end{aligned} \tag{9}$$

Similarly, for the solution $\psi(x,\lambda)$ following asiymptotic relation hold as $|\lambda|\to\infty$. For $0\le x<\frac{\pi}{2}$,

$$\begin{aligned}\psi(x,\lambda)&=R^+\cos\left[\lambda(\sigma(\pi)-\sigma(x))-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]\\ &+R^-\cos\left[\lambda(\beta\pi-(\sigma(\pi)-\sigma(x)))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\\ &+\frac{1}{\lambda}\left(\frac{H}{\beta}R^++\frac{\gamma}{\alpha}\right)\sin\left[\lambda(\sigma(\pi)-\sigma(x))-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]\\ &+\frac{1}{\lambda}\left(\frac{H}{\beta}R^-+\frac{\gamma}{\alpha}\right)\sin\left[\lambda(\beta\pi-(\sigma(\pi)-\sigma(x)))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]+O\left(\lambda^{-2}\exp(|\tau|(\sigma(\pi)-\sigma(x)))\right)\end{aligned} \tag{10}$$

$$\begin{aligned}\psi'(x,\lambda) &= R^{+}\left(\lambda\alpha-\tfrac{1}{\alpha}p(x)\right)\sin\left[\lambda(\sigma(\pi)-\sigma(x))-\tfrac{w^{+}(x)}{\sqrt{\rho(x)}}\right]\\ &-R^{-}\left(\lambda\alpha-\tfrac{1}{\alpha}p(x)\right)\sin\left[\lambda(\beta\pi-(\sigma(\pi)-\sigma(x)))+\tfrac{w^{-}(x)}{\sqrt{\rho(x)}}\right]\\ &+\tfrac{1}{\lambda}\left(\tfrac{H}{\beta}R^{+}+\tfrac{\gamma}{\alpha}\right)\left(\lambda\alpha-\tfrac{1}{\alpha}p(x)\right)\cos\left[\lambda(\sigma(\pi)-\sigma(x))-\tfrac{w^{+}(x)}{\sqrt{\rho(x)}}\right]\\ &+\tfrac{1}{\lambda}\left(\tfrac{H}{\beta}R^{-}+\tfrac{\gamma}{\alpha}\right)\left(\lambda\alpha-\tfrac{1}{\alpha}p(x)\right)\cos\left[\lambda(\beta\pi-(\sigma(\pi)-\sigma(x)))+\tfrac{w^{-}(x)}{\sqrt{\rho(x)}}\right]\\ &+O\left(\lambda^{-1}\exp\left(|\tau|(\sigma(\pi)-\sigma(x))\right)\right)\end{aligned} \tag{11}$$

where $R^{\pm}=\dfrac{1}{2}\left(\dfrac{1}{a}\pm\dfrac{\beta a}{\alpha}\right)$.

Define

$$\langle\varphi(x,\lambda),\psi(x,\lambda)\rangle:=\varphi(x,\lambda)\psi'(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda)$$

It is easy to verify that if $y(x)$ and $z(x)$ satisfy equations (1) and jump conditions (4), then $\langle y,z\rangle$ is independent of x, and

$$\langle y,z\rangle\big|_{x=\frac{\pi}{2}-0}=\langle y,z\rangle\big|_{x=\frac{\pi}{2}+0}$$

Denote

$$\Delta(\lambda)=\langle\varphi,\psi\rangle=V(\varphi)=-U(\psi) \tag{12}$$

The function $\Delta(\lambda)$ is called the characteristic function of L,which is entire in λ and it has an at most countable set of zeros $\{\lambda_n\}$,$n\in Z$. It follows from (3) and (4) that the characteristic function of the pencil L can be reduced

$$\Delta(\lambda)=\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda) \tag{13}$$

or

$$\Delta(\lambda)=\Delta_0(\lambda)+\int_0^{\sigma(\pi)}A(\pi,t)\cos\lambda t dt+\int_0^{\sigma(\pi)}B(\pi,t)\sin\lambda t dt \tag{14}$$

Where $\Delta_0(\lambda)=\varphi_0^{'}(\pi,\lambda)+H\varphi_0^{'}(\pi,\lambda)$. Denote by $G_\delta=\{\lambda:|\lambda-\lambda_n|\geq\delta, n\in Z\}$ with fixed $\delta>0$. Then exist a constant $C_\delta>0$ such that

$$|\Delta(\lambda)|\geq C_\delta(C+\beta(\lambda))\exp(|\tau|\sigma(\pi))\, for \lambda\in G_\delta \tag{15}$$

On here supposes that the function $q(x)$satisfies the additional condition

$$\int_0^{\pi}\left\{\left|y'(x)\right|^2+q(x)\left|y(x)\right|^2\right\}dx>0 \tag{16}$$

For all $y(x)\in W_2^2\left(\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]\right)$ such that $y(x)\neq 0$and

$$y'(0)\overline{y(0)}-y'(\pi)\overline{y(\pi)}=0. \tag{17}$$

Lemma 2.1. *The following statements hold:*

i) The zeros $\{\lambda_n\}_{n\geq 0}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem L.

ii) The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are corresponding eigenfunctions and exists a sequence $\{\beta_n\}$, $\beta_n\neq 0$, $n=0,1,2,...$, such that

$$\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n). \tag{18}$$

Next, we denote by $L_2((0,\pi);\rho(x))$ a space which has the inner product

$$(\varphi,\psi)=\int_0^{\pi}\rho(x)\varphi(x,\lambda)\psi(x,\lambda)dx$$

Then it is shown in [2] that the eigenvalues of the boundary values problem L are real, nonzero, simple and does not have associated functions. Additionaly, eigenfunctions correspondings to different eigenvalues of the problem L are orthogonal in the sense of the equality

$$(\lambda_1 + \lambda_2)(\rho(x)\, y_1, y_2) - 2(\rho(x)\, y_1, y_2) = 0$$

Lemma 2.2. *The eigenvalues $\{k_n\}_{n\geq 0}$ of the problem L are real and simple. The eigenfunctions corresponding to the different eigenvalues are orthogonal in the weighted space $L_2((0,\pi);\rho(x))$ and for sufficiently large values of n, the eigenvalue k_n has the following behavior*

$$k_n = k_n^0 + \frac{d_n}{k_n^0} + \frac{k_n}{k_n^0} \tag{19}$$

where, λ_n^0 are zeros of $\Delta_0(\lambda) = \varphi_0'(\pi,\lambda) + H\varphi_0(\pi,\lambda)$, d_n is bounded and $k_n \in \ell_2$,

$$k_n^0 = \frac{n\pi}{\sigma(\pi)} + \theta_n, \;\; \sup_n |\theta_n| < +\infty$$

Proof of lemmas similarly to the proof of [7], so we omit the proof. Let α_n $(n \geq 0)$ be the normalized constants, which are defined as $\alpha_n := \int\limits_0^\pi \rho(x)\,\varphi^2(x,\lambda_n)\,dx$ for all $n \geq 0$.

Lemma 2.3. *The following relation holds:*

$$\overset{\bullet}{\Delta}(k_n) = -2\alpha_n\beta_n k_n \tag{20}$$

where $\overset{\bullet}{\Delta}(k_n) = \left(\frac{d}{d\lambda}\Delta(\lambda)\right)_{k=k_n}$, $\beta_n = -[\varphi(\pi,k_n)]^{-1}$.

In particular, it follows from (19) that all eigenvalues k_n are simple.

Let be $\delta > 0$ and fixed. Define $G_\delta := \left\{k \in \mathbb{C} : \left|k - k_n^0\right| \geq \delta, n = 1,2,...\right\}$. The following inequality can be deduced using the asymptotic formula for $\Delta(\lambda)$,

$$\Delta_0(k) \geq c\,|k| \exp(|\tau|\,\sigma(\pi)), \;\; k \in G_\delta \tag{21}$$

for some pozitive constant c.

3. Main Results

Now we state the main result of this work. It is assumed in what follows that if a certain symbol s denotes an object related to L, then the corresponding symbol $\widetilde{s}$ with tilde denote the analogous object related to $\tilde{L}$.

Lemma 3.1. *If $\lambda_n = \widetilde{\lambda}_n$, $n = 0,1,2,...$ then $\sigma(\pi) = \widetilde{\sigma}(\pi)$.*

Proof of Lemma is easily obtained from the asymptotic expression of λ_n.

Lemma 3.2. *If $k_n = \widetilde{k}_n$, $n = 0,1,2,...$ then $a = \widetilde{a}$, $\alpha = \widetilde{\alpha}$, $\beta = \widetilde{\beta}$, $\rho(x) = \widetilde{\rho}(x)$, $h = \widetilde{h}$ and $H = \widetilde{H}$.*

Proof. Since, $k_n = \widetilde{k}_n$, $n = 0,1,2,...$, Lemma 2.2 requires $\sigma(\pi) = \widetilde{\sigma}(\pi)$ or $\alpha + \beta = \widetilde{\alpha} + \widetilde{\beta}$. $\Delta(k), \widetilde{\Delta}(k)$ are entire functions of order one by Hadamard factorization theorem, for $\lambda \in \mathbb{C}$

$$\Delta(k) \equiv C\widetilde{\Delta}(k). \tag{22}$$

Then from Lemma 2.3 and $\sigma(\pi) = \widetilde{\sigma}(\pi)$ we obtain $C = 1$.

On the other hand, (22) can be written as

$$\Delta_0(k) - C\widetilde{\Delta}_0(k) = \left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right] \tag{23}$$

Hence

$$\begin{aligned}
&\left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right] = \\
&= -r^{+}k\beta\sin k\sigma(\pi) + r^{-}k\beta\sin k(\alpha\pi - \sigma(\pi)) \\
&+ h\frac{\beta}{\alpha}\left[r^{+}\cos k\sigma(\pi) - r^{-}\cos k(\alpha\pi - \sigma(\pi))\right] \\
&+ H\{r^{+}\cos k\sigma(\pi) + r^{-}\cos k(\alpha\pi - \sigma(\pi)) \\
&+ \frac{h}{k\alpha}\left[r^{+}\sin k\sigma(\pi) + r^{-}\sin k(\alpha\pi - \sigma(\pi))\right]\} \\
&- \{\widetilde{r}^{+}k\beta\sin k\sigma(\pi) + \widetilde{r}^{-}k\beta\sin k(\alpha\pi - \sigma(\pi)) \\
&+ \widetilde{h}\frac{\beta}{\alpha}\left[\widetilde{r}^{+}\cos k\sigma(\pi) - \widetilde{r}^{-}\cos k(\alpha\pi - \sigma(\pi))\right]\} \\
&- \widetilde{H}\{\widetilde{r}^{+}\cos k\sigma(\pi) + \widetilde{r}^{-}\cos k(\alpha\pi - \sigma(\pi)) \\
&+ \frac{\widetilde{h}}{k\alpha}\left[\widetilde{r}^{+}\sin k\sigma(\pi) + \widetilde{r}^{-}\sin k(\alpha\pi - \sigma(\pi))\right]\}
\end{aligned} \tag{24}$$

if we multiply both sides of (24) with $\sin k\sigma(\pi)$ and integrate with respect to k in (ε, T) (ε is sufficiently small pozitive number) for any pozitive real number T, then we get

$$\begin{aligned}
&\int_{\varepsilon}^{T}\left(\left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right]\right)\sin k\sigma dk = \\
&\int_{\varepsilon}^{T}\{-r^{+}k\beta\sin k\sigma(\pi) + r^{-}k\beta\sin k(\alpha\pi - \sigma(\pi)) + h\frac{\beta}{\alpha}\left[r^{+}\cos k\sigma(\pi) - r^{-}\cos k(\alpha\pi - \sigma(\pi))\right] \\
&+H\left[r^{+}\cos k\sigma(\pi) - r^{-}\cos k(\alpha\pi - \sigma(\pi)) + \frac{h}{k\alpha}\left(r^{+}\sin k\sigma(\pi) + r^{-}\sin k(\alpha\pi - \sigma(\pi))\right)\right] \\
&-\left[\widetilde{r}^{+}k\beta\sin k\sigma(\pi) + \widetilde{r}^{-}k\beta\sin k(\alpha\pi - \sigma(\pi)) + \widetilde{h}\frac{\beta}{\alpha}\left(\widetilde{r}^{+}\cos k\sigma(\pi) - \widetilde{r}^{-}\cos k(\alpha\pi - \sigma(\pi))\right)\right] \\
&-\widetilde{H}\left[\widetilde{r}^{+}\cos k\sigma(\pi) + \widetilde{r}^{-}\cos k(\alpha\pi - \sigma(\pi)) + \frac{\widetilde{h}}{k\alpha}\left(\widetilde{r}^{+}\sin k\sigma(\pi) + \widetilde{r}^{-}\sin k(\alpha\pi - \sigma(\pi))\right)\right]\Bigg\}\sin k\sigma dk
\end{aligned}$$

Since

$$\Delta(k) - \Delta_0(k) = O\left(k^{-2}\exp(|\tau|\sigma(\pi))\right), \widetilde{\Delta}(k) - \widetilde{\Delta}_0(k) = O\left(k^{-2}\exp(|\tau|\sigma(\pi))\right)$$

for all k in (ε, T)

$$\frac{\beta}{4}\widetilde{r}^{+} - \frac{\beta}{4}r^{+} = O(\frac{1}{T^2})$$

By letting T tend to infinity we see that

$$r^{+} = \widetilde{r}^{+} \tag{25}$$

Similarly, if we multiply both sides of (24) with $\sin k(\alpha\pi - \sigma(\pi))$ and integrate again with respect to k in (ε, T), and by letting T tend to infinity, then we get

$$r^{-} = \widetilde{r}^{-} \tag{26}$$

Taking $a > 0$ into account, (25) and (26) implies that $a = \widetilde{a}, \alpha = \widetilde{\alpha}, \beta = \widetilde{\beta}$.

Considering that Lemma 3.2, and $a = \widetilde{a}$, if both sides of the last expression are multiplied by the $\cos k\sigma(\pi)$ and integrate with respect to k in (ε, T), then we get

$$h\frac{\beta}{\alpha}r^{+} + Hr^{+} = \widetilde{h}\frac{\beta}{\alpha}r^{+} + \widetilde{H}r^{+} \tag{27}$$

Similary, if we multiply both sides of the last expression are with $\cos k(\alpha\pi - \sigma(\pi))$ and integrate again with respect to k in (ε, T), and by letting T tend to infinity, then we get

$$h\frac{\beta}{\alpha}r^{-} - Hr^{-} = \widetilde{h}\frac{\beta}{\alpha}r^{-} - \widetilde{H}r^{-} \tag{28}$$

Finaly, from (27) and (28) implies that $h = \widetilde{h}$ and $H = \widetilde{H}$. □

Theorem 3.3. *If for any $n \in Z$, $\lambda_n = \tilde{\lambda}_n$,*

$$\frac{y'(c_1, \lambda_n)}{y(c_2, \lambda_n)} = \frac{\tilde{y}'(c_1, \lambda_n)}{\tilde{y}(c_2, \lambda_n)} \tag{29}$$

Then $p(x) = \tilde{p}(x)$ on $[0, \pi]$, $q(x) = \tilde{q}(x)$ a. e. on $[0, \pi]$, and $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.

Proof. Let $\varphi(x, \lambda)$ be the solution of the equations (1) satisfying the initial conditions $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = h$ and the jump conditions (4). Let $\tilde{\varphi}(x, \lambda)$be the solution of the equations

$$-\tilde{\varphi}''(x, \lambda) + [\tilde{q}(x) + 2\lambda\tilde{p}(x)]\tilde{\varphi}(x, \lambda) = \lambda^2\tilde{\rho}(x)\tilde{\varphi}(x, \lambda) \tag{30}$$

With the initial conditions

$$\tilde{\varphi}(0, \lambda) = 1, \; \tilde{\varphi}'(0, \lambda) = \tilde{h} \tag{31}$$

and the jump conditions (4). Multiplying (1) by $\tilde{\varphi}(x, \lambda)$ and (30) by $\varphi(x, \lambda)$, respectively, and subtracting, we get

$$\frac{d}{dx}\left[\tilde{\varphi}(x, \lambda)\varphi'(x, \lambda) - \tilde{\varphi}'(x, \lambda)\varphi(x, \lambda)\right] = \left[(q(x) - \tilde{q}(x)) + 2\lambda(p(x) - \tilde{p}(x))\right]\varphi(x, \lambda)\tilde{\varphi}(x, \lambda) \tag{32}$$

Integrating the above equality from 0 to c_1 with respect to x, using the initial conditions at $x = 0$ and Lemma 3.1, we have

$$\begin{aligned} H(\lambda) &= \int_0^{c_1}\left[(q(x) - \tilde{q}(x)) + 2\lambda(p(x) - \tilde{p}(x))\right]\varphi(x, \lambda)\tilde{\varphi}(x, \lambda)\,dx \\ &= \tilde{\varphi}(c_1, \lambda)\varphi'(c_1, \lambda) - \tilde{\varphi}'(c_1, \lambda)\varphi(c_1, \lambda) \end{aligned} \tag{33}$$

It follows from (6)-(7) that $H(\lambda)$ is an entire function of exponential type and there are some pozitive constant A and B such that

$$|H(\lambda)| \leq (A + B|\lambda|)\exp(|\tau|\sigma(\pi)) \;\; for\, all\; \lambda \in C \tag{34}$$

From the assumption (29) we have

$$H(\lambda_n) = 0\;,\;\; n \in Z \tag{35}$$

Define

$$F(\lambda) = \frac{H(\lambda)}{\Delta(\lambda)} \tag{36}$$

Which is entire function from the above arguments and it follows from (14) and (35) that

$$F(\lambda) = O(1)$$

For sufficiently large $|\lambda|$,$\lambda \in G_\delta$, thus, by liouville's theorem [4], we obtain for all λ that $F(\lambda) = C$.

Where c is a constant. Let us show that the constant $C = 0$. Based on (24) and (14), we can rewrite the equations $H(\lambda) = C\Delta(\lambda)$ in the form

$$\begin{aligned} &2\lambda \int_0^{c_1} (p(x) - \tilde{p}(x)) \varphi(x,\lambda) \tilde{\varphi}(x,\lambda)\, dx + \int_0^{c_1} (q(x) - \tilde{q}(x)) \varphi(x,\lambda) \tilde{\varphi}(x,\lambda)\, dx \\ &= C\left\{\Delta_0(\lambda) + \int_0^{\sigma(\pi)} A(\pi,t) \cos\lambda t dt + \int_0^{\sigma(\pi)} B(\pi,t) \sin\lambda t dt\right\} \end{aligned}$$

By use of Riemann-Lebesgue lemma [4], we see that the limit of the left-hand side of the above equality exists as $\lambda \to \infty$, $\lambda \in \mathrm{R}$ thus we obtain that $C = 0$. So we have $H(\lambda) = 0$ for all $\lambda \in \mathrm{C}$.
As already mentioned, if $H(\lambda) = 0$ for all $\lambda \in \mathrm{C}$, then from (33) we have
$\tilde{\varphi}(c_1,\lambda) \varphi'(c_1,\lambda) - \tilde{\varphi}'(c_1,\lambda) \varphi(c_1,\lambda) = 0$ for all $\lambda \in \mathrm{C}$
so
$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)} = \frac{\tilde{\varphi}(c_1,\lambda)}{\tilde{\varphi}'(c_1,\lambda)}$ for all $\lambda \in \mathrm{C}$.
The function $M(\lambda) := \frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)}$is the Weyl function of the boundary value problem for equation (1) on $(0, c_1)$with boundary conditoons $V(y) = 0$, $y'(c_1) = 0$ and without jump conditions.
By [2], the Weyl function uniquely species $p(x)$ and $q(x)$a.e. on $(0, c_1)$and the coefficients in boundary and jump conditions and $\rho(x)$. □

Theorem 3.4. *If for any $n \in \mathrm{Z}$, $\lambda_n = \tilde{\lambda}_n$, $\frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}}$, $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $\left(0, \frac{\alpha+\beta}{4}\pi\right)$, then $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ a.e. on $\left(\frac{\alpha+\beta}{4}\pi, \frac{\alpha+\beta}{2}\pi\right)$ and $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$*

.

Proof. Let the boundary value problems $L$and $\tilde{L}$ satisfy the conditions of Teorem 3.4, then by virtue of Lemma 2.4 and Lemma 3.2 $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$ and $\rho(x) = \tilde{\rho}(x)$. For brevity, denote $c_1 = \frac{\alpha+\beta}{4}\pi$, $c_2 = \frac{\alpha+\beta}{2}\pi$.
Let $\psi(x,\lambda)$, $\tilde{\psi}(x,\lambda)$ be the solutions of the equations

$$-\psi''(x,\lambda) + [q(x) + 2\lambda p(x)] \psi(x,\lambda) = \lambda^2 \rho(x) \psi(x,\lambda) \tag{37}$$

$$-\tilde{\psi}''(x,\lambda) + [\tilde{q}(x) + 2\lambda \tilde{p}(x)] \tilde{\psi}(x,\lambda) = \lambda^2 \tilde{\rho}(x) \tilde{\psi}(x,\lambda) \tag{38}$$

With the initial conditions, respectively

$$\psi(\pi,\lambda) = 1\,,\ \psi'(\pi,\lambda) = -H \tag{39}$$

$$\tilde{\psi}(\pi,\lambda) = 1\,,\ \tilde{\psi}'(\pi,\lambda) = -\tilde{H} \tag{40}$$

and the jump conditions (4). After multipliying (37) by $\tilde{\psi}(x,\lambda)$ and (38) by $\psi(x,\lambda)$, we subtract these equations from each other. Then by integrating on$[c_1, \pi]$ with respect to x, using the initial conditions (39) and (40)and jump conditions (4), we have

$$\int_{c_1}^{\pi} [(q(x) - \tilde{q}(x)) + 2\lambda (p(x) - \tilde{p}(x))] \psi(x,\lambda) \tilde{\psi}(x,\lambda)\, dx = \tilde{\psi}(c_1,\lambda) \psi'(c_1,\lambda) - \tilde{\psi}'(c_1,\lambda) \psi(c_1,\lambda) \tag{41}$$

From the hypothesis $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $(0, c_1)$.
Denote $Q(x) = q(x) - \tilde{q}(x)$, $P(x) = p(x) - \tilde{p}(x)$ and

$$F_0(\lambda) = 2\lambda \int_{c_1}^{\pi} P(x) \psi(x,\lambda) \tilde{\psi}(x,\lambda)\, dx + \int_{c_1}^{\pi} Q(x) \psi(x,\lambda) \tilde{\psi}(x,\lambda)\, dx \tag{42}$$

It follows from (10) and (41) that $F_0(\lambda)$ is an entire function of exponential type and there are some pozitive constants A_1 and B_1 such that

$$|F_0(\lambda)| \le (A_1 + B_1 |\lambda|) \exp(|\tau| \sigma(\pi)) \quad for\, all\, \lambda \in \mathrm{C} \tag{43}$$

It is clear from the properties $\psi(x,\lambda)$, $\tilde{\psi}(x,\lambda)$ and the boundary conditions (2)

$$F_0(\lambda_n) = 0\ ,\ n \in Z \tag{44}$$

Define

$$F(\lambda) := \frac{F_0(\lambda)}{\Delta(\lambda)}$$

Which is an entire function from the above arguments and it follows from (15) and (43) that

$$F(\lambda) = O(1)$$

For sufficiently large $|\lambda|, \lambda \in G_\delta$. Using Liouville's theorem [4], we obtain for all λ that $F(\lambda) = C$.
Where C is a constant. Let us Show that the constant $C = 0$. We can rewrite the equations $F_0(\lambda) = C\Delta(\lambda)$ as

$$\begin{aligned}
&2\lambda \int_{c_1}^{\pi} P(x)\psi(x,\lambda)\tilde{\psi}(x,\lambda)\,dx + \int_{c_1}^{\pi} Q(x)\psi(x,\lambda)\tilde{\psi}(x,\lambda)\,dx \\
&= -a^+C\left(\lambda\beta - \tfrac{1}{\beta}p(\pi)\right)\sin\left[\lambda\sigma(\pi) - \tfrac{w^+(\pi)}{\beta}\right] \\
&+a^-C\left(\lambda\beta - \tfrac{1}{\beta}p(\pi)\right)\sin\left[\lambda(\alpha\pi - \sigma(\pi)) + \tfrac{w^-(\pi)}{\beta}\right] \\
&+Ha^+C\cos\left[\lambda\sigma(\pi) - \tfrac{w^+(\pi)}{\beta}\right] + Ha^-C\cos\left[\lambda(\alpha\pi - \sigma(\pi)) + \tfrac{w^-(\pi)}{\beta}\right] \\
&+O(\exp(|\tau|\sigma(\pi)))
\end{aligned}$$

By use of Riemann-Lebesgue lemma [4], we see that the limit of the left-hand side of the above equality exists as $\lambda \to \infty$, $\lambda \in R$. Therefore, we get that $C = 0$. So, we have $F_0(\lambda) = 0$ for all $\lambda \in C$.
Then, from teh equality (41) we obtain
$\tilde{\psi}(c_1,\lambda)\psi'(c_1,\lambda) - \tilde{\psi}'(c_1,\lambda)\psi(c_1,\lambda) = 0$ for all $\lambda \in C$. Hence,

$$\frac{\psi(c_1,\lambda)}{\psi'(c_1,\lambda)} = \frac{\tilde{\psi}(c_1,\lambda)}{\tilde{\psi}'(c_1,\lambda)}. \tag{45}$$

Note that $M(\lambda) := -\frac{\psi(c_1,\lambda)}{\psi'(c_1,\lambda)}$is the Weyl function, defined [2], of the boundary value problem for equation (1) on the interval $(c_1,\pi)$with the boundary conditoons $V(y) = 0$, $y'(c_1) = 0$ and jump conditions (4). It has been show in [2] that the Weyl function species the function $p(x)$ and $q(x)$on (c_1,π), consequently on (c_1,c_2). Theorem is proved. □

Corollary.
If for any $n \in Z$, $\lambda_n = \tilde{\lambda}_n$, $\frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}}$, $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $(0,c_1)$, then $p(x) = \tilde{p}(x)$ on $(0,\pi)$ and $q(x) = \tilde{q}(x)$ a.e. on $(0,\pi)$ and $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.

Theorem 3.5. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \in Z$,, $\frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}}$, $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $\left(\frac{\alpha+\beta}{4}\pi, \frac{\alpha+\beta}{2}\pi\right)$, then $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ a.e. on $\left(0, \frac{\alpha+\beta}{4}\pi\right)$ and $\left(\frac{\alpha+\beta}{2}\pi, \pi\right)$ and$\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.*

Proof. By the Lemma 3.1 and the condition of Teorem 3.5, we have $h = \tilde{h}$, $H = \tilde{H}$, $a = \tilde{a}$, $\rho(x) = \tilde{\rho}(x)$ and $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on (c_1,c_2).
Let

$$-\varphi''(x,\lambda) + [q(x) + 2\lambda p(x)]\varphi(x,\lambda) = \lambda^2\rho(x)\varphi(x,\lambda) \tag{46}$$

$$-\tilde{\varphi}''(x,\lambda) + [\tilde{q}(x) + 2\lambda\tilde{p}(x)]\tilde{\varphi}(x,\lambda) = \lambda^2\tilde{\rho}(x)\tilde{\varphi}(x,\lambda) \tag{47}$$

With the initial conditions, respectively

$$\varphi(0,\lambda) = 1\,,\ \varphi'(0,\lambda) = h \tag{48}$$

$$\tilde{\varphi}(0,\lambda) = 1\,,\ \tilde{\varphi}'(0,\lambda) = \tilde{h} \tag{49}$$

and the jump conditions (4). Multipliying (46) by $\tilde{\varphi}(x,\lambda)$ and (47) by $\varphi(x,\lambda)$, we subtract these equations from each other. Then by integrating on$[0,c_2]$ with respect to x, using the initial conditions (48) and (49) and jump conditions (4), we have

$$\begin{aligned} H(\lambda) &= 2\lambda\int_0^{c_1} P(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx + \int_0^{c_1} Q(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx \\ &= \varphi'(c_1,\lambda)\tilde{\varphi}(c_1,\lambda) - \tilde{\varphi}'(c_1,\lambda)\varphi(c_1,\lambda) \end{aligned} \tag{50}$$

From the hypothesis $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on (c_1,c_2). Similarly to proof of Theorem 3.5, we have that $H(\lambda) = 0$ for all $\lambda \in C$. Then, from equality
$\varphi'(c_1,\lambda)\tilde{\varphi}(c_1,\lambda) - \tilde{\varphi}'(c_1,\lambda)\varphi(c_1,\lambda) = 0$ for all $\lambda \in C$.
so

$$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)} = \frac{\tilde{\varphi}(c_1,\lambda)}{\tilde{\varphi}'(c_1,\lambda)}.$$

The function $M(\lambda) := -\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)}$is the Weyl function of the boundary value problem for the equation (1) on $(0,c_1)$with boundary conditoons $V(y) = 0$, $y'(c_1) = 0$ and without jump conditions (4) (*see*[2]). By [2], the Weyl function uniquely species $p(x)$ and $q(x)$a.e. on $(0,c_1)$. Next, now using Theorem 3.6 we obtain $p(x) = \tilde{p}(x)$and $q(x) = \tilde{q}(x)$ a.e. on (c_2,π). Theorem is proved.
□

4. An interior inverse problems.

We cconsider the interior inverse problem for the same boundary problem L and obtain the corresponding result.

Theorem 4.1. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \in Z$, $\frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}}$, and*

$$\frac{y(c_1,\lambda_n)}{y'(c_1,\lambda_n)} = \frac{\tilde{y}(c_1,\lambda_n)}{\tilde{y}'(c_1,\lambda_n)} \tag{51}$$

, then $p(x) = \tilde{p}(x)$ on $[0,\pi]$, $q(x) = \tilde{q}(x)$ a.e. on $[0,\pi]$and $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.

Proof. Let $\varphi(x,\lambda)$ be the solution of the equations (1) satisfying the initial conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$ and jump conditions (4). Firstly, the assumption that $\lambda_n = \tilde{\lambda}_n$ and $\frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}}$ can determine $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$ by Lemma 3.1 the other hand from (50), we see that

$$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)} = \frac{\tilde{\varphi}(c_1,\lambda)}{\tilde{\varphi}'(c_1,\lambda)},$$

Then from (50), the entire function $H(\lambda)$ has zeros $\{\lambda_n\}$, $n \in Z$, i.e. $H(\lambda_n) = 0$. Similarly to the proof of Theorem4, we have that $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $(0,c_1)$. Once we get that $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$, by Corollary of Theorem 3.4 we have that $p(x) = \tilde{p}(x)$ on $[0,\pi]$, $q(x) = \tilde{q}(x)$ a.e. on $[0,\pi]$. Theorem is proved. □

Theorem 4.2. *Let $m(n)$be a sequence of integers such that $\inf\limits_{n\in Z} \frac{m(n)}{\lambda_n} \le 1$*
(i) If for any $n \in Z$,

$$\lambda_{m(n)} = \tilde{\lambda}_{m(n)}, \frac{y\left(c_1,\lambda_{m(n)}\right)}{y'\left(c_1,\lambda_{m(n)}\right)} = \frac{\tilde{y}\left(c_1,\lambda_{m(n)}\right)}{\tilde{y}'\left(c_1,\lambda_{m(n)}\right)} \text{ and } \frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}} \tag{52}$$

Then $p(x) = \tilde{p}(x)$ on $(0,c_1)$ and $q(x) = \tilde{q}(x)$ a.e. on $(0,c_1)$ and $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.

(ii) *If for any* $n \in \mathrm{Z}$,

$$\lambda_{m(n)} = \tilde{\lambda}_{m(n)}, \frac{y\left(c_2, \lambda_{m(n)}\right)}{y'\left(c_2, \lambda_{m(n)}\right)} = \frac{\tilde{y}\left(c_2, \lambda_{m(n)}\right)}{\tilde{y}'\left(c_2, \lambda_{m(n)}\right)} \; and \; \frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}} \tag{53}$$

Then $p(x) = \tilde{p}(x)$ *on* (c_2, π)*and* $q(x) = \tilde{q}(x)$ *a.e. on* (c_2, π) *and* $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.

Proof. (i) from the assumption (52) and (50) we have

$$\varphi'\left(c_1, \lambda_{m(n)}\right) \tilde{\varphi}\left(c_1, \lambda_{m(n)}\right) - \tilde{\varphi}'\left(c_1, \lambda_{m(n)}\right) \varphi\left(c_1, \lambda_{m(n)}\right) = 0$$

Which means

$$H\left(\lambda_{m(n)}\right) = 0\,,\; n \in \mathrm{Z} \tag{54}$$

Next, we shall show that $H(\lambda) = 0$ on the whole λ-plane. From (50) and (6) on has

$$|H(\lambda)| \le (A + Br)\, e^{2c_1 r |\sin\theta|} \tag{55}$$

For some pozitive costants A and B, where $\lambda = re^{i\theta}$. Moreover, we see that the entire function $H_1(\lambda)$ is a function of exponential type less than $2c_1$.
Define the indicator of function $H_1(\lambda)$ by

$$h(\theta) = \lim_{r\to\infty} \sup \frac{\ln\left|H_1\left(re^{i\theta}\right)\right|}{r} \tag{56}$$

One obtain the following estimate from (55)and (56) that $h(\theta) \le 2c_1 |\sin\theta|$.
Let us denote by $n(r)$the number of zeros of $H_1(\lambda)$ in the disk $|\lambda| \le r$. From the equations (4.4), the assimption of (52) and known asymtotic expreession of the eigenvalues λ_n, we have the following estimate for the number of zeros of $H_1(\lambda)$ in the disk $|\lambda| \le r$.

$$n(r) = 1 + 2\left[\sigma r (1 + \varepsilon(r))\right] = 2\sigma r (1 + \varepsilon(r))\,.$$

Here $\varepsilon(r) \to 0 \; for \; r \to \infty$, $\sigma$is number such that $\sigma > \frac{\alpha+\beta}{2} = \frac{2c_1}{\pi}$and $[x]$is the integer part of x. It follows that in the case under consideration

$$\lim_{r\to\infty} \frac{n(r)}{r} = 2\sigma > \frac{4c_1}{\pi} = \frac{c_1}{\pi}\int_0^{2\pi} |\sin\theta|\, d\theta \ge \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\, d\theta \tag{57}$$

To complate the proof we have to recall the following theorem [4]: the set of zeros of every entire function of the exponential type, not identically zero, satisfy the inequality

$$\lim_{r\to\infty} \inf \frac{n(r)}{r} \le \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\, d\theta \tag{58}$$

Inequalities (57) and (58) implay that $H_1(\lambda) \equiv 0$ on the whole λ-plane. As already mentioned, if $H_1(\lambda) \equiv 0$, then from (52) we have

$$\tilde{\varphi}(c_1, \lambda)\, \varphi'(c_1, \lambda) - \tilde{\varphi}'(c_1, \lambda)\, \varphi(c_1, \lambda) = 0$$

so
$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)} = \frac{\tilde{\varphi}(c_1,\lambda)}{\tilde{\varphi}'(c_1,\lambda)}$ on the whole λ-plane.
The function $M(\lambda) := \frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)}$is the Weyl function of the boundary value problem for the equation (1) on $(0, c_1)$with boundary conditoons $U(y) = 0$, $y'(c_1) = 0$ and without jump conditions (4) (*see*[2]). By [2], the Weyl function uniquely species $p(x)$ and $q(x)$a.e. on $(0, c_1)$ and coefficient h.
(ii)

To prove that $p(x) = \tilde{p}(x)$ on $(c_2, \pi)$and $q(x) = \tilde{q}(x)$ a.e. on (c_2, π) and $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$. We will consider the supplementary problem L

$$\begin{cases} -y'' + [q_1(x) + 2\lambda p_1(x)]\, y = \lambda^2 \rho(x)\, y\ ,\ x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\ y(0) - Hy(0) = 0 \\ y(\pi) - hy(\pi) = 0 \\ y\left(\frac{\pi}{2} + 0\right) = a^{-1} y\left(\frac{\pi}{2} - 0\right) \\ y'\left(\frac{\pi}{2} + 0\right) = a y'\left(\frac{\pi}{2} - 0\right) + \gamma\left(\frac{\pi}{2} - 0\right) \end{cases}$$

Where $q_1(x) = q(\pi - x)$ and $p_1(x) = p(\pi - x)$. A direct calculation implies that $\hat{y}_n := y_n(\pi - x)$ is the solution to the supplementary problem $\hat{L}$ and $\hat{y}_n(\pi - x) = y_n(c_2)$. Note that $\pi - c_2 \in \left(0, \frac{\pi}{2}\right)$. Thus the assmption conditions for $\hat{L}$ in the case (i) are still satisfied. Repeting the above arguments we can obtain the proof of this Theorem 4.2. □

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An inverse coefficient problem for quasilinear pseudo-parabolic of heat conduction of Poly(methyl methacrylate) (PMMA)

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Abstract. In this research, we consider a coefficient problem of an inverse problem of a quasilinear pseudo-parabolic equation with periodic boundary condition. It proved the existence, uniqueness and continuously dependence upon the data of the solution by iteration method.

1. Introduction

Consider the equation

$$u_t - u_{xx} - \varepsilon u_{xxt} - a(t)u = f(x,t,u),\ (x,t) \in \ \Gamma, \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x),\ \ x \in [0,\pi], \tag{2}$$

the periodic boundary condition

$$u(0,t) = u(\pi,t),\ u_x(0,t) = u_x(\pi,t),\ 0 \le t \le T, \tag{3}$$

and the overdetermination data

$$E(t) = \int_0^\pi xu(x,t)dx, 0 \le t \le T, \tag{4}$$

for a quasilinear parabolic equation with the nonlinear source term $f = f(x,t,u)$.

Here $\Gamma := \{0 < x < \pi,\ 0 < t < T\}$. The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,\pi]$ and $\overline{\Gamma} \times (-\infty,\infty)$, respectively.

The inverse problem of determining unknown coefficient in a quasi-linear parabolic equation has generated an increasing amount of interest from engineers and scientist [1–11].

Definition 1.1. *The pair $\{a(t),u(x,t)\}$ from the class $C[0,T] \times (C^{2,1}(\Gamma) \cap C^{1,0}(\overline{\Gamma}))$ for which conditions (1)-(4) are satisfied is called the classical solution of the inverse problem (1)-(4).*

The paper organized as follows:

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2. Existence and Uniqueness of the Solution of the Inverse Problem

The main result on the existence and the uniqueness of the solution of the inverse problem (1)-(4) is presented as follows:

We have the following assumptions on the data of the problem (1)-(4).

(A1) $E(t) \in C^1[0,T]$.

(A2) $\varphi(x) \in C^2[0,\pi]$, $\varphi(0) = \varphi(\pi)$, $\varphi'(0) = \varphi'(\pi)$,

(A3) Let the function $f(x,t,u)$ is continuous with respect to all arguments in $\overline{\Gamma} \times (-\infty,\infty)$ and satisfies the following condition

(1)

$$\left| \frac{\partial^{(n)} f(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} f(x,t,\tilde{u})}{\partial x^n} \right| \le b(t,x)\,|u-\tilde{u}| \ ,n = 0,1,2$$

where $b(x,t) \in L_2(\Gamma),\ b(x,t) \ge 0,$

(2) $f(x,t,u) \in C^2[0,\pi]$, $t \in [0,T]$,

(3) $f(x,t,u)|_{x=0} = f(x,t,u)|_{x=\pi}\,,\ f_x(0,t,u)|_{x=0} = f_x(\pi,t,u)|_{x=\pi}\,,$

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1)-(3) for arbitrary $a(t) \in C[0,T]$:

$$u(x,t) = \frac{u_0(t)}{2} + \sum_{k=1}^{\infty} \left[u_{ck}(t)\cos 2kx + u_{sk}(t)\sin 2kx\right],$$

$$\begin{aligned}
u_0(t) &= \varphi_0 e^{-\int_0^t a(\tau)d\tau} + \frac{2}{\pi}\int_0^t\int_0^\pi f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right)^{-\int_0^t a(\tau)d\tau} d\xi d\tau,\\
u_{ck}(t) &= \varphi_{ck}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2} - -\int_0^t a(\tau)d\tau}\\
&+ \frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^\pi f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right)\cos 2k\xi\, e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2} - -\int_\tau^t a(\tau)d\tau} d\xi d\tau,\\
u_{sk}(t) &= \varphi_{sk}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2} - -\int_0^t a(\tau)d\tau}\\
&+ \frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^\pi f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right)\sin 2k\xi e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2} - -\int_\tau^t a(\tau)d\tau} d\xi d\tau.
\end{aligned}$$

$$\begin{aligned}
u(x,t) &= \varphi_0 e^{--\int_0^t a(\tau)d\tau} + \int_0^t f_0(\tau,u)\, d\tau\\
&+ \sum_{k=1}^{\infty}\cos 2kx\left[\varphi_{ck} e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2} - -\int_0^t a(\tau)d\tau} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t f_{ck}(\tau,u) e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2} - -\int_\tau^t a(\tau)d\tau} d\tau\right]\\
&+ \sum_{k=1}^{\infty}\sin 2kx\left[\varphi_{sk} e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2} - -\int_0^t a(\tau)d\tau} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t f_{sk}(\tau,u)\, e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2} - -\int_\tau^t a(\tau)d\tau} d\tau\right],
\end{aligned} \tag{5}$$

where $\varphi_0 = \frac{2}{\pi}\int_0^\pi \varphi(x)dx, \varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(x)\cos 2kxdx, \varphi_{sk} = \frac{2}{\pi}\int_0^\pi \varphi(x)\sin 2kxdx,$

$f_0(t,u)=\frac{2}{\pi}\int\limits_0^\pi f(x,t,u)dx,\ f_{ck}(t,u)=\frac{2}{\pi}\int\limits_0^\pi f(x,t,u)\cos 2kxdx,\ f_{sk}(t,u)=\frac{2}{\pi}\int\limits_0^\pi f(x,t,u)\sin 2kxdx(\ k=1,2,3,\dots .)$

Under the condition (A1)-(A3), differentiating (4), we obtain

$$E^{'}(t)=\int\limits_0^\pi xu_t(x,t)dx, 0\le t\le T. \tag{6}$$

(5) and (6) yield

$$\begin{aligned} a(t) &= \frac{1}{E(t)}\left[-E^{'}(t)+\frac{\pi^2}{2}f_0(t,u)\right] \\ & \frac{1}{E(t)}\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{-(2k)^2t}{1+\varepsilon(2k)^2}\left(\varphi_{sk}\ e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_0^t a(\tau)d\tau}+\frac{1}{1+\varepsilon(2k)^2}\int\limits_0^t f_{ck}(\tau,u)\ e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_\tau^t a(\tau)d\tau}d\tau\right) \\ & -\frac{1}{E(t)}\sum_{k=1}^{\infty}f_{sk}(t,u) \end{aligned} \tag{7}$$

Definition 2.1. *Denote the set*

$\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),k=1,...,n\}$, *of continuous on* $[0,T]$ *functions satisfying the condition* $\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)|+\max\limits_{0\le t\le T}|u_{sk}(t)|\right)<\infty$, *by* **B**. *Let*

$\|u(t)\|_B=\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)|+\max\limits_{0\le t\le T}|u_{sk}(t)|\right)$, *be the norm in* **B**.

It can be shown that **B** *is Banach space.*

Theorem 2.2. *Let the assumptions (A1)-(A3) be satisfied. Then the inverse problem (1)-(4) has a unique solution.*

Proof. Iterations for the Fourier coefficients of (5) are defined as follows:

$$\begin{aligned} u_0^{(N+1)}(t) &= u_0^{(0)}(t)+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi f\left(\xi,\tau,u^{(N)}(\xi,\tau)\right)\ e^{--\int\limits_\tau^t a^{(N)}(\tau)d\tau}d\xi d\tau, \\ u_{ck}^{(N+1)}(t) &= u_{ck}^{(0)}(t)+\frac{2}{\pi(1+\varepsilon(2k)^2)}\int\limits_0^t\int\limits_0^\pi\ f(\xi,\tau,u^{(N)}(\xi,\tau))\cos 2k\xi\ e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_\tau^t a^{(N)}(\tau)d\tau}d\xi d\tau, \\ u_{sk}^{(N+1)}(t) &= u_{sk}^{(0)}(t)+\frac{2}{\pi(1+\varepsilon(2k)^2)}\int\limits_0^t\int\limits_0^\pi\ f(\xi,\tau,u^{(N)}(\xi,\tau))\sin 2k\xi e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_\tau^t a^{(N)}(\tau)d\tau}d\xi d\tau, \end{aligned} \tag{8}$$

$$u_0^{(0)}(t)=\varphi_0 e^{--\int\limits_\tau^t a^{(0)}(\tau)d\tau}, u_{ck}^{(0)}(t)=\varphi_{ck}e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_0^t a^{(0)}(\tau)d\tau}, u_{sk}^{(0)}(t)=\varphi_{sk}e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_0^t a^{(0)}(\tau)d\tau}.$$

Applying Cauchy inequality, Hölder inequality, Bessel inequality and using Lipschitzs condition and taking the maximum of both side, we have:

$$\begin{aligned}
\left\|u^{(1)}(t)\right\|_{\mathbf{B}} &= \max_{0\le t\le T}\left\|u_0^{(1)}(t)\right\|_B+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left\|u_{ck}^{(1)}(t)\right\|_B+\max_{0\le t\le T}\left\|u_{sk}^{(1)}(t)\right\|_B\right)\\
&\le \frac{\|\varphi_0\|}{2}+\sum_{k=1}^{\infty}\left(\|\varphi_{ck}\|+\|\varphi_{sk}\|\right)\\
&\quad +(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\,\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B\\
&\quad +(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\,\|f(x,t,0)\|_{L_2(\Gamma)}.
\end{aligned}$$

From the conditions of the theorem $u^{(1)}(t)\in\mathbf{B}$.
Same estimations for the step N,

$$\begin{aligned}
\left\|u^{(N+1)}(t)\right\|_{B} &= \max_{0\le t\le T}\left\|u_0^{(N)}(t)\right\|_B+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left\|u_{ck}^{(N)}(t)\right\|_B+\max_{0\le t\le T}\left\|u_{sk}^{(N)}(t)\right\|_B\right)\\
&\le \frac{\|\varphi_0\|}{2}+\sum_{k=1}^{\infty}\left(\|\varphi_{ck}\|+\|\varphi_{sk}\|\right)\\
&\quad +(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\,\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(N)}(t)\right\|_B\\
&\quad +(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\,\|f(x,t,0)\|_{L_2(\Gamma)}.
\end{aligned}$$

Since $u^{(N)}(t)\in\mathbf{B}$ and from the conditions of the theorem, we have $u^{(N+1)}(t)\in\mathbf{B}$,

$$\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),\ k=1,2,...\}\in\mathbf{B}.$$

By same estimations,

$$\begin{aligned}
\left\|a^{(1)}(t)\right\|_{C[0,T]} &\le \left\|\frac{E'(t)}{E(t)}\right\|+\frac{\pi^2}{4\sqrt{6}E(t)}\sum_{k=1}^{\infty}\left\|\varphi_{ck}'''\right\|\\
&\quad +\frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B\\
&\quad +\frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)M
\end{aligned}$$

Same estimations for the step N,

$$\begin{aligned}
\left\|a^{(N+1)}(t)\right\|_{C[0,T]} &\le \left\|\frac{E'(t)}{E(t)}\right\|+\frac{\pi^2}{4\sqrt{6}E(t)}\sum_{k=1}^{\infty}\left\|\varphi_{ck}'''\right\|\\
&\quad +\frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(N)}(t)\right\|_B\\
&\quad +\frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)M
\end{aligned}$$

Now we prove that the iterations $u^{(N+1)}(t), a^{(N+1)}$ converge **B** and $C[0,T]$, respectively.(*as* $N \to \infty$)

$$u^{(1)}(t)-u^{(0)}(t)=\frac{(u_0^{(1)}(t)-u_0^{(0)}(t))}{2}+\sum_{k=1}^{\infty}[(u_{ck}^{(1)}(t)-u_{ck}^{(0)}(t))+(u_{sk}^{(1)}(t)-u_{sk}^{(0)}(t))]$$

Applying Cauchy inequality, Bessel inequality, Hölder inequality, Lipschitzs condition in the last equation, taking maximum of both side of the last inequality :

$$\begin{aligned}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B &\le (\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\left\|b(x,t)\right\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B\\ &+(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\left\|f(x,t,0)\right\|_{L_2(\Gamma)}.\end{aligned}$$

$$A=(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})(\left\|b(x,t)\right\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}}+\left\|f(x,t,0)\right\|_{L_2(\Gamma)}).$$

Applying Cauchy inequality, Hölder Inequality, Lipschitzs condition and Bessel inequality to the last equation and taking maximum of both side of the last inequality, we obtain

$$\begin{aligned}\left\|a^{(1)}(t)-a^{(0)}(t)\right\|_{C[0,T]} &\le \frac{\pi}{\left\|E(t)\right\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)\left\|b(x,t)\right\|_{L_2(\Gamma)}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B\\ &+\left(\frac{\pi TM}{\left\|E(t)4\sqrt{3}\right\|}+\frac{\pi^2 T}{\left\|E(t)4\sqrt{6}\right\|}\sum_{k=1}^{\infty}\left|\varphi_{ck}'''\right|\right)\left\|a^{(1)}(t)-a^{(0)}(t)\right\|_{C[0,T]}\end{aligned}$$

where

$$\begin{aligned}B &= \frac{\pi}{\left\|E(t)\right\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)\\ C &= \left(\frac{\pi TM}{\left\|E(t)4\sqrt{3}\right\|}+\frac{\pi^2 T}{\left\|E(t)4\sqrt{6}\right\|}\sum_{k=1}^{\infty}\left|\varphi_{ck}'''\right|\right)\end{aligned}$$

$$\left\|a^{(1)}(t)-a^{(0)}(t)\right\|_{C[0,T]}\le\frac{B}{1-C}\left\|b(x,t)\right\|_{L_2(\Gamma)}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B$$

$$\begin{aligned}\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_B &\le (\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\left\|b(x,t)\right\|_{L_2(\Gamma)}\left\|u^{(1)}-u^{(0)}\right\|_B\\ &+(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\frac{BT}{1-C}M\left\|b(x,t)\right\|_{L_2(\Gamma)}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B\end{aligned}$$

$$\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_B\le\left\{(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\left(1+\frac{BT}{1-C}\right)\right\}A\left\|b(x,t)\right\|_{L_2(\Gamma)},$$

For the step N :

$$\left\|a^{(N+1)}(t)-a^{(N)}(t)\right\|_{C[0,T]}\le\frac{B}{1-C}\left\|b(x,t)\right\|_{L_2(\Gamma)}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B$$

$$\left\| u^{(N+1)}(t) - u^{(N)}(t) \right\|_B \le \left\{ \left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{3}} \right) \left(1 + \frac{BT}{1-C} \right) \right\}^N \frac{A}{\sqrt{N!}} \|b(x,t)\|^N_{L_2(\Gamma)}.$$

By the Weierstrass M test we deduce from (9) that the series $\sum\limits_{N=0}^{\infty} \left| u^{(N+1)}(t) - u^{(N)}(t) \right|$ is uniformly convergent to an element of B. However,the general term of the sequence $\left\{ u^{(N+1)}(t) \right\}$ may be written as

$$u^{(N+1)}(t) = u^{(0)}(t) + \sum_{n=0}^{N} \left| u^{(n+1)}(t) - u^{(n)}(t) \right|,$$

so the sequence $\left\{ u^{(N+1)}(t) \right\}$ is uniformly convergent to an element of **B** because the sum on the right is the N th partial sum of the aforementioned uniformly convergent series. So $u^{(N+1)} \to u^{(N)}$, $N \to \infty$, then $a^{(N+1)} \to a^{(N)}$, $N \to \infty$.

Therefore $u^{(N+1)}(t)$ and $a^{(N+1)}(t)$ converge in **B** and $C[0,T]$, respectively.

Now let us show that there exists u and a such that

$$\lim_{N\to\infty} u^{(N+1)}(t) = u(t), \ \lim_{N\to\infty} a^{(N+1)}(t) = a(t).$$

$$\begin{aligned} \left\| u - u^{(N+1)} \right\|_B \ \le \ & \left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{3}} \right) \|b(x,t)\|_{L_2(\Gamma)} \left\| u(t) - u^{(N+1)}(t) \right\|_B \\ & + \left\{ \left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{3}} \right) \left(1 + \frac{BT}{1-C} \right) \right\}^N \frac{A}{\sqrt{N!}} \|b(x,t)\|_{L_2(\Gamma)} \\ & + \left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{3}} \right) M \left\| a(\tau) - a^{(N)}(\tau) \right\|_{C[0,T]}, \end{aligned} \tag{9}$$

$$\left\| a(\tau) - a^{(N+1)}(\tau) \right\|_{C[0,T]} \le \frac{B}{1-C} \|b(x,t)\|_{L_2(\Gamma)} \left\| u(t) - u^{(N+1)}(t) \right\|_B. \tag{10}$$

Let us consider (10) in (9) and apply Gronwall's inequality to (9) and taking maximum of both side of the last inequality, we have

$$\begin{aligned} \left\| u(t) - u^{(N+1)}(t) \right\|^2_{\mathbf{B}} \ \le \ & \\ & 2 \left[\frac{A}{\sqrt{N!}} \left(\left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{3}} \right) \left(1 + \frac{BT}{1-C} \right) \right)^{N+1} \|b(x,t)\|_{L_2(\Gamma)} \right]^2 \\ & \times \exp 2 \left(1 + \left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{3}} \right) \left(1 + \frac{BT}{1-C} \right) \right)^2 \|b(x,t))\|^2_{L_2(\Gamma)}. \end{aligned}$$

We obtain $u^{(N+1)} \to u$, $a^{(N+1)} \to a$, $N \to \infty$.

For the uniqueness, we assume that the problem (1)-(4) has two solution pair (a,u), (b,v). Applying Cauchy inequality, Hölder Inequality, Lipschitzs condition and Bessel inequality to $|u(t) - v(t)|$ and $|a(t) - b(t)|$, we obtain

$$\begin{aligned} \|u(t) - v(t)\|_B \ \le \ & \left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{3}} \right) M \|a(t) - b(t)\|_{C[0,T]} \\ & + \left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{3}} \right) \left(\int_0^t \int_0^{\pi} b^2(\xi,\tau) |u(\tau) - v(\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}, \end{aligned}$$

$$\|a(t)-b(t)\|_{C[0,T]} \le \frac{B}{1-C}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,|u(\tau)-v(\tau)|^2\,d\xi d\tau\right)^{\frac{1}{2}},$$

$$\|u(t)-v(t)\|_B \le \left[\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\left(1+\frac{B}{1-C}\right)\right]\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,|u(\tau)-v(\tau)|^2\,d\xi d\tau\right)^{\frac{1}{2}}, \tag{11}$$

applying Gronwall's inequality to (11) we have
$u(t)=v(t)$. Hence $a(t)=b(t)$. □

This completes the proof of Theorem 2.2.

3. Continuous Dependence of (a,u) upon the data

Theorem 3.1. *Under assumption (A1)-(A3) the solution (r,u) of the problem (1)-(4) depends continuously upon the data* φ, E.

Proof. Let $\Phi=\{\varphi,\ a,\ f\}$ and $\overline{\Phi}=\{\overline{\varphi},\ \overline{a},\ f\}$ be two sets of the data, which satisfy the assumptions $(A_1)-(A_3)$. Suppose that there exist positive constants M_i, $i=1,2$ such that

$$\|a\|_{C^1[0,T]} \le M_1, \|\overline{a}\|_{C^1[0,T]} \le M_1,\ \|\varphi\|_{C^3[0,\pi]} \le M_2, \|\overline{\varphi}\|_{C^3[0,\pi]} \le M_2.$$

Let us denote $\|\Phi\| = (\|a\|_{C^1[0,T]}+\|\varphi\|_{C^3[0,\pi]}+\|f\|_{C^{3,0}(\overline{D})})$.
By using same estimations to $u-\overline{u}$, we obtain

$$\begin{aligned}\|u-\overline{u}\| &\le M_3\left\|\Phi-\overline{\Phi}\right\| \\ &\quad + M_4\left(\int_0^t\int_0^\pi r^2(\tau)b^2(\xi,\tau)\,\|u(\tau)-\overline{u}(\tau)\|^2\,d\xi d\tau\right)^{\frac{1}{2}}\end{aligned} \tag{12}$$

applying Gronwall's inequality to the last equation, we obtain

$$\begin{aligned}\|u-\overline{u}\|^2 &\le 2M_3^2\left\|\Phi-\overline{\Phi}\right\|^2 \\ &\quad \times\exp\left(2M_4^2\int_0^t\int_0^\pi r^2(\tau)b^2(\xi,\tau)d\xi d\tau\right).\end{aligned}$$

For $\Phi\to\overline{\Phi}$ then $u\to\overline{u}$. Hence $a\to\overline{a}$. □

4. Numerical Procedure for the nonlinear problem (1)-(4)

We construct an iteration algorithm for the linearization of the problem (1)-(4):

$$\frac{\partial u^{(n)}}{\partial t}-\frac{\partial^2 u^{(n)}}{\partial x^2}-\varepsilon\frac{\partial^3 u^{(n)}}{\partial x^2\partial t}-a(t)u = f(x,t,u^{(n-1)}),\quad (x,t)\in D \tag{13}$$

$$u^{(n)}(0,t) = u^{(n)}(\pi,t),\quad t\in[0,T] \tag{14}$$

$$u_x^{(n)}(0,t) = u_x^{(n)}(\pi,t)=0,\ t\in[0,T] \tag{15}$$

$$u^{(n)}(x,0) = \varphi(x),\quad x\in[0,\pi]. \tag{16}$$

Let $u^{(n)}(x,t)=v(x,t)$ and $f(x,t,u^{(n-1)})=\widetilde{f}(x,t)$. Then the problem (13)-(16) can be written as a linear problem:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2}+\varepsilon\frac{\partial^3 v}{\partial x^2\partial t}+r(t)\widetilde{f}(x,t) \quad (x,t)\in D && (17)\\ v(0,t) &= v(\pi,t), \quad t\in[0,T] && (18)\\ v_x(0,t) &= v_x(\pi,t), \quad t\in[0,T] && (19)\\ v(x,0) &= \varphi(x), \quad x\in[0,\pi]\ . && (20)\end{aligned}$$

After linearization, we use the finite difference method to solve (17)-(20).

We subdivide the intervals $[0,\pi]$ and $[0,T]$ into subintervals N_x and N_t of equal lengths $h=\frac{\pi}{N_x}$ and $\tau=\frac{T}{N_t}$, respectively. We choose the implicit scheme which is absolutely stable and has a second-order accuracy in h and a first-order accuracy in τ. The implicit scheme for (17)-(20) is as follows:

$$\frac{1}{\tau}\left(v_i^{j+1}-v_i^j\right)=\frac{1}{2h^2}\left(v_{i-1}^j-2v_i^j+v_{i+1}^j\right)+\varepsilon\frac{1}{2h^2\tau}\left[(v_{i-1}^{j+1}-2v_i^{j+1}+v_{i+1}^{j+1})-(v_{i-1}^j-2v_i^j+v_{i+1}^j)\right]-a^j v_i^{j+1}=\widetilde{f}_i^j, \tag{21}$$

$$v_i^0=\phi_i, \tag{22}$$

$$v_0^j=v_{N_x+1}^j, \tag{23}$$

$$\frac{v_1^j+v_{N_x}^j}{2}=v_{N_x+1}^j, \tag{24}$$

where $1\le i\le N_x$ and $0\le j\le N_t$ are the indices for the spatial and time steps respectively, $v_i^j=v(x_i,t_j)$, $\phi_i=\varphi(x_i)$, $\widetilde{f}_i^j=\widetilde{f}(x_i,t_j)$, $x_i=ih$, $t_j=j\tau$. At the level $t=0$, adjustment should be made according to the initial condition and the compatibility requirements.

Now, let us construct the predicting-correcting mechanism. First, integrating the equation (1) with respect to x from 0 to π and using (3) and (4), we obtain

$$a(t)=\frac{-E'(t)+\int\limits_0^\pi x\widetilde{f}(x,t)dx+v_t(\pi,t)}{E(t)}. \tag{25}$$

The finite difference approximation of (25) is

$$a^j=\frac{-\left(E^{j+1}-E^j\right)/\tau+(f_{in})^j+\left(v_{N_x}^{j+1}-v_{N_x}^j\right)/\tau}{E^j},$$

where $E^j=E(t_j)$, $j=0,1,...,N_t$.

For $j=0$,

We denote the values of a^j, v_i^j at the s-th iteration step .and the values of ϕ_i provide us to start our computation. We denote the values of p^j, v_i^j at the s-th iteration step a$r^{j(s)}$, $v_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $a^{j+1(0)}=a^j$, $v_i^{j+1(0)}=v_i^j$, $j=0,1,2,....N_t$, $i=1,2,...,N_x$. At each $(s+1)$-th iteration step we first determine $a^{j+1(s+1)}$ from the formula

$$a^{j+1(s+1)}=\frac{-\left(E^{j+1(s+1)}-E^{j(s+1)}\right)/\tau+(f_{in})^{j(s+1)}+\left(v_{N_x}^{j+1(s+1)}-v_{N_x}^{j(s+1)}\right)/\tau}{E^{j(s+1)}}.$$

Then from (21)-(24) we obtain

$$\frac{1}{\tau}\left(v_i^{j+1(s+1)}-v_i^{j+1(s)}\right) = \frac{1}{h^2}\left(v_{i-1}^{j+1(s+1)}-2v_i^{j+1(s+1)}+v_{i+1}^{j+1(s+1)}\right) \tag{26}$$

$$+\varepsilon\frac{1}{2h^2\tau}\left[(v_{i-1}^{j+1(s+1)}-2v_i^{j+1(s+1)}+v_{i+1}^{j+1(s+1)})-(v_{i-1}^{j+1(s)}-2v_i^{j+1(s)}+v_{i+1}^{j+1(s)})\right]$$

$$+a^j v_i^{j+1(s+1)} = \widetilde{f}_i^{j+1}, \tag{27}$$

$$v_0^{j(s)} = v_{N_x+1}^{j(s)}, \tag{28}$$

$$\frac{v_1^{j(s)}+v_{N_x}^{j(s)}}{2} = v_{N_x+1}^{j(s)}, \tag{29}$$

The system of equations (26)-(29) can be solved by the Gauss elimination method and $v_i^{j+1(s+1)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $a^{j+1(s+1)}$, $v_i^{j+1(s+1)}(i=1,2,...,N_x)$ as a^{j+1}, $v_i^{j+1}(i=1,2,...,N_x)$, on the $(j+1)$-th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

5. Conclusions

The inverse problem regarding the simultaneously identification of the time-dependent source and the temperature distribution in one-dimensional quasilinear pseudo parabolic equation with periodic boundary and integral overdetermination conditions has been considered. This inverse problem has been investigated from both theoretical and numerical points of view. In the theoretical part of the article, the conditions for the existence, uniqueness and continuous dependence upon the data of the problem have been established. The problem is solved implicit difference scheme and an example is given.

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A new study on focal surface of a given surface

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Abstract. Focal surfaces are special cases of line congruences. With the aid of the definiton of a focal surface of a given surface M, we obtain a new type of focal surface in Galilean 3-space $\mathbb{G}_3$. We show that the focal surface we found is not the same type of surface as the given surface. We present the visualizations of the focal surface and the given surface with an example. Lastly, by searching the curvature functions, we give the minimality conditions of the focal surface.

1. Introduction

The concept of line congruences is first defined in the area of visualization by Hagen et al in 1991 [8]. Actually, line congruences are surfaces which are obtained from by transforming one surface to another by lines. Focal surface is one of these congruences. For a given surface M with the parametrization $X(u, v)$, the line congruence is defined as

$$C(u, v, z) = X(u, v) + zE(u, v). \tag{1}$$

Here $E(u, v)$ is the set of unit vectors and z is a distance. For each pair (u, v), the equation (1), expresses a line of the congruence and called as generatrix. On every generatrix of C, there are two points called as focal points and the focal surface is the locus of the focal points. If $E(u, v) = N(u, v)$, the unit normal vector field of the surface, then C is a normal congruence. In this case, the parametric equation of the focal surface $C = X^*(u, v)$ of $X(u, v)$ is given as

$$X^*(u, v) = C(u, v, z) = X(u, v) + {\kappa_i}^{-1} N(u, v); \quad i = 1, 2 \tag{2}$$

where κ_is; $(i = 1, 2)$ are the principal curvature functions of $X(u, v)$ [7]. Focal surfaces are the subject of many studies such as [7, 15–17, 23].

Galilean geometry is a non-Euclidean geometry and associated with Galilei principle of relativity. This principle can be explained briefly as "in all inertial frames, all law of physics are the same." (Except for the Euclidean geometry in some cases), Galilean geometry is the easiest of all Klein geometries, and it is revelant to the theory of relativity of Galileo and Einstein. One can have a look at the studies [20, 24] for Galilean geometry. Recently, many works related to Galilean geometry have been done by several authors in [2, 6, 21].

Tubular surfaces are special cases of canal surfaces which are the envelopes of a family of spheres. In canal surfaces, center of the spheres are on a given space curve (spine curve), and the radius of the spheres are different. As to tubular surfaces, the radius functions are constant. These surfaces have been widely studied in recent times [4, 10, 11, 13, 14, 18]. In Galilean 3-space, tubular surfaces are studied in [5].

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2. Preliminaries

In Galilean 3-space $\mathbb{G}_3$, we can give the following basic concepts.

The vector $a = (a_1, a_2, a_3)$ is isotropic if $a_1 = 0$ and non-isotropic otherwise. Thus, for the standard coordinates (x, y, z), the x-axis is non-isotropic while the others are isotropic. The yz-plane, i.e. $x = 0$, is Euclidean and the xy-plane and xz-plane are isotropic. The scalar product of the vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ and the length of the vector $a = (a_1, a_2, a_3)$ in $\mathbb{G}_3$ are respectively defined as

$$\langle a, b\rangle = \begin{cases} a_1b_1, & \text{if } \ a_1 \neq 0 \ \vee b_1 \neq 0 \\ a_2b_2 + a_3b_3, & \text{if } \ a_1 = 0 \ \wedge b_1 = 0, \end{cases} \tag{3}$$

$$\|a\| = \begin{cases} |a_1|, & \text{if } \ a_1 \neq 0 \\ a_2^2 + a_3^2, & \text{if } \ a_1 = 0. \end{cases} \tag{4}$$

The cross product of the vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{G}_3$ is also defined as

$$a \wedge b = \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{5}$$

[19]. An admissible unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ is given with the parametrization

$$\alpha(u) = (u, y(u), z(u)). \tag{6}$$

The associated Frenet frame vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ on the curve is given as

$$\begin{aligned} \mathbf{t}(u) &= (1, y'(u), z'(u)), \\ \mathbf{n}(u) &= \frac{1}{\kappa(u)}(0, y''(u), z''(u)), \\ \mathbf{b}(u) &= \frac{1}{\kappa(u)}(0, -z''(u), y''(u)), \end{aligned} \tag{7}$$

where $\kappa(u) = \sqrt{(y''(u))^2 + (z''(u))^2}$ and $\tau(u) = \frac{\det(\alpha'(u), \alpha''(u), \alpha'''(u))}{\kappa^2(u)}$ are the curvature and the torsion of the curve, respectively. Thus, the famous Frenet formulas can be written as

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n}, \\ \mathbf{n}' &= \tau\mathbf{b}, \\ \mathbf{b}' &= -\tau\mathbf{n}. \end{aligned} \tag{8}$$

Let M be a surface parametrized with

$$X(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)) \tag{9}$$

in $\mathbb{G}_3$. To represent the partial derivatives, we use

$$x_{,i} = \frac{\partial x}{\partial u_i}, \quad x_{,ij} = \frac{\partial^2 x}{\partial u_i \partial u_j}, 1 \leq i, j \leq 2. \tag{10}$$

If $x_{,i} \neq 0$ for some $i = 1, 2$, then the surface is admissible (i.e. having not any Euclidean tangent planes). The first fundamental form I of the surface M is defined as

$$I = (g_1 d_{u_1} + g_2 d_{u_2})^2 + \varepsilon(h_{11} d_{u_1}^2 + 2h_{12} d_{u_1} d_{u_2} + h_{22} d_{u_2}^2), \tag{11}$$

where $g_i = x_{,i}$, $h_{ij} = y_{,i}\, y_{,j} + z_{,i}\, z_{,j}$; $i, j = 1, 2$ and

$$\varepsilon = \begin{cases} 0, & \text{if} \quad d_{u_1} : d_{u_2} \quad \text{is non-isotropic,} \\ 1, & \text{if} \quad d_{u_1} : d_{u_2} \quad \text{is isotropic.} \end{cases} \tag{12}$$

Let a function W is given by

$$W = \sqrt{(x_{,1}\, z_{,2} - x_{,2}\, z_{,1})^2 + (x_{,2}\, y_{,1} - x_{,1}\, y_{,2})^2}. \tag{13}$$

Then, the unit normal vector field is given as

$$N = \frac{1}{W}(0, -x_{,1}\, z_{,2} + x_{,2}\, z_{,1}\,, x_{,1}\, y_{,2} - x_{,2}\, y_{,1}\,). \tag{14}$$

Similarly, the second fundamental form II of the surface M is defined as

$$II = L_{11} d_{u_1}^2 + 2L_{12} d_{u_1} d_{u_2} + L_{22} d_{u_2}^2, \tag{15}$$

where

$$L_{ij} = \frac{1}{g_1} \left\langle g_1(0, y_{,ij}\,, z_{,ij}\,) - g_{i,j}(0, y_{,1}\,, z_{,1}\,), N \right\rangle, \quad g_1 \neq 0 \tag{16}$$

or

$$L_{ij} = \frac{1}{g_2} \left\langle g_2(0, y_{,ij}\,, z_{,ij}\,) - g_{i,j}(0, y_{,2}\,, z_{,2}\,), N \right\rangle, \quad g_2 \neq 0.$$

The Gaussian and the mean curvatures of M are defined as

$$K = \frac{L_{11}L_{22} - L_{12}^2}{W^2}, \quad H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2W^2}. \tag{17}$$

A surface is called as flat (resp. minimal) if its Gaussian (resp. mean) curvatures vanish [2, 20]. The principal curvatures κ_1 and κ_2 of the surface M are given as

$$\kappa_1 = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{W^2}, \quad \kappa_2 = \frac{L_{11}L_{22} - L_{12}^2}{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}, \tag{18}$$

respectively [22].

3. Focal Surface of Tubular Surface in $\mathbb{G}_3$

A tubular surface M in $\mathbb{G}_3$ at a distance r from the points of spine curve $\alpha(u) = (u, y(u), z(u))$ is given with

$$M : X(u, v) = \alpha(u) + r(\cos v\mathbf{n} + \sin v\mathbf{b}). \tag{19}$$

Writing the Frenet vectors of $\alpha(u)$ in (19), the parametrization can be given as

$$M : X(u, v) = (u, y(u), z(u)) + \frac{r}{\kappa}\left[\cos v(0, y''(u), z''(u)) + \sin v(0, -z''(u), y''(u))\right]. \tag{20}$$

From (20),

$$g_1 = u_{,1} = 1, \quad g_2 = u_{,2} = 0. \tag{21}$$

The tangent vectors X_u, X_v and the normal vector N of M are given by

$$\begin{aligned} X_u &= \mathbf{t} - r\tau \sin v\mathbf{n} + r\tau \cos v\mathbf{b}, \\ X_v &= -r \sin v\mathbf{n} + r \cos v\mathbf{b}, \end{aligned} \tag{22}$$

and

$$N = -\cos v\mathbf{n} - \sin v\mathbf{b}. \tag{23}$$

Here $W = r$. The coefficients of the second fundamental form are obtained as

$$L_{11} = -\kappa\cos v + r\tau^2, \qquad L_{12} = r\tau, \qquad L_{22} = r. \tag{24}$$

From, (21) and (24), the curvature functions of M are obtained as

$$K = \frac{-\kappa\cos v}{r}, \qquad H = \frac{1}{2r} \tag{25}$$

[5].

Corollary 3.1. *[5] Tubular surfaces are constant mean curvature surfaces in Galilean space.*

By the equation (18), we obtain the principal curvatures κ_1, κ_2 of M as

$$\kappa_1 = -\kappa\cos v \text{ and } \kappa_2 = \frac{1}{r}. \tag{26}$$

For the function $\kappa_2 = \frac{1}{r}$, the focal surface degenerates to a curve. Thus, we obtain the focal surface M^* of M for the function $\kappa_1 = -\kappa\cos v$ as

$$M^* : X^*(u,v) = \alpha(u) + \left(r + \frac{1}{\kappa(u)\cos v}\right)(\cos v\mathbf{n} + \sin v\mathbf{b}), \tag{27}$$

where $\kappa \neq 0$.

Corollary 3.2. *The focal surface M^* of M is not a canal surface.*

Proposition 3.3. *If the spine curve $\alpha(u)$ is a straight line or equivalently M is flat, we cannot construct the focal surface of M.*

Example 3.4. *Let us consider the cylindrical helix $\alpha(u) = (u, \cos u, \sin u)$ in $\mathbb{G}_3$. The Frenet frame vectors of the spine curve $\alpha(u)$ is given by*

$$\begin{aligned} \mathbf{t}(u) &= (1, -\sin u, \cos u), \\ \mathbf{n}(u) &= (0, -\cos u, -\sin u), \\ \mathbf{b}(u) &= (0, \sin u, -\cos u). \end{aligned}$$

The tubular surface M has the following parametrization

$$X(u,v) = (u, \cos u - r\cos(u+v), \sin u - r\sin(u+v)).$$

[5]. Then from the equation (27), we write the parametrization of the focal surface M^* of M as in the following:

$$X^*(u,v) = (u, -r\cos(u+v) + \tan v\sin u, -r\sin(u+v) - \tan v\cos u).$$

By using the maple programme, we plot the graph of the tubular surface and its focal surface for the value $r = 2$ in $\mathbb{G}_3$.

For the focal surface M^*, the tangent space is spanned by the vectors

$$\begin{aligned} (X^*)_u &= \mathbf{t}(u) + \lambda_1(u,v)\mathbf{n}(u) + \lambda_2(u,v)\mathbf{b}(u), \\ (X^*)_v &= -r\sin v\mathbf{n}(u) + \lambda_3(u,v)\mathbf{b}(u), \end{aligned} \tag{28}$$

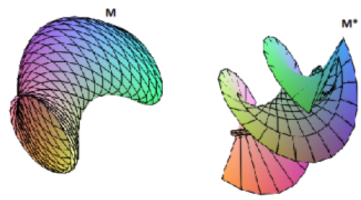

Figure 1: Tubular surface M and the focal surface M^*

where

$$\begin{aligned}\lambda_1(u,v) &= \frac{-\kappa'(u)}{(\kappa(u))^2} - r\tau(u)\sin v - \frac{\tau(u)}{\kappa(u)}\tan v,\\ \lambda_2(u,v) &= \frac{-\kappa'(u)}{(\kappa(u))^2}\tan v + r\tau(u)\cos v + \frac{\tau(u)}{\kappa(u)},\\ \lambda_3(u,v) &= \frac{1}{\kappa(u)\cos^2 v} + r\cos v.\end{aligned} \tag{29}$$

Thus, from (28), $W^* = ((\lambda_3(u,v))^2 + (r\sin v)^2)^{\frac{1}{2}}$ and the unit normal vector field N^* of M^* is

$$N^* = \frac{-\lambda_3(u,v)\mathbf{n}(u) - r\sin v\mathbf{b}(u)}{W^*}. \tag{30}$$

Further, we get

$$g_1^* = u_{,1} = 1, \quad g_2^* = u_{,2} = 0. \tag{31}$$

The second partial derivatives of X^* are

$$\begin{aligned}(X^*)_{uu} &= \lambda_4(u,v)\mathbf{n}(u) + \lambda_5(u,v)\mathbf{b}(u),\\ (X^*)_{uv} &= \lambda_6(u,v)\mathbf{n}(u) + \lambda_7(u,v)\mathbf{b}(u),\\ (X^*)_{vv} &= -r\cos v\mathbf{n}(u) + \lambda_8(u,v)\mathbf{b}(u),\end{aligned} \tag{32}$$

where

$$\begin{aligned}\lambda_4(u,v) &= \kappa(u) + (\lambda_1(u,v))_u - \tau(u)\lambda_2(u,v),\\ \lambda_5(u,v) &= (\lambda_2(u,v))_u + \tau(u)\lambda_1(u,v),\\ \lambda_6(u,v) &= (\lambda_1(u,v))_v\,,\\ \lambda_7(u,v) &= (\lambda_2(u,v))_v\\ \lambda_8(u,v) &= (\lambda_3(u,v))_v\,.\end{aligned} \tag{33}$$

Thus from the equations (30)-(33), the coefficients of the second fundamental form become

$$\begin{aligned}L_{11}^* &= \frac{-\lambda_3(u,v)\lambda_4(u,v) - \lambda_5(u,v)r\sin v}{W^*},\\ L_{12}^* &= \frac{-\lambda_3(u,v)\lambda_6(u,v) - \lambda_7(u,v)r\sin v}{W^*},\\ L_{22}^* &= \frac{\lambda_3(u,v)r\cos v - \lambda_8(u,v)r\sin v}{W^*}.\end{aligned} \tag{34}$$

By using the equations (31) and (34), we give the following theorems:

Theorem 3.5. *Let M be a tubular surface given with the parametrization (19) and M^* be the focal surface of M with the parametrization (27) in $\mathbb{G}_3$. Then, the Gaussian and the mean curvatures of M^* are*

$$\begin{aligned} K^* &= \frac{1}{(W^*)^4}\left\{\begin{array}{c} -\lambda_3^2\lambda_4 r\cos v + \lambda_3\lambda_4\lambda_8 r\sin v - \lambda_3\lambda_5 r^2\sin v\cos v \\ +\lambda_5\lambda_8 r^2\sin^2 v - \lambda_3^2\lambda_6^2 - \lambda_7^2 r^2\sin^2 v - 2\lambda_3\lambda_6\lambda_7 r\sin v \end{array}\right. \\ H^* &= \frac{\lambda_3 r\cos v - \lambda_8 r\sin v}{2(W^*)^3}. \end{aligned} \tag{35}$$

Corollary 3.6. *If the focal surface M^* is minimal, then*

$$r = -\frac{1}{\kappa(u)\cos^3 v}.$$

Proof. Let M^* be the focal surface of M with the parametrization (27) in $\mathbb{G}_3$. If M^* is minimal, then $\lambda_3 r\cos v - \lambda_8 r\sin v = 0$. Since the functions $\cos v$ and $\sin v$ are linearly independent, $\lambda_3 = \lambda_8 = 0$ i.e. $\lambda_3 = (\lambda_3)_v = 0$ which corresponds to the last equation. □

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Continuous Dependence on Data for a Solution of determination of an unknown source of Heat Conduction of Poly(methyl methacrylate) (PMMA)

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Abstract. In this paper,we consider a coefficient problem of an inverse problem of a quasilinear parabolic equation with periodic boundary and integral over determination conditions. It showed the stability of the solution by iteration method and examined numerical solution.

1. Introduction

The inverse problem of determining unknown coefficient in a quasi-linear parabolic equation has generated an increasing amount of interest from engineers and scientist during the last few decades.Inverse Problem is a research area dealing with inversion of models or data. An inverse problem is a mathematical framework that is used to obtain information about a physical object or system from observed measurements. It is called an inverse problem because it starts with the results and then calculates the causes. This is the inverse of a direct problem, which starts with the causes and then calculates the results. Thus, inverse problems are some of the most important and well-studied mathematical problems in science and mathematics because they provide us about parameters that we cannot directly observe[1–3]. There are many different applications including medical imaging, geophysics, computer vision, astronomy, nondestructive testing, and many others. Nevertheless the inverse coefficient problems with periodic boundary and integral over determination conditions are not investigated by many researchers because of the difficulties of these conditions [1–3, 5–8]. This kind of conditions arise from many important applications in heat transfer, life sciences, etc. The inverse problem of unknown coefficients in a quasi-linear parabolic equations with periodic boundary conditions was studied by Kanca and Baglan [9, 10]. Over the last years, considerable efforts have been put into develop either approximate analytical solution and numerical solution to non-local boundary value problems [3]. Cannon implemented implicit finite difference scheme to obtain numerical solution of the one dimensional non-local boundary value problems [1]. Liu studied non-local boundary value problems and concluded that the presence of integral terms in boundary conditions can greatly complicate the application of standard numerical techniques such as finite difference schemes and finite element techniques [4]. Several researchers have discussed numerical solutions for non-local boundary value problems in one dimension.The one-dimensional case of this problem has been the guiding force behind considerable research in numerical methods such as finite difference method and finite element method.

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Explicit and implicit finite difference schemes were used by many researchers to obtain numerical solutions of one-dimensional non-local boundary value problem. Finite difference method to a class of parabolic inverse problems is investigated. This method is very effective for solving various kinds of partial differential equations.

Consider the equation

$$u_t = u_{xx} + l(t) f(x,t,u),\ (x,t) \in\ D, \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x),\ \ x \in [0,\pi], \tag{2}$$

the periodic boundary condition

$$u(0,t) = u(\pi,t),\ u_x(0,t) = u_x(\pi,t),\ 0 \le t \le T, \tag{3}$$

and the over determination data

$$g(t) = u(\pi,t), 0 \le t \le T, \tag{4}$$

for a quasilinear parabolic equation with the nonlinear source term $f = f(x,t,u)$.

Here $D := \{0 < x < \pi,\ 0 < t < T\}$.The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,\pi]$ and $\bar{D} \times (-\infty,\infty)$, respectively.

The problem of finding the pair $\{l(t), u(x,t)\}$ in (1)-(4) will be called an inverse problem.

Definition 1.1. *The pair $\{l(t), u(x,t)\}$ from the class $C[0,T] \times (C^{2,1}(D) \cap C^{1,0}(\overline{D}))$ for which conditions (1)-(4) are satisfied is called the classical solution of the inverse problem (1)-(4).*

The paper organized as follows:

In Section 2, the existence and uniqueness of the solution of the inverse problem (1)-(4) is proved by using the Fourier method and iteration method. In Section 3, the continuous dependence upon the data of the inverse problem is shown. In Section 4, the numerical procedure for the solution of the inverse problem is given.

2. Existence and Uniqueness of the Solution of the Inverse Problem

The main result on the existence and the uniqueness of the solution of the inverse problem (1)-(4) is presented as follows:

We have the following assumptions on the data of the problem (1)-(4).

(A1) $g(t) \in C^1[0,T], l(t) \in C[0,T]$.

(A2) $\varphi(x) \in C^3[0,\pi]$, $\varphi(0) = \varphi(\pi)$, $\varphi'(0) = \varphi'(\pi)$, $\varphi''(0) = \varphi''(\pi)$,

(A3) Let the function $f(x,t,u)$ is continuous with respect to all arguments in $\bar{D} \times (-\infty,\infty)$ and satisfies the following condition

(1)

$$\left| \frac{\partial^{(n)} f(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} f(x,t,\tilde{u})}{\partial x^n} \right| \le b(t,x) |u - \tilde{u}|\ \ , n = 0,1,2,$$

where $b(x,t) \in L_2(D),\ b(x,t) \ge 0,$

(2) $f(x,t,u) \in C^3[0,\pi]$, $t \in [0,T]$,

(3) $f(x,t,u)|_{x=0} = f(x,t,u)|_{x=\pi}$, $f_x(0,t,u)|_{x=0} = f_x(\pi,t,u)|_{x=\pi}$, $f_{xx}(0,t,u)|_{x=0} = f_{xx}(\pi,t,u)|_{x=\pi}$,

(4) $\int\limits_0^\pi f(x,t,u)\,dx \neq 0,\ \ \forall t \in [0,T]$.

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1)-(3) for arbitrary $l(t) \in C[0,T]$:

$$u(x,t) = \frac{u_0(t)}{2} + \sum_{k=1}^{\infty} \left[u_{ck}(t)\cos 2kx + u_{sk}(t)\sin 2kx \right],$$

$$\begin{aligned}
u_0(t) &= \varphi_0 + \frac{2}{\pi}\int_0^t\int_0^\pi l(\tau) f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right) d\xi d\tau,\\
u_{ck}(t) &= \varphi_{ck}\, e^{-(2k)^2 t} + \frac{2}{\pi}\int_0^t\int_0^\pi l(\tau) f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right)\cos 2k\xi\; e^{-(2k)^2(t-\tau)} d\xi d\tau,\\
u_{sk}(t) &= \varphi_{sk}\, e^{-(2k)^2 t} + \frac{2}{\pi}\int_0^t\int_0^\pi l(\tau) f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right)\sin 2k\xi\; e^{-(2k)^2(t-\tau)} d\xi d\tau.
\end{aligned}$$

$$\begin{aligned}
u(x,t) &= \varphi_0 + \int_0^t l(\tau) f_0(\tau,u)\, d\tau \\
&\quad + \sum_{k=1}^{\infty} \cos 2kx \left[\varphi_{ck}\, e^{-(2k)^2 t} + \int_0^t l(\tau) f_{ck}(\tau,u)\, e^{-(2k)^2(t-\tau)} d\tau \right] \\
&\quad + \sum_{k=1}^{\infty} \sin 2kx \left[\varphi_{sk}\, e^{-(2k)^2 t} + \int_0^t l(\tau) f_{sk}(\tau,u)\, e^{-(2k)^2(t-\tau)} d\tau \right],
\end{aligned} \tag{5}$$

where $\varphi_0 = \frac{2}{\pi}\int_0^\pi \varphi(x)dx$, $\varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(x)\cos 2kx dx$, $\varphi_{sk} = \frac{2}{\pi}\int_0^\pi \varphi(x)\sin 2kx dx$.
Under the condition (A1)-(A3), differentiating (4), we obtain

$$u_t(\pi,t) = g'(t), 0 \le t \le T. \tag{6}$$

(5) and (6) yield

$$l(t) = \frac{g'(t) + \sum_{k=1}^{\infty}(4k^2)\left(\varphi_{ck} e^{-(2k)^2 t} + \int_0^t l(\tau) f_{ck}(\tau,u)\, e^{-(2k)^2(t-\tau)} d\tau\right)}{f_0(t) + \sum_{k=1}^{\infty} f_{ck}(t)}. \tag{7}$$

Definition 2.1. *Denote the set*
$\{u(t)\} = \{u_0(t), u_{ck}(t), u_{sk}(t), k = 1,...,n\}$, *of continuous on* $[0,T]$ *functions satisfying the condition* $\max_{0\le t\le T}\frac{|u_0(t)|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)| + \max_{0\le t\le T}|u_{sk}(t)|\right) < \infty$, *by* $\mathbf{B}_1$. *Let*
$\|u(t)\| = \max_{0\le t\le T}\frac{|u_0(t)|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)| + \max_{0\le t\le T}|u_{sk}(t)|\right)$, *be the norm in* $\mathbf{B}_1$.
It can be shown that $\mathbf{B}_1$ *are the Banach spaces.*

3. Continuous Dependence of (l,u) upon the data

Theorem 3.1. *Under assumption (A1)-(A3) the solution* (l,u) *of the problem (1)-(4) depends continuously upon the data* φ, g.

Proof. Let $\Phi=\{\varphi,\ g,\ f\}$ and $\overline{\Phi}=\{\overline{\varphi},\ \overline{g},\ f\}$ be two sets of the data, which satisfy the assumptions $(A_1)-(A_3)$. Suppose that there exist positive constants M_i, $i=1,2$ such that

$$\|g\|_{C^1[0,T]}\le M_1, \|\overline{g}\|_{C^1[0,T]}\le M_1, \|\varphi\|_{C^3[0,\pi]}\le M_2, \|\overline{\varphi}\|_{C^3[0,\pi]}\le M_2.$$

Let us denote $\|\Phi\|=(\|g\|_{C^1[0,T]}+\|\varphi\|_{C^3[0,\pi]}+\|f\|_{C^{3,0}(\overline{D})})$. Let (l,u) and $\left(\overline{l},\overline{u}\right)$ be solutions of inverse problems (1)-(4) corresponding to the data $\Phi=\{\varphi,\ g,\ f\}$ and $\overline{\Phi}=\{\overline{\varphi},\ \overline{g},\ f\}$ respectively. According to (5)

$$\begin{aligned}
u-\overline{u} \;=\;& \frac{(\varphi_0-\overline{\varphi_0})}{2}+\sum_{k=1}^{\infty}\cos 2k\xi\ \ (\varphi_{ck}-\overline{\varphi_{ck}})\,e^{-(2k)^2t}+\sum_{k=1}^{\infty}\sin 2k\xi\ \ (\varphi_{sk}-\overline{\varphi_{sk}})\,e^{-(2k)^2t}\\
&+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)\,[f(\xi,\tau,u(\xi,\tau))-f(\xi,\tau,\overline{u}(\xi,\tau))]\,d\xi d\tau\right)\\
&+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^\pi \left(l(\tau)-\overline{l}(\tau)\right)f(\xi,\tau,\overline{u}(\xi,\tau))d\xi d\tau\right)\\
&+\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)\,[f(\xi,\tau,u(\xi,\tau))-f(\xi,\tau,\overline{u}(\xi,\tau))]\cos 2k\xi\ e^{-(2k)^2(t-\tau)}d\xi d\tau\\
&+\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi \left(l(\tau)-\overline{l}(\tau)\right)[f(\xi,\tau,u(\xi,\tau))-f(\xi,\tau,\overline{u}(\xi,\tau))]\cos 2k\xi\ e^{-(2k)^2(t-\tau)}d\xi d\tau\\
&+\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)\,[f(\xi,\tau,u(\xi,\tau))-f(\xi,\tau,\overline{u}(\xi,\tau))]\cos 2k\xi\ e^{-(2k)^2(t-\tau)}d\xi d\tau\\
&+\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi \left(l(\tau)-\overline{l}(\tau)\right)[f(\xi,\tau,u(\xi,\tau))-f(\xi,\tau,\overline{u}(\xi,\tau))]\cos 2k\xi\ e^{-(2k)^2(t-\tau)}d\xi d\tau.
\end{aligned}$$

By using same estimations, we obtain:

$$\begin{aligned}
|u-\overline{u}| \;\le\;& M_3\left\|\Phi-\overline{\Phi}\right\|\\
&+M_4\left(\int_0^t\int_0^\pi l^2(\tau)b^2(\xi,\tau)\,|u(\tau)-\overline{u}(\tau)|^2\,d\xi d\tau\right)^{\frac{1}{2}}
\end{aligned} \tag{8}$$

$$\begin{aligned}
|a-\overline{a}| \;\le\;& M_5\left\|\Phi-\overline{\Phi}\right\|\\
&+M_6\,|r(t)|\left|u(t)-\overline{u(t)}\right|,
\end{aligned}$$

applying Gronwall's inequality to (8), we obtain:

$$\begin{aligned}
|u-\overline{u}|^2 \;\le\;& 2M_3^2\left\|\Phi-\overline{\Phi}\right\|^2\\
&\times\exp 2M_4^2\left(\int_0^t\int_0^\pi l^2(\tau)b^2(\xi,\tau)d\xi d\tau\right).
\end{aligned}$$

For $\Phi\to\overline{\Phi}$ then $u\to\overline{u}$. Hence $l\to\overline{l}$. $\square$

4. Numerical Procedure for the nonlinear problem (1)-(4)

We construct an iteration algorithm for the linearization of the problem (1)-(4):

$$\begin{aligned} \frac{\partial u^{(n)}}{\partial t} &= \frac{\partial^2 u^{(n)}}{\partial x^2} + l(t) f(x,t,u^{(n-1)}), \quad (x,t) \in D & (9)\\ u^{(n)}(0,t) &= u^{(n)}(\pi,t), \quad t \in [0,T] & (10)\\ u_x^{(n)}(0,t) &= u_x^{(n)}(\pi,t) = 0,\ t \in [0,T] & (11)\\ u^{(n)}(x,0) &= \varphi(x)\ , \quad x \in [0,\pi]\,. & (12)\end{aligned}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $f(x,t,u^{(n-1)}) = \widetilde{f}(x,t)$. Then the problem (9)-(12) can be written as a linear problem:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + l(t)\widetilde{f}(x,t) \quad (x,t) \in D & (13)\\ v(0,t) &= v(\pi,t), \quad t \in [0,T] & (14)\\ v_x(0,t) &= v_x(\pi,t), \quad t \in [0,T] & (15)\\ v(x,0) &= \varphi(x), \quad x \in [0,\pi]\ . & (16)\end{aligned}$$

We use the method of the linearization then we use the finite difference method to solve (13)-(16).

We subdivide the intervals $[0,\pi]$ and $[0,T]$ into subintervals N_x and N_t of equal lengths $h = \frac{\pi}{N_x}$ and $\tau = \frac{T}{N_t}$, respectively. We choose the implicit scheme which is absolutely stable and has a second-order accuracy in h and a first-order accuracy in τ. The implicit scheme for (13)-(16) is as follows:

$$\frac{1}{\tau}\left(v_i^{j+1} - v_i^j\right) = \frac{1}{2h^2}\left(v_{i-1}^{j+1} - 2v_i^{j+1} + v_{i+1}^{j+1}\right) + \frac{1}{2h^2}\left(v_{i-1}^j - 2v_i^j + v_{i+1}^j\right) + l^j \widetilde{f}_i^j, \tag{17}$$

$$v_i^0 = \phi_i, \tag{18}$$

$$v_0^j = v_{N_x+1}^j, \tag{19}$$

$$\frac{v_1^j + v_{N_x}^j}{2} = v_{N_x+1}^j, \tag{20}$$

where $1 \le i \le N_x$ and $0 \le j \le N_t$ are the indices for the spatial and time steps respectively, $v_i^j = v(x_i,t_j)$, $\phi_i = \varphi(x_i)$, $\widetilde{f}_i^j = \widetilde{f}(x_i,t_j)$, $x_i = ih$, $t_j = j\tau$. At the level $t = 0$, adjustment should be made according to the initial condition and the compatibility requirements.

Now, let us construct the predicting-correcting mechanism. First, integrating the equation (1) with respect to x from 0 to 1 and using (3) and (4), we obtain

$$l(t) = \frac{g'(t) - v_{xx}(\pi,t)}{\widetilde{f}(x,t)}. \tag{21}$$

The finite difference approximation of (21) is

$$l^j = \frac{-\left(\left(g^{j+1} - g^j\right)/\tau\right) + \frac{1}{2h^2}\left(v_{N_x-1}^{j+1} - 2v_{N_x}^{j+1} + v_{N_x+1}^{j+1}\right) + \frac{1}{2h^2}\left(v_{N_x-1}^j - 2v_{N_x}^j + v_{N_x+1}^j\right)}{(\widetilde{fi})^j}.$$

and the values of ϕ_i provide us to start our computation. We denote the values of l^j, v_i^j at the s-th iteration step .and the values of ϕ_i provide us to start our computation. We denote the values of l^j, v_i^j at the s-th iteration step $l^{j(s)}$,

$v_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $l^{j+1(0)} = l^j$, $v_i^{j+1(0)} = v_i^j$, $j = 0,1,2,\ldots N_t$, $i = 1,2,\ldots,N_x$. At each $(s+1)$-th iteration step we first determine $l^{j+1(s+1)}$ from the formula

$$l^{j+1(s+1)} = \frac{-\left(\left(g^{j+2} - g^{j+1}\right)/\tau\right) + \frac{1}{2h^2}\left(v_{N_x-1}^{j+1(s)} - 2v_{N_x}^{j+1(s)} + v_{N_x+1}^{j+1(s)}\right) + \frac{1}{2h^2}\left(v_{N_x-1}^{j(s)} - 2v_{N_x}^{j(s)} + v_{N_x+1}^{j(s)}\right)}{(\widetilde{f}_i)^{j+1}}.$$

Then from (17)-(20) we obtain

$$\begin{aligned} \frac{1}{\tau}\left(v_i^{j+1(s+1)} - v_i^{j+1(s)}\right) &= \frac{1}{h^2}\left(v_{i-1}^{j+1(s+1)} - 2v_i^{j+1(s+1)} + v_{i+1}^{j+1(s+1)}\right) \\ &\quad + l^{j+1(s+1)}\widetilde{f}_i^{j+1}, \end{aligned} \tag{22}$$

$$v_0^{j(s)} = v_{N_x+1}^{j(s)}, \tag{23}$$

$$\frac{v_1^{j(s)} + v_{N_x}^{j(s)}}{2} = v_{N_x+1}^{j(s)}. \tag{24}$$

The system of equations (22)-(24) can be solved by the Gauss elimination method and $v_i^{j+1(s+1)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $l^{j+1(s+1)}$, $v_i^{j+1(s+1)}(i = 1,2,\ldots,N_x)$ as l^{j+1}, $v_i^{j+1}(i = 1,2,\ldots,N_x)$, on the $(j+1)$-th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

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A note on Hopf bifurcation and steady state analysis for a predator-prey model

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Abstract. This paper is concerned with the Hopf bifurcation and steady state analysis of a predator-prey model. Firstly, by analyzing the characteristic equation, the local stability of the nonnegative equilibriums is discussed. Then the Hopf bifurcation around the positive equilibrium is obtained, and the direction and the stability of the Hopf bifurcation are investigated. Finally, some numerical simulations are given to support the theoretical results.

1. Introduction

Mathematical ecology is a subject field in which dynamic systems are involved in species, populations, and how these groups interact with the environment. This subject field primarily studies how species population size changes over time and space. Since Lotka–Volterra's groundbreaking work in the 1920s, the predator-prey model has become one of the most important research topics in mathematical ecology for nearly a century. Species compete, evolve and disperse for the purpose of finding resources to sustain their struggle for their existence. Depending on their specific settings of applications, they can take the forms of resource-consumer, plant-herbivore, parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions, etc. Mathematicians used the theory of dynamics to analyze the differential equations based on a predator-prey model. There are some scholars who applied bifurcation theory in dynamics based on models and we can find them in [2]-[11] etc.

In this paper, we consider a predator-prey model satisfies the following differential equations in [1]

$$\frac{dH}{d\tau} = rH\left(1 - \frac{H}{K}\right) - \alpha\frac{PH}{H+\beta}, \tag{1}$$

$$\frac{dP}{d\tau} = \gamma P\left(-1 + \delta\frac{H}{H+\beta}\right), \tag{2}$$

where H is the prey density and P is the predator density. The parameters are $r,\ K,\ \alpha,\ \beta,\ \gamma,\ \delta > 0$, $H(0) > 0$ and $P(0) > 0$.

The rest of the paper is organized as follows. Basic properties of the model are given in Section 2. Sufficient conditions for the existence of the Hopf bifurcation are obtained in Section 3. In Section 4, the numerical examples are given to illustrate the validity of our results.

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2. Preliminary

In this section, firstly, we make the following change of variables to put the model in dimensionless form:

$$x = \frac{H}{K}, \quad y = \frac{\alpha}{rK}P, \quad t = r\tau.$$

Thus (1)-(2) can be written as

$$\frac{dx}{dt} = x\Big(1 - x - \frac{y}{x+b}\Big), \tag{3}$$

$$\frac{dy}{dt} = cy\Big(-1 + a\frac{x}{x+b}\Big). \tag{4}$$

We introduce the basic properties of the nonnegative constant solutions for the system (3)-(4). It is obvious that $\vec{u_1} = (x_1, y_1) = (0,0)$ and $\vec{u_2} = (x_2, y_2) = (1,0)$ are constant steady states of (3)-(4). Furthermore, $\vec{u_3} = (x_3, y_3) = \Big(\frac{b}{a-1}, \frac{ab(a-b-1)}{(a-1)^2}\Big)$ is a constant steady state of (3)-(4).

It is clear that when $a < b+1$, (3)-(4) has no positive equilibrium.

In the following, we discuss the local stability of equilibrium $\vec{u_i} = (x_i, y_i)$ $(i = 1,2,3)$. By directly calculating, the Jacobian matrix at $\vec{u_i}$ is

$$J_i \triangleq J(\vec{u_i}) = \begin{pmatrix} 1 - 2x_i - \frac{by_i}{(x_i+b)^2} & -\frac{x_i}{x_i+b} \\ abc\frac{y_i}{(x_i+b)^2} & c\Big(\frac{ax_i}{x_i+b} - 1\Big) \end{pmatrix}.$$

Theorem 2.1. For system (3)-(4), the following statements are hold.
(i) For all $a,\ b,\ c > 0$, the constant equilibrium solution $\vec{u_1}$ is a saddle point which is unstable.
(ii) The constant equilibrium solution $\vec{u_2}$ is stable when $a < b+1$ and it is unstable for $a > b+1$.
(iii) In the case $a < b+1$, there is no limit cycle since there is no positive equilibrium.

3. Existence of Hopf Bifurcation

In this section, we restrict $a > b+1$ and only study the Hopf bifurcation around $\vec{u_3}$. Taking a as the bifurcation parameter, we study the existence of Hopf bifurcation for (3)-(4) and so the direction and the stability of Hopf bifurcation are investigated.

Now, we investigate the results of Hopf bifurcation for (3)-(4). We primarily get the Jacobian matrix of (3)-(4) at $\vec{u_3}$

$$J_3 = \begin{pmatrix} -\frac{2b}{a-1} + \frac{b+1}{a} & -\frac{1}{a} \\ c(a-b-1) & 0 \end{pmatrix}.$$

The characteristic equation of J_3 is

$$\lambda^2 - traceJ_3\lambda + detJ_3 = 0, \tag{5}$$

where

$$traceJ_3 = -\frac{2b}{a-1} + \frac{b+1}{a}, \qquad detJ_3 = \frac{c}{a}(a-b+1) > 0.$$

Let $(\tilde{x}, \tilde{y}) = (x, y) - (x_3, y_3)$. For convenience, we denote $(\tilde{x}, \tilde{y})$ as (x, y). Then the model (3)-(4) is changed to

$$\frac{dx}{dt} = (x + x_3)\Big(1 - (x + x_3) - \frac{y + y_3}{x + x_3 + b}\Big), \tag{6}$$

$$\frac{dy}{dt} = c(y + y_3)\Big(-1 + a\frac{x + x_3}{x + x_3 + b}\Big). \tag{7}$$

Theorem 3.1. The model (3)-(4) undergoes a Hopf bifurcation at (x_3, y_3) for $a = a^H = \frac{b+1}{1-b}$.

Proof. Since we assume that $a > b+1$, it should be $0 < b < 1$. Clearly, if $a = a^H = \frac{b+1}{1-b}$ holds, then $\pm i\sqrt{bc}$ is a pair of imaginary eigenvalues of J_3. Let $\alpha(a) \pm iw(a)$ be the roots of (5) in the neighborhood of a^H. So we obtain

$$\alpha(a) = \frac{traceJ_3}{2} = \frac{b+1}{2a} - \frac{b}{a-1}, \qquad w(a) = \sqrt{4\frac{c}{a}(a-b-1) - \left(\frac{b+1}{a} - \frac{2b}{a-1}\right)^2}$$

and

$$\alpha'(a) = -\frac{b+1}{4a^2} + \frac{b}{(a-1)^2}.$$

It is clear that $traceJ_3(a^H) = 0,\ detJ_3(a^H) > 0$ and $\alpha'(a^H) \neq 0$. It follows from the Hopf bifurcation theorem [1] that the model (3)-(4) undergoes a Hopf bifurcation at (x_3, y_3, a^H).

Now, we use a computational method to test whether the Hopf bifurcation is supercritical or subcritical. To study the system around the point $a = a^H$ we expand the right hand side of the system (6)-(7) using the Maclaurin series and we rewrite the system (6)-(7) as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = J_3 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F(x,y,a) \\ G(x,y,a) \end{pmatrix}, \tag{8}$$

where

$$F = \left(\frac{by_3}{(x_3+b)^3} - 1\right)x^2 - \frac{b}{(x_3+b)^2}xy + \frac{b}{(x_3+b)^3}x^2y - \frac{by_3}{(x_3+b)^5}x^3$$

and

$$G = -\frac{abcy_3}{(x_3+b)^3}x^2 + \frac{abc}{(x_3+b)^2}xy - \frac{abc}{(x_3+b)^3}x^2y + \frac{abcy_3}{(x_3+b)^5}x^3.$$

Next, we make the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \tag{9}$$

where

$$P = \begin{pmatrix} \frac{1-b}{bc(b+1)}w(a) & 0 \\ 0 & \frac{b+1}{1-b)}w(a) \end{pmatrix},$$

and substitute it into (8). To avoid the abuse of mathematical notation, we still denote $(\tilde{x}, \tilde{y})$ by (x, y). Then we obtain the normal form of (8) as follows

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -w(a) \\ w(a) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y,a) \\ g(x,y,a) \end{pmatrix}, \tag{10}$$

where

$$f(x,y,a) = \frac{bc(b+1)}{(1-b)w(a)}F\left(\frac{1-b}{bc(b+1)}w(a)x, \frac{b+1}{1-b}w(a)y\right),$$

$$g(x,y,a) = \frac{1-b}{(b+1)w(a)}G\left(\frac{1-b}{bc(b+1)}w(a)x, \frac{b+1}{1-b}w(a)y\right).$$

To determine the stability of periodic solutions, we need to calculate the sign of the following coefficient

$$\begin{aligned} \gamma &= \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) \\ &+ \frac{1}{16w(a^H)}\left[f_{xy}\left(f_{xx} + f_{yy}\right) - g_{xy}\left(g_{xx} + g_{yy}\right) - f_{xx}g_{xx} + f_{yy}g_{yy}\right], \end{aligned} \tag{11}$$

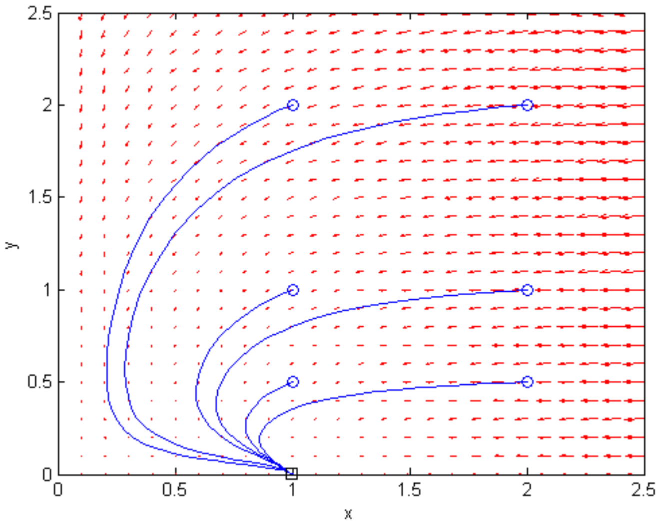

Figure 1: When $a < b+1$, there is no positive equilibrium. The constant equilibrium $\vec{u_2} = (1,0)$ is locally stable.

where all the partial derivatives are evaluated at the bifurcation point $(0, 0, a^H)$. Then, by computing we obtain

$$\begin{aligned} \gamma &= -12\frac{b^2}{c^{1/2}(b+1)^5} - \frac{1-b}{(b+1)^4} - \frac{b^{3/2}c^{1/2}}{(1-b)^2}\left(\frac{2b^3(b+1)}{(1-b)^2} - 1\right) \\ &+ \frac{(1-b)^2}{b^{1/2}c^{1/2}(b+1)^3} + \frac{1-b}{4b^{1/2}c^{1/2}(b+1)}\left(\frac{2b^3(b+1)}{(1-b)^2} - 1\right). \end{aligned} \tag{12}$$

Therefore, we have the following result.

Theorem 3.2. If $\gamma < 0$, the direction of Hopf bifurcation is supercritical. This means that for $a < a^H$ the positive equilibrium (x_3, y_3) is a stable spiral but for $a > a^H$ there exists a stable periodic solution and (x_3, y_3) is unstable. If $\gamma > 0$, the direction of Hopf bifurcation is subcritical. In this situation, when $a < a^H$ the positive equilibrium (x_3, y_3) is stable and there exists an unstable periodic solution but when $a > a^H$, (x_3, y_3) is unstable.

4. Numerical Simulations

In this section, some numerical simulations are presented, which support and complement the results given in the previous section. There are three parameters $a,\ b,\ c$ in our model (3)-(4). We fix $b = 0.5,\ c = 1$ and obtain the following numerical simulations which illustrate the main theoretical results.

Example 4.1. We take $a = 1,\ b = 0.5,\ c = 1$. Then $a < b+1$ and model (3)-(4) has no positive equilibrium. From Fig. 1, we see that $\vec{u_2} = (1,0)$ is locally stable.

Example 4.2. We take $a = 2.5,\ b = 0.5,\ c = 1$. Then $a > b+1$ and there exists unique positive equilibrium $\vec{u_3} = (x_3, y_3)$. When $a = 2.5,\ b = 0.5, a < a^H$. From Fig. 2, we see that (x_3, y_3) is a stable spiral.

Example 4.3. We take $a = 3.5,\ b = 0.5,\ c = 1$, then $a > a^H$. We observe that there exists a stable periodic solution and the positive equilibrium (x_3, y_3) is unstable. This is seem from Fig 3.

In Example 4.2 and Example 4.3, we fix $b = 0.5,\ c = 1$, then we derive $\gamma < 0$. From the numerical simulations (see Fig. 2 and Fig. 3), we can say that there exists a supercritical Hopf bifurcation and this supports our theorical results.

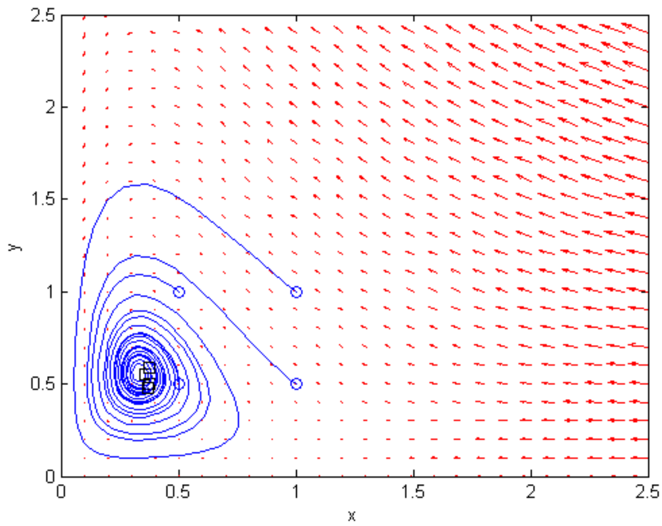

Figure 2: When $a > b + 1$ and $0 < b < 1$, we have a bifurcation parameter and a bifurcation value a and a^H, respectively. If $a < a^H$, (x_3, y_3) is a stable spiral.

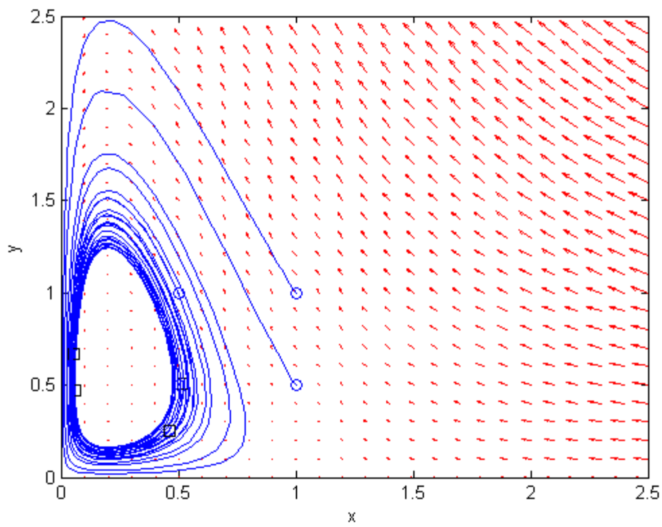

Figure 3: If $a > a^H$, there exist stable periodic orbits and (x_3, y_3) is unstable.

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Neighborhoods of Certain Classes of Analytic Functions Defined by Normalized Function $az^2 J''_\vartheta(z) + bzJ'_\vartheta(z) + cJ_\vartheta(z)$

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Abstract. In this paper, we introduce a new subclass of analytic functions in the open unit disk $\mathcal{U}$ with negative coefficients defined by normalized of the $az^2 J''_\vartheta(z) + bzJ'_\vartheta(z) + cJ_\vartheta(z)$ function, where $J_\vartheta(z)$ is called the Bessel function of the first kind of order ϑ. The object of the present paper is to determine coefficient inequalities, inclusion relations and neighborhoods properties for functions $f(z)$ belonging to this subclass.

1. Introduction

Let $\mathcal{A}$ be a class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Denote by $\mathcal{A}(n)$ the class of functions consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

which are analytic in $\mathcal{U}$.
We recall that the convolution (or Hadamard product) of two functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z) \quad (z \in \mathcal{U}).$$

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Note that $f * g \in \mathcal{A}$.
Next, following the earlier investigations by Goodman [8], Ruscheweyh [16], Silverman [18] and Altıntaş et al. [1, 2] (see also [4]-[7], [10], [12], [14]-[16]), we define the $(n, \delta)-$neighborhood of a function $f \in \mathcal{A}(n)$ by

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n\,|a_n - b_n| \leq \delta \right\}. \tag{3}$$

For $e(z) = z$, we have

$$\mathcal{N}_{n,\delta}(e) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n\,|b_n| \leq \delta \right\}. \tag{4}$$

A function $f \in \mathcal{A}(n)$ is $\alpha-$starlike of complex order γ, denoted by $f \in \mathcal{S}_n^*(\beta, \gamma)$ if it satisfies the following condition

$$\Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \beta \qquad \left(\gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \beta < 1, z \in \mathcal{U}\right),$$

and a function $f \in \mathcal{A}(n)$ is $\beta-$convex of complex order γ, denoted by $f \in C_n(\beta, \gamma)$ if it satisfies the following condition

$$\Re\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > \beta \qquad \left(\gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \beta < 1, z \in \mathcal{U}\right).$$

The Bessel function of the first kind of order ϑ is defined by [13, p.217]

$$J_\vartheta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \vartheta + 1)}\left(\frac{z}{2}\right)^{2n+\vartheta} \qquad (z \in \mathbb{C}). \tag{5}$$

We know that it has all its zeros real for $\vartheta > -1$. Here now we consider mainly the general function

$$N_\vartheta(z) = az^2 J_\vartheta''(z) + bzJ_\vartheta'(z) + cJ_\vartheta(z)$$

studied by Mercer [11]. Here, as in [11], $q = b - a$ and

$$(c = 0 \text{ and } q \neq 0) \ \ or \ \ (c > 0 \text{ and } q > 0).$$

From (5), we have the power series representation

$$N_\vartheta(z) = \sum_{n=0}^{\infty} \frac{Q(2n + \vartheta)(-1)^n}{n!\Gamma(n + \vartheta + 1)}\left(\frac{z}{2}\right)^{2n+\vartheta} \qquad (z \in \mathbb{C}) \tag{6}$$

where $Q(\vartheta) = a\vartheta(v - 1) + b\vartheta + c \quad (a, b, c \in \mathbb{R})$. Lastly, Baricz, Çağlar and Deniz [3] obtained sufficient and necessary conditions for the starlikeness of a normalized form of N_ϑ by using results of Mercer [11], Ismail and Muldoon [9] and Shah and Trimble [17].
Note that N_ϑ is not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization for the function $N_\vartheta(z)$:

$$\tilde{N}_\vartheta(z) = \frac{2^\vartheta\Gamma(\vartheta + 1)z^{1-\frac{\vartheta}{2}}}{Q(\vartheta)} N_\vartheta\left(\sqrt{z}\right). \tag{7}$$

In the rest of this paper, the quadratic $Q(\vartheta) = a\vartheta(\vartheta - 1) + b\vartheta + c$ will always provide on $(a, b, c \in \mathbb{R})$ $(c = 0 \text{ and } q \neq 0)$ or $(c > 0 \text{ and } q > 0)$. Moreover, ϑ_0 is the largest real root of the quadratic $Q(\vartheta)$ defined according to the above conditions.
Easily, we can write

$$\tilde{N}_\vartheta(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta + 1)\,Q(\vartheta + 2(n - 1))}{4^{n-1}(n - 1)!\Gamma(\vartheta + n)\,Q(\vartheta)} z^n \quad (z \in \mathcal{U}). \tag{8}$$

In terms of Hadamard product and $\tilde{N}_{\vartheta}(z)$ given by (8), a new operator $\tilde{N}_{\vartheta} : \mathcal{A} \to \mathcal{A}$ can be defined as follows:

$$\tilde{N}_{\vartheta} f(z) = \left(\tilde{N}_{\vartheta} * f\right)(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)\, Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)\, Q(\vartheta)} a_n z^n \quad (z \in \mathcal{U}). \tag{9}$$

If $f \in \mathcal{A}(n)$ is given by (2) then we have

$$\tilde{N}_{\vartheta} f(z) = z - \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)\, Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)\, Q(\vartheta)} a_n z^n \quad (z \in \mathcal{U}). \tag{10}$$

Finally, by using the differential operator defined by (10), we investigate the subclasses $\mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ and $\mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$ of $\mathcal{A}(n)$ consisting of functions f as following:

Definition 1.1. *The subclass $\mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ of $\mathcal{A}(n)$ is defined as the class of functions f such that*

$$\left|\frac{1}{\gamma}\left(\frac{z\left[\tilde{N}_{\vartheta} f(z)\right]'}{\tilde{N}_{\vartheta} f(z)} - 1\right)\right| < \beta \qquad (z \in \mathcal{U}) \tag{11}$$

where $\gamma \in \mathbb{C}\setminus\{0\}$ and $0 \le \beta < 1$.

Definition 1.2. *Let $\mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$ denote the subclass of $\mathcal{A}(n)$ consisting of f which satisfy the inequality*

$$\left|\frac{1}{\gamma}\left[(1-\mu)\frac{\tilde{N}_{\vartheta} f(z)}{z} + \mu\left(\tilde{N}_{\vartheta} f(z)\right)' - 1\right]\right| < \beta \qquad (z \in \mathcal{U}) \tag{12}$$

where $\gamma \in \mathbb{C}\setminus\{0\}$ and $0 \le \beta < 1$, $0 \le \mu \le 1$.

In this paper, we obtain the coefficient inequalities, inclusion relations and neighborhood properties of the subclasses $\mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ and $\mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$.

2. Coefficient inequalities for the classes $\mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ and $\mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$

Theorem 2.1. *Let $f \in \mathcal{A}(n)$. Then $f \in \mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)\, Q(\vartheta)} \left[n-1+\beta\left|\gamma\right|\right] a_n \le \beta\left|\gamma\right| \tag{13}$$

where $\gamma \in \mathbb{C}\setminus\{0\}$ and $0 \le \beta < 1$.

Proof. Let $f \in \mathcal{A}(n)$. Then, by (11) we can write

$$\Re\left\{\frac{z\left[\tilde{N}_{\vartheta} f(z)\right]'}{\tilde{N}_{\vartheta} f(z)} - 1\right\} > -\beta\left|\gamma\right| \qquad (z \in \mathcal{U}). \tag{14}$$

Using (2) and (10), we have,

$$\Re\left\{\frac{-\sum\limits_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)Q(\vartheta)}\left[n-1\right] a_n z^n}{z - \sum\limits_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)Q(\vartheta)} a_n z^n}\right\} > -\beta\left|\gamma\right| \qquad (z \in \mathcal{U}). \tag{15}$$

Since (15) is true for all $z \in \mathcal{U}$, choose values of z on the real axis. Letting $z \to 1$, through the real values, the inequality (15) yields the desired inequality

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)Q(\vartheta)}\left[n-1+\beta\left|\gamma\right|\right]a_n \le \beta\left|\gamma\right|.$$

Conversely, supposed that inequality (13) holds true and $|z| = 1$, we obtain

$$\begin{aligned}
\left|\frac{z\left[\Psi_{\lambda,\mu}f(z)\right]'}{\Psi_{\lambda,\mu}f(z)}-1\right| &\le \left|\frac{\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)Q(\vartheta)}[n-1]a_n z^n}{z-\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)Q(\vartheta)}a_n z^n}\right| \\
&\le \frac{\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)Q(\vartheta)}[n-1]a_n}{1-\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)Q(\vartheta)}a_n} \\
&\le \beta\left|\gamma\right|.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z) \in \mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$, which establishes the required result. □

Theorem 2.2. *Let $f \in \mathcal{A}(n)$. Then $f \in \mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\vartheta+1)Q(\vartheta+2(n-1))}{4^{n-1}(n-1)!\Gamma(\vartheta+n)Q(\vartheta)}\left[1+\mu(n-1)\right]a_n \le \beta\left|\gamma\right| \tag{16}$$

for $\gamma \in \mathbb{C}\setminus\{0\}$, $0 \le \beta < 1$ and $0 \le \mu \le 1$.

Proof. We omit the proofs since it is similar to Theorem 2.1. □

3. Inclusion relations involving $\mathcal{N}_{n,\delta}(e)$ of the classes $\mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ and $\mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$

Theorem 3.1. *If*

$$\delta = \frac{-8\beta\left|\gamma\right|\Gamma(\vartheta+2)Q(\vartheta)}{\left(1+\beta\left|\gamma\right|\right)\Gamma(\vartheta+1)Q(\vartheta+2)} \qquad \left(\left|\gamma\right|<1\right) \tag{17}$$

then $\mathcal{M}_{\vartheta}^{n}(\beta,\gamma) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. Let $f(z) \in \mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$. By Theorem 2.1, we have

$$\frac{-\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)Q(\vartheta)}(1+\beta\left|\gamma\right|)\sum_{n=2}^{\infty}a_n \le \beta\left|\gamma\right|,$$

which implies

$$\sum_{n=2}^{\infty}a_n \le \frac{\beta\left|\gamma\right|}{\frac{-\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)Q(\vartheta)}(1+\beta\left|\gamma\right|)}. \tag{18}$$

Using (13) and (18), we get

$$\begin{aligned}
\frac{-\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)Q(\vartheta)}\sum_{n=2}^{\infty}na_n &\le \beta\left|\gamma\right| + \frac{-\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)Q(\vartheta)}(1-\beta\left|\gamma\right|)\sum_{n=2}^{\infty}a_n \\
&\le \frac{2\beta\left|\gamma\right|}{1+\beta\left|\gamma\right|} = \delta.
\end{aligned}$$

That is,

$$\sum_{n=2}^{\infty} na_n \leq \frac{-8\beta\left|\gamma\right|\Gamma(\vartheta+2)\,Q(\vartheta)}{\left(1+\beta\left|\gamma\right|\right)\Gamma(\vartheta+1)Q(\vartheta+2)} = \delta.$$

Thus, by the definition given by (4), $f(z) \in \mathcal{N}_{n,\delta}(e)$, which completes the proof. □

Theorem 3.2. *If*

$$\delta = \frac{-8\beta\left|\gamma\right|\Gamma(\vartheta+2)\,Q(\vartheta)}{(1+\mu)\,\Gamma(\vartheta+1)Q(\vartheta+2)} \quad \left(\left|\gamma\right| < 1\right) \tag{19}$$

then $\mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. For $f(z) \in \mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$ and making use of the condition (16), we obtain

$$\frac{-\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)\,Q(\vartheta)}(1+\mu)\sum_{n=2}^{\infty} a_n \leq \beta\left|\gamma\right|$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{-4\beta\left|\gamma\right|\Gamma(\vartheta+2)\,Q(\vartheta)}{(1+\mu)\Gamma(\vartheta+1)Q(\vartheta+2)}. \tag{20}$$

Thus, using (16) along with (20), we also get

$$\begin{aligned} -\mu\frac{\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)\,Q(\vartheta)}\sum_{n=2}^{\infty} na_n &\leq \beta\left|\gamma\right| + (1-\mu)\frac{\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)\,Q(\vartheta)}\sum_{n=2}^{\infty} a_n \\ &\leq \beta\left|\gamma\right| + (\mu-1)\frac{\beta\left|\gamma\right|}{1+\mu}. \end{aligned}$$

Hence,

$$\sum_{n=2}^{\infty} na_n \leq \frac{-8\beta\left|\gamma\right|\Gamma(\vartheta+2)\,Q(\vartheta)}{(1+\mu)\,\Gamma(\vartheta+1)Q(\vartheta+2)} = \delta$$

which in view of (4), completes the proof of theorem. □

4. Neighborhood properties for the classes $\mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ and $\mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$

Definition 4.1. *For $0 \leq \eta < 1$ and $z \in \mathcal{U}$, a function $f(z) \in \mathcal{M}_{\lambda,\mu}^{n}(\alpha,\gamma)$ if there exists a function $g(z) \in \mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ such that*

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \eta. \tag{21}$$

For $0 \leq \eta < 1$ and $z \in \mathcal{U}$, a function $f(z) \in \mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$ if there exists a function $g(z) \in \mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$ such that the inequality (21) holds true.

Theorem 4.2. *If $g(z) \in \mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ and*

$$\eta = 1 - \frac{\delta\left(1+\beta\left|\gamma\right|\right)\Gamma(\vartheta+1)Q(\vartheta+2)}{2\left[\left(1+\beta\left|\gamma\right|\right)\Gamma(\vartheta+1)\,Q(\vartheta+2) + 4\beta\left|\gamma\right|\Gamma(\vartheta+2)\,Q(\vartheta)\right]} \tag{22}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$.

Proof. Let $f(z) \in \mathcal{N}_{n,\delta}(g)$. Then,

$$\sum_{n=2}^{\infty} n\,|a_n - b_n| \le \delta, \tag{23}$$

which yields the coefficient inequality,

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{\delta}{2} \qquad (n \in \mathbb{N}).$$

Since $g(z) \in \mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ by (18), we have

$$\sum_{n=2}^{\infty} b_n \le \frac{-4\beta\left|\gamma\right|\Gamma(\vartheta+2)\,Q(\vartheta)}{(1+\beta\left|\gamma\right|)\Gamma(\vartheta+1)Q(\vartheta+2)}, \tag{24}$$

and so

$$\begin{aligned}
\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum\limits_{n=2}^{\infty}|a_n - b_n|}{1 - \sum\limits_{n=2}^{\infty} b_n} \\
&\le \frac{\delta}{2}\,\frac{\frac{\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)Q(\vartheta)}(1+\beta\left|\gamma\right|)}{\frac{\Gamma(\vartheta+1)Q(\vartheta+2)}{4\Gamma(\vartheta+2)Q(\vartheta)}(1+\beta\left|\gamma\right|)+\beta\left|\gamma\right|} \\
&= 1-\eta.
\end{aligned}$$

Thus, by definition, $f(z) \in \mathcal{M}_{\vartheta}^{n}(\beta,\gamma)$ for η given by (22), which establishes the desired result. □

Theorem 4.3. *If $g(z) \in \mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$ and*

$$\eta = 1 - \frac{\delta\,(1+\mu)\,\Gamma(\vartheta+1)Q(\vartheta+2)}{2\left[(1+\mu)\,\Gamma(\vartheta+1)\,Q(\vartheta+2) + 4\beta\left|\gamma\right|\Gamma(\vartheta+2)\,Q(\vartheta)\right]} \tag{25}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{R}_{\vartheta}^{n}(\beta,\gamma,\mu)$.

Proof. We omit the proofs since it is similar to Theorem 4.2. □

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Weak Structures on Pythagorean Fuzzy Soft Topological Spaces

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Abstract. In this paper, we initiate the topological structures of pythagorean fuzzy soft semi-open sets and pythagorean fuzzy soft semi-closed sets. Furthermore, the concept of pythagorean fuzzy soft semi-interior, pythagorean fuzzy soft semi-closure are presented. Also some related properties are investigated.

1. Introduction

Molodtsov [20] has presented soft-set theory as a new mathematical method for working with complexity, imprecise and uncertainly defined objects, and overcoming incompatibility with parameterization methods, where theories such as fuzzy sets, intuitionistic fuzzy sets theory, rough set theory fall short. The soft set theory proved useful in a number of areas, not restricted to decision-making [8, 26], data analysis [6, 34], forecasting [29] and so on. Topological structures for soft sets are studied and explored in [1, 2, 10, 11]. In [21] Molodtsov et al. listed a variety of directions for the implementation of soft sets, such as smoothness of functions, game theory, operational analysis, Riemann integration, Perron integration, probability and calculation theory for modeling problems in architecture, computer science, economics, social sciences.

The concept of fuzzy sets was initiated by Zadeh [33]. Intuitionistic fuzzy set (IFS) and intuitionistic L-fuzzy sets (ILFS) were initiated and discussed by Atanassov [3] to generalize the idea of fuzzy set. Maji et al. developed the idea of intuitionistic fuzzy soft sets [18], a generalization of fuzzy soft sets [17] and standard soft sets[19]. Coker [7] has presented and researched the concept of intuitionistic fuzzy topological spaces. Li et al. [16] initiated intuitionistic fuzzy topological constructs of intuitionistic fuzzy soft sets. They discussed the notions of intuitionistic fuzzy soft open (closed) sets, intuitionistic fuzzy soft interior (closure) and intuitionistic fuzzy soft base in intuitionistic fuzzy soft topological spaces. Recently, Hussain [12] initiated the idea of an intuitionistic fuzzy soft boundary and discussed the features and properties of the intuitionistic fuzzy soft boundary in general as well as the intuitionistic fuzzy soft interior and intuitionistic fuzzy soft closure. He also studied some weak structures on intuitionistic fuzzy soft topological spaces [13].

Yager [30, 31] introduced Pythagorean fuzzy set(PFS) as an expansion of Atanassov's intuitionistic fuzzy set and provided Pythagorean membership ratings for multi-criteria decision-making (MCDM) implementations. The main features of the PFS are that the sum of the membership degree and non-membership squares for each alternative is less than or equal to 1. Obviously, PFSs have more power than IFSs to model

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the vagueness of realistic MCDM issues. The Pythagorean fuzzy soft set theory was defined by Peng et al.[24], and its significant properties were studied. Pythagorean fuzzy topology introduced by Olgun et al [22]. Also Riaz et al. [27] and Yolcu and Ozturk [32] studied on Pythagorean fuzzy soft topological spaces. Pythagorean fuzzy set theory is one of the most studied topics of recent times [4, 5, 9, 14, 23, 25, 28].

In this paper, we initiate and define the topological structures of pythagorean fuzzy soft semi-open sets and pythagorean fuzzy, soft semi-closed sets. We also investigate the properties of pythagorean fuzzy soft semi-interior, pythagorean fuzzy soft semi-closure, and discuss the relationship between them.

2. Preliminaries

Definition 2.1. *[33] Let X be an universe. A fuzzy set F in X,* $F = \{(x, \mu_F(x)) : x \in X\}$, *where* $\mu_F : X \to [0, 1]$ *is the membership function of the fuzzy set F;* $\mu_F(x) \in [0, 1]$ *is the membership of* $x \in X$ *in f. The set of all fuzzy sets over X will be denoted by* $FS(X)$.

Definition 2.2. *[3] An intuitionistic fuzzy set F in X is* $F = \{(x, \mu_F(x), v_F(x)) : x \in X\}$, *where* $\mu_F : X \to [0, 1]$, $v_F : X \to [0, 1]$ *with the condition* $0 \leq \mu_F(x) + v_F(x) \leq 1$, $\forall x \in X$. *The numbers* $\mu_F, v_F \in [0, 1]$ *denote the degree of membership and non-membership of x to F, respectively. The set of all intuitionistic fuzzy sets over X will be denoted by* $IFS(X)$.

Definition 2.3. *[20] Let E be a set of parameters and X be the universal set. A pair* (F, E) *is called a soft set over X, where F is a mapping* $F : E \to \mathcal{P}(X)$. *In other words, the soft set is a parameterized family of subsets of the set X.*

Definition 2.4. *[17] Let E be a set of parameters and X be the universal set. A pair* (F, E) *is called a fuzzy soft set over X, If* $F : E \to FS(X)$ *is a mapping from E into set of all fuzzy sets in X, where* $FS(X)$ *is set of all fuzzy subset of X.*

Definition 2.5. *[18] Let X be an initial universe E be a set of parameters. A pair* (F, E) *is called an intuitionistic fuzzy soft set over X, where F is a mapping given by,* $F : E \to IFS(X)$.

In general, for every $e \in E$, $F(e)$ *is an intuitionistic fuzzy set of X and it is called intuitionistic fuzzy value set of parameter e. Clearly,* $F(e)$ *can be written as a intuitionistic fuzzy set such that* $F(e) = \{(x, \mu_F(x), v_F(x)) : x \in X\}$

Definition 2.6. *[30] Let X be a universe of discourse. A pythagorean fuzzy set (PFS) in X is given by,* $P = \{(x, \mu_P(x), v_P(x)) : x \in X\}$ *where,* $\mu_P : X \to [0, 1]$ *denotes the degree of membership and* $v_p : X \to [0, 1]$ *denotes the degree of nonmembership of the element* $x \in X$ *to the set P with the condition that* $0 \leq (\mu_P(x))^2 + (v_P(x))^2 \leq 1$.

Definition 2.7. *[24] Let X be the universal set and E be a set of parameters. The pythagorean fuzzy soft set is defined as the pair* (F, E) *where,* $F : E \to PFS(X)$ *and* $PFS(X)$ *is the set of all Pythagorean fuzzy subsets of X. If* $\mu_F^2(x) + v_F^2(x) \leq 1$ *and* $\mu_F(x) + v_F(x) \leq 1$, *then pythagorean fuzzy soft sets degenerate into intuitionistic fuzzy soft sets.*

Definition 2.8. *[24] Let* $A, B \subseteq E$ *and* (F, A), (G, B) *be two pythagorean fuzzy soft sets over X.* (F, A) *is said to be pythagorean fuzzy soft subset of* (G, B) *denoted by* $(F, A)\widetilde{\subseteq}(G, B)$ *if,*

1. $A \subseteq B$
2. $\forall e \in A$, *F(e) is a pythagorean fuzzy subset of G(e) that is,* $\forall x \in U$ *and* $\forall e \in A$, $\mu_{F(e)}(x) \leq \mu_{G(e)}(x)$ *and* $v_{F(e)}(x) \geq v_{G(e)}(x)$. *If* $(F, A)\widetilde{\subseteq}(G, B)$ *and* $(G, B)\widetilde{\subseteq}(F, A)$ *then* (F, A), (G, B) *are said to be equal.*

Definition 2.9. *[24] Let* (F, E) *two pythagorean fuzzy soft sets over X. The complement of* (F, E) *is denoted by* $(F, E)^c$ *and is defined by*

$$(F, E)^c = \{(e, (x, v_{F(e)}(x), \mu_{F(e)}(x)) : x \in X) : e \in E\}$$

Definition 2.10. *[15] a) A pythagorean fuzzy soft set (F,E) over the universe X is said to be null pythagorean fuzzy soft set if $\mu_{F(e)}(x) = 0$ and $v_{F(e)}(x) = 1$; $\forall e \in E$, $\forall x \in X$. It is denoted by $\widetilde{0}_{(X,E)}$.*

b) A pythagorean fuzzy soft set (F,E) over the universe X is said to be absolute pythagorean fuzzy soft set if $\mu_{F(e)}(x) = 1$ and $v_{F(e)}(x) = 0$; $\forall e \in E$, $\forall x \in X$. It is denoted by $\widetilde{1}_{(X,E)}$.

Definition 2.11. *[15] Let (F,A) and (G,B) be two pythagorean fuzzy soft sets over the universe set X, E be a parameter set and $A, B \subseteq E$.Then,*

a) Extended union of (F,A) and (G,B) is denoted by $(F,E)\widetilde{\cup}_E(G,B) = (H,C)$ where $C = A \cup B$ and (H,C) defined by;

$$(H,C) = \{(e,(x,\mu_{H(e)}(x), v_{H(e)}(x)) : x \in X) : e \in E\}$$

where

$$\mu_{H(e)}(x) = \begin{cases} \mu_{F(e)}(x), \text{ if } e \in A - B \\ \mu_{G(e)}(x), \text{ if } e \in B - A \\ \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \text{ if } e \in A \cap B \end{cases}$$

$$v_{H(e)}(x) = \begin{cases} v_{F(e)}(x), \text{ if } e \in A - B \\ v_{G(e)}(x), \text{ if } e \in B - A \\ \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \text{ if } e \in A \cap B \end{cases}$$

b) Extended intersection of (F,A) and (G,B) is denoted by $(F,E)\widetilde{\cap}_E(G,B) = (H,C)$ where $C = A \cup B$ and (H,C) defined by;

$$(H,C) = \{(e,(x,\mu_{H(e)}(x), v_{H(e)}(x)) : x \in X) : e \in E\}$$

where

$$\mu_{H(e)}(x) = \begin{cases} \mu_{F(e)}(x), \text{ if } e \in A - B \\ \mu_{G(e)}(x), \text{ if } e \in B - A \\ \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \text{ if } e \in A \cap B \end{cases}$$

$$v_{H(e)}(x) = \begin{cases} v_{F(e)}(x), \text{ if } e \in A - B \\ v_{G(e)}(x), \text{ if } e \in B - A \\ \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \text{ if } e \in A \cap B \end{cases}$$

Let X be an initial universe and $PFS(X)$ denote the family of pythagorean fuzzy sets over X and $PFSS(X,E)$ be the family of all pythagorean fuzzy soft sets over X with parameters in E.

Definition 2.12. *[32]Let $X \neq \emptyset$ be a universe set and $\widetilde{\tau} \subset PFSS(X,E)$ be a collection of pythagorean fuzzy soft sets over X, then τ is said to be on pythagorean fuzzy soft topology on X if*

(i) $\widetilde{0}_{(X,E)}$, $\widetilde{1}_{(X,E)}$ belong to $\widetilde{\tau}$,
(ii) The union of any number of pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,
(iii) The intersection of any two pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triple $(X,\widetilde{\tau},E)_p$ is called an pythagorean fuzzy soft tpological space over X. Every member of τ is called a pythagorean fuzzy soft open set in X.

Definition 2.13. *[32] a) Let X be an initial universe set, E be the set of parameters and $\widetilde{\tau} = \left\{\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)}\right\}$. Then $\widetilde{\tau}$ is called a pythagorean fuzzy soft indiscrete topology on X and $(X,\widetilde{\tau},E)_p$ is said to be a pythagorean fuzzy soft indiscrete space over X.*

b) Let X be an initial universe set, E be the set of parameters and $\widetilde{\tau}$ be the collection of all pythagorean fuzzy soft sets which can be defined over X. Then $\widetilde{\tau}$ is called a pythagorean fuzzy soft discrete topology on X and $(X,\widetilde{\tau},E)_p$ is said to be a pythagorean fuzzy soft discrete space over X.

Definition 2.14. *[32] Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X. A pythagorean fuzzy soft set (F,E) over X is said to be a pythagorean fuzzy soft closed set in X, if its complement $(F,E)^c$ belongs to $\widetilde{\tau}$.*

Proposition 2.15. *[32]Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X. Then, the following properties hold.*

(i) $\widetilde{0}_{(X,E)},\widetilde{1}_{(X,E)}$ are pythagorean fuzzy soft closed set over X.

(ii) The intersection of any number of pythagorean fuzzy soft closed set is a pythagorean fuzzy soft closed set over X.

(iii) The union of any two pythagorean fuzzy soft closed set is a pythagorean fuzzy soft closed set over X.

Definition 2.16. *[32]Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and (F,E) be a pythagorean fuzzy soft sets over X. The pythagorean fuzzy soft closure of (F,E) denoted by $pcl(F,E)$ is the intersection of all pythagorean fuzzy soft closed super sets of (F,E).*

Clearly $pcl(F,E)$ is the smallest pythagorean fuzzy soft closed set over X which contain (F,E).

Definition 2.17. *[32]Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(H,E)\in PFSS(X,E)$. The pythagorean fuzzy soft interior of (H,E), denoted by $pint(H,E)$, is the union of all the pythagorean fuzzy soft open sets contained in (H,E).*

3. Main Results

Definition 3.1. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\subset}PFSS(X,E)$. If there exists a pythagorean fuzzy soft open set (G,E) such that $(G,E)\widetilde{\subset}(F,E)\widetilde{\subset}pcl(G,E)$, then (F,E) is called pythagorean fuzzy soft semi-open set over X.*

Definition 3.2. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\subset}PFSS(X,E)$. (F,E) is pythagorean fuzzy soft semi-closed set if and only if its complement $(F,E)^c$ is pythagorean fuzzy soft semi-open set.*

Remark 3.3. *It is obvious that a pythagorean fuzzy soft open set is pythagorean fuzzy soft semi-open set. But the converse is not true in general. This is shown in following example.*

Example 3.4. *Let $X=\{x_1,x_2\}$, $E=\{e_1,e_2\}$ and*

$$\widetilde{\tau}=\left\{\widetilde{0}_{(X,E)},\widetilde{1}_{(X,E)},(F_1,E),(F_2,E),(F_3,E)\right\}$$

where $(F_1,E),(F_2,E),(F_3,E)$ pythagorean fuzzy soft sets over X, defined as;

$$\begin{aligned}
(F_1,E) &= \left\{\begin{array}{l}(e_1,\{(x_1,0.4,0.6),(x_2,0.3,0.7)\})\\(e_2,\{(x_1,0.5,0.4),(x_2,0.7,0.6)\})\end{array}\right\}\\
(F_2,E) &= \left\{\begin{array}{l}(e_1,\{(x_1,0.5,0.5),(x_2,0.4,0.5)\})\\(e_2,\{(x_1,0.6,0.4),(x_2,0.8,0.3)\})\end{array}\right\}\\
(F_3,E) &= \left\{\begin{array}{l}(e_1,\{(x_1,0.7,0.3),(x_2,0.7,0.2)\})\\(e_2,\{(x_1,0.8,0.2),(x_2,0.9,0.2)\})\end{array}\right\}
\end{aligned}$$

It is clear that $\widetilde{\tau}$ is a pythagorean fuzzy soft topological spaces and $(X,\widetilde{\tau},E)_p$ is pythagorean fuzzy soft topological spaces. The pythagorean fuzzy soft closed sets as follow;

$$\begin{aligned}
\left(\widetilde{0}_{(X,E)}\right)^c &= \widetilde{1}_{(X,E)} \\
\left(\widetilde{1}_{(X,E)}\right)^c &= \widetilde{0}_{(X,E)} \\
(F_1,E)^c &= \left\{ \begin{array}{l} (e_1,\{(x_1,0.6,0.4),(x_2,0.7,0.3)\}) \\ (e_2,\{(x_1,0.4,0.5),(x_2,0.6,0.7)\}) \end{array} \right\} \\
(F_2,E)^c &= \left\{ \begin{array}{l} (e_1,\{(x_1,0.5,0.5),(x_2,0.5,0.4)\}) \\ (e_2,\{(x_1,0.4,0.6),(x_2,0.3,0.8)\}) \end{array} \right\} \\
(F_3,E)^c &= \left\{ \begin{array}{l} (e_1,\{(x_1,0.3,0.7),(x_2,0.3,0.7)\}) \\ (e_2,\{(x_1,0.3,0.8),(x_2,0.2,0.9)\}) \end{array} \right\}
\end{aligned}$$

Now we consider a pythagorean fuzzy soft set (G,E) over X defined by,

$$(G,E) = \left\{ \begin{array}{l} (e_1,\{(x_1,0.6,0.4),(x_2,0.7,0.3)\}) \\ (e_2,\{(x_1,0.7,0.3),(x_2,0.8,0.2)\}) \end{array} \right\}$$

Then there exist a pythagorean fuzzy open set (F_2,E) such that $(F_2,E)\widetilde{\subset}(G,E)\widetilde{\subset}pcl(F_2,E) = \widetilde{1}_{(X,E)}$.

Hence (G,E) is a pythagorean fuzzy soft semi-open set, but (G,E) is not pythagorean fuzzy soft open set.

Proposition 3.5. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\subset}PFSS(X,E)$. Then (F,E) is pythagorean fuzzy soft semi-open set if and only if $(F,E)\widetilde{\subset}pcl\,(pint(F,E))$.*

Proof. ($\Rightarrow$)Suppose that (F,E) is pythagorean fuzzy soft semi-open set, then there exists a pythagorean fuzzy soft open set (G,E) such that $(G,E)\widetilde{\subset}(F,E)\widetilde{\subset}pcl(G,E)$. Now $(G,E)\widetilde{\subset}pint(F,E)$ implies that $pcl(G,E)\widetilde{\subset}pcl\,(pint(F,E))$. Therefore (

$$F,E)\widetilde{\subset}pcl(G,E)\widetilde{\subset}pcl\,(pint(F,E))$$

.

($\Leftarrow$) Suppose that $(F,E)\widetilde{\subset}pcl\,(pint(F,E))$. Take $(G,E) = pint(F,E)$, we have

$$(G,E)\widetilde{\subset}(F,E)\widetilde{\subset}pcl(G,E)$$

This complete this proof. □

Theorem 3.6. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X . Then an arbitrary union of pythagorean fuzzy soft semi-open sets is pythagorean fuzzy soft semi-open set.*

Proof. Let $\{(F_i,E) : i \in I\}$ be a collection of pythagorean fuzzy soft semi-open sets and $(G,E) = \underset{i\in I}{\cup}(F_i,E)$. Since each (F_i,E) is PFS semi-open, then there exist a pythagorean fuzzy soft open set (H_i,E) such that $(H_i,E)\widetilde{\subset}(F_i,E)\widetilde{\subset}pcl(H_i,E)$ and so $\underset{i\in I}{\cup}(H_i,E)\widetilde{\subset}\underset{i\in I}{\cup}(F_i,E)\widetilde{\subset}\underset{i\in I}{\cup}pcl(H_i,E)\widetilde{\subset}pcl\left(\underset{i\in I}{\cup}(H_i,E)\right)$. Let $(H,E) = \underset{i\in I}{\cup}(H_i,E)$. Then (H,E) is pythagorean fuzzy soft open and $(H,E)\widetilde{\subset}\underset{i\in I}{\cup}(F_i,E)\widetilde{\subset}pcl(H,E)$. Therefore, $\underset{i\in I}{\cup}(F_i,E)$ is a pythagorean fuzzy soft semi-open set. □

Corollary 3.7. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X . Then the family of all pythagorean fuzzy soft semi-open sets are a pythagorean fuzzy soft supra topology on X.*

Proposition 3.8. *Let (F,E) be a pythagorean fuzzy soft semi-open set and (G,E) be a pythagorean fuzzy soft set in $(X,\widetilde{\tau},E)_p$. Suppose $(F,E)\widetilde{\subset}(G,E)\widetilde{\subset}pcl(F,E)$. Then (G,E) is a pythagorean fuzzy soft semi-open set over X.*

Proof. (F,E) be a pythagorean fuzzy soft semi-open set implies that there exist a pythagorean fuzzy soft open set (H,E) such that $(H,E)\widetilde{\subset}(F,E)\widetilde{\subset}pcl(H,E)$. Now $(H,E)\widetilde{\subset}(G,E)$ and $pcl(F,E)\widetilde{\subset}pcl(H,E)$ implies that $(G,E)\widetilde{\subset}pcl(H,E)$. Therefore $(H,E)\widetilde{\subset}(G,E)\widetilde{\subset}pcl(H,E)$. Hence (G,E) is a pythagorean fuzzy soft semi-open set in X. □

Proposition 3.9. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset}PFSS(X, E)$. Then (F, E) is pythagorean fuzzy soft semi-closed if and only if there exist a pythagorean fuzzy soft closed set (G, E) such that $pint(G, E)\widetilde{\subset}(F, E)\widetilde{\subset}(G, E)$.*

Proof. This proof is clear that from the definition of pythagorean fuzzy soft semi-closed set. □

Proposition 3.10. *Every pythagorean fuzzy soft closed set is pythagorean fuzzy soft semi-closed set in a pythagorean fuzzy soft topological spaces $(X, \widetilde{\tau}, E)_p$.*

Proof. Straightforward. □

Remark 3.11. *The converse of Proposition 3.10 may not be provide in general. It is shown in following example.*

Example 3.12. *Consider the Example 3.4.*

$$(G, E)^c = \left\{ \begin{array}{l} (e_1, \{(x_1, 0.4, 0.6), (x_2, 0.3, 0.7)\}) \\ (e_2, \{(x_1, 0.3, 0.7), (x_2, 0.2, 0.8)\}) \end{array} \right\}$$

is pythagorean fuzzy soft semi-closed set. But $(G, E)^c$ is not pythagorean fuzzy soft closed set.

Theorem 3.13. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E) \in PFSS(X, E)$. Then (F, E) is pythagorean fuzzy soft semi-closed set if and only if $pint\,(pcl(F, E))\,\widetilde{\subset}(F, E)$.*

Proof. ($\Rightarrow$) Suppose that (F, E) is a pythagorean fuzzy soft closed set, then by Proposition 3.9, there exists a pythagorean fuzzy soft closed set (G, E) such that $pint(G, E)\widetilde{\subset}(F, E)\widetilde{\subset}(G, E)$. This follows that $pcl(F, E)\widetilde{\subset}pcl(G, E) = (G, E)$. Thus $pint\,(pcl(F, E))\,\widetilde{\subset}pint(G, E)$. Therefore, $pint\,(pcl(F, E))\,\widetilde{\subset}pint(G, E)\widetilde{\subset}(F, E)$.

($\Leftarrow$) Suppose that (F, E) be a pythagorean fuzzy soft set in $(X, \widetilde{\tau}, E)_p$ such that $pint\,(pcl(F, E))\,\widetilde{\subset}(F, E)$. We take $pcl(F, E) = (G, E)$. Then $pint(G, E)\widetilde{\subset}(F, E)\widetilde{\subset}(G, E)$. This implies that (F, E) is a pythagorean fuzzy soft semi-closed set. □

Theorem 3.14. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X . Then an arbitrary intersection of pythagorean fuzzy soft semi-closed sets is pythagorean fuzzy soft semi-closed set.*

Proof. Suppose that $\{(F_i, E) : i \in I\}$ be a collection of pythagorean fuzzy soft semi-closed sets. Since each $i \in I$, (F_i, E) is a pythagorean fuzzy soft semi-closed set, then by Proposition 3.9, there exist a pythagorean fuzzy soft closed set (G_i, E) such that $pint(G_i, E)\widetilde{\subset}(F_i, E)\widetilde{\subset}(G_i, E)$. This follows that $\bigcap_{i\in I}(pint(G_i, E))\,\widetilde{\subset} \bigcap_{i\in I}(F_i, E)\widetilde{\subset} \bigcap_{i\in I}(G_i, E)$. We take $\bigcap_{i\in I}(G_i, E) = (G, E)$. Then by Theorem 2.15, (G, E) is a pythagorean fuzzy soft closed set and hence $\bigcap_{i\in I}(F_i, E)$ is a pytahgorean fuzzy soft semi-closed set. □

Theorem 3.15. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X, (F, E) be a pythagoren fuzzy soft semi-closed set and (G, E) be a pythagorean fuzzy soft set over X. If $pint(F, E)\widetilde{\subset}(G, E)\widetilde{\subset}(F, E)$, then (G, E) is a pythagorean fuzzy soft semi-closed set.*

Proof. Since (F, E) is a pythagorean fuzzy soft semi-closed set, then by Prosoposition 3.9, tehere exists an pythagorean fuzzy soft closed set (H, E) such that $pint(H, E)\widetilde{\subset}(F, E)\widetilde{\subset}(H, E)$. Then $(G, E)\widetilde{\subset}(H, E)$. Also $pint\,(pint(H, E)) = pint(H, E)\widetilde{\subset}pint(F, E)$. This implies that $pint(H, E)\widetilde{\subset}(G, E)$.

Therefore, $pint(H, E)\widetilde{\subset}(G, E)\widetilde{\subset}(H, E)$. Hence (G, E) is a pythagorean fuzzy soft semi-closed set. □

Definition 3.16. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E) \in PFSS(X, E)$.*

1. *The pythagorean fuzzy soft semi-interior of (F, E), denoted by $spint(F, E)$, is the union of all the pythagorean fuzzy soft semi-open sets contained in (F, E).*
   Clearly, $spint(F, E)$ is the largest pythagorean fuzzy soft semi-open set over X contained in (F, E).

2. *The pythagorean fuzzy soft semi-closure of* (F,E)*, denoted by* $spcl(F,E)$*, is the intersection of all the pythagorean fuzzy soft semi-closed sets contains* (F,E)*.*
   Clearly, $spcl(F,E)$ *is the smallest pythagorean fuzzy soft semi-closed set over* X *which contains* (F,E)*.*

Remark 3.17. *It is clear that, If* (F,E) *be a pythagorean fuzzy soft set, then*

$$pint(F,E)\widetilde{\subset}spint(F,E)\widetilde{\subset}(F,E)\widetilde{\subset}spcl(F,E)\widetilde{\subset}pcl(F,E)$$

.

Theorem 3.18. *Let* $(X,\widetilde{\tau},E)_p$ *be a pythagorean fuzzy soft topological space over* X *and* $(F,E),(G,E)\in PFSS(X,E)$*.Then,*

1. $spint(\widetilde{0}_{(X,E)}) = spcl(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(X,E)}$ *and* $spint(\widetilde{1}_{(X,E)}) = spcl(\widetilde{1}_{(X,E)}) = \widetilde{1}_{(X,E)}$,
2. (F,E) *is a pythagorean fuzzy soft semi-open(semi-closed) set if and only if* $spint(F,E) = (F,E)$ $(spcl(F,E) = (F,E))$*.*
3. $spint\left(spint(F,E)\right) = (F,E)$.
4. $(F,E)\widetilde{\subset}(G,E)$ *implies* $spint(F,E)\widetilde{\subset}spint(G,E)$ *and* $spcl(F,E)\widetilde{\subset}spcl(G,E)$,
5. *(i)* $spint(F,E)\widetilde{\cap}_E spint(G,E) = spint\Big((F,E)\widetilde{\cap}_E(G,E)\Big)$
   (ii) $spcl(F,E)\widetilde{\cap}_E spcl(G,E)\widetilde{\supseteq}spcl\Big((F,E)\widetilde{\cap}_E(G,E)\Big)$
6. $spint(F,E)\widetilde{\cup}_E spint(G,E)\widetilde{\subseteq}spint\Big((F,E)\widetilde{\cup}_E(G,E)\Big)$
   $spcl(F,E)\widetilde{\cup}_E spcl(G,E) = spcl\Big((F,E)\widetilde{\cup}_E(G,E)\Big)$

Proof. *(1)-(4) follow directly from the definition of pythagorean fuzzy soft semi-interior and pythagorean fuzzy soft semi-closure .*

(5) (i) By (4), we have $\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}(F,E)$, $\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}(G,E)$ *implies*

$$spint\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}spint(F,E),\ spint\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}spint(G,E),$$

so that $spint\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}spint(F,E)\widetilde{\cap}_E spint(G,E)$*. Also, since* $spint(F,E)\widetilde{\subseteq}(F,E)$ *and* $spint(G,E)\widetilde{\subseteq}(G,E)$ *implies* $spint(F,E)\widetilde{\cap}_E spint(G,E)\widetilde{\subseteq}\Big((F,E)\widetilde{\cap}_E(G,E)\Big)$.

Thus $spint(F,E)\widetilde{\cap}_E spint(G,E)$ *is a pythagorean fuzzy soft semi-open subsets of* $\Big((F,E)\widetilde{\cap}_E(G,E)\Big)$.
Hence $spint(F,E)\widetilde{\cap}_E spint(G,E)\widetilde{\subseteq}spint\Big((F,E)\widetilde{\cap}_E(G,E)\Big)$*. Thus* $spint(F,E)\widetilde{\cap}_E spint(G,E) = \Big((F,E)\widetilde{\cap}_E(G,E)\Big)$.

(ii) By (4), we have $\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}(F,E)$, $\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}(G,E)$ *implies*

$$spcl\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}spcl(F,E),\ spcl\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}spcl(G,E),$$

so that $spcl\Big((F,E)\widetilde{\cap}_E(G,E)\Big)\widetilde{\subseteq}spcl(F,E)\widetilde{\cap}_E spcl(G,E)$.

(6) The proof is similar to (5) by using property that $(F,E)\widetilde{\subseteq}\Big((F,E)\widetilde{\cup}_E(G,E)\Big)$, $(G,E)\widetilde{\subseteq}\Big((F,E)\widetilde{\cup}_E(G,E)\Big)$. □

Theorem 3.19. *Let* $(X,\widetilde{\tau},E)_p$ *be a pythagorean fuzzy soft topological space over* X *and* $(F,E)\in PFSS(X,E)$*.Then,*

1. $\left(spint(F,E)\right)^c = spcl\left((F,E)^c\right)$.
2. $\left(pcl(F,E)\right)^c = spint\left((F,E)^c\right)$.
3. $spint\left(pint(F,E)\right) = pint\left(spint(F,E)\right) = pint(F,E)$.
4. $spcl\left(pcl(F,E)\right) = pcl\left(spcl(F,E)\right) = pcl(F,E)$.

Proof. (1) $spint(F,E)\widetilde{\subseteq}(F,E)$ implies that $(F,E)^c\widetilde{\subseteq}\left(spint(F,E)\right)^c$. Now by Theorem 3.18 (2), and since $\left(spint(F,E)\right)^c$ is a pythagorean fuzzy soft semi-closed set, we have $spcl\left((F,E)^c\right)\widetilde{\subseteq}spcl\left(\left(spint(F,E)\right)^c\right) = \left(spint(F,E)\right)^c$. For the reverse inclusion, $(F,E)^c\widetilde{\subseteq}spcl\left((F,E)^c\right)$ implies that $\left(spcl\left((F,E)^c\right)\right)^c\widetilde{\subseteq}\left((F,E)^c\right)^c = (F,E)$. $spcl\left((F,E)^c\right)$ being pythagorean fuzzy soft semi-closed implies that $(spcl\,((F,E)^c))^c$ is pythagorean fuzzy soft semi-open. Thus $(spcl\,((F,E)^c))^c\,\widetilde{\subseteq}\,spint(F,E)$ and hence $(spint(F,E))^c\,\widetilde{\subseteq}\,\left((spcl\,((F,E)^c))^c\right)^c = spcl\,((F,E)^c)$.

(2) It is similar to (1).

(3) By Remark 3.3, $pint(F,E)$ is a pythagorean fuzzy soft open set implies that it is pythagorean fuzzy soft semi-open set. Therefore, by Theorem 3.18(2), $spint\,(pint(F,E)) = pint(F,E)$. $pint(F,E)\widetilde{\subseteq}spint(F,E) = (F,E)$. This implies that $spint(pint(F,E)) = pint(F,E)$.

(4) $pcl(F,E)$ is pythagorean fuzzy soft closed set implies that it is pythagorean fuzzy soft semi-closed. Therefore $spcl\,(pcl(F,E)) = pcl(F,E)$. Then $(F,E)\widetilde{\subseteq}spcl(F,E)\widetilde{\subseteq}pcl(F,E)$. Hence $spcl(F,E)\widetilde{\subseteq}pcl\,(spcl(F,E))\,\widetilde{\subseteq}spcl(F,E)$. This implies that $pcl\,(spcl(F,E)) = pcl(F,E)$. □

4. Conclusion

In this study,we presented topological structures of pythagorean fuzzy soft semi-open and pythagorean fuzzy soft semi-closed sets. We also investigated and explored some properties of pythagorean fuzzy soft semi-interior and pythagorean fuzzy soft semi-closure and discussed relationship between them. We hope that the findings in this paper will enhance and promote the further study in the pythagorean fuzzy soft set theory.

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On Pythagorean Fuzzy Soft Boundary

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Abstract. The aim of this paper is to initiate the concept of pythagorean fuzzy soft (PFS) boundary. The characterizations and properties of PFS boundary are discussed and investigated in general as well as in terms of PFS interior and PFS closure.

1. Introduction

Many complicated ideas in the fields of economics, architecture, management , medical research, etc. require unknown data. These problems, which we face in our day-to-day lives, can not be solved by classical mathematical methods due to a large number of uncertainties. Decision-making is a vital activity for all those professions where professionals apply their expertise of a particular field to take effective decisions. However, owing to the various pressures of day-to-day life, decision-makers can not be able to offer their decisions in precisely crisp shape. Thus, in order to deal with it, they tend to use the fuzzy set theory [26] to offer their preferences under the ambiguous and imprecise existence. In this theory, the calculation of each element is achieved with the aid of the degree of membership. However, with increasing uncertainty, there is often a degree of hesitancy between the priorities of decision-making and thus the study performed in those conditions is not optimal. To fix this, the essential extension of the fuzzy set theory named as intuitionistic fuzzy set (IFS) proposed by Atanassov [1] inserted the degree of non-membership v in the analysis along with the degree of membership μ in such a way that $\mu + v \leq 1$. D. Coker [5] has developed and studied the concept of IF topological spaces and Hussain [8] studied intutionistic fuzzy soft boundary.

Intuitionistic fuzzy set theory is based on a limitation on decision-makers that they have assigned their desires only to the setting where the $\mu + v \leq 1$ limit is reached. However, if an expert gives 0.8 as membership and 0.3 as non-membership to an object, then it is obvious that $0.8 + 0.3 \nleq 1$ and therefore the above intuitionistic fuzzy set theory can not solve these problems. To overcome these problems, Yager [23, 24] introduced the concept of Pythagorean Fuzzy sets which is a generalization of intuitionistic fuzzy sets, by relaxing the condition $\mu + v \leq 1$ to $\mu^2 + v^2 \leq 1$. Thus, the pythagorean fuzzy sets treats far more information than the intuitionistic fuzzy sets. After that, some different studies are investigated using aggregation operators of pythagorean fuzzy sets.

In 1999, Molodtsov [13] introduced soft sets to address the lack of a parametrization tool when handling vagueness. Soft set theory is one of the most popular theories of recent times. Therefore, many researchers have made successful studies on soft set structure [16–19]. A soft set is a parameterized family of sets which

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has been extended into different hybrid structures such as fuzzy soft sets [11], intuitionistic fuzzy soft sets [12] and Pythagorean fuzzy soft sets [20]. Since the Pythagorean fuzzy set is extremely capable of dealing with vagueness or ambiguity, the parameterized Pythagorean fuzzy set family, which is Pythagorean fuzzy soft set, can also perform well. Recently, many studies on pythagorean fuzzy theory and pythagorean fuzzy soft theory have been conducted by researchers [2–4, 6, 7, 9, 14, 22]. Pythagorean fuzzy topological structure introduced by Olgun et al [15]. Also, Riaz et al. [21], Yolcu and Ozturk [25] studied Pythagorean fuzzy soft topological spaces.

In this paper, we initiate the concept of pythagorean fuzzy soft boundary. We discuss and explore the characterizations and properties of pythagorean fuzzy soft boundary in general as well as in terms of pythagorean fuzzy soft interior and pythagorean fuzzy soft closure. Examples and counterexamples are also presented to validate the discussed results.

2. Preliminaries

Definition 2.1. *[26] Let X be a universe. A fuzzy set F in X, $F = \{(x, \mu_F(x)) : x \in X\}$, where $\mu_F : X \to [0,1]$ is the membership function of the fuzzy set F; $\mu_F(x) \in [0,1]$ is the membership of $x \in X$ in f. The set of all fuzzy sets over X will be denoted by $FS(X)$.*

Definition 2.2. *[1] An intuitionistic fuzzy set F in X is $F = \{(x, \mu_F(x), v_F(x)) : x \in X\}$, where $\mu_F : X \to [0,1]$, $v_F : X \to [0,1]$ with the condition $0 \leq \mu_F(x) + v_F(x) \leq 1$, $\forall x \in X$. The numbers $\mu_F, v_F \in [0,1]$ denote the degree of membership and non-membership of x to F, respectively. The set of all intuitionistic fuzzy sets over X will be denoted by $IFS(X)$.*

Definition 2.3. *[13] Let E be a set of parameters and X be the universal set. A pair (F,E) is called a soft set over X, where F is a mapping $F : E \to \mathcal{P}(X)$. In other words, the soft set is a parameterized family of subsets of the set X.*

Definition 2.4. *[11] Let E be a set of parameters and X be the universal set. A pair (F,E) is called a fuzzy soft set over X, If $F : E \to FS(X)$ is a mapping from E into the set of all fuzzy sets in X, where $FS(X)$ is the set of all fuzzy subset of X.*

Definition 2.5. *[12] Let X be an initial universe E be a set of parameters. A pair (F,E) is called an intuitionistic fuzzy soft set over X, where F is a mapping given by, $F : E \to IFS(X)$.*

In general, for every $e \in E$, $F(e)$ is an intuitionistic fuzzy set of X and it is called an intuitionistic fuzzy value set of parameter e. Clearly, $F(e)$ can be written as a intuitionistic fuzzy set such that $F(e) = \{(x, \mu_F(x), v_F(x)) : x \in X\}$

Definition 2.6. *[23] Let X be a universe of discourse. A pythagorean fuzzy set (PFS) in X is given by, $P = \{(x, \mu_P(x), v_P(x)) : x \in X\}$ where, $\mu_P : X \to [0,1]$ denotes the degree of membership and $v_p : X \to [0,1]$ denotes the degree of nonmembership of the element $x \in X$ to the set P with the condition that $0 \leq (\mu_P(x))^2 + (v_P(x))^2 \leq 1$.*

Definition 2.7. *[20] Let X be the universal set and E be a set of parameters. The pythagorean fuzzy soft set is defined as the pair (F,E) where, $F : E \to PFS(X)$ and $PFS(X)$ is the set of all Pythagorean fuzzy subsets of X. If $\mu_F^2(x) + v_F^2(x) \leq 1$ and $\mu_F(x) + v_F(x) \leq 1$, then pythagorean fuzzy soft sets degenerate into intuitionistic fuzzy soft sets.*

Definition 2.8. *[20] Let $A, B \subseteq E$ and (F,A), (G,B) be two pythagorean fuzzy soft sets over X. (F,A) is said to be pythagorean fuzzy soft subset of (G,B) denoted by $(F,A)\widetilde{\subseteq}(G,B)$ if,*

1. $A \subseteq B$
2. *$\forall e \in A$, $F(e)$ is a pythagorean fuzzy subset of $G(e)$ that is, $\forall x \in U$ and $\forall e \in A$, $\mu_{F(e)}(x) \leq \mu_{G(e)}(x)$ and $v_{F(e)}(x) \geq v_{G(e)}(x)$. If $(F,A)\widetilde{\subseteq}(G,B)$ and $(G,B)\widetilde{\subseteq}(F,A)$ then (F,A), (G,B) are said to be equal.*

Definition 2.9. *[20] Let* (F,E) *two pythagorean fuzzy soft sets over X. The complement of* (F,E) *is denoted by* $(F,E)^c$ *and is defined by*

$$(F,E)^c = \{(e,(x,v_{F(e)}(x),\mu_{F(e)}(x)) : x \in X) : e \in E\}$$

Definition 2.10. *[10] a) A pythagorean fuzzy soft set* (F,E) *over the universe X is said to be a null pythagorean fuzzy soft set if* $\mu_{F(e)}(x) = 0$ *and* $v_{F(e)}(x) = 1$; $\forall e \in E$, $\forall x \in X$. *It is denoted by* $\widetilde{0}_{(X,E)}$.

b) A pythagorean fuzzy soft set (F,E) *over the universe X is said to be an absolute pythagorean fuzzy soft set if* $\mu_{F(e)}(x) = 1$ *and* $v_{F(e)}(x) = 0$; $\forall e \in E$, $\forall x \in X$. *It is denoted by* $\widetilde{1}_{(X,E)}$.

Definition 2.11. *[10] Let* (F,A) *and* (G,B) *be two pythagorean fuzzy soft sets over the universe set X, E be a parameter set and* $A, B \subseteq E$.*Then,*

a) Extended union of (F,A) *and* (G,B) *is denoted by* $(F,E)\widetilde{\cup}_E(G,B) = (H,C)$ *where* $C = A \cup B$ *and* (H,C) *defined by;*

$$(H,C) = \{(e,(x,\mu_{H(e)}(x),v_{H(e)}(x)) : x \in X) : e \in E\}$$

where

$$\mu_{H(e)}(x) = \begin{cases} \mu_{F(e)}(x), & \text{if } e \in A - B \\ \mu_{G(e)}(x), & \text{if } e \in B - A \\ \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, & \text{if } e \in A \cap B \end{cases}$$

$$v_{H(e)}(x) = \begin{cases} v_{F(e)}(x), & \text{if } e \in A - B \\ v_{G(e)}(x), & \text{if } e \in B - A \\ \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, & \text{if } e \in A \cap B \end{cases}$$

b) Extended intersection of (F,A) *and* (G,B) *is denoted by* $(F,E)\widetilde{\cap}_E(G,B) = (H,C)$ *where* $C = A \cup B$ *and* (H,C) *defined by;*

$$(H,C) = \{(e,(x,\mu_{H(e)}(x),v_{H(e)}(x)) : x \in X) : e \in E\}$$

where

$$\mu_{H(e)}(x) = \begin{cases} \mu_{F(e)}(x), & \text{if } e \in A - B \\ \mu_{G(e)}(x), & \text{if } e \in B - A \\ \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, & \text{if } e \in A \cap B \end{cases}$$

$$v_{H(e)}(x) = \begin{cases} v_{F(e)}(x), & \text{if } e \in A - B \\ v_{G(e)}(x), & \text{if } e \in B - A \\ \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, & \text{if } e \in A \cap B \end{cases}$$

Let X be an initial universe and $PFS(X)$ denote the family of pythagorean fuzzy sets over X and $PFSS(X,E)$ the family of all pythagorean fuzzy soft sets over X with parameters in E.

Definition 2.12. *[25]Let* $X \neq \emptyset$ *be a universe set and* $\widetilde{\tau} \subset PFSS(X,E)$ *be a collection of pythagorean fuzzy soft sets over X, then* τ *is said to be on pythagorean fuzzy soft topology on X if*

(i) $\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)}$ belong to $\widetilde{\tau}$,

(ii) The union of any number of pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,

(iii) The intersection of any two pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triple $(X,\widetilde{\tau},E)_p$ is called an pythagorean fuzzy soft topological space over X. Every member of τ is called a pythagorean fuzzy soft open set in X.

Definition 2.13. *[25] Let* $(X,\widetilde{\tau},E)_p$ *be a pythagorean fuzzy soft topological space over X. A pythagorean fuzzy soft set* (F,E) *over X is said to be a pythagorean fuzzy soft closed set in X, if its complement* $(F,E)^c$ *belongs to* $\widetilde{\tau}$.

Definition 2.14. *[25]Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and (F,E) be a pythagorean fuzzy soft sets over X. The pythagorean fuzzy soft closure of (F,E) denoted by $pcl(F,E)$ is the intersection of all pythagorean fuzzy soft closed super sets of (F,E).*

Clearly $pcl(F,E)$ is the smallest pythagorean fuzzy soft closed set over X which contain (F,E).

Theorem 2.15. *[25]Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E) \in PFSS(X,E)$. Then the following propeties hold.*

(i) $pcl(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(X,E)}$ and $pcl(\widetilde{1}_{(X,E)}) = \widetilde{1}_{(X,E)}$,
(ii) $(F,E)\,\widetilde{\subseteq}\, pcl\,(F,E)$,
(iii) (F,E) is a pythagorean fuzzy soft closed set $\Leftrightarrow pcl(F,E) = (F,E)$,
(iv) $pcl\,(pcl\,(F,E)) = pcl(F,E)$,
(v) $(F,E)\,\widetilde{\subseteq}\,(G,E) \Rightarrow pcl\,(F,E)\,\widetilde{\subseteq}\, pcl\,(G,E)$,
(vi) $pcl\left((F,E)\,\widetilde{\cup}_E\,(G,E)\right) = pcl\,(F,E)\,\widetilde{\cup}_E pcl\,(G,E)$.

Definition 2.16. *[25]Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(H,E) \in PFSS(X,E)$. The pythagorean fuzzy soft interior of (H,E), denoted by $pint(H,E)$, is the union of all the pythagorean fuzzy soft open sets contained in (H,E).*

Theorem 2.17. *[25]Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(H,E) \in PFSS(X,E)$. Then the following properties hold.*
(i) $pint(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(X,E)}$ *and* $pint(\widetilde{1}_{(X,E)}) = \widetilde{1}_{(X,E)}$,
(ii) $pint\,(H,E)\,\widetilde{\subseteq}\,(H,E)$,
(iii) (H,E) *is a pythagorean fuzzy soft open set* $\Leftrightarrow pint(H,E) = (H,E)$,
(iv) $pint\,(pint\,(H,E)) = pint(H,E)$,
(v) $(H,E)\,\widetilde{\subseteq}\,(G,E) \Rightarrow pint\,(H,E)\,\widetilde{\subseteq}\, pint\,(G,E)$,
(vi) $pint\left((H,E)\,\widetilde{\cap}_E\,(G,E)\right) = pint\,(H,E)\,\widetilde{\cap}_E pint\,(G,E)$.

3. Pythagorean Fuzzy Soft Boundary

Definition 3.1. *The difference of two pythagorean fuzzy soft sets (F,E) and (G,E) over X, denoted by $(F,E)\widetilde{\setminus}(G,E)$ and defined by $(F,E)\widetilde{\setminus}(G,E) = (F,E)\widetilde{\cap}_E(G,E)^c$*

Example 3.2. *Let (F,E) and (G,E) be two pythagorean fuzzy soft set defined as follows;*

$$(F,E) = \left\{ \begin{array}{c} (e_1, \{(x_1, 0.3, 0.5), (x_2, 0.2, 0.6)\}) \\ (e_2, \{(x_1, 0.4, 0.1), (x_2, 0.5, 0.6)\}) \end{array} \right\}$$

$$\begin{array}{rcl} (G,E) & = & \left\{ \begin{array}{c} (e_1, \{(x_1, 0.4, 0.8), (x_2, 0.9, 0.2)\}) \\ (e_2, \{(x_1, 0.6, 0.3), (x_2, 0.7, 0.4)\}) \end{array} \right\} \\ (G,E)^c & = & \left\{ \begin{array}{c} (e_1, \{(x_1, 0.8, 0.4), (x_2, 0.2, 0.9)\}) \\ (e_2, \{(x_1, 0.3, 0.6), (x_2, 0.4, 0.7)\}) \end{array} \right\} \end{array}$$

Then $(F,E)\widetilde{\setminus}(G,E) = (F,E)\widetilde{\cap}_E(G,E)^c$ and we find

$$(F,E)\widetilde{\cap}_E(G,E)^c = \left\{ \begin{array}{c} (e_1, \{(x_1, 0.3, 0.5), (x_2, 0.2, 0.9)\}) \\ (e_2, \{(x_1, 0.3, 0.6), (x_2, 0.4, 0.7)\}) \end{array} \right\}.$$

Definition 3.3. *[21] Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset}PFSS(X, E)$. Then the pythagorean fuzzy soft boundary of (F, E), denoted by $Fr(F, E)$ and defined as $Fr(F, E) = pcl(F, E)\widetilde{\cap}_E pcl((F, E)^c)$.*

Example 3.4. *Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and*

$$\widetilde{\tau} = \left\{\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)}, (F_1, E), (F_2, E)\right\}$$

where $(F_1, E), (F_2, E)$ are pythagorean fuzzy soft sets over X, defined as;

$$\begin{aligned} (F_1, E) &= \left\{ \begin{array}{l} (e_1, \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.4)\}) \\ (e_2, \{(x_1, 0.7, 0.3), (x_2, 0.5, 0.2)\}) \end{array} \right\} \\ (F_2, E) &= \left\{ \begin{array}{l} (e_1, \{(x_1, 0.7, 0.2), (x_2, 0.9, 0.2)\}) \\ (e_2, \{(x_1, 0.8, 0.2), (x_2, 0.7, 0.1)\}) \end{array} \right\} \end{aligned}$$

Then $(X, \widetilde{\tau}, E)_p$ is a pythagorean fuzzy soft topological spaces on X.The members of $\widetilde{\tau}$ obviously pythagorean fuzzy open sets. Now, we find pythagorean fuzzy closed sets;

$$\begin{aligned} \widetilde{0}^c_{(X,E)} &= \widetilde{1}_{(X,E)} \\ \widetilde{1}^c_{(X,E)} &= \widetilde{0}_{(X,E)} \end{aligned}$$

$$(F_1, E)^c = \left\{ \begin{array}{l} (e_1, \{(x_1, 0.2, 0.6), (x_2, 0.4, 0.8)\}) \\ (e_2, \{(x_1, 0.3, 0.7), (x_2, 0.2, 0.5)\}) \end{array} \right\}$$

$$(F_2, E)^c = \left\{ \begin{array}{l} (e_1, \{(x_1, 0.2, 0.7), (x_2, 0.2, 0.9)\}) \\ (e_2, \{(x_1, 0.2, 0.8), (x_2, 0.1, 0.7)\}) \end{array} \right\}$$

We consider the pythagorean fuzzy soft set $(G, E)\widetilde{\subset}PFSS(X, E)$.

$$(G, E) = \left\{ \begin{array}{l} (e_1, \{(x_1, 0.1, 0.8), (x_2, 0.2, 0.9)\}) \\ (e_2, \{(x_1, 0.1, 0.9), (x_2, 0.1, 0.7)\}) \end{array} \right\}$$

so that

$$(G, E)^c = \left\{ \begin{array}{l} (e_1, \{(x_1, 0.8, 0.1), (x_2, 0.9, 0.2)\}) \\ (e_2, \{(x_1, 0.9, 0.1), (x_2, 0.7, 0.1)\}) \end{array} \right\}$$

Obviously, $\widetilde{0}^c_{(X,E)}, \widetilde{1}^c_{(X,E)}, (F_1, E)^c, (F_2, E)^c$ are all pythagorean fuzzy soft closed sets over $(X, \widetilde{\tau}, E)_p$. Then

$$\widetilde{1}^c_{(X,E)}, (F_1, E)^c, (F_2, E)^c \widetilde{\supset} (G, E).$$

Therefore $pcl(F, E) = \widetilde{1}^c_{(X,E)}\widetilde{\cap}_E(F_1, E)^c\widetilde{\cap}_E(F_2, E)^c = (F_2, E)^c$. Also we find $pcl((F, E)^c) = \widetilde{1}_{(X,E)}$.
So, $Fr(F, E) = pcl(F, E)\widetilde{\cap}_E pcl((F, E)^c) = (F_2, E)^c\widetilde{\cap}_E\widetilde{1}_{(X,E)} = (F_2, E)^c$, Hence

$$Fr(F, E) = \left\{ \begin{array}{l} (e_1, \{(x_1, 0.2, 0.7), (x_2, 0.2, 0.9)\}) \\ (e_2, \{(x_1, 0.2, 0.8), (x_2, 0.1, 0.7)\}) \end{array} \right\}.$$

Theorem 3.5. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset}PFSS(X, E)$. Then the following properties hold;*

1. $(Fr(F, E))^c = pint(F, E)\widetilde{\cup}_E pint((F, E)^c)$
2. $pcl(F, E) = pint(F, E)\widetilde{\cup}_E Fr(F, E)$

3. $Fr(F,E) = pcl(F,E)\widetilde{\setminus} pint(F,E)$
4. $pint(F,E) = (F,E)\widetilde{\setminus} Fr(F,E)$
5. $Fr\left(pcl(F,E)\right)\widetilde{\subset} Fr(F,E)$
6. $Fr(F,E)\widetilde{\cap}_E pint(F,E) = \widetilde{0}_{(X,E)}$.
7. $pcl(pint(F,E)) = (F,E)\widetilde{\setminus} Fr(F,E)$

Proof. (1)

$$\begin{aligned}
pint(F,E)\widetilde{\cup}_E pint\left((F,E)^c\right) &= \left((pint(F,E))^c\right)^c \widetilde{\cup}_E \left((pint\left((F,E)^c\right))^c\right)^c \\
&= \left[(pint(F,E))^c \widetilde{\cap}_E \left(pint\left((F,E)^c\right)\right)^c\right]^c \\
&= \left[pcl\left((F,E)^c\right) \widetilde{\cap}_E pcl(F,E)\right]^c \\
&= (Fr(F,E))^c .
\end{aligned}$$

(2)

$$\begin{aligned}
pint(F,E)\widetilde{\cup}_E Fr(F,E) &= pint(F,E)\widetilde{\cup}_E \left(pcl(F,E)\widetilde{\cap}_E pcl\left((F,E)^c\right)\right) \\
&= \left[pint(F,E)\widetilde{\cup}_E pcl(F,E)\right]\widetilde{\cap}_E \left[pint(F,E)\widetilde{\cup}_E pcl\left((F,E)^c\right)\right] \\
&= pcl(F,E)\widetilde{\cap}_E \left[pint(F,E)\widetilde{\cup}_E \left(pint(F,E)\right)^c\right] \\
&= pcl(F,E)\widetilde{\cap}_E \left(pint(F,E)\widetilde{\cup}_E \left(pint(F,E)\right)^c\right) \\
&= pcl(F,E)\widetilde{\cap}_E \widetilde{1}_{(X,E)} \\
&= pcl(F,E).
\end{aligned}$$

(3)

$$\begin{aligned}
Fr(F,E) &= pcl(F,E)\widetilde{\cap}_E pcl\left((F,E)^c\right) \\
&= pcl(F,E)\widetilde{\cap}_E \left(pint(F,E)\right)^c \\
&= pcl(F,E)\widetilde{\setminus} pint(F,E).
\end{aligned}$$

(4)

$$\begin{aligned}
(F,E)\widetilde{\setminus} Fr(F,E) &= (F,E)\widetilde{\cap}_E Fr\left((F,E)^c\right) \\
&= (F,E)\widetilde{\cap}_E \left(pint(F,E)\widetilde{\cup}_E pint\left((F,E)^c\right)\right) (by\ (1)) \\
&= \left[(F,E)\widetilde{\cap}_E pint(F,E)\right]\widetilde{\cup}_E \left[(F,E)\widetilde{\cap}_E pint\left((F,E)^c\right)\right] \\
&= pint(F,E)\widetilde{\cup}_E \widetilde{0}_{(X,E)}. \\
&= pint(F,E).
\end{aligned}$$

(5)

$$\begin{aligned}
Fr\left(pcl(F,E)\right) &= pcl\left(pcl(F,E)\right)\widetilde{\setminus} pint\left(pcl(F,E)\right) \\
&= pcl(F,E)\widetilde{\setminus} pint\left(pcl(F,E)\right) \\
&\widetilde{\subset} pcl(F,E)\widetilde{\setminus} pint(F,E) \\
&= Fr(F,E).
\end{aligned}$$

(6) is similar to (3)

(7) can be easily obtained from the definition of a pythagorean fuzzy soft boundary. □

Remark 3.6. *By (3) of above Theorem 3.5, it is clear that* $Fr(F,E) = Fr\left((F,E)^c\right)$.

Theorem 3.7. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset} PFSS(X, E)$. Then;*

1. *(F, E) is a pythagorean fuzzy open set over X if and only if $(F, E)\widetilde{\cap}_E Fr(F, E) = \widetilde{0}_{(X,E)}$,*
2. *(F, E) is a pythagorean fuzzy closed set over X if and only if $Fr(F, E)\widetilde{\subset}(F, E)$,*
3. *If (G, E) be a pythagorean fuzzy closed (respt. open) set of an pythagorean fuzzy soft topological space with $(F, E)\widetilde{\subset}(G, E)$, then $Fr(F, E)\widetilde{\subset}(G, E)$ (respt. $Fr(F, E)\widetilde{\subset}(G, E)^c$).*

Proof. (1) Let (F, E) be an pythagorean fuzzy soft open set over X. Then $pint(F, E) = (F, E)$ implies that $(F, E)\widetilde{\cap}_E Fr(F, E) = pint(F, E)\widetilde{\cap}_E Fr(F, E) = \widetilde{0}_{(X,E)}$.

Conversely, let $(F, E)\widetilde{\cap}_E Fr(F, E) = \widetilde{0}_{(X,E)}$. Then $(F, E)\widetilde{\cap}_E pcl(F, E)\widetilde{\cap}_E pcl((F, E)^c) = \widetilde{0}_{(X,E)}$ or $(F, E)\widetilde{\cap}_E pcl((F, E)^c) = \widetilde{0}_{(X,E)}$ or $pcl((F, E)^c)\widetilde{\subset}(F, E)^c$, which implies $(F, E)^c$ is a pythagorean fuzzy closed and hence (F, E) is pythagorean fuzzy open set.

(2) Let (F, E) be an pythagorean fuzzy soft closed set over X. Then $pcl(F, E) = (F, E)$. Now $Fr(F, E) = pcl(F, E)\widetilde{\cap}_E pcl((F, E)^c)\widetilde{\subset} pcl(F, E) = (F, E)$. That is, $Fr(F, E)\widetilde{\subset}(F, E)$.

Conversely, $Fr(F, E)\widetilde{\subset}(F, E)$. Then $Fr(F, E)\widetilde{\cap}_E(F, E)^c = \widetilde{0}_{(X,E)}$. Since $Fr(F, E) = Fr((F, E)^c) = \widetilde{0}_{(X,E)}$, we have $Fr((F, E)^c)\widetilde{\cap}_E(F, E)^c = \widetilde{0}_{(X,E)}$. By (1), $(F, E)^c$ is pythagorean fuzzy open set and hence (F, E) is pythagorean fuzzy closed set.

(3) $(F, E)\widetilde{\subset}(G, E)$ follows that $pcl(F, E)\widetilde{\subset} pcl(G, E)$. Since (G, E) is pythagorean fuzzy soft closed, then we get $Fr(F, E) = pcl(F, E)\widetilde{\cap}_E pcl((F, E)^c)\widetilde{\subset} pcl(G, E)\widetilde{\cap}_E pcl((F, E)^c)\widetilde{\subset} pcl(G, E) = (G, E)$. Similarly for the other inclusion. □

Theorem 3.8. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E), (G, E)\widetilde{\subset} PFSS(X, E)$. Then the following properties hold.*

1. $Fr\left((F, E)\widetilde{\cup}_E(G, E)\right)\widetilde{\subset} Fr\left((F, E)\widetilde{\cap}_E((G, E)^c)\right)\widetilde{\cup}_E\left[Fr(G, E)\widetilde{\cap}_E pcl((F, E)^c)\right]$,
2. $Fr\left((F, E)\widetilde{\cap}_E(G, E)\right)\widetilde{\subset} Fr\left((F, E)\widetilde{\cap}_E pcl(G, E)\right)\widetilde{\cup}_E\left[Fr(G, E)\widetilde{\cap}_E pcl(F, E)\right]$.

Proof. (1)

$$\begin{aligned}
Fr\left((F, E)\widetilde{\cup}_E(G, E)\right) &= pcl\left((F, E)\widetilde{\cup}_E(G, E)\right)\widetilde{\cap}_E pcl\left(\left((F, E)\widetilde{\cup}_E(G, E)\right)^c\right)\\
&= \left(pcl(F, E)\widetilde{\cup}_E pcl(G, E)\right)\widetilde{\cap}_E pcl\left((F, E)^c\widetilde{\cap}_E(G, E)^c\right)\\
&\widetilde{\subset} \left(pcl(F, E)\widetilde{\cup}_E pcl(G, E)\right)\widetilde{\cap}_E\left[pcl((F, E)^c)\widetilde{\cap}_E pcl((G, E)^c)\right]\\
&= \left(pcl(F, E)\widetilde{\cup}_E pcl(G, E)\right)\widetilde{\cap}_E pcl((G, E)^c)\widetilde{\cup}_E pcl(G, E)\\
&\quad \widetilde{\cap}_E\left[pcl((F, E)^c)\widetilde{\cap}_E pcl((G, E)^c)\right]\\
&= \left(Fr(F, E)\widetilde{\cap}_E pcl((G, E)^c)\right)\widetilde{\cup}_E\left(Fr(G, E)\widetilde{\cap}_E pcl((F, E)^c)\right)\\
&\widetilde{\subset} Fr(F, E)\widetilde{\cup}_E Fr(G, E).
\end{aligned}$$

(2)

$$\begin{aligned}
Fr\left((F, E)\widetilde{\cap}_E(G, E)\right) &= pcl\left((F, E)\widetilde{\cap}_E(G, E)\right)\widetilde{\cap}_E pcl\left(\left((F, E)\widetilde{\cap}_E(G, E)\right)^c\right)\\
&\quad \widetilde{\subset}\left(pcl(F, E)\widetilde{\cup}_E pcl(G, E)\right)\widetilde{\cap}_E pcl\left((F, E)^c\widetilde{\cup}_E(G, E)^c\right)\\
&= \left(pcl(F, E)\widetilde{\cup}_E pcl(G, E)\right)\widetilde{\cap}_E\left[pcl((F, E)^c)\widetilde{\cup}_E pcl((G, E)^c)\right]\\
&= \left[\left(pcl(F, E)\widetilde{\cap}_E pcl(G, E)\right)\widetilde{\cap}_E pcl((G, E)^c)\right]\widetilde{\cup}_E[(pcl(F, E)\\
&\quad \widetilde{\cap}_E pcl(G, E))\widetilde{\cap}_E pcl((G, E)^c)]\\
&= \left(Fr(F, E)\widetilde{\cap}_E Fr(G, E)\right)\widetilde{\cup}_E\left(pcl(F, E)\widetilde{\cap}_E Fr((G, E))\right)
\end{aligned}$$

□

Corollary 3.9. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E),(G,E)\widetilde{\subset}PFSS(X,E)$. Then $Fr\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subset}Fr(F,E)\widetilde{\cap}_E Fr(G,E)$.*

Theorem 3.10. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\subset}PFSS(X,E)$. Then we have $Fr\left(Fr\left(Fr(F,E)\right)\right) = Fr\left(Fr(F,E)\right)$.*

Proof.

$$\begin{aligned} Fr\left(Fr\left(Fr(F,E)\right)\right) &= pcl\left(Fr\left(Fr(F,E)\right)\right)\widetilde{\cap}_E pcl\left(\left(Fr\left(Fr(F,E)\right)\right)^c\right) \\ &= \left(Fr\left(Fr(F,E)\right)\right)\widetilde{\cap}_E pcl\left(\left(Fr\left(Fr(F,E)\right)\right)^c\right) \end{aligned}$$

Now consider

$$\begin{aligned} \left(Fr\left(\left(Fr(F,E)\right)\right)^c\right) &= \left[pcl\left(Fr(F,E)\right)\widetilde{\cap}_E\left(Fr(F,E)\right)^c\right]^c \\ &= \left[Fr(F,E)\widetilde{\cap}_E pcl\left(\left(Fr(F,E)\right)^c\right)\right]^c \\ &= \left(Fr(F,E)\right)^c\widetilde{\cup}_E\left(pcl\left(\left(Fr(F,E)\right)^c\right)\right)^c \end{aligned}$$

Therefore

$$\begin{aligned} pcl\left(\left(Fr\left(Fr(F,E)\right)\right)^c\right) &= pcl\left(\left[pcl\left(\left(Fr(F,E)\right)^c\right)\widetilde{\cup}_E\left(pcl\left(\left(Fr(F,E)\right)^c\right)\right)^c\right]\right) \\ &= pcl\left(pcl\left(\left(Fr(F,E)\right)^c\right)\right)\widetilde{\cup}_E pcl\left(\left(pcl\left(\left(Fr(F,E)\right)^c\right)\right)^c\right) \\ &= (G,E)\widetilde{\cup}_E\left(\left(pcl\left(\left(Fr(G,E)\right)^c\right)\right)^c\right) = \widetilde{1}_{(X,E)} \end{aligned}$$

where $(G,E) = pcl\left(\left(pcl\left(\left(Fr(F,E)\right)^c\right)\right)\right)$. From the above equations, we have

$$Fr\left(Fr\left(Fr(F,E)\right)\right) = Fr\left(Fr(F,E)\right)\widetilde{\cap}_E\widetilde{1}_{(X,E)} = Fr\left(Fr(F,E)\right)$$

□

Theorem 3.11. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E),(G,E)\widetilde{\subset}PFSS(X,E)$. Then the following properties hold.*

1. $\left((F,E)\widetilde{\setminus}pint(G,E)\right)\widetilde{\subset}pint(F,E)\widetilde{\setminus}pint(G,E)$
2. $Fr\left(pint(F,E)\right)\widetilde{\subset}Fr(F,E)$.

Proof. (1)

$$\begin{aligned} \left((F,E)\widetilde{\setminus}pint(G,E)\right) &= \left((F,E)\widetilde{\cap}_E pint\left((G,E)^c\right)\right) \\ &= pint(F,E)\widetilde{\cap}_E pint\left((G,E)^c\right) \\ &= pint(F,E)\widetilde{\cap}_E\left(pcl(G,E)\right)^c \\ &= pint(F,E)\widetilde{\setminus}pcl(G,E) \\ &\widetilde{\subset} pint(F,E)\widetilde{\setminus}pint(G,E). \end{aligned}$$

(2)

$$\begin{aligned} Fr\left(pint(F,E)\right) &= pcl\left(pint(F,E)\right)\widetilde{\cap}_E pcl\left(\left(pint(F,E)\right)^c\right) \\ &\widetilde{\subset} pcl\left(pint(F,E)\right)\widetilde{\cap}_E pcl\left(pcl\left((F,E)^c\right)\right) \\ &\widetilde{\subset} pcl(F,E)\widetilde{\cap}_E pcl\left((F,E)^c\right) = Fr(F,E). \end{aligned}$$

□

Theorem 3.12. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset} PFSS(X, E)$. Then $Fr(F, E) = \widetilde{0}_{(X,E)}$ if and only if (F, E) is both a pythagorean fuzzy soft closed and pythagorean fuzzy soft open set.*

Proof. *Suppose that $Fr(F, E) = \widetilde{0}_{(X,E)}$. Firstly, we show that (F, E) is a pythagorean fuzzy soft closed set.*

$$\begin{aligned} Fr(F, E) &= \widetilde{0}_{(X,E)} \Rightarrow pcl(F, E)\widetilde{\cap}_E pcl\left((F, E)^c\right) = \widetilde{0}_{(X,E)} \\ &\Rightarrow pcl(F, E)\widetilde{\subset}\left(pcl\left((F, E)^c\right)\right)^c = pint(F, E) \\ &\Rightarrow pcl(F, E)\widetilde{\subset}(F, E) \Rightarrow pcl(F, E) = (F, E) \end{aligned}$$

This implies that (F, E) is pythagorean fuzzy soft closed set.

Now, we prove that (F, E) is a pythagorean fuzzy soft open set.

$$Fr(F, E) = \widetilde{0}_{(X,E)} \Rightarrow pcl(F, E)\widetilde{\cap}_E pcl\left((F, E)^c\right)$$

or

$$\begin{aligned} (F, E)\widetilde{\cap}_E \left(pint(F, E)\right)^c &= \widetilde{0}_{(X,E)} \Rightarrow (F, E)\widetilde{\subset} pint(F, E) \\ &\Rightarrow pint(F, E) = (F, E) \end{aligned}$$

This implies that (F, E) is pythagorean fuzzy soft open set.

Conversely, suppose that (F, E) is both pythagorean fuzzy soft open and pythagorean fuzzy soft closed set. Then

$$\begin{aligned} Fr(F, E) &= pcl(F, E)\widetilde{\cap}_E pcl\left((F, E)^c\right) \\ &= pcl(F, E)\widetilde{\cap}_E \left(pint(F, E)\right)^c \\ &= (F, E)\widetilde{\cap}_E (F, E)^c = \widetilde{0}_{(X,E)}. \end{aligned}$$

□

4. Conclusion

In this paper, we introduced the concept of the pythagorean fuzzy soft boundary. We discussed and investigated the characteristics and properties of pythagorean fuzzy soft boundary in general as well as pythagorean fuzzy soft interior and pythagorean fuzzy soft closure. Examples and counterexamples are also given to verify the findings discussed. We will research more topological structures in pythagorean fuzzy soft sets in future studies. We hope that this study will be useful for the paper to be done in the theory of pythagorean fuzzy soft.

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Mathematical Modeling of the Effect of CO2 Laser Power on Texture Size on Polyoxmethylene (POM) Sheet

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Abstract. Variation of the groove size depending on the laser power has been modeled in this proposed mathematical model. It is obtained by polymerization of Polyoxmethylene formaldehyde, a semi-crystalline polymer, and is among the hardest and strongest thermoplastics. Polyoxmethylene can be used in slip-friction pairs without lubrication. Polyoxmethylene materials are widely used for materials in tribological applications. They also show good friction properties.

1. Introduction

The tribologic properties wettability, hydrophobization and adhesion properties surface can be improved by surface texturing. The polymer surfaces have been modified with many commercial methods [9]. Chemical and physical modification can be applied to the polymer surfaces. There are some disadvantages in the chemical processing of polymer materials. Since chemical processes are very difficult to control in chemical surface treatment, the application areas of this method are also very limited. In addition, the measures to be taken to prevent environmental pollution by chemical methods are costly and increase the number of processes. One of the foremost disadvantages in the processing of polymer surfaces by mechanical methods is the wear of the tools used during the process. In addition to increasing the cost of wear, it also decreases the sensitivity of the process as the processing time increases.

Most of the materials can be easily processed with a laser. Many polymer materials can be processed precisely with the appropriate laser selection. Material processing precision is continuous and does not change over time. The selection of suitable parameters is very important in laser material processing. For each material and the desired product, the effects of many parameters such as laser power, frequency, the wavelength should be investigated and the most appropriate parameter selection should be made. The material processing time is short since high energy can be transferred very precisely to a small area by laser in a very short time.

In laser material processing, when the laser beam hits the surface, the material surface heats first. When the laser application time increases, if the energy is high enough, melting, evaporation or burning occurs respectively. The ablation mechanism in laser material processing has not been fully explained. In addition to the process parameters of the laser used in material processing with laser, the thermophysical properties

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of the material such as specific heat, absorption coefficient and thermal conduction significantly affect the quality of the processed material.

The effects of laser parameters on the surface texture have been investigated in many studies in order to obtain surface textures such as grooves and small cavities in the desired shape and size [10, 11]. Many studies have been carried out to obtain suitable laser and parameters for many different materials in order to obtain the desired surface properties [8, 12]. In addition to optimization studies to determine the material properties to be obtained by selecting the appropriate parameters, mathematical modeling studies are also carried out for the product to be obtained The applicability of mathematical modeling in laser material processing of polymer has also been proven by experimental results [12–15].

In this study, mathematical modeling of the width of micro-sized grooves created with laser on a Polyoxmethylene (POM) sheet has been made. Fourier method was used in the mathematical modeling of the heat distribution on the surface of the Polyoxymethylene. To obtain a mathematical model, the effects of the laser power on the groove width of Polyoxymethylene sheet were investigated. A mathematical model has been obtained by using the thermophysical properties of Polyoxymethylene and laser parameters.

The following problem of parabolic equations with various boundary conditions was studied [1–7].

The heat distribution equation on surface can be written as below;

$$\frac{\partial T(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 T(x,t)}{\partial x^2} \tag{1}$$

where, T is the temperature as a function of time "t" and distance "x", α is the thermal diffusivity of the investigate material.

$$\alpha^2 = \frac{\lambda}{c\rho},$$

where λ denotes the thermal conductivity, c specific heat, ρ density.

Let $t_p > 0$ be a fixed number and denote by $D = \{(x.t) : 0 < x < l, 0 < t < t_p\}$, where t_p is the pulse duration.

The initial condition can be written as;

$$T(x,0) = T_0, \quad 0 < x < l$$

where T_0 is the initial temperature of the material. It was assumed that all the energy absorbed by the surface was transmitted to the material. Thus, the boundary condition ($x = 0$) on the surface can be written as follows:

$$\frac{\partial T(0,t)}{\partial t} = 0, \quad \frac{\partial T(l,t)}{\partial t} = 0.$$

This problem is called a parabolic problem. Classical solution of the problem (1)-(3) is $T(x,t) \in C^{2,1}(D) \cap C^{1,0}(D)$. The heat source problem has been investigated with parabolic equation in many studies. Then the following solution is obtained using Fourier method.

$$T(x,t) = \sum_{k=1}^{\infty} (T_{ck}(t) \cos \frac{2\pi\alpha k}{l} x + T_{sk}(t) \sin \frac{2\pi\alpha k}{l} x) e^{-(\frac{2\pi\alpha k}{l})^2 t} \tag{2}$$

The laser intensity within the material can be found using the Beer-Lambert's Law:

$\frac{dI(x)}{dx} = -aI$

Where $I(x)$ is the laser intensity as a function of distance from laser spot and α is the absorption coefficient of the material respectively. Although absorption coefficient is changed within the material but it was taken as constant in our study. Laser intensity as a function of distance within material can be written as;

$I = I_0 e^{-\int_b^z a dx}$

Actually most of the beam intensities have Gaussian distribution. We made one more assumption that our laser beam is top-hat beam that means intensity is homogeneously distributed in spot area.

The heat generation from the laser beam absorbed by the material is defined as,
$S = -dI/dx$
Using Leibniz rule yields, the heat source can be written as;
$S = I_0 e^{-\int_b^z a dx}$.
The temperature distribution as a function was obtained as given below;

$$\begin{aligned} T(x,t) &= \sum_{k=1}^{\infty}\left(\varphi_{ck} e^{-(\frac{2\pi\alpha k}{l})^2 t} + \int_0^t \int_0^\pi S(x,t)\cos\frac{2\pi\alpha k}{l} x e^{-(\frac{2\pi\alpha k}{l})^2(t-\tau)} dx d\tau\right)\cos\frac{2\pi\alpha k}{l}x \\ &+ \sum_{k=1}^{\infty}\left(\varphi_{sk} e^{-(\frac{2\pi\alpha k}{l})^2 t} + \int_0^t \int_0^\pi S(x,t)\sin\frac{2\pi\alpha k}{l} x e^{-(\frac{2\pi\alpha k}{l})^2(t-\tau)} dx d\tau\right)\sin\frac{2\pi\alpha k}{l}x \end{aligned} \tag{3}$$

2. Material and Experimental Setup

The surfaces of 5 mm thick Polyoxymethylene sheets were used to ablation process. Some physical and thermal properties of Polyoxymethylene which were used in mathematical modeling have been listed in Table 1.

In the ablation process commercial 130 W CO2 laser was used with different power at constant scan speed. Laser spot diameter is 160 μm.

Table 1 Some physical and thermal properties of Polyoxymethylene

Properties	Value	Unit
Density	1410	kg/m^3
Thermal Capacity	1.5	$kJ/kg.K$
Melting point	165	$°C$
Heat Deflection Temperature	110	$°C$
Tensile module of elasticity	2800	MPa
Thermal Conductivity	0.31	W/mK

3. Results and Discussion

In this study, mathematical model has been proposed for the groove formation on Polyoxymethylene sheet with various power and constant scan speed. Groove widths were measured from optical microscope images of ablated surfaces of Polyoxymethylene sheets.

For 26 Watts of laser power, from the optical microscope images, the Heat Deflection Zone boundary and molten zone boundary distances were measured as 1434 μm and 1252 μm respectively. Temperatures at Heat Deflection boundary and molten zone boundary are 383 K and 438 K respectively. These values are used in temperature distribution equation obtain the Fourier coefficients which are depends on the material properties. The coefficients in the temperature distribution equation (2) were calculated as φ_{ck} (=451,32) and φ_{sk} (-205.15). These are the coefficients depend on the thermal properties of Polyoxymethylene. Then, in order to verify the validity of mathematical model, new grooves were obtained using 39, 52, 65, 78, 91 and 104 Watts. These coefficients were used to calculate temperature distribution for the Polyoxymethylene and variour laser beam powers.

Table 2 Laser Powers and groove widths measured from images.

Lazer Powerwatt	Molten Zone width (μm)	Heat Deflection Zone width (μm)
26	1252	1434
39	1319	1513
52	1367	1568
65	1404	1611
78	1434	1646
91	1459	1676
104	1482	1702

Each laser powers and the coefficients obtained previously were used in the temperature distribution equation to calculate the temperatures for each speed of laser beam. The calculated temperatures for boundaries are given in Table 3.

Table 3 Melting and Heat Deflection Temperatures calculated with mathematical model, real values and percent error between them.

Powerwatt		T(x,t) (K)	T(x,t) (K) (Calculated)	error
39	Melting	438	441.316	0.76
39	Heat Deflection	383	385.021	0.53
52	Melting	438	442.256	0.97
52	Heat Deflection	383	385.895	0.76
65	Melting	438	443.462	1.25
65	Heat Deflection	383	387.021	1.05
78	Melting	438	444.891	1.57
78	Heat Deflection	383	388.105	1.33
91	Melting	438	446.114	1.85
91	Heat Deflection	383	389.206	1.62
104	Melting	438	447.365	2.14
104	Heat Deflection	383	390.170	1.86

4. Conclusion

By texturing the surfaces, the mechanical properties of my surfaces can be changed. The properties of material surfaces can be improved by many methods such as mechanical and chemical methods. While texturing surfaces with a laser have many advantages, it can require complex processes to be controlled. Mathematical modeling of the heat distribution of the surface to be obtained with laser texture can be known in advance the properties of the product to be obtained. This saves time and material.

Grooves were formed on the Polyoxymethylene material surface with seven different laser beam power. Measurements were made from the images of the groove obtained by using a 26-watt laser beam. The measurement results were applied to the proposed mathematical model and the φ_{ck} and φ_{sk} coefficients in the mathematical model were calculated. These coefficients are coefficients depending on the properties of Polyoxymethylene. These coefficients were applied for grooves obtained using 39, 52, 65, 78, 91 and 104 W laser beams. Heat deflection and melting point values obtained in the mathematical model are quite compatible with the actual values. As the laser power increased, the error rate increased acceptably.

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Mathematical Modeling of the Effect of CO2 Laser Parameter on Shape and Geometry of Polymer Plate

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Abstract. In recent years, the use of polymer-based materials is in almost every aspect of daily life [1]. PMMA can be used in many areas from aircraft to the medical industry with their good chemical stability, high strength, high corrosion and aging resistance, insulation performance, and smooth surface [2]. In this study, grooves were formed on Polymethyl Methacrylate (PMMA) Plates with different scanning speeds with CO2 laser. Since the scan speed of the laser is increased, the interaction time between the laser beam and the material decreases then the amount of energy transferred to the material also decreases. Measurements were made from high-resolution optical microscope images of the grooves created on PMMA. In this study, the distribution of heat energy transferred to the material was modeled mathematically. The change to groove size depending on the laser scan speed is modeled. To validate the mathematical model, the surfaces of the PMMA plate were ablated with different scan speed at constant power. The CO2 laser that has 10600 nm wavelengths and 130 Watts maximum power was used in the ablation.

1. Introduction

Polymeric materials can be divided into two groups; Thermoplastics and thermosets. The main difference between the two is their reaction to heating. Thermoplastics can be reheated, coated and cooled as required. No chemical treatment is required during this process. Thermosets, on the other hand, cannot be reshaped after being heated and shaped. It becomes very strong and durable in the first forming. PMMA is classified as thermoplastic. PMMA has various performance benefits such as high strength, shrink-resistance, and easy flexibility. Polymer materials are frequently preferred in the industry as they can be processed easily. Although it can be processed by mechanical and chemical methods, laser processing of polymer materials has superior properties compared to other methods. Due to the difficulty of controlling chemical reactions and their negative effects on the environment, the application area of the chemical method is very limited. Although mechanical processing is one of the frequently used methods, it has disadvantages such as abrasion of the abrasive elements used and the inability to obtain a product with the same precision.

The tribology, wettability, adhesion and hydrophobization properties have been improving by surface texturing. Many different methods have been developed for texturing the surfaces of polymers with

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different specialties [3]. Many laser parameters such as wavelength, frequency, power and spot size can be selected in accordance with the material and the desired surface structure. In addition to these features, lasers are preferred in many areas today because they are compact and do not require additional systems other than ambient gas.

Although the ablation mechanism in laser material processing is strictly dependent on material properties and process parameters, it is very difficult to obtain a surface structure with the desired precision. The effective thermophysical properties in the ablation mechanism are thermal conduction, absorption coefficient and specific heat. Besides the laser properties such as the wavelength, frequency and power of the laser used, process parameters such as scan speed, overlap rate, number of pulses and beam size determine the ablation and therefore the quality of the processed material.

Regular textures such as micro-sized cavities and grooves created on the polymeric material surface improve the friction and adhesion behavior of the materials. The geometries, density and orientation of the microstructures created on the surface play an important role in increasing the surface performance. [4,5]. For these reasons, many optimization studies have been carried out in order to obtain the desired texture on the surface of many kinds of materials. [6,7,8]. In addition to optimization studies, mathematical modeling of the heat distribution in the material can be obtained from data about the geometry of the cavities to be obtained by laser. [9,10,11]. In this study, the mathematical modeling of the heat distribution for the width of the grooves created by laser on the PMMA plate was made. In the mathematical model, the Fourier method with a homogenous approach was used. To obtain a numerical model, the effects of the laser scan speed on the groove size of PMMA sheet were investigated and a simple mathematical model of the heat distribution on surface is proposed.

The heat distribution equation on surface can be written as below;

$$\frac{\partial T(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 T(x,t)}{\partial x^2}, \tag{1}$$

where T is the temperature as a function of time t and distance x, α is the thermal diffusivity of the investigate material.

$$\alpha^2 = \frac{\lambda}{c\rho}$$

where, λ denotes the thermal conductivity, c specific heat ρ density.

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This problem is called a parabolic problem. Classical solution of the problem (1)-(3) is $T(x,t) \in C^{2,1}(D) \cap C^{1,0}(D)$. The heat source problem has been investigated with parabolic equation in many studies. Then the following solution is obtained using Fourier method.

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The laser intensity within the material can be found using the Beer-Lambert's Law:
$\frac{dI(x)}{dx} = -al$

Where $I(x)$ is the laser intensity as a function of distance from laser spot and α is the absorption coefficient of the material respectively. Although absorption coefficient is changed within the material but it was taken as constant in our study. Laser intensity as a function of distance within material can be written as;

$I = I_0 e^{-\int_b^z a dx}$

Actually most of the beam intensities have Gaussian distribution. We made one more assumption that our laser beam is top-hat beam that means intensity is homogeneously distributed in spot area.

The heat generation from the laser beam absorbed by the material is defined as,

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2. Material and Experimental Setup

The surfaces of 10 mm thick PMMA sheets to be used were polished before ablation to cleaning and increase the transparency of the surfaces. Some physical and thermal properties of PMMA sheet which were used in ablation and mathematical modeling have been listed in Table 1. In the ablation process commercial 130 W CO_2 laser was used with different scan speeds at constant power. Laser spot diameter is 160 μm the laser beam intensity $6.5 \times 10^9 W/m^2$.

Table 1 Some physical and thermal properties of PMMA

Properties	Value	Unit
Density	1180	kg/m^3
Coefficient of Thermal Expansion	75	$(.10^{-6}K^{-1})$
Melting point	130	$°C$
Heat Deflection Temperature	95	$°C$
Specific heat	69	$J.K^{-1}kg^{-1}$
Thermal Conductivity	0.18	$W.m^{-1}.K^{-1}$

3. Results and Discussion

In this study, mathematical model has been proposed for the groove formation on PMMA sheet with various scan speeds and constant power. Groove sizes were measured from optical microscope images of ablated surfaces of PMMA sheets.

The Heat Deflection Zone boundary and molten zone boundary distances were calculated as 2059 μm and 1733 μm respectively. Temperatures at Heat Deflection boundary and molten zone boundary are 368 K and 403 K respectively. Fourier coefficients in the mathematical model were obtained using these boundary temperatures.

The coefficients in the temperature distribution equation φ_c and φ_s were calculated as 321.45 and -201.15 respectively. These coefficients depend on the thermo physical properties of PMMA. Then, in order to verify the validity of mathematical model, new grooves were obtained using 100, 150, 200, 250, 300, 350 mm/s scan speeds. To verify the mathematical model, these coefficients were used to calculate the melting and heat deflection temperatures for the same material and different scan speeds. The calculated temperatures for boundaries (melting and heat deflection region) are given in Table 3.

Table 2 Laser scan speeds and groove widths measured from images.

Scan Speed mm/s	Molten Zone width (μm)	Heat Deflection Zone width (μm)
50	1733	2059
100	1707	2027
150	1677	1991
200	1642	1949
250	1677	1897
300	1707	1830
350	1733	1735

Table 3 The calculated melting and heat deflection temperatures temperatures for boundaries.

Scan Speed mm/s		T(x,t) (K)	T(x,t) (K) (Calculated)	error
100	Melting	403	416.69	3.40
100	Heat Deflection	368	377.89	2.69
150	Melting	403	423.73	5.14
150	Heat Deflection	368	382.84	4.03
200	Melting	403	429.12	6.48
200	Heat Deflection	368	389.68	5.89
250	Melting	403	438.25	8.75
250	Heat Deflection	368	396.47	7.74
300	Melting	403	445.54	10.45
300	Heat Deflection	368	406.17	10.37
350	Melting	403	453.59	12.55
350	Heat Deflection	368	413.71	12.42

4. Conclusion

It can be used for different purposes such as improving the mechanical properties of the materials by laser processing the surfaces of polymer materials, as well as using them in electronic devices. It is very important for the quality of the product to control the dimensions of the geometries to be obtained by laser on the material. By modeling the heat dissipation mechanism in material processing with laser, the dimensions of the shape to be obtained on the material can be controlled. Applicable mathematical modeling plays an important role in explaining this mechanism. In accordance with the purpose of the study, applicable mathematical modeling has been created and the applicability of this model has been proven.

In this study, grooves were formed on Polymethyl Methacrylate (PMMA) Plates with different scanning speeds with CO2 laser. Since the scan speed of the laser is increased, the interaction time between the laser beam and the material decreases then the amount of energy transferred to the material also decreases. Measurements were made from high-resolution optical microscope images of the grooves created on PMMA. In this study, the distribution of heat energy transferred to the material was modeled mathematically. The change to groove size depending on the laser scan speed is modeled. The heat distribution that causes the formation of grooves is modeled with the Fourier method. First, material-specific coefficients were calculated with the proposed mathematical model. In order to prove the validity of these coefficients, 7 different grooves obtained with 7 different scanning speeds were examined. The results obtained show that the proposed mathematical model is reliable.

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The Hosoya Polynomial of the Schreier Graphs of the Grigorchuk Group and the Basilica Group

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Abstract. The Grigorchuk group was first introduced by R. Grigorchuk in 1980. Also the Basilica group was introduced in 2002 by R. Grigorchuk and A. Zuk. In the following years, it was shown that these groups have deep connections with profinite group theory and complex dynamics. These groups have been proven to provide the self-similarity property, reflecting the fractalness of some limit objects associated with them. The Schreier graph codifies the intangible structure of a group. It establishes an equivalence relationship created by cosets. The Schreier graphs of the Grigorchuk group and the Basilica group are a combination of cycles arranged in a tree-like form due to the recursive expression of the generators of these groups. In this work, we study the Hosoya polynomial of these graphs and try to characterize them.

1. Introduction

The Hosoya polynomial of a graph was presented in 1988 by H. Hosoya [8]. The concept *distance* is one of the basic elements used in graph theory. This important concept has gained a wide place among the applications of graph theory in other disciplines. Hosoya polynomial is also defined with the help of this important concept. The main contribution of the Hosoya polynomial is that it provides important information for graph invariants defined with the help of the concept of distance. The value at point 1 of the first derivative of the Hosoya polynomial of a graph gives us the Wiener index, which is an important topological index [5]. The Hosoya polynomial has gained an important place in chemical graph theory studies [4].

In this work, we study the Hosoya polynomial of Schreier graphs associated with the motion of two automorphism groups of a binary rooted tree. These are the Grigorchuk group and the Basilica group. The Tutte polynomial of these graphs was calculated in 2010 [3]. The Grigorchuk group was first introduced by R. Grigorchuk in 1980. It gives a fairly simple solution to the Burnside problem and the first example of a finitely generated group of intermediate growth, see [7]. Also the Basilica group was introduced in 2002 by R. Grigorchuk and A. Zuk [6]. To the work of V. Nekrashevych, it was seen that this group can be defined as the iterated monodromy group of the complex polynomial $z^2 - 1$ [9]. Thus, a compact limit space that is homeomorphic to the Basilica fractal can be associated with it. It is also the first example of amenable group that does not belong to subexponentially amenable groups [2]. It is proved that these groups are very

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closely related to complex dynamics and profinite group theory [1]. These groups provide a self-similarity property that reflects the fractalness of some limit objects associated with them [9].

We are here doing some calculations over the Schreier graphs of the Grigorchuk group and the Basilica group. Moreover, we reckon the Wiener index of some these graphs. We carry out these calculations by deleting the loops in the graphs. The Schreier graph codifies the intangible structure of a group. It establishes an equivalence relationship created by cosets. The Schreier graphs of the Grigorchuk group and the Basilica group are a combination of cycles formed in a tree-like way. Because the recursive expressions of the generators of these groups cause these graphs to have a cactus structure.

2. Preliminaries

Definition 2.1. ([5]) *Let $G = (V, E)$ be connected and distance-based graph. The distance $d(u, v)$ between any two vertices u and v is the minimum of the lenghts of paths between u and v. The topological diameter $d(G)$ of a graph G (i.e. the longest topological distance in G) is defined as*

$$d(G) = \max_{u,v \in V(G)} \{d(u, v)\}.$$

Definition 2.2. ([10]) *Let $D_k = \{(u, v)|\ u, v \in V(G)$ and $d(u, v) = k\}$ be a set and we denote the number of elements of D_k by $|D_k|$ i.e. $d(G, k) = |D_k|$, $k \geq 0$.*

Let $d(G, k)$, $k \geq 0$, be the number of vertex pairs at distance k. The Hosoya polynomial of G is defined as follows:

$$H(G, y) = \sum_{k=0}^{d(G)} d(G, k) y^k$$

where $d(G, 0) = n$ such that n is the number of vertices in G.

The Grigorchuk group and the Basilica group are a self-similar group of automorphisms of the rooted binary tree generated by some elements which are the trivial and the non-trivial permutations in the symmetric group on 2 elements $Sym(2)$ [3]. The Schreier graphs of these groups are recursively constructed within the framework of certain rules, see [3] for more detailed information. The symbol Γ_n indicates the Schreier graphs of the Grigorchuk group, for $n = 1, 2, 3, \ldots$, as seen in Figure 1 [3]. The symbol B_n indicates the Schreier graphs of the Basilica group, for $n = 1, 2, 3, \ldots$, as seen in Figures 2 and 4 [3].

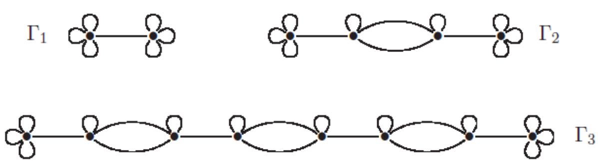

Figure 1: Some the Schreier graphs of the Grigorchuk group

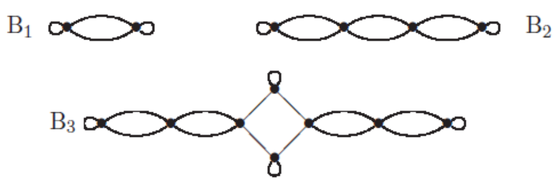

Figure 2: Some the Schreier graphs of the Basilica group

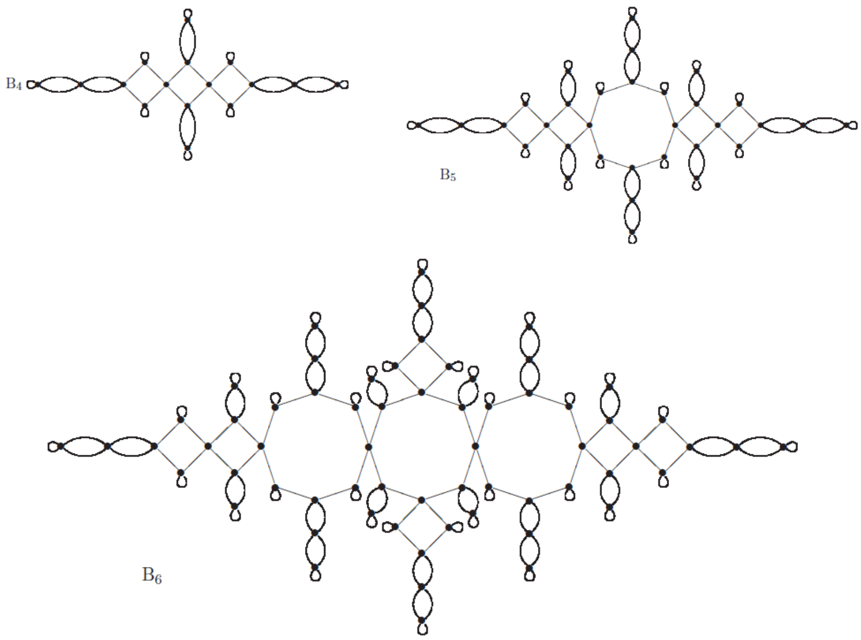

Figure 3: Some the Schreier graphs of the Basilica group

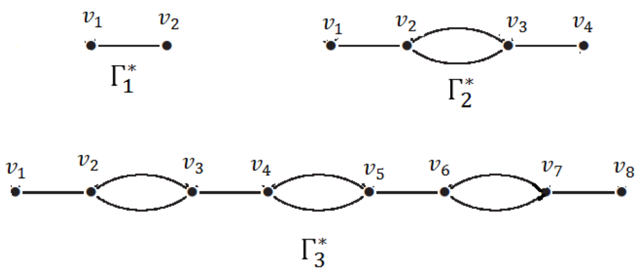

Figure 4: Some the Schreier graphs without loops of the Grigorchuk group

Since many calculations are inconclusive for graphs containing loops, we will consider the graphs obtained by deleting loops from these graphs, as seen in Figures 4 and 5.

The graphs Γ_n^* and B_n^* contain the values specified in the table below.

$d(\Gamma_1^*) = 1$	$d(\Gamma_2^*) = 3$	$d(\Gamma_3^*) = 7$	$d(\Gamma_4^*) = 15$	$d(\Gamma_5^*) = 31$
$d(B_1^*) = 1$	$d(B_2^*) = 3$	$d(B_3^*) = 6$	$d(B_4^*) = 10$	$d(B_5^*) = 16$

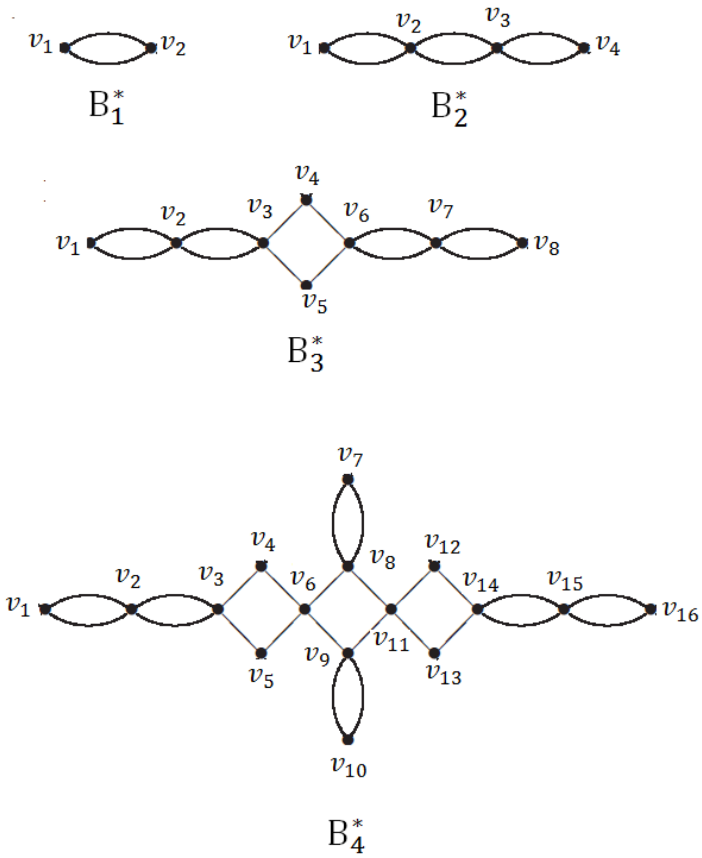

Figure 5: Some the Schreier graphs without loops of the Basilica group

3. Main Results

In this section, we will compute the Hosoya polynomials of the graphs Γ_n^* and B_n^*, $(n = 1, 2, 3, \dots)$.

For Γ_1^*:

$$D_0 = \{v_1, v_2\} \Rightarrow |D_0| = d(\Gamma_1^*, 0) = 2,$$
$$D_1 = \{(v_1, v_2)\} \Rightarrow |D_1| = d(\Gamma_1^*, 1) = 1,$$
$$\Rightarrow H(\Gamma_1^*, y) = 2y^0 + 1y^1$$
$$H(\Gamma_1^*, y) = 2 + y \tag{1}$$

For Γ_2^*:

$$D_0 = \{v_1, v_2, v_3, v_4\} \Rightarrow |D_0| = d(\Gamma_2^*, 0) = 4,$$
$$D_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\} \Rightarrow |D_1| = d(\Gamma_2^*, 1) = 3,$$
$$D_2 = \{(v_1, v_3), (v_2, v_4)\} \Rightarrow |D_2| = d(\Gamma_2^*, 2) = 2,$$
$$D_3 = \{(v_1, v_4)\} \Rightarrow |D_3| = d(\Gamma_2^*, 3) = 1,$$

$$\Rightarrow H(\Gamma_2^*, y) = 4y^0 + 3y^1 + 2y^2 + 1y^3$$
$$H(\Gamma_2^*, y) = 4 + 3y + 2y^2 + y^3$$

For Γ_3^*:

$$D_0 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \Rightarrow |D_0| = d(\Gamma_3^*, 0) = 8,$$
$$D_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_8)\} \Rightarrow |D_1| = d(\Gamma_3^*, 1) = 7,$$
$$D_2 = \{(v_1, v_3), (v_2, v_4), (v_3, v_5), (v_4, v_6), (v_5, v_7), (v_6, v_8)\} \Rightarrow |D_2| = d(\Gamma_3^*, 2) = 6,$$
$$D_3 = \{(v_1, v_4), (v_2, v_5), (v_3, v_6), (v_4, v_7), (v_5, v_8)\} \Rightarrow |D_3| = d(\Gamma_3^*, 3) = 5,$$
$$D_4 = \{(v_1, v_5), (v_2, v_6), (v_3, v_7), (v_4, v_8)\} \Rightarrow |D_4| = d(\Gamma_3^*, 4) = 4,$$
$$D_5 = \{(v_1, v_6), (v_2, v_7), (v_3, v_8)\} \Rightarrow |D_5| = d(\Gamma_3^*, 5) = 3,$$
$$D_6 = \{(v_1, v_7), (v_2, v_8)\} \Rightarrow |D_6| = d(\Gamma_3^*, 6) = 2,$$
$$D_7 = \{(v_1, v_8)\} \Rightarrow |D_7| = d(\Gamma_3^*, 7) = 1,$$
$$\Rightarrow H(\Gamma_3^*, y) = 8y^0 + 7y^1 + 6y^2 + 5y^3 + 4y^4 + 3y^5 + 2y^6 + 1y^7$$
$$H(\Gamma_3^*, y) = 8 + 7y + 6y^2 + 5y^3 + 4y^4 + 3y^5 + 2y^6 + y^7$$

Theorem 3.1. *The Hosoya polynomial of the Schreier graphs of the Grigorchuk group is defined as*

$$H(\Gamma_n^*, y) = \sum_{i=1}^{2^n} i y^{2^n - i}, \tag{2}$$

where $n = 1, 2, 3, \ldots$.

Proof. We will make the proof of the theorem by the induction method on n. Firstly, it is clear that the expression is $H(\Gamma_1^*, y) = y + 2$ for $n = 1$ and it is obvious. It follows from the equation (1). Then, for $n = k$, let us assume that the expression, i.e. the equation

$$H(\Gamma_k^*, y) = y^{2^k-1} + 2y^{2^k-2} + 3y^{2^k-3} + \cdots + 2^{k-1}y^{2^k-2^{k-1}} + 2^k y^{2^k-2^k}$$
$$H(\Gamma_k^*, y) = y^{2^k-1} + 2y^{2^k-2} + 3y^{2^k-3} + \cdots + 2^{k-1}y^{2^k-2^{k-1}} + 2^k,$$

is true. The correctness of the expression will now be shown for $n = k + 1$. For $n = 1, 2, 3, \ldots$ Γ_n^* has a linear shape formed by alternating bridges and 2-cycles. Moreover, for $n = 1, 2, 3, \ldots$ there are 2^n vertices and $3.2^{n-1} - 2$ edges in Γ_n^* and the diameter of Γ_n^* is equal to $2^n - 1$. It means that there are $\frac{2^{k+1}}{2}$ bridges among the edges in Γ_{k+1}^* and the remaining $(3.2^{n-1} - 2 - \frac{2^{k+1}}{2})$ edges in Γ_{k+1}^* are two by two parallel. For $n = k + 1$, there must be 2^{k+1} terms in the expansion of the expression. Therefore by the concept of distance in graphs and the definition of the Hosoya polynomial, for $n = k + 1$ it is obtained that

$$H(\Gamma_{k+1}^*, y) = y^{2^{k+1}-1} + 2y^{2^{k+1}-2} + 3y^{2^{k+1}-3} + \cdots + 2^k y^{2^{k+1}-2^k} + 2^{k+1}.$$

Thus the proof is completed. □

Proposition 3.2. *The Wiener index of the Schreier graphs of the Grigorchuk group is defined as*

$$W(\Gamma_n^*) = \sum_{i=1}^{2^n} i(2^n - i),$$

where $n = 1, 2, 3, \ldots$.

Proof. By applying the equation (2), The Wiener index of the Schreier graphs of the Grigorchuk group is obtained. It is reckoned as the first derivative of the polynomial of $H(\Gamma_n^*, y)$ at $y = 1$, i.e.,

$$\begin{aligned} \Big(H(\Gamma_n^*, y)\Big)' &= \Big(\sum_{i=1}^{2^n} iy^{2^n-i}\Big)' \\ &= \sum_{i=1}^{2^n} i(2^n-i)y^{2^n-i-1}, \\ \Big(H(\Gamma_n^*, 1)\Big)' &= \sum_{i=1}^{2^n} i(2^n-i) = W(\Gamma_n^*). \end{aligned}$$

So the proof is completed. □

Now let us give a few examples of calculating the Hosoya polynomials of B_n^*. According to the definition of the Hosoya polynomial, the following results are obtained by applying the method applied in the above calculations.

For $n = 1$,

$$H(B_1^*, y) = 2 + y.$$

For $n = 2$,

$$H(B_2^*, y) = 4 + 3y + 2y^2 + y^3.$$

For $n = 3$,

$$H(B_3^*, y) = 8 + 8y + 8y^2 + 6y^3 + 3y^4 + 2y^5 + y^6$$

For $n = 4$,

$$H(B_4^*, y) = 16 + 18y + 24y^2 + 24y^3 + 17y^4 + 14y^5 + 11y^6 + 6y^7 + 3y^8 + 2y^9 + y^{10}.$$

For $n = 5$,

$$H(B_5^*, y) = 32+36y+49y^2+62y^3+62y^4+64y^5+55y^6+42y^7+36y^8+30y^9+18y^{10}+14y^{11}+11y^{12}+6y^{13}+3y^{14}+2y^{15}+y^{16}.$$

Conclusion 3.3. *In the calculations for the Hosoya polynomial of the Schreier graphs of the Basilica group, as can be seen in the examples given above, the following can be stated: in the expansion of polynomials to be obtained for each n, although some values such as the number of terms, the degree of the terms, some of its beginning and last terms are known a general characterization of these polynomials is not possible in this way. Because there is no clarity for the coefficients of the polynomials. However, it is predicted that this problem can be solved by conducting a study on the array of the shape of the graph obtained for each n as different from the method followed here. This prediction stands as an open problem.*

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The Signatures and Boundary Components of The Groups $\hat{\Gamma}_{0,n}(N)$

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Abstract. In this paper, we established the group $\hat{\Gamma}_{0,n}(N)$ by group $\Gamma_{0,n}(N)$ extending with reflection. Then, we obtain boundary components in signature of the group and we get some calculation for link periods $2, 3, \infty$. And then, we constitute chain of reflections with fixed points via Extended Hoore-Uzzell Theorem in the group. Finally, The number of boundary components in the signature of some groups $\hat{\Gamma}_{0,p}(p)$ and $\hat{\Gamma}_{0,p}(p^2)$, p is a prime number, and the number of link periods was found.

1. Introduction and Preliminaries

Modular group and its congruence subgroups have an important role on discrete group theory. Many authors studied at this area such as Akbaş [1], Beşenk [3], Jones [6], Kader [7], Tekcan [10], etc.

Non-euclidean crystallographic groups (written NEC group) have an important role on discrete group theory and firstly defined by Wilkie [11]. And then Bujalance [4], Jones [6], Macbeath [8], etc. studied. So in this paper, we research signatures and boundary components of a special groups. And now we give some basic definitions and theorems for understanding our paper.

Definition 1.1. *[5] Let*

$$T(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, \quad \Delta = ad - bc > 0; \tag{1}$$

then dividing the numerator and denominator by $\sqrt{\Delta}$ *we obtain*

$$T(z) = \frac{\left(a/\sqrt{\Delta}\right)z + \left(b/\sqrt{\Delta}\right)}{\left(c/\sqrt{\Delta}\right)z + \left(d/\sqrt{\Delta}\right)}$$

and as $\left(a/\sqrt{\Delta}\right)\left(d/\sqrt{\Delta}\right) - \left(b/\sqrt{\Delta}\right)\left(c/\sqrt{\Delta}\right) = 1$, *this shows that* $T \in PSL(2,\mathbb{R})$. *We can show the elements of* $PSL(2,\mathbb{R})$ *as follows,*

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{R} \quad and \quad ad - bc = 1.$$

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Remark 1.2. *This set is a group of all linear fractional transformations. It is the automorphism group of the upper half plane* $\mathbb{H} := \{z \in \mathbb{C} : Im(z) > 0\}$.

Definition 1.3. *[5] The modular group* $\Gamma = PSL(2,\mathbb{Z})$ *is the subgroup of* $PSL(2,\mathbb{R})$.

Definition 1.4. *[11] The group G consist of all transformations of one or other of the two forms:*

$$w = \frac{az+b}{cz+d}, \quad ad-bc = 1 \quad a,b,c,d \in \mathbb{R}, \tag{2}$$

$$w = \frac{a\bar{z}+b}{c\bar{z}+d}, \quad ad-bc = -1 \quad a,b,c,d \in \mathbb{R}. \tag{3}$$

Those of the form (2) *preserve orientation, and form a subgroup* $LF(2,\mathbb{R})$ *of index2-the hyperbolic group; Those of the form* (3) *do not preserve orientation. G maps* $\mathbb{H}$ *into itself. The topology on G comes from the numbers* $a,b,c,d \in \mathbb{R}$.

Definition 1.5. *[11] Firstly, we assume that* $T \in PSL(2,\mathbb{R}) \setminus I$ *and* $T(z) = \frac{az+b}{cz+d}$. *Then*

1. *Hyperbolic if* $|a+d| > 2$ *with two fixed points on the real axis,*
2. *Elliptic if* $|a+d| < 2$ *with one fixed point in* $\mathbb{H}$,
3. *Parabolic if* $|a+d| = 2$ *with one fixed point multiplicity two on the real axis.*

Secondly, we assume that $S \in \overline{PSL}(2,\mathbb{Z})$ *and* $S(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$. *Then*

1. *Glide reflection if* $a+d \neq$ *with two fixed points on the real axis.*
2. *Reflection if* $a+d = 0$ *with hyperbolic line perpendicular to* $\mathbb{R}$.

Definition 1.6. *[11] A non-Euclidean crystallographic (written N. E. C.) group is a discrete subgroup of G.*

Theorem 1.7. *[5] Finite-order elements different from the unit of G are either elliptic or reflection transformations.*

Definition 1.8. *[9] We suppose that* Λ *is a NEC group and* $x \in \mathbb{R} \cup \{\infty\}$. *In this case, if there is a parabolic element* $g \in \Lambda$ *such that* $g(x) = x$, *then x is called "cusp point (cusp representative)". Hence, the expression of* Λx *which it is orbit* Λ *of x is called cusp and denoted by* $[x]$. *Moreover, if there is a reflection* $S \in \Lambda$ *such that* $S([x]) = [x]$, *then* $[x]$ *is called "real cusp".*

Remark 1.9. *Throughout this article we will study at finite generated NEC group* Λ *provided that the orbital space* $\mathbb{H}^*/\Lambda$ *is compact. Here,* $\mathbb{H}^* = \mathbb{H} \cup \mathcal{B}$, *and* $\mathcal{B} := \left\{[x] : x \in \mathbb{R}_\infty\right\}$.

Remark 1.10. *We can write the following table for generators and relations of NEC group* Λ *[8],[11]*

Table 2.1 : *Generators and relations of NEC group* Λ

Generators	x_i ; $i = 1,\dots,r$	
	e_i ; $i = 1,\dots,k$	
	c_{ij} ; $i = 1,\dots,k$ and $j = 0,1,\dots,s_i$	
	a_i, b_i ; $i = 1,\dots,g$	(I. kind)
	d_i ; $i = 1,\dots,g$	(II. kind)
Relations	$x_i^{m_i} = 1$; $i = 1,\dots,r$	
	$c_{is_i} = e_i^{-1} c_{i0} e_i$; $i = 1,\dots,k$	
	$c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1$	
	$x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1$	(I. kind)
	$x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_g^2 = 1$	(II. kind)

Here, let $\mathbb{N}_2 := \{2,3,\dots\}$. *If* $m_i \in \mathbb{N}_2$, *then* x_i *is an elliptic element. If* $m_i = \infty$, *then* x_i *is a parabolic element. If* $n_{ij} \in \mathbb{N}_2$, *then the combination of the two reflections is an elliptical element. And if* $n_{ij} = \infty$, *this combination is either a parabolic element or a hyperbolic element. It is clear that the numbers* $m_i, n_{ij} \in \mathbb{N}_2 \cup \{\infty\}$ *are the order of the direction-protecting elements of* Λ.

Definition 1.11. *[4] The representation*

$$\sigma(\Lambda) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})$$

is called a NEC signature of Λ for NEC group Λ given at Table 2.1. We say shortly $\sigma(\Lambda)$ or signature of Λ. Moreover, it is called some notions at the signature $\sigma(\Lambda)$ as follow:
(1.) *Number $g \in \mathbb{N}$ in the signature is called genus of orbit space's $\mathbb{H}^*/\Lambda$. And it is topologically invariant of surface.*
(2.) *If orbit space $\mathbb{H}^*/\Lambda$ can be directable, then $sgn\sigma(\Lambda) = " + "$ or indirectable, then $sgn\sigma(\Lambda) = " - "$.*
(3.) *For $i = 1, 2, \cdots, r$, the numbers $m_i \in \mathbb{N}_2$ is called natural period of Λ.*
(4.) *For $i = 1, 2, \cdots, r$, the numbers $m_i \in \mathbb{N}_2 \cup \{\infty\}$ is called special period of Λ.*
(5.) *The set $C = \{C_1, C_2, \cdots, C_k\}$ is called boundary component of Λ.*
(6.) *For $i = 1, 2, \cdots, k$, the notion $C_i = (n_{i_1}, n_{i_2}, \ldots, n_{is_i})$ are called i-th boundary component of signature or i-th periodic-cycles.*
(7.) *For $i = 1, 2, \cdots, k$, the numbers $n_{i_1}, n_{i_2}, \cdots, n_{is_i} \in \mathbb{N}_2 \cup \{\infty\}$ are called period of i-th boundary component or link period of Λ.*

Theorem 1.12. *[5] (Extended Hoare-Uzzell Theorem) Let G be a NEC group with signature*

$$\sigma(G) = \Big(g; \mp; [m_1, \cdots, m_r]; \{(n_{11}, \cdots, n_{1s_1}), \cdots, (n_{k1}, \cdots, n_{ks_k})\}\Big)$$

and H a subgroup of finite index. Each fixed point of a reflection c_i of the permutation representation of G on the H-cosets gives a reflection in H.

Let c_i, c_{i+1} be two reflections, with $c_i c_{i+1}$ having order $n_i \leq \infty$. Let $y_i = c_i c_{i+1}$ have an orbit (cycle) of length r_i. Then: either

- ***a)*** *this orbit contains no fixed points of c_i or c_{i+1} in which case there exists another orbit of the same length, and these two together induce an ordinary period n_i/r_i.*

or

- ***b)*** *this orbit contains two fixed points of c_i and c_{i+1} (one fixed by each if r_i is odd, two by one and one by the other if r_i is even): and there is a relation between two induced reflections as, $c_i \backsim^{n_i/r_i} c_{i+1}$. Combining these relations makes up period cycles with link periods n_i/r_i.*

Lemma 1.13. *[6] Let T, K be $\in \hat{\Gamma}_0(N)$*

$$T = \begin{pmatrix} r & -k \\ s & -t \end{pmatrix} \text{ and } K = \begin{pmatrix} x & -m \\ y & -n \end{pmatrix} \in \hat{\Gamma}$$

then,

$$\frac{r}{s} \approx \frac{x}{y} \iff ry - sx \equiv 0 \ mod \ N \ \ (ry - sx = \mp N).$$

Here the relation $"\approx"$ is on $\hat{\mathbb{Q}}$ that $\hat{\Gamma}_0(N)$ is a reduced $\hat{\Gamma}$ invariant equivalence relation,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \ : \ c \equiv 0 \ mod \ N \right\}, \ \hat{\Gamma}_0(N) := \Big\langle \Gamma_0(N), z \to -\bar{z} \Big\rangle,$$

$X_0(N) = \mathbb{H}^*/\Gamma_0(N)$ *and* $\hat{X}_0(N) = \mathbb{H}^*/\hat{\Gamma}_0(N)$.

Theorem 1.14. *[1] Let the numbers $N \in \mathbb{Z}^+$ and r are divisor number of N. We can write the followings for the group $\hat{\Gamma}_0(N)$:*

I. case: *If N is odd, then the number of boundary component of $X_0(N)$ is 2^{r-1} and there are 2 cusps in each boundary component.*

II. case: ***a)*** *Let $2 \| N$.*

i) *If* $N = 2$*, then there is only one boundary component. And there are* 2 *cusps belonging to it.*

ii) *If* $N = 2m$*,* $m > 1$*, then there are* 2^{r-2} *boundary component. And there are* 4 *cusps belonging to each boundary components.*

b) *Let* $2^2 \| N$*.*

i) *If* $N = 4$*, then there is only one boundary component. And there are* 3 *cusps belonging to it.*

ii) *If* $N > 4$*, then there are* 2^{r-2} *boundary component. And there are* 6 *cusps belonging to each boundary components.*

c) *If* $2^3 | N$*, then the number of boundary component are* 2^{r-1}*. And there are* 4 *cusps in each boundary component.*

2. Main Results

2.1. Signature of the Extended Congruence Subgroup

Let we consider the following extended congruence subgroup for $N \in \mathbb{Z}^+$

$$\hat{\Gamma}_0(N) = \left\langle \Gamma_0(N), z \to -\bar{z} \right\rangle = \Gamma_0(N) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_0(N).$$

Thus, $\hat{\Gamma}_\infty < \hat{\Gamma}_0(N) < \hat{\Gamma}$. If we take $u = \dfrac{r}{s}, v = \dfrac{x}{y} \in \hat{\mathbb{Q}}$, then there are $T, K \in \hat{\Gamma}$ such that $T(\infty) = u$ and $K(\infty) = v$

$$T = \begin{pmatrix} r & -k \\ s & -t \end{pmatrix} \text{ and } K = \begin{pmatrix} x & -m \\ y & -n \end{pmatrix}.$$

Now we consider the special subgroup of $\hat{\Gamma}_0(N)$ for $N \in \mathbb{Z}^+$, namely,

$$\hat{\Gamma}_{0,n}(N) = \left\langle \Gamma_{0,n}(N), z \to -\bar{z} \right\rangle = \Gamma_{0,n}(N) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_{0,n}(N).$$

Let we calculate in the signature of the group

$$\hat{\Gamma}_{0,n}(N) = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \hat{\Gamma}_0(N) \; : \; a \equiv \mp d \; mod \; n \right\}.$$

And also let we determine the orbit space $Y_0(N) = \mathbb{H}^* / \Gamma_{0,n}(N)$ and $\hat{Y}_0(N) = \mathbb{H}^* / \hat{\Gamma}_{0,n}(N)$ for $\Gamma_{0,n}(N)$ and $\hat{\Gamma}_{0,n}(N)$, respectively.

Theorem 2.1. *Let* $\hat{\Gamma}$ *be an extended modular group and*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{\Gamma}, \qquad c_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, c_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then,

a.) c_1 *leaves fixed to* $\hat{\Gamma}_{0,n}(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff N | 2cd$ *and* $(ad + bc)^2 \equiv 1 \; mod \; n$*,*

b.) c_2 *leaves fixed to* $\hat{\Gamma}_{0,n}(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff N | d^2 - c^2$ *and* $(bd - ac)^2 \equiv 1 \; mod \; n$*,*

c.) c_3 *leaves fixed to* $\hat{\Gamma}_{0,n}(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff N | 2cd - c^2$ *and* $(ad - ac + bc)^2 \equiv 1 \; mod \; n$*.*

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{\Gamma}$ and $\hat{\Gamma} = PSL(2, \mathbb{Z}) \cup \overline{PSL}(2, \mathbb{Z})$.

a)

$$\hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} c_1 = \hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} ad+bc & -2ab \\ 2cd & -bc-ad \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff N|2cd \text{ and } (ad+bc)^2 \equiv 1 \bmod n.$$

b)

$$\hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} c_2 = \hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff \begin{pmatrix} b & a \\ d & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} bd-ac & a^2-b^2 \\ d^2-c^2 & ac-bd \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff N|d^2-c^2 \text{ and } (bd-ac)^2 \equiv 1 \bmod n.$$

c)

$$\hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} c_3 = \hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} a & a-b \\ c & c-d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} ad-ac+bc & a^2-2ab \\ 2cd-c^2 & -bc+ac-ad \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff N|2cd-c^2 \text{ and } (ad-ac+bc)^2 \equiv 1 \bmod n.$$

So, the proof is completed.

Lemma 2.2. *Elliptic and parabolic elements generated with reflections of c_1, c_2, c_3 in $\hat{\Gamma}$ are determined as follows:*

a.) $T_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $T_1^2 = T_2^3 = T_3^\infty = I.$

b.) $T_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T_5 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, T_6 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ *and* $T_4^2 = T_5^3 = T_6^\infty = I.$

Proof. We know

$$c_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, c_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, (c_1c_2)^2 = (c_2c_3)^3 = (c_1c_3)^\infty = I.$$

Then,

$$\textbf{a)}\ T_1 = c_1c_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$T_2 = c_2c_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$
$$T_3 = c_1c_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this case, we obtain the relation $T_1^2 = T_2^3 = T_3^\infty = I$. Then,

$$\textbf{b)}\ \begin{aligned} T_4 &= c_2c_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ T_5 &= c_3c_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\ T_6 &= c_3c_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

So, we have $T_4^2 = T_5^3 = T_6^\infty = I$.

Remark 2.3. *The combinations of these transformations can also be used.*

$$(c_2c_3)^2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } (c_3c_1)^k = \begin{pmatrix} 1 & -k \\ -1 & 1 \end{pmatrix}$$

Lemma 2.4. *[1] $ad \equiv 1 \bmod s$ provides $a \equiv d \bmod s$ if and only if s is the integer divisor of* 24.

Proof. "$\Longrightarrow$": Let $ad \equiv 1 \ mod\ s$ provides the congruence $a \equiv d \ mod\ s$ and $U_s := \{a \in \mathbb{Z}_s \mid (a,s) = 1\}$. Here, $a^2 \equiv 1 \ mod\ s$ reduces to finding s for each $a \in U_s$ that satisfies the congruence. In this case, we assume that $s = 2^\alpha.3^\beta q_1^{\alpha_1} \dots q_k^{\alpha_k}, (q_i \in \mathbb{P}, q_i \neq 2, q_i \neq 3)$. So, we have $U_s \cong U_{2^\alpha} \times U_{3^\beta} \times U_{q_1^{\alpha_1}} \times \dots \times U_{q_k^{\alpha_k}}$. If p is odd prime number and $n \geq 1$, then U_{p^n} is cyclic. The order of these groups are $\varphi(3^\beta), \varphi(q_1^{\alpha_1}), \dots, \varphi(q_k^{\alpha_k})$, respectively. Here φ is an Euler function. Because each of these groups has two members with an order of 2. So β should be 1, and $q_i^{\alpha_i}$ does not exist. Thus, it is determined as $s = 2^\alpha 3^\beta$, either $\beta = 0$ or $\beta = 1$. On the other hand, if $\alpha \geq 3$, then $U_{2^\alpha} := \{\mp 5^t \ : \ 0 \leq t \leq 2^{\alpha-2}\}$. Here, mth order of 5 is exactly $2^{\alpha-2}$. If $\alpha > 3$, then m will be at least 4. But it is a contradiction because each elements of U_{2^α} have got 2nd order. So it should be $\alpha \leq 3$. Consequently, we obtain $s|24$.

"$\Longleftarrow$": Let $ad \equiv 1 \ mod\ s$ and $s|24$. In this case, due to $\varphi(24) = 8$ we determine the integer a and d such that $a, d \in \{1, 5, 7, 11, 13, 17, 19, 23\}$. That is, the counting number less than 24 and prime between 24 is 8, and let's make the selection according to the cluster above. In this case, we get $a^2 \equiv d^2 \equiv 1 \ mod\ s$. Thus, we obtain $a \equiv d \ mod\ s$.

$$\begin{aligned} &\alpha = 1 \Longrightarrow U_{2^1} := \{a \in \mathbb{Z}_2 \ : \ (a,2) = 1\} = \{1\} \text{ and } a^2 \equiv 1 \ mod\ 2, \\ &\alpha = 2 \Longrightarrow U_{2^2} := \{a \in \mathbb{Z}_4 \ : \ (a,4) = 1\} = \{1,3\} \text{ and } a^2 \equiv 1 \ mod\ 4, \\ &\alpha = 3 \Longrightarrow U_{2^3} := \{a \in \mathbb{Z}_8 \ : \ (a,8) = 1\} = \{1,3,5,7\} \text{ and } a^2 \equiv 1 \ mod\ 8, \\ &\alpha = 4 \Longrightarrow U_{2^4} := \{a \in \mathbb{Z}_{16} \ : \ (a,16) = 1\} = \{1,3,5,7,9,11,13,15\} \text{ and } a^2 \equiv 1 \ mod\ 16. \end{aligned}$$

Now, the order U_{16} is 4, but it does not. Namely, counting number α and β exist such that $0 \leq \beta \leq 1$ for $s = 2^\alpha 3^\beta$.

Theorem 2.5. *Let $n, N \in \mathbb{Z}^+$ and $n|N$. Then,*

a) $n|24 \Longleftrightarrow \Gamma_{0,n}(N) = \Gamma_0(N)$,

b) $n|24 \Longleftrightarrow \hat{\Gamma}_{0,n}(N) = \hat{\Gamma}_0(N)$.

Proof. **a)** "$\Longrightarrow$:" Let $n|24$. Thus, $\exists k \in \mathbb{Z}$ such that $24 = nk$. It is clear that $\Gamma_{0,n}(N) \subset \Gamma_0(N)$ from $\Gamma_{0,n}(N) \leq \Gamma_0(N)$. Now let we show $\Gamma_0(N) \subset \Gamma_{0,n}(N)$.

We take $T = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$. In this case, we have $detT = ad - bcN = 1$ and $ad \equiv 1 \ mod\ n$. We obtain $a \equiv d \ mod\ n$ from Lemma 2.4 for $n|24$ and $ad \equiv 1 \ mod\ n$. That is, $a^2 \equiv 1 \ mod\ n$ and thus $T \in \Gamma_{0,n}(N)$.

$''\Longleftarrow''$ Let $\Gamma_{0,n}(N)=\Gamma_0(N)$. We take $\begin{pmatrix} a & b \\ cN & d \end{pmatrix}\in\Gamma_{0,n}(N)=\Gamma_0(N)$. From this $ad-bcN=1$ and we obtain $ad\equiv 1 \ mod\ N$. Thus, $ad\equiv 1\ mod\ n$ from $n|N$. Furthermore, it should be $a\equiv d\ mod\ n$ from $T\in\Gamma_{0,n}(N)$ and $n|24$ from Lemma 2.4.

b) The proof is clear according to case of a) from $\hat{\Gamma}_{0,n}(N)=\Gamma_{0,n}(N)\cup R\Gamma_{0,n}(N)$ and $R(z)=-\bar{z}$ for $\Gamma_{0,n}(N)$. Now we prove for $R\Gamma_{0,n}(N)$.

$''\Longrightarrow''$: Let $n|24$, and $T=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ cN & d \end{pmatrix}\in R\Gamma_{0,n}(N)$. Thus, $\begin{pmatrix} a & b \\ -cN & -d \end{pmatrix}\in R\Gamma_{0,n}(N)$ and $-ad+bcN=-1$. If we use $-ad\equiv -1\ mod\ n$ and $n|24$ with Lemma 2.4, then $a\equiv d\ mod\ n$.

$''\Longleftarrow''$ Let $\hat{\Gamma}_{0,n}(N)=\hat{\Gamma}_0(N)$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ cN & d \end{pmatrix}\in R\Gamma_{0,n}(N)$. In this case, $-ad+bcN=-1$ and $a\equiv d\ mod\ n$. So, we also obtain $-ad\equiv -1\ mod\ n$ and $a\equiv d\ mod\ n$. And we have the same result $n|24$ from Lemma 2.4.

2.2. Boundary Components in the Signature

Theorem 2.6. *Let $p\in\mathbb{P}$. Then, it can be given for the boundary components in the signature of the group $\hat{\Gamma}_{0,p}(p)$ as follows:*
a) *If $p=2$, then the group's signature has one boundary component and there is one 2 valued link period and two cusp in this component.*
b) *If $p=3$, then the group's signature has one boundary component and there is one 3 valued link period and two cusp in this component.*
c) *If $p=5$, then the group's signature has one boundary component and there are two cusp in this component.*

Proof. **a)** Let $N=p=2$. Then from Theorem 2.5, we have $\hat{\Gamma}_{0,2}(2)=\hat{\Gamma}_0(2)$, and instead of the second terms of Theorem 2.1, only the first conditions can be examined.

$$c_1 \quad \text{reflection leaves fixed to the elements} \quad \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix},\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix},\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix},$$
$$c_2 \quad \text{reflection leaves fixed to the elements} \quad \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix},$$
$$c_3 \quad \text{reflection leaves fixed to the elements} \quad \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

The chain $\mathfrak{T}_1$ is below from Theorem 1.14 and Lemma 2.2 for boundary components;

$$^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}\overset{1}{\sim}{}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\overset{\infty}{\sim}{}^{c_1}\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}\overset{2}{\sim}{}^{c_2}\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}$$
$$\overset{1}{\sim}{}^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\overset{\infty}{\sim}{}^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\overset{1}{\sim}{}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}.$$

So, there is a boundary component in the group's signature. There is a 2-valued link period in the signature. And there are also two cusps in it.

b) Let $N=p=3$. From Theorem 2.5 we have $\hat{\Gamma}_{0,3}(3)=\hat{\Gamma}_0(3)$. And thus instead of the second terms of Theorem 2.1, only the first conditions can be examined.

$$c_1 \text{ reflection leaves fixed to the elements } \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix},$$

$$c_2 \text{ reflection leaves fixed to the elements } \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix},$$

c_3 reflection leaves fixed to the elements $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$.

The chain $\mathfrak{T}_2$ is below from Theorem 1.14 and Lemma 2.2 for boundary components;

$$^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \underset{\sim}{1} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \underset{\sim}{\infty} {}^{c_3}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \underset{\sim}{3} {}^{c_2}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$$

$$\underset{\sim}{1} {}^{c_2}\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix} \underset{\sim}{1} {}^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \underset{\sim}{\infty} {}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

So, there is a boundary component in the group's signature. There is a 3-valued link period in the boundary component. And there are also two cusps in the boundary component.

c) Let we research the group $\hat{\Gamma}_{0,5}(5)$ for $N = p = 5$.

i) The reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 5c & d \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ c & 5d \end{pmatrix}$. Here the condition of Theorem 2.1-a) satisfies. Indeed, we have $N|5cd$ and $(ad + 5bc)^2 \equiv 1 \ mod\ 5$ due to $ad - 5bc = \pm 1$. And then we get $(5ad + bc)^2 \equiv 1 \ mod\ 5$.

$$(ad)^2 \equiv 1 \ mod\ 5 \Longrightarrow ad \equiv \pm 1 \ mod\ 5 \Longrightarrow \begin{cases} a = 1 \text{ and } d = 1;4 \\ a = 2 \text{ and } d = 2;3 \\ a = 3 \text{ and } d = 2;3 \\ a = 4 \text{ and } d = 1;4 \end{cases}$$

So, $a \equiv -d \ mod\ 5$. Similarly, the same situation occurs with $(bc)^2 \equiv 1 \ mod\ 5$. Thus, the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} \pm 1 & k \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} k & \pm 1 \\ 1 & 0 \end{pmatrix}$. So, we have

$$\begin{pmatrix} a & b \\ 5c & d \end{pmatrix}\begin{pmatrix} \mp 1 & k \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 5c & d \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} a & -ak \mp b \\ 5c & -5kc \mp d \end{pmatrix} \in \hat{\Gamma}_{0,5}(5)$$

and

$$\begin{pmatrix} a & b \\ c & 5d \end{pmatrix}\begin{pmatrix} k & \mp 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & 5d \end{pmatrix}\begin{pmatrix} 0 & \pm 1 \\ -1 & k \end{pmatrix} = \begin{pmatrix} -b & \mp a + bk \\ -5d & \mp c + 5kd \end{pmatrix} \in \hat{\Gamma}_{0,5}(5).$$

In this case, the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 5c & d \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ c & 5d \end{pmatrix}$. Moreover, these elements $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} \pm 1 & k \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} k & \pm 1 \\ 1 & 0 \end{pmatrix}$ are in the same coset class. Thus, the reflection c_1 without breaking generality leaves fixed to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}$.

ii) From Theorem 2.1, the reflection c_2 leaves fixed to

$$\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 5|\, d^2 - c^2 \\ (bc - ad)^2 \equiv 1 \ mod\ 5. \end{cases}$$

From this, we have $5|(d-c)(d+c)$. And $5|d - c$ or $5|d + c$. Therefore $d - c \equiv 0 \ mod\ 5$ or $d + c \equiv 0 \ mod\ 5$. According to this, we can take either $c = d = 1$ or $c = -1, d = 1$.

The reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}$. So,

$$\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -1 & k \end{pmatrix} = \begin{pmatrix} a - b & -at + bk \\ 0 & k - t \end{pmatrix} \in \hat{\Gamma}_{0,5}(5)$$

and

$$\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 1 & k \end{pmatrix} = \begin{pmatrix} a+b & -at+bk \\ 0 & t+k \end{pmatrix} \in \hat{\Gamma}_{0,5}(5).$$

Hence the reflection c_2 leaves fixed to $\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix}$.

iii) From Theorem 2.1, the reflection c_3 leaves fixed to

$$\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 5|2cd - c^2 \\ (ad - ac + bc)^2 \equiv 1 \ mod\ 5. \end{cases}$$

Here, there are two important conditions. Hence, it can be taken either $c = 0, d = 1$ or $c = 2, d = 1$.

The reflection c_3 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}$. In this case, we have

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & k \end{pmatrix} = \begin{pmatrix} a & -at+bk \\ 0 & k \end{pmatrix} \in \hat{\Gamma}_{0,5}(5)$$

and

$$\begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -2 & k \end{pmatrix} = \begin{pmatrix} a-2b & -at+bk \\ 0 & -2t+k \end{pmatrix} \in \hat{\Gamma}_{0,5}(5).$$

So, the reflection c_3 leaves fixed to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$. The chain $\mathfrak{T}_3$ is below from the conditions *i*), *ii*), *iii*) with Theorem 1.14 and Lemma 2.2;

$$^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$$

$$\overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix} \overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

Hence, there is a boundary component in the signature. There are two ∞-valued link period in the boundary component.

Corollary 2.7. *We obtain the following results:*

a) *For the signature of* $\hat{\Gamma}_{0,1}(1) = \hat{\Gamma}_0(1)$; $C = \{(2, 3, \infty)\}$,

b) *For the signature of* $\hat{\Gamma}_{0,2}(2)$; $C = \{(\infty, 2, \infty)\}$,

c) *For the signature of* $\hat{\Gamma}_{0,3}(3)$; $C = \{(\infty, 3, \infty)\}$,

d) *For the signature of* $\hat{\Gamma}_{0,5}(5)$; $C = \{(\infty, \infty)\}$.

Theorem 2.8. *Let* $p \in \mathbb{P}$. *Then we can give the follows for the signature of the group* $\hat{\Gamma}_{0,p}(p^2)$ *in the boundary component,*

a) *If* $p = 2$, *then there is a boundary component in the signature and there are* 3 *cusp in the boundary component.*

b) *If* $p = 3$, *then there is a boundary component in the signature and there are* 2 *cusp in the boundary component.*

c) *If* $p = 5$, *then there is a boundary component in the signature and there are* 2 *cusp in the boundary component.*

Proof. **a)** Let $n = p = 2$ and $N = 2^2$. Then $\hat{\Gamma}_{0,2}(4) = \hat{\Gamma}_0(4)$ from Theorem 2.5, and hence instead of the second terms of Theorem 2.1, only the first conditions can be examined.

The reflection c_1 leaves fixed to the elements $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 1 & 2 \end{pmatrix}$,

The reflection c_2 leaves fixed to the elements $\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}$,

The reflection c_3 leaves fixed to the elements $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$.

So, the chain $\mathfrak{T}_4$ is below from Theorem 1.14 and Lemma 2.2

$$
{}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \overset{\infty}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 2 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}
$$
$$
\overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}
$$
$$
\overset{1}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{\infty}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.
$$

Hence, there is a boundary component in the group's signature, and there are 3 cusps in the boundary component.

b) Let $n = p = 3$ and $N = 3^2$. we have $\hat{\Gamma}_{0,3}(9) = \hat{\Gamma}_0(9)$ from Theorem 2.5, and instead of the second terms of Theorem 2.1, only the first conditions can be examined.

$$
\text{The reflection } c_1 \text{ leaves fixed to the elements } \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix},
$$
$$
\text{the reflection } c_2 \text{ leaves fixed to the elements } \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix},
$$
$$
\text{the reflection } c_3 \text{ leaves fixed to the elements } \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}.
$$

The chain $\mathfrak{T}_5$ is below from Theorem 1.14 and Lemma 2.2

$$
{}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}
$$
$$
\overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix} \overset{\infty}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.
$$

Hence there is a boundary component, and there are 2 cusps in the boundary component.

c) Let $n = p = 5$ and $N = 5^2$. Now we research the group $\hat{\Gamma}_{0,5}(25)$.

i) According to Theorem 2.1,

$$
\text{The reflection } c_1 \text{ leaves fixed to } \hat{\Gamma}_{0,5}(5^2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 25|2cd \\ (ad+bc)^2 \equiv 1 \ mod\ 5. \end{cases}
$$

In this case, the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 25c & d \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ c & 25d \end{pmatrix}$. Here, it satisfies Theorem 2.1-a). Indeed, firstly we have $N|25cd$ and $(ad+25bc)^2 \equiv 1 \ mod\ 5$ from $N = 25$ and $ad - 25bc = \pm 1$. Secondly, we have $N|25cd$ and $(25ad+bc)^2 \equiv 1 \ mod\ 5$ from $N = 25$ and $25ad - bc = \pm 1$. Hence the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} \mp 1 & k \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} k & \mp 1 \\ 1 & 0 \end{pmatrix}$. In this case, we obtain

$$
\begin{pmatrix} a & b \\ 25c & d \end{pmatrix}\begin{pmatrix} \mp 1 & k \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 25c & d \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & \mp 1 \end{pmatrix} = \begin{pmatrix} a & -ak \mp b \\ 25c & -25kc \mp d \end{pmatrix} \in \hat{\Gamma}_{0,5}(25)
$$

and

$$
\begin{pmatrix} a & b \\ c & 25d \end{pmatrix}\begin{pmatrix} k & \mp 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & 25d \end{pmatrix}\begin{pmatrix} 0 & \mp 1 \\ -1 & k \end{pmatrix} = \begin{pmatrix} -b & \mp a + bk \\ -25d & \mp c + 25kd \end{pmatrix} \in \hat{\Gamma}_{0,5}(25).
$$

From this, the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 25c & d \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ c & 25d \end{pmatrix}$. So, these elements and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} \mp 1 & k \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} k & \mp 1 \\ 1 & 0 \end{pmatrix}$ elements are in the same coset class. Therefore, the reflection c_1 leaves fixed to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}$.

ii) According to Theorem 2.1, the reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 25|d^2 - c^2 \\ (bd - ac)^2 \equiv 1 \ mod \ 5. \end{cases}$
From this, $25|d^2 - c^2 \Longrightarrow 5|(d-c)(d+c) \Longrightarrow$ if and only if $5|d-c$ or only $5|d+c$. So, we obtain $d - c \equiv 0 \ mod \ 5^2$ or $d + c \equiv 0 \ mod \ 5^2$. Hence we can take either $c = d = 1$ or $c = -1, d = 1$.
The reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$. Because of $25|1^2 - 1^2$ and $(a1 - b1)^2 \equiv 1 \ mod \ 5$, it satisfies Theorem 2.1. Then, the reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}$. In this case, we have

$$\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -1 & k \end{pmatrix} = \begin{pmatrix} a-b & -at+bk \\ 0 & k-t \end{pmatrix} \in \hat{\Gamma}_{0,5}(25)$$

and

$$\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -t \\ -1 & k \end{pmatrix} = \begin{pmatrix} -a-b & -at+bk \\ 0 & k+t \end{pmatrix} \in \hat{\Gamma}_{0,5}(25).$$

Hence, the reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}$. These elements $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} k & t \\ 1 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} k & t \\ -1 & 1 \end{pmatrix}$ are in the same coset. Thus, the reflection c_2 leaves fixed to $\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix}$.

iii) According to Theorem 2.1 the reflection c_3 leaves fixed to

$$\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 25|2cd - c^2 \\ (ad - ac + bc)^2 \equiv 1 \ mod \ 5. \end{cases}$$

In this case, there are either $c = 0, d = 1$ or $c = 2, d = 1$.

The reflection c_3 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}$. These elements satisfy the condition of Theorem 2.1-*c*). Thereby, we get

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & k \end{pmatrix} = \begin{pmatrix} a & -at+bk \\ 0 & k \end{pmatrix} \in \hat{\Gamma}_{0,5}(25)$$

and

$$\begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -2 & k \end{pmatrix} = \begin{pmatrix} a-2b & -at+bk \\ 0 & -2t+k \end{pmatrix} \in \hat{\Gamma}_{0,5}(25).$$

And these elements are also in the same coset. From this the reflection c_3 leaves fixed to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$. Hence, the chain $\mathfrak{T}_6$ is below from Theorem 1.14 and Lemma 2.2

$$^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix} \overset{\infty}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

Consequently, there is a boundary component in the group' s signature, and there are 2 cusps in the boundary component.

Corollary 2.9. *We obtain the following results:*

a) *For the signature of* $\hat{\Gamma}_{0,2}(4)$; $C = \{(\infty, \infty, \infty)\}$,

b) *For the signature of* $\hat{\Gamma}_{0,3}(9)$; $C = \{(\infty, \infty)\}$,

c) *For the signature of* $\hat{\Gamma}_{0,5}(25)$; $C = \{(\infty, \infty)\}$.

Corollary 2.10. *There are not* 2 *and* 3*-valued link periods in the signature of the group* $\hat{\Gamma}_{0,5}(5^{\alpha})$ *for* $\alpha \in \mathbb{Z}$ *and* $\alpha \geq 1$. *Then there is only one boundary component and there are two cusps in the group' s signature. Namely, the set of boundary component is* $C = \{(\infty, \infty)\}$.

3. Conclusions

Considering the investigations done so far, we can get more general results as in the Table 3.1 by using Theorem 2.5 as we did before, based on Theorem 1.14

It should be noted that there are no 2 and 3-valued link periods except the groups $\hat{\Gamma}, \hat{\Gamma}_{0,2}(2), \hat{\Gamma}_{0,3}(3)$. In all other cases there is a ∞-valued link period. These ∞-valued link periods appear to be associated with parabolic transformations and even with fixed points they left constant.

Table 3.1 : Boundary components of the signatures of the some groups $\hat{\Gamma}_{0,n}(N)$

The Group Name	The set of boundary component in the signature
$\hat{\Gamma}_{0,4}(4)$	$\{(\infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,4}(8)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,4}(16)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,4}(24)$	$\{(\infty, \infty, \infty, \infty), (\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,2}(6)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,6}(6)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,6}(12)$	$\{(\infty, \infty, \infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,6}(18)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,6}(24)$	$\{(\infty, \infty, \infty, \infty), (\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,8}(8)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,8}(16)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,8}(24)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,12}(12)$	$\{(\infty, \infty, \infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,12}(24)$	$\{(\infty, \infty, \infty, \infty), (\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,24}(24)$	$\{(\infty, \infty, \infty, \infty), (\infty, \infty, \infty, \infty)\}$

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The Pell-Fibonacci Sequence Modulo *m*

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Abstract. In [6], Deveci defined the Pell-Fibonacci sequence as follows:

$$P-F(n+4)=3P-F(n+3)-3P-F(n+1)-P-F(n)$$

for $n \geq 0$ with initial constants $P-F(0)=P-F(1)=P-F(2)=0, P-F(3)=1$. Also, he derived the permanental and determinantal representations of the Pell-Fibonacci numbers and he obtained miscellaneous properties of the Pell-Fibonacci numbers by the aid of the generating function and the generating matrix of the Pell-Fibonacci sequence. The linear recurrence sequences appear in modern research in many fields from mathematics, physics, computer, architecture to nature and art; see, for example, [2, 4, 13, 18]. In this paper, we obtain the cyclic groups which are produced by generating matrix of the Pell-Fibonacci sequence when read modulo m. Furthermore, we research the Pell-Fibonacci sequence modulo m, and then we derive the relationship between the order of the cyclic groups obtained and the periods of the Pell-Fibonacci sequence modulo m.

1. Introduction

In [6], Deveci defined the Pell-Fibonacci sequence which is directly related to the Pell and Fibonacci numbers as follows:

$$P-F(n+4)=3P-F(n+3)-3P-F(n+1)-P-F(n) \tag{1}$$

for $n \geq 0$ with initial constants $P-F(0)=P-F(1)=P-F(2)=0, P-F(3)=1$.

Then by an inductive argument, he gave the generating matrix of Pell-Fibonacci sequence as follows:

$$M_3=\begin{bmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix M_3 is said to be Pell-Fibonacci matrix.Then, he obtained that

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$$(M_3)^n = \begin{bmatrix} x^3_{n+4} & F_{n+2} + x^3_{n+3} - x^3_{n+4} & F_{n+3} + x^3_{n+4} - x^3_{n+5} & -x^3_{n+3} \\ x^3_{n+3} & F_{n+1} + x^3_{n+2} - x^3_{n+3} & F_{n+2} + x^3_{n+3} - x^3_{n+4} & -x^3_{n+2} \\ x^3_{n+2} & F_n + x^3_{n+1} - x^3_{n+2} & F_{n+1} + x^3_{n+2} - x^3_{n+3} & -x^3_{n+1} \\ x^3_{n+1} & F_{n-1} + x^3_n - x^3_{n+1} & F_n + x^3_{n+1} - x^3_{n+2} & -x^3_n \end{bmatrix} \quad (2)$$

for $n \geq 1$. It is important to note that $\det M_3 = 1$.

The linear recurrence sequences appear in modern research in many fields from mathematics, physics, computer, architecture to nature and art; see, for example, [2, 4, 13, 18]. Many authors have studied some special linear recurrence sequences in algebraic structures. Some of these proved that the lengths of the periods of the recurring sequences obtained by the reducing sequences by a modulo m are equal to the lengths of the ordinary recurrences in cyclic groups; see for example, [1, 3, 5, 7–15, 17, 20]. Wall [19] proved that the lengths of the periods of the recurring sequences obtained by reducing Fibonacci sequences by a modulo m are equal to the lengths of the ordinary 2-step Fibonacci recurrences in cyclic groups. Lü and Wang [16] obtained the rules for the orders of the cyclic groups generated by reducing the k-generalized Fibonacci matrix modulo m . Ozkan et al. [17] proved two original theorem concerning Wall number of the 3-step Fibonacci sequences and they gave conjectures concerning 3-step Fibonacci sequence.In this paper, we obtain the cyclic groups which are produced by generating matrix of the Pell-Fibonacci sequence when read modulo m. Also, we study the Pell-Fibonacci sequence modulo m. Finally, we derive the relationship between the order of the cyclic groups obtained and the periods of the Pell-Fibonacci sequence modulo m.

2. The Pell-Fibonacci Sequence Modulo m

For given a matrix $A = \left[a_{ij}\right]$ of integers, $A\,(\text{mod } m)$ means that the entries of A are reduced modulo m, that is, $A\,(mod\ m) = \left(a_{ij}\ (mod\ m)\right)$. Let us consider the set $\langle A\rangle_m = \left\{A^i\,(modm)\middle|\ i \geq 0\right\}$. If $\gcd(m, \det A) = 1$, then the set $\langle A\rangle_m$ is a cyclic group. Let the notation $|\langle A\rangle_m|$ denote the order of the set $\langle B\rangle_m$.

Since $\det M_3 = 1$, it is clear that the set $\langle M_3\rangle_m$ is a cyclic group for every positive integer m.

Theorem 2.1. ***(Wall [19]).*** *The number $k(s, p^n)$ divides $k(s, p^n)p^{n-1}$, and the two quantities are equal provided $k(s, p) = k(s, p^2)$*

Theorem 2.2. *Let p be a prime and let $\langle M_3\rangle_{p^m}$ be a cyclic groups. If u is the largest positive integer such that $\left|\langle M_3\rangle_p\right| = \left|\langle M_3\rangle_{p^u}\right|$, then $\left|\langle M_3\rangle_{p^v}\right| = p^{v-u}.\left|\langle M_3\rangle_p\right|$ for every $v \geq u$. In particular, if $\left|\langle M_3\rangle_p\right| \neq \left|\langle M_3\rangle_{p^2}\right|$, then $\left|\langle M_3\rangle_{p^v}\right| = p^{v-1}.\left|\langle M_3\rangle_p\right|$ for every $v \geq 2$.*

Proof. Let us consider the cyclic group $\langle M_3\rangle_{p^m}$. Suppose that s is a positive integer and $\left|\langle M_3\rangle_{p^m}\right|$ is denoted by $L_{P-F}\,(p^m)$. If $(M_3)^{L_{P-F}\left(p^{s+1}\right)} \equiv I\left(modp^{s+1}\right)$, then, we can write $(M_3)^{L_{P-F}\left(p^{s+1}\right)} \equiv I\,(modp^s)$ where I is a 4×4 identity matrix. Thus we get that $L_{P-F}\,(p^s)$ divides $L_{P-F}\left(p^{s+1}\right)$. Furthermore, if we denote $(M_3)^{L_{P-F}\left(p^s\right)} = I + \left(m^{(s)}_{ij} \cdot p^s\right)$, then by the binomial expansion, we may write

$$(M_3)^{L_{P-F}\left(p^s\right)\cdot p} = \left(I + \left(m^{(s)}_{ij} \cdot p^s\right)\right)^p = \sum_{i=0}^{p} \binom{p}{i} \left(m^{(s)}_{ij} \cdot p^s\right)^i \equiv I\left(modp^{s+1}\right)$$

This yields that $L_{P-F}\left(p^{s+1}\right)$ divides $L_{P-F}\left(p^{s}\right)\cdot p$. Thus, $L_{P-F}\left(p^{s+1}\right)=L_{P-F}\left(p^{s}\right)$ or $L_{P-F}\left(p^{s+1}\right)=L_{P-F}\left(p^{s}\right)\cdot p$. It is easy to see that the latter holds if and only if there is an $m_{ij}^{(s)}$ which is not divisible by p. Since u is the largest positive integer such that $L_{P-F}\left(p^{s}\right)=L_{P-F}\left(p^{u}\right)$, we have $L_{P-F}\left(^{u}\right)\neq L_{P-F}\left(p^{u+1}\right)$. Then there is an $m_{ij}^{(u+1)}$ which is not divisible by p. Thus we get that $L_{P-F}\left(p^{u+1}\right)\neq L_{P-F}\left(p^{u+2}\right)$. The proof is finished by induction on u. □

Reducing the Pell-Fibonacci sequence $\{P-F(n)\}$ by a modulo m, we obtain the following repeating sequence:

$$\{P-F^{m}(n)\}=\{P-F^{m}(0),P-F^{m}(1),\ldots,P-F^{m}(i),\ldots\}$$

where $P-F^{m}(i)=P-F(i)\,(modm)$. It has the same recurrence relation as in (1).

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. For example, the sequence $a,b,c,d,b,c,d,b,c,d,\ldots$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a,b,c,d,a,b,c,d,a,b,c,d,\ldots$ is simply periodic with period 4.

Theorem 2.3. *For every positive integer m, the Pell-Fibonacci sequence modulo m $\{P-F^{m}(n)\}$ is simply periodic.*

Proof. Let us consider set

$$X=\left\{(x_0,x_1,x_2,x_3)\,\middle|\,\text{x}_\text{i}\text{'s are integers such that}\quad 0\leq x_i\leq m-1\right\}.$$

Since $|X|=m^4$, there are m^4 distinct 4-tuples of elements of Z_m. Then it is easy to see that at least one of the 4-tuples appears twice in the sequence $\{P-F^{m}(n)\}$. Therefore, the subsequence following this 4-tuple repeats; hence, the sequence is periodic. Let

$$P-F^{m}(i+1)\equiv P-F^{m}(j+1),\ldots,P-F^{m}(i+3)\equiv P-F^{m}(j+3)$$

such that $i>j$, then $i\equiv j\,(mod4)$. From the definition of the Pell-Fibonacci sequence we can easily obtain

$$P-F^{m}(i)\equiv P-F^{m}(j),P-F^{m}(i-1)\equiv P-F^{m}(j-1),\ldots,P-F^{m}(i-j)\equiv P-F^{m}(0)$$

which implies that the $\{P-F^{m}(n)\}$ is a simply periodic sequence. □

The period of the sequence $\{P-F^{m}(n)\}$ is denoted by $h_{P-F}(m)$.

Example 2.4. *Some term of the Pell-Fibonacci sequence $\{P-F(n)\}$ are as follows:*

$$\{0,0,0,1,3,9,24,62,156,387,951,2323,5652,13716,33228,\ldots\}.$$

Reducing he Pell-Fibonacci sequence $\{P-F(n)\}$ by a modulo 2, the sequence becomes:

$$\{0,0,0,1,1,1,0,0,0,1,\ldots\}.$$

So, we obtained that the period of the sequence $\left\{P - F^2(n)\right\}$ *is* 6.

Similarly, Since the sequence becomes as shown:

$$\{0,0,0,1,0,0,0,2,0,0,0,1\ldots\}$$

for $m = 3$, *we have* $h_{P-F}(3) = 8$.

It is easily seen from equation (2) that $h_{P-F}(m) = |\langle M_3\rangle_m|$ for every positive integer m.

Theorem 2.5. *If m has the prime factorization* $m = \prod_{i=1}^{u} (p_i)^{s_i}$, $(u \geq 1)$ *where* p_i*'s are distinct primes. Then*

$$h_{P-F}(m) = lcm\left[h_{P-F}\left((p_1)^{s_1}\right), h_{P-F}\left((p_2)^{s_2}\right), \ldots, h_{P-F}\left((p_u)^{s_u}\right)\right].$$

Proof. Since $h_{P-F}\left((p_i)^{s_i}\right)$ is the length of the period of the sequence $\left\{P - F^{(p_i)^{s_i}}(n)\right\}$, the sequence repeats only after blocks of length $\lambda \cdot h_{P-F}\left((p_i)^{s_i}\right)$, $(\lambda \in \mathbb{N})$. Since $h_{P-F}(m)$, is period of the sequence $\{P - F^m(n)\}$, the sequence $\left\{h_{P-F}\left((p_i)^{s_i}\right)\right\}$ repeats after $h_{P-F}(m)$ terms for all values i. Thus $h_{P-F}(m)$ is the form $\lambda \cdot h_{P-F}\left((p_i)^{s_i}\right)$ for all values i, and since any such number gives a period of $\{P - F^m(n)\}$. So we get

$$h_{P-F}(m) = lcm\left[h_{P-F}\left((p_1)^{s_1}\right), h_{P-F}\left((p_2)^{s_2}\right), \ldots, h_{P-F}\left((p_u)^{s_u}\right)\right].$$

□

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The Padovan- Padovan p-Sequences in Groups

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Abstract. Erdag and Deveci [13] defined the Padovan-Padovan p-sequence and they studied properties of this sequence. Then, Akuzum and Deveci [1] studied the Padovan-Padovan p-sequence modulo m. Also, they discussed the connections between the order the cyclic groups obtained and the periods of the Padovan-Padovan p-sequence according to modulo m. In this paper, we redefine the Padovan-Padovan p-sequence by means of the elements of the groups and then, we examine this sequence in the finite groups in detail. Also, we obtain the lengths of the periods of the Padovan-Padovan 4-sequence in the semidihedral group SD_{2^m} as applications of the results obtained.

1. Introduction

Erdag and Deveci [13] defined the Padovan-Padovan p-sequence as shown:

$$Pa^{P,p}_{n+p+5} = 2Pa^{P,p}_{n+p+3} + Pa^{P,p}_{n+p+2} - Pa^{P,p}_{n+p+1} - Pa^{P,p}_{n+p} + Pa^{P,p}_{n+3} - Pa^{P,p}_{n+1} - Pa^{P,p}_{n}$$

for $p\,(4, 5, 6, \dots)$ and $n \geq 0$ with initial constants $Pa^{P,p}_{0} = \cdots = Pa^{P,p}_{p+3} = 0$, $Pa^{P,p}_{p+4} = 1$.

Also, they gave the Padovan-Padovan p-matrix as shown:

$$C_p = \begin{bmatrix} 0 & 2 & 1 & -1 & -1 & 0 & \cdots & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+5)\times(p+5)}.$$

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Then by an inductive argument, they derived that

$$\left(C_p\right)^n = \begin{bmatrix} Pa_{n+p+4}^{P,p} & Pa_{n+p+5}^{P,p} & -Pa_{n+p+4}^{P,p} + Pa_{n+p+3}^{P,p} + Pap\,(n+p+1) + Pap\,(n+1) & -Pa_{n+p+5}^{P,p} + Pap\,(n+p+4) & -Pa_{n+p+3}^{P,p} + Pap\,(n+3) & Pap\,(n+4) & \cdots & \\ Pa_{n+p+3}^{P,p} & Pa_{n+p+4}^{P,p} & -Pa_{n+p+3}^{P,p} + Pa_{n+p+2}^{P,p} + Pap\,(n+p) + Pap\,(n) & -Pa_{n+p+4}^{P,p} + Pap\,(n+p+3) & -Pa_{n+p+2}^{P,p} + Pap\,(n+2) & Pap\,(n+3) & \cdots & \\ Pa_{n+p+2}^{P,p} & Pa_{n+p+3}^{P,p} & -Pa_{n+p+2}^{P,p} + Pa_{n+p+1}^{P,p} + Pap\,(n+p-1) + Pap\,(n-1) & -Pa_{n+p+3}^{P,p} + Pap\,(n+p+2) & -Pa_{n+p+1}^{P,p} + Pap\,(n+1) & Pap\,(n+2) & \cdots & C_p^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ Pa_{n+1}^{P,p} & Pa_{n+2}^{P,p} & -Pa_{n+1}^{P,p} + Pa_{n}^{P,p} + Pap\,(n-2) + Pap\,(n-p-2) & -Pa_{n+2}^{P,p} + Pap\,(n+1) & -Pa_{n}^{P,p} + Pap\,(n-p) & Pap\,(n-p+1) & \cdots & \\ Pa_{n}^{P,p} & Pa_{n+1}^{P,p} & -Pa_{n}^{P,p} + Pa_{n-1}^{P,p} + Pap\,(n-3) + Pap\,(n-p-3) & -Pa_{n+1}^{P,p} + Pap\,(n) & -Pa_{n-1}^{P,p} + Pap\,(n-p-1) & Pap\,(n-p) & \cdots & \end{bmatrix},$$

where C_p^* is a matrix as follows:

$$C_p^* = \begin{bmatrix} Pap\,(n+p) & -Pa_{n+p+4}^{P,p} + Pap\,(n+p+1) & -Pa_{n+p+5}^{P,p} + Pap\,(n+p+2) & -Pa_{n+p+3}^{P,p} \\ Pap\,(n+p-1) & -Pa_{n+p+3}^{P,p} + Pap\,(n+p) & -Pa_{n+p+4}^{P,p} + Pap\,(n+p+1) & -Pa_{n+p+2}^{P,p} \\ Pap\,(n+p-2) & -Pa_{n+p+2}^{P,p} + Pap\,(n+p-1) & -Pa_{n+p+3}^{P,p} + Pap\,(n+p) & -Pa_{n+p+1}^{P,p} \\ \vdots & \vdots & \vdots & \vdots \\ Pap\,(n-3) & -Pa_{n+1}^{P,p} + Pap\,(n-2) & -Pa_{n+2}^{P,p} + Pap\,(n-1) & -Pa_{n}^{P,p} \\ Pap\,(n-4) & -Pa_{n}^{P,p} + Pap\,(n-3) & -Pa_{n+1}^{P,p} + Pap\,(n-2) & -Pa_{n-1}^{P,p} \end{bmatrix}.$$

Akuzum and Deveci [1] obtained the following repeating sequence, reducing the Padovan-Padovan p-sequences $\left\{Pa_n^{P,p}\right\}$ by a modulus m:

$$\left\{Pa_n^{P,p,m}\right\} = \left\{Pa_o^{P,p,m}, Pa_1^{P,p,m}, Pa_2^{P,p,m}, \ldots, Pa_i^{P,p,m}, \ldots\right\}$$

where $Pa_i^{P,p,m} = Pa_i^{P,p}\,(modm)$.

It is well-known that a sequence is periodic if, after certain points, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.

Theorem 1. (Akuzum and Deveci [1]). The sequence $\left\{Pa_n^{P,p,m}\right\}$ is simply periodic for every positive integer m.

The linear recurrence sequences in groups were firstly studied by Wall [15] who calculated the periods of the Fibonacci sequences in cyclic groups. In the mid-eighties, Wilcox [16] extended the problem to abelian groups and Campbell et al. [5] expanded the theory to some finite simple groups. Further, the concept extended to some special linear recurrence sequences by several authors; see for example, [2–4, 6–14]. In this paper, we redefine the Padovan-Padovan p-sequence by means of the elements of the groups and then, we examine this sequence in the finite groups in detail. Also, we obtain the lengths of the periods of the Padovan-Padovan 4-sequence in the semidihedral group SD_{2^m} as applications of the results obtained.

2. The Padovan- Padovan p-Sequences in Groups

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times G \cdots \times G}_{j}$ such that $\left(x_0, x_2, \ldots, x_{j-1}\right) \in X$ if and only if G is generated by $x_0,\ x_1,\ \ldots,\ x_{j-1}$. We call $\left(x_0, x_2, \ldots, x_{j-1}\right)$ a generating j-tuple for G.

Definition 2.1. *For a j-tuple $\left(x_0, x_1, \ldots, x_{j-1}\right) \in X$, we define the Padovan-Padovan p-orbit $PA^p\left(G : x_0, x_1, \ldots, x_{j-1}\right) = \left\{a_p\left(n\right)\right\}$ as shown:*

$$a_p\left(n+p+5\right) = \left(a_p\left(n\right)\right)^{-1}\left(a_p\left(n+3\right)\right)\left(a_p\left(n+p\right)\right)^{-1}\left(a_p\left(n+p+1\right)\right)^{-1}\left(a_p\left(n+p+2\right)\right)\left(a_p\left(n+p+3\right)\right)^{2}$$

where $n \geq 0$ and

$$\begin{cases} a_p(0) = x_0, a_p(1) = x_1, \ldots, a_p(j-1) = x_j, a_p(j) = e, \ldots, a_p(p+4) = e & \text{if} \quad j < p+4, \\ a_p(0) = x_0, a_p(1) = x_1, a_p(2) = x_2, \ldots, a_p(p+4) = x_{p+4} & \text{if} \quad j = p+4. \end{cases}$$

Theorem 2.2. *If G is a finite group, then a Padovan-Padovan p-orbit of the group G is simply periodic.*

Proof. Suppose that t is the order of the group G. Since there are t^{p+5} distinct $p+5$-tuples of elements of G, at least one of the $p+5$-tuples appears twice in a Padovan-Padovan p-orbit of the group G. Because of the repeating, the Padovan-Padovan p-orbit of the group G is periodic. Since the orbit $PA^p\left(G : x_0, x_1, \ldots, x_{j-1}\right)$ is periodic, there exist natural numbers i and j, with $i \equiv j\,(mod p+5)$, such that

$$a_p\left(i\right) = a_p\left(j\right), a_p\left(i+1\right) = a_p\left(j+1\right), \ldots, a_p\left(i+p+5\right) = a_p\left(j+p+5\right).$$

By the definition of the Padovan-Padovan p-orbit, it is clear that

$$a_p\left(n\right) = \left(a_p\left(n+3\right)\right)\left(a_p\left(n+p\right)\right)^{-1}\left(a_p\left(n+p+1\right)\right)^{-1}\left(a_p\left(n+p+2\right)\right)\left(a_p\left(n+p+3\right)\right)^{2}\left(a_p\left(n+p+5\right)\right)^{-1}.$$

Therefore, we obtain $a_p\left(i\right) = a_p\left(j\right)$, and hence

$$a_p\left(i-j\right) = a_p\left(0\right), a_p\left(i-j+1\right) = a_p\left(1\right), \ldots, a_p\left(i-j+p+5\right) = a_p\left(p+5\right),$$

which implies that the Padovan-Padovan p-orbit is simply periodic. □

We denote the length of the period of Padovan-Padovan p-orbit $PA^p\left(G : x_0, x_1, \ldots, x_{j-1}\right)$ by $hPA^p\left(G : x_0, x_1, \ldots, x_{j-1}\right)$.

In **[1]**, Akuzum and Deveci denoted the period of the sequence $\left\{Pa_n^{P,p,m}\right\}$ by $h_p\left(m\right)$.

Now we give the lengths of the periods of the Padovan-Padovan 4-orbit of the semidihedral group SD_{2^m} as applications of the results obtained.

The semidihedral group SD_{2^m}, $(m \geq 4)$ is defined by the presentation

$$SD_{2^m} = \left\langle x, y : x^{2^{m-1}} = y^2 = e,\ \ yxy = x^{2^{m-2}-1} \right\rangle.$$

Note that $|SD_{2^m}| = 2^m$, $|x| = 2^{m-1}$ and $\left|y\right| = 2$.

Theorem 2.3. *The length of the period of the Padovan-Padovan 4-orbit of the semidihedral group SD_{2^m} is $2^{m-2} \cdot h_4(2)$.*

Proof. We consider the length of the period of the the Padovan-Padovan 4-orbit in the semidihedral group by the aid of the period $h_4(2) = 14$. The orbit $PA^4(SD_{2^m} : x, y)$ is

$$a_4(0) = x,\ a_4(1) = y,\ a_4(2) = e, \ldots, a_4(8) = e.$$

Thus, we also have

$$\begin{aligned} &a_4(28i) = x^{4ir_1+1},\ a_4(28i+1) = x^{8ir_2}y,\ a_4(28i+2) = e,\ a_4(28i+3) = x^{8ir_3},\ a_4(28i+4) = x^{8ir_4},\\ &a_4(28i+5) = x^{-4ir_5},\ a_4(28i+6) = x^{8ir_6},\ a_4(28i+7) = x^{4ir_7},\ a_4(28i+8) = x^{4ir_8}, \end{aligned}$$

where $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8$ are positive integers such that $\gcd(r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8) = 1$ So we need the smallest $i \in \mathbb{N}$ such that $4i = 2^{m-1}.k$ $(k \in \mathbb{N})$. If we choose $i = 2^{m-3}$,we obtain

$$\begin{aligned} &a_4\left(2^{m-2}14\right) = x,\ a_4\left(2^{m-2}14+1\right) = y,\ a_4\left(2^{m-2}14+2\right) = e,\ a_4\left(2^{m-2}14+3\right) = e,\ a_4\left(2^{m-2}14+4\right) = e,\\ &a_4\left(2^{m-2}14+5\right) = e,\ a_4\left(2^{m-2}14+6\right) = e,\ a_4\left(2^{m-2}14+7\right) = e,\ a_4\left(2^{m-2}4+8\right) = e. \end{aligned}$$

Since the elements succeeding $a_4\left(2^{m-2}14\right)$, $a_4\left(2^{m-2}14+1\right)$,$a_4\left(2^{m-2}14+2\right)$,. . .,$a_4\left(2^{m-2}14+8\right)$ depend on x, y, e for their values and $h_4(2) = 14$, the cycle begins again with the $2^{m-2}.h_4(2)nd$ element. Thus it is verifed that the length of the period of the Padovan-Padovan 4-orbit of the semidihedral group SD_{2^m} is $2^{m-2} \cdot h_4(2)$. □

Example 2.4. *For $m = 4$, we consider the length of the period of the Padovan-Padovan 4-orbit in the semidihedral group SD_{2^4}. Since $h_4(2) = 14$, we have the sequence*

$$\begin{aligned} &a_4(0) = x,\ a_4(1) = y,\ a_4(2) = e,\ a_4(3) = e,\ a_4(4) = e,\ a_4(5) = e,\ a_4(6) = e,\ a_4(7) = e,\ a_4(8) = e, \ldots,\\ &a_4(28) = x^5,\ a_4(29) = y,\ a_4(30) = e,\ a_4(31) = e,\ a_4(32) = e, a_4(33) = x^4,\ a_4(34) = e,\ a_4(35) = x^4,\ a_4(36) = x^4, \ldots,\\ &a_4(56) = x^5,\ a_4(57) = y,\ a_4(58) = e,\ a_4(59) = e,\ a_4(60) = e, a_4(61) = x,\ a_4(62) = e,\ a_4(63) = e,\ a_4(64) = e, \ldots. \end{aligned}$$

Since $a_4(0) = a_4(56)$,$a_4(1) = a_4(57)$,$a_4(2) = a_4(58)$,$a_4(3) = a_4(59)$,$a_4(4) = a_4(60)$, $a_4(5) = a_4(61)$,$a_4(6) = a_4(61)$, $a_4(7) = a_4(62)$, $a_4(8) = a_4(63)$ the length of the period of the the Padovan-Padovan 4-orbit $PA^4(SD_{2^4} : x, y)$ is 56.

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New Integral Inequalities of Ostrowski Type for Quasi-Convex Functions with Applications

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Abstract. In this paper some new Ostrowski-type inequalities for functions whose derivatives in absolute values are quasi-convex are established. Some applications to special means of real numbers and applications for P.D.F.'s are given. We also give some applications of our results to get new error bounds for the sum of the midpoint formula.

1. Introduction

We recall that the notion of quasi-convex functions as following.

Definition 1.1. *(See [7]) A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if*

$$f(tx + (1 - t)y) \leq \max \{f(x), f(y)\}, \text{ for all } x, y \in [a, b].$$

It is to be noted that any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see e.g. [2]-[6]).

Let $f : I \subset [0, \infty] \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $\left|f'(x)\right| \leq M$, then the following inequality holds (see [8]).

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right] \tag{1}$$

This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve and extend the inequality (1), see [2] and the references therein.

In [4], Alomari and Darus proved several inequalities of Ostrowski type for quasi-convex functions, we will mention some them as following.

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Theorem 1.2. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is quasi-convex function on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq \frac{(x-a)^2}{2(b-a)} \max\left\{ \left|f'(x)\right|, \left|f'(a)\right| \right\} + \frac{(b-x)^2}{2(b-a)} \max\left\{ \left|f'(x)\right|, \left|f'(b)\right| \right\}
\end{aligned}$$

for each $x \in [a,b]$.

Theorem 1.3. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq \left(\frac{(b-x)^{p+1}}{(b-a)(p+1)} \right)^{\frac{1}{p}} \left(\max\left\{ \left|f'(b)\right|^q, \left|f'(x)\right|^q \right\} \right)^{\frac{1}{q}} \\
&+ \left(\frac{(x-a)^{p+1}}{(b-a)(p+1)} \right)^{\frac{1}{p}} \left(\max\left\{ \left|f'(a)\right|^q, \left|f'(x)\right|^q \right\} \right)^{\frac{1}{q}}
\end{aligned}$$

for each $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

The main aim of this paper is to establish some new inequalities of Ostrowski type for quasi-convex functions and to give some deduced results to the celebrated Hadamard integral inequality. Based on these results, we obtain several applications for special means of real numbers, numerical integration and P.D.F.

2. Main Results

To prove our results we need the following Lemma:

Lemma 2.1. *(See [1]) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$ and $f' \in L([a,b])$. Then*

$$\begin{aligned}
&f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \\
&= \frac{(x-a)^2}{4(b-a)} \left(\int_0^1 t f'\left(t\frac{a+x}{2} + (1-t)a \right) dt \right. \\
&\left. + \int_0^1 (1+t) f'\left(tx + (1-t)\frac{a+x}{2} \right) dt \right) \\
&- \frac{(b-x)^2}{4(b-a)} \left(\int_0^1 (2-t) f'\left(t\frac{b+x}{2} + (1-t)x \right) dt \right. \\
&\left. + \int_0^1 (1-t) f'\left(tb + (1-t)\frac{b+x}{2} \right) dt \right).
\end{aligned}$$

By using the Lemma 2.1 the following results can be obtained:

Theorem 2.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is quasi-convex function on $[a, b]$, then one has the following inequality:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (2)\\
&\le \frac{(x-a)^2}{8(b-a)} \max\left\{ \left| f'\left(\frac{a+x}{2}\right)\right|, \left|f'(a)\right| \right\}\\
&+ \frac{3(x-a)^2}{8(b-a)} \max\left\{ \left|f'(x)\right|, \left| f'\left(\frac{a+x}{2}\right)\right| \right\}\\
&+ \frac{3(b-x)^2}{8(b-a)} \max\left\{ \left| f'\left(\frac{b+x}{2}\right)\right|, \left|f'(x)\right| \right\}\\
&+ \frac{(b-x)^2}{8(b-a)} \max\left\{ \left|f'(b)\right|, \left| f'\left(\frac{b+x}{2}\right)\right| \right\},
\end{aligned}$$

for all $x \in [a, b]$.

Proof. From the integral identity that is given in Lemma 2.1 and by using the properties of modulus, we can write

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (3)\\
&\le \frac{(x-a)^2}{4(b-a)} \left(\int_0^1 t \left| f'\left(t\frac{a+x}{2} + (1-t)a\right)\right| dt \right.\\
&\left. + \int_0^1 (1+t) \left| f'\left(tx + (1-t)\frac{a+x}{2}\right)\right| dt \right)\\
&- \frac{(b-x)^2}{4(b-a)} \left(\int_0^1 (2-t) \left| f'\left(t\frac{b+x}{2} + (1-t)x\right)\right| dt \right.\\
&\left. + \int_0^1 (1-t) \left| f'\left(tb + (1-t)\frac{b+x}{2}\right)\right| dt \right).
\end{aligned}$$

By using quasi-convexity of $|f'|$, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (4)\\
&\le \frac{(x-a)^2}{4(b-a)} \max\left\{ \left| f'\left(\frac{a+x}{2}\right)\right|, \left|f'(a)\right| \right\} \int_0^1 t\,dt\\
&+ \frac{(x-a)^2}{4(b-a)} \max\left\{ \left|f'(x)\right|, \left| f'\left(\frac{a+x}{2}\right)\right| \right\} \int_0^1 (1+t)\,dt\\
&+ \frac{(b-x)^2}{4(b-a)} \max\left\{ \left| f'\left(\frac{b+x}{2}\right)\right|, \left|f'(x)\right| \right\} \int_0^1 (2-t)\,dt\\
&+ \frac{(b-x)^2}{4(b-a)} \max\left\{ \left|f'(b)\right|, \left| f'\left(\frac{b+x}{2}\right)\right| \right\} \int_0^1 (1-t)\,dt,
\end{aligned}$$

for all $x \in [a, b]$.

By using the facts that

$$\int_0^1 (1+t)\,dt = \int_0^1 (2-t)\,dt = \frac{3}{2}$$
$$\int_0^1 t dt = \int_0^1 (1-t)\,dt = \frac{1}{2}$$

we get the inequality (2). This completes the proof of the theorem. □

An immediate consequence of Theorem 2.2 is the following:

Corollary 2.3. *If all the assumptions of Theorem 2.2 are satisfied and if we choose* $x = \frac{a+b}{2}$, *we get the following inequality:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\leq \frac{b-a}{32}\left[\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|,\left|f'(a)\right|\right\}\right. \\ &+3\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'\left(\frac{3a+b}{4}\right)\right|\right\} \\ &+3\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\} \\ &\left.+\max\left\{\left|f'(b)\right|,\left|f'\left(\frac{a+3b}{4}\right)\right|\right\}\right]\end{aligned} \tag{5}$$

Additionally,

1. *If* $|f'|$ *is increasing, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\leq \frac{b-a}{32}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+3\left|f'\left(\frac{a+b}{2}\right)\right|+3\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'(b)\right|\right].\end{aligned} \tag{6}$$

2. *If* $|f'|$ *is decreasing, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\leq \frac{b-a}{32}\left[\left|f'(a)\right|+3\left|f'\left(\frac{3a+b}{4}\right)\right|+3\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right].\end{aligned} \tag{7}$$

Corollary 2.4. *If all the assumptions of Theorem 2.2 are satisfied and if we choose* $x = a$ *and* $x = b$, *respectively, we get the following inequalities:*

$$\begin{aligned}&\left|f(a)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\leq \frac{3(b-a)}{8}\max\left\{\left|f'(a)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\} \\ &\quad+\frac{(b-a)}{8}\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(b)\right|\right\}\end{aligned}$$

and

$$\begin{aligned}&\left|f(b)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\\&\le\frac{(b-a)}{8}\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(a)\right|\right\}\\&+\frac{3(b-a)}{8}\max\left\{\left|f'(b)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\}.\end{aligned}$$

Additionally, if we add these inequalities and by choosing $|f'|$ is increasing and decreasing, respectively, then we obtain:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|\le\frac{(b-a)}{4}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right]$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|\le\frac{(b-a)}{4}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(a)\right|\right].$$

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following theorem.

Theorem 2.5. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$ for some fixed $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \qquad (8)\\&\le\frac{1}{4(b-a)(p+1)^{\frac{1}{p}}}\left\{(x-a)^2\left(\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\&+(x-a)^2\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(x)\right|^q,\left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\&+(b-x)^2\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(x)\right|^q\right\}\right)^{\frac{1}{q}}\\&\left.+(b-x)^2\left(\max\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{b+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

for all $x\in[a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and by using the Hölder integral inequality, we get

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (9)\\
&\le \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left| f'\left(t\frac{a+x}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+ \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 (1+t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left| f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+ \frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (2-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left| f'\left(t\frac{b+x}{2}+(1-t)x\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+ \frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left| f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{\frac{1}{q}},
\end{aligned}$$

for all $x \in [a,b]$.
Since $\left|f'\right|^q$ is quasi-convex on $[a,b]$, we know

$$\int_0^1 \left| f'\left(t\frac{a+x}{2}+(1-t)a\right)\right|^q dt \le \max\left\{\left| f'\left(\frac{a+x}{2}\right)\right|^q, \left|f'(a)\right|^q\right\}. \qquad (10)$$

Similarly,

$$\int_0^1 \left| f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt \le \max\left\{\left|f'(x)\right|^q, \left| f'\left(\frac{a+x}{2}\right)\right|^q\right\}, \qquad (11)$$

$$\int_0^1 \left| f'\left(t\frac{b+x}{2}+(1-t)x\right)\right|^q dt \le \max\left\{\left| f'\left(\frac{b+x}{2}\right)\right|^q, \left|f'(x)\right|^q\right\} \qquad (12)$$

and

$$\int_0^1 \left| f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt \le \max\left\{\left|f'(b)\right|^q, \left| f'\left(\frac{b+x}{2}\right)\right|^q\right\}. \qquad (13)$$

Using these inequalities in (9) and by making use of the necessary computations, the desired result is obtained. □

The following corollary is an immediate consequence of Theorem 2.5:

Corollary 2.6. *Suppose all the assumptions of Theorem 2.5 are satisfied. If we choose* $x = \frac{a+b}{2}$*, we get the following inequality:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (14)\\
&\le \frac{b-a}{16(p+1)^{\frac{1}{p}}}\left\{\left(\max\left\{\left| f'\left(\frac{3a+b}{4}\right)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\
&+ \left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left| f'\left(\frac{a+b}{2}\right)\right|^q, \left| f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&+ \left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left| f'\left(\frac{a+3b}{4}\right)\right|^q, \left| f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\left.+ \left(\max\left\{\left|f'(b)\right|^q, \left| f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Additionally,

1. *If* $|f'|^q$ *is increasing, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{15}$$
$$\leq \frac{b-a}{16(p+1)^{\frac{1}{p}}}\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.$$
$$\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right\}.$$

2. *If* $|f'|^q$ *is decreasing, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{16}$$
$$\leq \frac{b-a}{16(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right|^q+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right.$$
$$\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right].$$

A more general inequality can be given as follows:

Theorem 2.7. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^q$ *is quasi-convex on* $[a,b]$ *for some fixed* $q \geq 1$*, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{17}$$
$$\leq \frac{(x-a)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{3(x-a)^2}{8(b-a)}\left(\max\left\{\left|f'(x)\right|^q,\left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{3(b-x)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(x)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}},$$

for all $x \in [a,b]$.

Proof. Suppose that $q \geq 1$. From Lemma 2.1 and by using the well-known power-mean inequality, we have

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (18) \\ & \leq \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{a+x}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\ & + \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 (1+t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1+t)\left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\ & + \frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (2-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (2-t)\left|f'\left(t\frac{b+x}{2}+(1-t)x\right)\right|^q dt\right)^{\frac{1}{q}} \\ & + \frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{\frac{1}{q}}, \end{aligned}$$

for all $x \in [a,b]$.
By making use of the similar computations the proof of the theorem is completed. □

Corollary 2.8. *If all the assumptions of Theorem 2.7 are satisfied and if we choose $x = \frac{a+b}{2}$, we get the inequality:*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (19) \\ & \leq \frac{b-a}{32}\left\{\left(\max\left\{|f'(a)|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ & + 3\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ & + 3\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ & \left. + \left(\max\left\{|f'(b)|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Additionally,

1. *If $|f'|^q$ is increasing, then (6) holds.*
2. *If $|f'|^q$ is decreasing, then (7) holds.*

3. Applications to Special Means

Let consider the means for arbitrary real numbers $a, b \in \mathbb{R}$. We denote by

1. The arithmetic mean:
$$A(a,b) = \frac{a+b}{2}; a, b \in \mathbb{R}.$$
2. The harmonic mean:
$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}; a, b \in \mathbb{R}, a, b \neq 0.$$

3. The logarithmic mean:

$$L(a,b) = \frac{\ln|b| - \ln|a|}{b-a}; a,b \in \mathbb{R}, |a| \neq |b|, a,b \neq 0.$$

4. Generalized log-mean:

$$L_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}}; a,b \in \mathbb{R}, n \in \mathbb{Z} \setminus \{-1,0\}, a \neq b.$$

Now, it is time to give some applications to special means of real numbers by using the results of Section 2.

Proposition 3.1. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then*

$$\begin{aligned} &\left|A^n(a,b) - L_n^n(a,b)\right| \\ &\leq n\left(\frac{b-a}{32}\right)\left[\max\left\{\left|\frac{3a+b}{4}\right|^{n-1}, |a|^{n-1}\right\}\right. \\ &+ 3\max\left\{\left|\frac{3a+b}{4}\right|^{n-1}, \left|\frac{a+b}{2}\right|^{n-1}\right\} \\ &+ 3\max\left\{\left|\frac{a+3b}{4}\right|^{n-1}, \left|\frac{a+b}{2}\right|^{n-1}\right\} \\ &\left.+ \max\left\{\left|\frac{a+3b}{4}\right|^{n-1}, |b|^{n-1}\right\}\right]. \end{aligned} \tag{20}$$

Proof. The assertion follows from Corollary 2.3 when applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$. □

Proposition 3.2. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned} &\left|A^n(a,b) - L_n^n(a,b)\right| \\ &\leq n\frac{b-a}{16(p+1)^{\frac{1}{p}}}\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{q(n-1)}, |a|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right. \\ &+ \left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|\frac{3a+b}{4}\right|^{q(n-1)}, \left|\frac{a+b}{2}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}} \\ &+ \left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|\frac{a+b}{2}\right|^{q(n-1)}, \left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}} \\ &\left.+ \left(\max\left\{|b|^{q(n-1)}, \left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{21}$$

Proof. The assertion follows from Corollary 2.6 when applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$. □

Proposition 3.3. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then $q \geq 1$, we have*

$$\begin{aligned}
&\left|A^n(a,b) - L_n^n(a,b)\right| \qquad (22)\\
&\leq n\left(\frac{b-a}{32}\right)\left\{\left(\max\left\{|a|^{q(n-1)}, \left|\frac{3a+b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right.\\
&+3\left(\max\left\{\left|\frac{a+b}{2}\right|^{q(n-1)}, \left|\frac{3a+b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\\
&+3\left(\max\left\{\left|\frac{a+b}{2}\right|^{q(n-1)}, \left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\\
&\left.+\left(\max\left\{|b|^{q(n-1)}, \left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. The assertion follows from Corollary 2.8 when applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$. □

Proposition 3.4. *Let $a, b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$. Then*

$$\begin{aligned}
&\left|A^{-1}(a,b) - L^{-1}(a,b)\right| \qquad (23)\\
&\leq \frac{b-a}{32}\left[\max\left\{\left|\frac{3a+b}{4}\right|^{-2}, |a|^{-2}\right\} + 3\max\left\{\left|\frac{3a+b}{4}\right|^{-2}, \left|\frac{a+b}{2}\right|^{-2}\right\}\right.\\
&\left.+3\max\left\{\left|\frac{a+3b}{4}\right|^{-2}, \left|\frac{a+b}{2}\right|^{-2}\right\} + \max\left\{\left|\frac{a+3b}{4}\right|^{-2}, |b|^{-2}\right\}\right].
\end{aligned}$$

Proof. It is a direct consequence of Corollary 2.3 when applied to the function, $f(x) = \frac{1}{x}$, $x \in [a, b] \setminus \{0\}$. □

Proposition 3.5. *Let $a, b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$, then for all $p > 1$, we have*

$$\begin{aligned}
&\left|A^{-1}(a,b) - L^{-1}(a,b)\right| \qquad (24)\\
&\leq \frac{b-a}{16(p+1)^{\frac{1}{p}}}\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q}, |a|^{-2q}\right\}\right)^{\frac{1}{q}}\right.\\
&+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q}, \left|\frac{a+b}{2}\right|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\\
&+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|\frac{a+b}{2}\right|^{-2q}, \left|\frac{a+3b}{4}\right|^{-2q}\right\}\right)^{\frac{1}{q}}\\
&\left.+\left(\max\left\{|b|^{-2q}, \left|\frac{a+3b}{4}\right|^{-2q}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. It follows directly from Corollary 2.6 for the function, $f(x) = \frac{1}{x}$, $x \in [a, b] \setminus \{0\}$. □

Proposition 3.6. *Let $a, b \in \mathbb{R}$, $a < b$, $0 \notin [a,b]$. Then for all $q \geq 1$, we have the inequality*

$$\begin{aligned}
&\left|A^{-1}(a,b) - L^{-1}(a,b)\right| \qquad (25)\\
&\leq \frac{b-a}{32}\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q}, |a|^{-2q}\right\}\right)^{\frac{1}{q}}\right.\\
&+3\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q}, \left|\frac{a+b}{2}\right|^{-2q}\right\}\right)^{\frac{1}{q}}\\
&+3\left(\max\left\{\left|\frac{a+b}{2}\right|^{-2q}, \left|\frac{a+3b}{4}\right|^{-2q}\right\}\right)^{\frac{1}{q}}\\
&\left.+\left(\max\left\{|b|^{-2q}, \left|\frac{a+3b}{4}\right|^{-2q}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. It follows directly from Corollary 2.8 for the function, $f(x) = \frac{1}{x}$, $x \in [a,b] \setminus \{0\}$. □

4. Application to the Midpoint Formula

Let d be a division of the interval $[a,b]$, *i.e.* $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. Let consider the quadrature formulae

$$\int_a^b f(x)dx = M(f,d) + E(f,d),$$

where

$$M(f,d) = \sum_{i=0}^{n-1}(x_{i+1}-x_i) f\left(\frac{x_i + x_{i+1}}{2}\right)$$

is the midpoint version and the approximation error $E(f,d)$ of the integral $\int_a^b f(x)dx$. The midpoint formula satisfy

$$\left|E(f,d)\right| \leq \frac{K}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3. \qquad (26)$$

If f is not twice differentiable (or the second derivative of f is not bounded on (a,b) then (26) cannot be applied. Following results give some new estimates for the sum of remainders $E(f,d)$in terms of the first derivative.

Proposition 4.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|$ is quasi-convex on $[a,b]$, then for every division d of $[a,b]$, we have:*

$$\begin{aligned}
&\left|E(f,d)\right| \qquad (27)\\
&\leq \frac{1}{32}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left[\max\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|, \left|f'(x_i)\right|\right\}\right.\\
&+3\max\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|, \left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\}\\
&+3\max\left\{\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|, \left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\}\\
&\left.+\max\left\{\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|, \left|f'(x_{i+1})\right|\right\}\right].
\end{aligned}$$

Proof. By applying Corollary 2.3 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division d, we have

$$\begin{aligned} &\left| f\left(\frac{x_i + x_{i+1}}{2}\right) - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)\, dx \right| \\ &\leq \frac{1}{32} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\max\left\{ \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right|, \left| f'(x_i)\right| \right\} \right. \\ &+ 3\max\left\{ \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right|, \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right| \right\} \\ &+ 3\max\left\{ \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right|, \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right| \right\} \\ &\left. + \max\left\{ \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right|, \left| f'(x_{i+1})\right| \right\} \right] \end{aligned} \tag{28}$$

which completes the proof. □

Corollary 4.2. *Suppose all the assumptions of Proposition 4.1 are satisfied. Additionally,*

1. *If* $|f'|$ *is increasing, then*

$$\begin{aligned} &\left| E(f,d) \right| \\ &\leq \frac{1}{32} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\left| f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right| \right. \\ &\left. + 3\left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right| + 3\left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right| + \left| f'(x_{i+1})\right| \right]. \end{aligned} \tag{29}$$

2. *If* $|f'|$ *is decreasing, then*

$$\begin{aligned} &\left| E(f,d) \right| \\ &\leq \frac{1}{32} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\left| f'(x_i)\right| + 3\left| f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right| \right. \\ &\left. + 3\left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right| + \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right| \right]. \end{aligned} \tag{30}$$

Proposition 4.3. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is quasi-convex on* $[a,b]$ *for some fixed* $q > 1$, *then for every division* d *of* $[a,b]$, *we have*

$$\begin{aligned} &\left| E(f,d) \right| \\ &\leq \frac{1}{16(p+1)^{\frac{1}{p}}} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left\{ \left(\max\left\{ \left| f'(x_i)\right|^q, \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right|^q \right\} \right)^{\frac{1}{q}} \right. \\ &+ \left(2^{p+1} - 1\right)^{\frac{1}{p}} \left(\max\left\{ \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^q, \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right|^q \right\} \right)^{\frac{1}{q}} \\ &+ \left(2^{p+1} - 1\right)^{\frac{1}{p}} \left(\max\left\{ \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^q, \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right|^q \right\} \right)^{\frac{1}{q}} \\ &\left. + \left(\max\left\{ \left| f'(x_{i+1})\right|^q, \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right|^q \right\} \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{31}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof is similar to the proof of Proposition 4.1, by applying similar argument to the Corollary 2.6. □

Corollary 4.4. *Suppose all the conditions of Proposition 4.3 are satisfied. Additionally,*

1. *If* $\left|f'\right|^q$ *is increasing, then*

$$\begin{aligned}&\left|E(f,d)\right| \\ &\leq \frac{1}{16(p+1)^{\frac{1}{p}}}\sum_{i=0}^{n-1}(x_{i+1}-x_i) \\ &\times\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right. \\ &\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|+\left|f'(x_{i+1})\right|\right\}.\end{aligned} \tag{32}$$

2. *If* $\left|f'\right|^q$ *is decreasing, then*

$$\begin{aligned}&\left|E(f,d)\right| \\ &\leq \frac{1}{16(p+1)^{\frac{1}{p}}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left\{\left|f'(x_i)\right|+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|\right. \\ &\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|\right\}.\end{aligned} \tag{33}$$

Proposition 4.5. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $\left|f'\right|^q$ *is quasi-convex on* $[a,b]$ *for some fixed* $q \geq 1$*, then for every division* d *of* $[a,b]$*, we have*

$$\begin{aligned}&\left|E(f,d)\right| \\ &\leq \frac{1}{32}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left\{\left(\max\left\{\left|f'(x_i)\right|^q,\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ &+3\left(\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q,\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &+3\left(\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q,\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &\left.+\left(\max\left\{\left|f'(x_{i+1})\right|^q,\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{34}$$

Proof. The proof is similar to the proof of Proposition 4.1, now by applying to Corollary 2.8. □

Corollary 4.6. *Under the assumptions of Proposition 4.5, if*

1. $\left|f'\right|^q$ *is increasing, then (29) holds.*
2. $\left|f'\right|^q$ *is decreasing, then (30) holds.*

5. APPLICATIONS FOR P.D.F's

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=\Pr(X\le x)=\int_a^x f(t)dt$.

Theorem 5.1. *Under the assumptions of Theorem 2.2, we have the inequality;*

$$\begin{aligned}&\left|\Pr(X\le x)-\frac{1}{b-a}\left(b-E(x)\right)\right|\\ &\le\frac{(x-a)^2}{8(b-a)}\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|,\left|f'(a)\right|\right\}\\ &+\frac{3(x-a)^2}{8(b-a)}\max\left\{\left|f'(x)\right|,\left|f'\left(\frac{a+x}{2}\right)\right|\right\}\\ &+\frac{3(b-x)^2}{8(b-a)}\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|,\left|f'(x)\right|\right\}\\ &+\frac{(b-x)^2}{8(b-a)}\max\left\{\left|f'(b)\right|,\left|f'\left(\frac{b+x}{2}\right)\right|\right\},\end{aligned}$$

where $E(x)$ is the expectation of X.

Proof. The proof is immediate follows from the fact that;

$$E(x)=\int_a^b tdF(t)=b-\int_a^b F(t)\,dt.$$

□

Theorem 5.2. *Under the assumptions of Theorem 2.5, we have the inequality;*

$$\begin{aligned}&\left|\Pr(X\le x)-\frac{1}{b-a}\left(b-E(x)\right)\right|\\ &\le\frac{1}{4(b-a)(p+1)^{\frac{1}{p}}}\left\{(x-a)^2\left(\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\ &+(x-a)^2\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(x)\right|^q,\left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\ &+(b-x)^2\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(x)\right|^q\right\}\right)^{\frac{1}{q}}\\ &\left.+(b-x)^2\left(\max\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{b+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where $E(x)$ is the expectation of X.

Proof. Likewise the proof of the previous theorem, by using the fact that;

$$E(x)=\int_a^b tdF(t)=b-\int_a^b F(t)\,dt$$

the proof is completed. □

Theorem 5.3. *Under the assumptions of Theorem 2.7, we have inequality;*

$$\begin{aligned}
&\left|\Pr(X \le x) - \frac{1}{b-a}(b - E(x))\right| \\
&\le \frac{(x-a)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+ \frac{3(x-a)^2}{8(b-a)}\left(\max\left\{\left|f'(x)\right|^q, \left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+ \frac{3(b-x)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q, \left|f'(x)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+ \frac{(b-x)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}$$

where $E(x)$ is the expectation of X.

Proof. The proof is similar to the proof of the previous result. □

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Integral Inequalities for Different Kinds of Convexity via Classical Inequalities

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Abstract. The main purpose of this study is to prove new integral inequalities for product of different classes of convex functions via some classical inequalities such as general Cauchy inequality and reverse Minkowski inequality.

1. INTRODUCTION

The function $f:[a,b]\to\mathbb{R}$, is said to be convex, if we have

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. This definition is well-known in the literature and a huge amount of the researchers interested in this definition. We can define starshaped functions on $[0,b]$ which satisfy the condition

$$f(tx)\leq tf(x)$$

for $t\in[0,1]$.

Because of the importance of convex functions in inequality theory, integral inequalities including convex function classes have an important place in the literature of mathematical inequalities. Especially in recent years, many researchers have done many studies in this field. Interested readers can find different aspects of this subjects in references.

The concept of $m-$convexity has been introduced by Toader in [5], an intermediate between the ordinary convexity and starshaped property, as following:

Definition 1.1. *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $m-$convex, where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is $m-$concave if $-f$ is $m-$convex.

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Several papers have been written on $m-$convex functions and we refer the papers [1], [2], [3], [7], [8] and [9].

In [4], Miheşan gave definition of $(\alpha, m)-$convexity as following;

Definition 1.2. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \leq 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)-$convexity reduces to $m-$convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$. In [6], Set *et al.* proved some inequalities related to $(\alpha, m)-$convex functions.

The following inequality which well known in the literature as Minkowski inequality is given as;

Let $p \geq 1$, $0 < \int_a^b f(x)^p dx < \infty$, and $0 < \int_a^b g(x)^p dx < \infty$. Then

$$\left(\int_a^b (f(x) + g(x))^p \, dx \right)^{\frac{1}{p}} \leq \left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx \right)^{\frac{1}{p}}. \tag{1}$$

The reverse of this inequality was given by Bougoffa in [16], as the following;

Theorem 1.3. *Let f and g be positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \qquad \forall x\, [a, b].$$

Then

$$\left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx \right)^{\frac{1}{p}} \leq c \left(\int_a^b (f(x) + g(x))^p \, dx \right)^{\frac{1}{p}}. \tag{2}$$

where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Definition 1.4. *[See [10]] Let $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be an $s-$convex function in the second sense if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{3}$$

for all $x, y \in \mathbb{R}_+$ and $t \in [0, 1]$.

In [11], $s-$convexity introduced by Breckner as a generalization of convex functions. Also, Breckner proved the fact that the set valued map is $s-$convex only if the associated support function is $s-$convex function in [12]. Several properties of $s-$convexity in the first sense are discussed in the paper [10]. Obviously, $s-$convexity means just convexity when $s = 1$.

Theorem 1.5. *[See [14]] Suppose that $f : [0, \infty) \to [0, \infty)$ is an $s-$convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1[0, 1]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (4). The above inequalities are sharp.

Some new Hermite-Hadamard type inequalities based on concavity and $s-$convexity established by Kırmacı *et al.* in [15]. For related results see the papers [13], [14] and [15].

This paper organized as follows.

In Section 2, we prove some inequalities for $m-$convex and $s-$convex functions and in Section 3, we give some new inequalities for $(\alpha, m)-$convex functions by using some classical inequalities and fairly elementary analysis.

2. RESULTS FOR $m-$CONVEX AND $s-$CONVEX FUNCTIONS

We will start with the following Theorem which is involving $m-$convex functions.

Theorem 2.1. *Suppose that $f, g : [a, b] \to [0, \infty), 0 \leq a < b < \infty$, are m_1-convex and m_2-convex functions, respectively, where $m_1, m_2 \in (0, 1]$. If $f, g \in L_1[a, b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\, g^{\frac{b-x}{b-a}}(x)\, dx \leq \frac{1}{3}\left[f(b) + m_2 g\left(\frac{a}{m_2}\right)\right] + \frac{1}{6}\left[g(b) + m_1 f\left(\frac{a}{m_1}\right)\right]. \tag{5}$$

Proof. From m_1-convexity and m_2-convexity of f and g, we can write

$$f^t(tb + (1-t)a) \leq \left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right)\right]^t$$

and

$$g^{(1-t)}(tb + (1-t)a) \leq \left[tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right)\right]^{(1-t)}.$$

Since f, g are non-negative, we have

$$\begin{aligned} & f^t(tb + (1-t)a)\, g^{(1-t)}(tb + (1-t)a) \\ \leq\ & \left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right)\right]^t \left[tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right)\right]^{(1-t)}. \end{aligned} \tag{6}$$

Recall the General Cauchy Inequality (see [17], Theorem 3.1), let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for every positive real numbers x and y, we always have

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}.$$

By using the General Cauchy Inequality in (6), we get

$$\begin{aligned} & f^t(tb + (1-t)a)\, g^{(1-t)}(tb + (1-t)a) \\ \leq\ & t\left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right)\right] + (1-t)\left[tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right)\right]. \end{aligned}$$

By integrating with respect to t over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 f^t(tb + (1-t)a)\, g^{(1-t)}(tb + (1-t)a)\, dt \\ \leq\ & \frac{1}{3}\left[f(b) + m_2 g\left(\frac{a}{m_2}\right)\right] + \frac{1}{6}\left[g(b) + m_1 f\left(\frac{a}{m_1}\right)\right]. \end{aligned}$$

Hence, by taking into account the change of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, we obtain the required result. $\square$

Corollary 2.2. *If we choose $m_1 = m_2 = 1$ in Theorem 3, we have the inequality;*

$$\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\, g^{\frac{b-x}{b-a}}(x)\, dx \le \frac{1}{3}\left[f(b)+g(a)\right]+\frac{1}{6}\left[g(b)+f(a)\right].$$

Another result for $m-$convex functions is emboided in the following Theorem.

Theorem 2.3. *Suppose that $f, g : [0, b] \to \mathbb{R}$, $b > 0$, are m_1-convex and m_2-convex functions, respectively, where $m_1, m_2 \in (0, 1]$. If $f \in L_1[a, b]$, then the following inequality holds:*

$$\begin{aligned}
&\frac{g(b)}{(b-a)^2}\int_a^b (x-a)f(x)\,dx + m_2\frac{g\left(\frac{a}{m_2}\right)}{(b-a)^2}\int_a^b (b-x)f(x)\,dx \\
&+\frac{f(b)}{(b-a)^2}\int_a^b (x-a)g(x)\,dx + m_1\frac{f\left(\frac{a}{m_1}\right)}{(b-a)^2}\int_a^b (b-x)g(x)\,dx \\
\le\ &\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{3}f(b)\,g(b) + \frac{m_1}{6}f\left(\frac{a}{m_1}\right)g(b) \\
&+\frac{m_2}{6}f(b)\,g\left(\frac{a}{m_2}\right) + \frac{m_1m_2}{3}f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right).
\end{aligned}\tag{7}$$

Proof. Since f and g are m_1-convex and m_2-convex functions, respectively, we can write

$$f(tb+(1-t)a) \le tf(b) + m_1(1-t)\,f\left(\frac{a}{m_1}\right)$$

and

$$g(tb+(1-t)a) \le tg(b) + m_2(1-t)\,g\left(\frac{a}{m_2}\right).$$

By using the elementary inequality, $e \le f$ and $p \le r$, then $er + fp \le ep + fr$ for $e, f, p, r \in \mathbb{R}$, then we get

$$\begin{aligned}
&f(tb+(1-t)a)\left[tg(b)+m_2(1-t)\,g\left(\frac{a}{m_2}\right)\right] \\
&+g(tb+(1-t)a)\left[tf(b)+m_1(1-t)\,f\left(\frac{a}{m_1}\right)\right] \\
\le\ &f(tb+(1-t)a)\,g(tb+(1-t)a) \\
&+\left[tg(b)+m_2(1-t)\,g\left(\frac{a}{m_2}\right)\right]\left[tf(b)+m_1(1-t)\,f\left(\frac{a}{m_1}\right)\right].
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&tf(tb+(1-t)a)\,g(b) + m_2(1-t)\,f(tb+(1-t)a)\,g\left(\frac{a}{m_2}\right) \\
&+tf(b)\,g(tb+(1-t)a) + m_1(1-t)\,f\left(\frac{a}{m_1}\right)g(tb+(1-t)a) \\
\le\ &f(tb+(1-t)a)\,g(tb+(1-t)a) + t^2f(b)\,g(b) + m_1t(1-t)\,f\left(\frac{a}{m_1}\right)g(b) \\
&+m_2t(1-t)\,f(b)\,g\left(\frac{a}{m_2}\right) + m_1m_2(1-t)^2\,f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right).
\end{aligned}$$

By integrating this inequality with respect to t over $[0, 1]$ and by using the change of the variable $tb+(1-t)a = x$, $(b-a)dt = dx$, the proof is completed. □

Corollary 2.4. *If we choose $m_1 = m_2 = 1$ in Theorem 4, we have the inequality;*

$$\begin{aligned}
&\frac{g(b)}{(b-a)^2}\int_a^b (x-a)f(x)\,dx + \frac{g(a)}{(b-a)^2}\int_a^b (b-x)f(x)\,dx \\
&+\frac{f(b)}{(b-a)^2}\int_a^b (x-a)g(x)\,dx + \frac{f(a)}{(b-a)^2}\int_a^b (b-x)g(x)\,dx \\
\le\ & \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b).
\end{aligned}$$

Following inequality also holds for $m-$convex functions.

Theorem 2.5. *Suppose that $f, g : [a,b] \to [0,\infty), 0 \le a < b < \infty$, are m_1-convex and m_2-convex functions, respectively, where $m_1, m_2 \in (0,1]$. If $f, g \in L_1[a,b]$ and f, g satisfy following condition*

$$0 < m \le \frac{f(x)}{g(x)} \le M, \qquad \forall x \in [a,b]$$

then the following inequality holds:

$$\begin{aligned}
&\frac{1}{c}\left[\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx\right)^{\frac{1}{p}}\right] \\
\le\ & \left(\frac{2^{p-1}(b-a)}{p+1}\right)^{\frac{1}{p}}\left([f(b)+g(b)]^p - \left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right)\right]^p\right)^{\frac{1}{p}}
\end{aligned}$$

where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$ and $p \ge 1$.

Proof. Since f and g are m_1-convex and m_2-convex functions, respectively, we can write

$$f(tb + (1-t)a) \le tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \tag{8}$$

and

$$g(tb + (1-t)a) \le tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right). \tag{9}$$

By adding (8) and (9), we get

$$\begin{aligned}
f(tb+(1-t)a) + g(tb+(1-t)a) &\le tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \\
&\quad + tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right).
\end{aligned} \tag{10}$$

For $p \ge 1$, taking $p-$th power of both sides of the inequality (10) and by using the elementary inequality, $(c+d)^p \le 2^{p-1}(c^p + d^p)$, then we get

$$\begin{aligned}
&[f(tb+(1-t)a) + g(tb+(1-t)a)]^p \\
\le\ & 2^{p-1}\left(t^p[f(b)+g(b)]^p + (1-t)^p\left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right)\right]^p\right).
\end{aligned}$$

Integrating with respect to t over $[0,1]$ and by using the change of the variable $tb+(1-t)a=x$ and $(b-a)dt=dx$, we obtain

$$\frac{1}{b-a}\int_a^b (f(x)+g(x))^p\,dx \le \frac{2^{p-1}}{p+1}\left([f(b)+g(b)]^p-\left[m_1 f\left(\frac{a}{m_1}\right)+m_2 g\left(\frac{a}{m_2}\right)\right]^p\right). \tag{11}$$

By taking $\frac{1}{p}-$th power of both sides of the inequality (11) and by using the inequality (2), we get the desired inequality. Which completes the proof. □

We will give an inequality for $s-$convex functions in the following theorem. In the next theorem we will also make use of the Beta function of Euler type, which is for $x,y>0$ defined as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Theorem 2.6. *Suppose that $f,g:[0,\infty)\to[0,\infty)$ are s_1-convex and s_2-convex functions, respectively, where $s_1,s_2\in[0,1]$. Then the following inequality holds:*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\,g^{\frac{b-x}{b-a}}(x)\,dx &\le \frac{1}{s_1+2}f(b)+\beta(2,s_1+1)f(a)\\ &+\frac{1}{s_2+2}g(b)+\beta(2,s_2+1)g(a).\end{aligned}$$

Proof. Since f and g are s_1-convex and s_2-convex functions, respectively, we can write

$$f^t(tb+(1-t)a)\le\left[t^{s_1}f(b)+(1-t)^{s_1}f(a)\right]^t$$

and

$$g^{(1-t)}(tb+(1-t)a)\le\left[t^{s_2}g(b)+(1-t)^{s_2}g(a)\right]^{(1-t)}.$$

Since f,g are non-negative, we have

$$\begin{aligned}&f^t(tb+(1-t)a)\,g^{(1-t)}(tb+(1-t)a)\\ \le\ &\left[t^{s_1}f(b)+(1-t)^{s_1}f(a)\right]^t\left[t^{s_2}g(b)+(1-t)^{s_2}g(a)\right]^{(1-t)}.\end{aligned}\tag{12}$$

By using the General Cauchy Inequality in (12), we get

$$\begin{aligned}&f^t(tb+(1-t)a)\,g^{(1-t)}(tb+(1-t)a)\\ \le\ &t\left[t^{s_1}f(b)+(1-t)^{s_1}f(a)\right]+(1-t)\left[t^{s_2}g(b)+(1-t)^{s_2}g(a)\right].\end{aligned}$$

By integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 f^t(tb+(1-t)a)\,g^{(1-t)}(tb+(1-t)a)\,dt\\ \le\ &\int_0^1\left[t^{s_1+1}f(b)+t(1-t)^{s_1}f(a)+t^{s_2+1}g(b)+t(1-t)^{s_2}g(b)\right]dt.\end{aligned}$$

Hence, by taking into account the change of the variable $tb+(1-t)a=x$, $(b-a)dt=dx$, we obtain the required result. □

Corollary 2.7. *If we choose $s_1 = s_2 = 1$ in Theorem 6, we have the inequality;*

$$\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\, g^{\frac{b-x}{b-a}}(x)\, dx \leq \frac{1}{3}\left[f(b)+g(b)\right]+\frac{1}{6}\left[f(a)+g(a)\right].$$

3. RESULTS FOR $(\alpha, m)-$CONVEX FUNCTIONS

Similar results to Section 2 are given in this section, but now for $(\alpha, m)-$convex functions.

Theorem 3.1. *Suppose that $f,g : [a,b] \to [0,\infty)$, $0 \leq a < b < \infty$, are $(\alpha_1, m_1)-$convex and $(\alpha_2, m_2)-$convex functions, respectively, where $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$. If $f,g \in L_1[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\, g^{\frac{b-x}{b-a}}(x)\, dx \\
\leq\ & \frac{1}{\alpha_1+2} f(b) + \frac{m_1}{2(\alpha_1+2)} f\left(\frac{a}{m_1}\right) \\
&+\frac{1}{(\alpha_2+1)(\alpha_2+2)} g(b) + \frac{m_2\left(\alpha_2^2+3\alpha\right)}{2(\alpha_2+1)(\alpha_2+2)} g\left(\frac{a}{m_2}\right).
\end{aligned}$$

Proof. Since f and g are $(\alpha_1, m_1)-$convex and $(\alpha_2, m_2)-$convex functions, respectively, we can write

$$f^t(tb+(1-t)a) \leq \left[t^{\alpha_1} f(b) + m_1(1-t^{\alpha_1}) f\left(\frac{a}{m_1}\right)\right]^t$$

and

$$g^{(1-t)}(tb+(1-t)a) \leq \left[t^{\alpha_2} g(b) + m_2(1-t^{\alpha_2}) g\left(\frac{a}{m_2}\right)\right]^{(1-t)}.$$

Since f,g are non-negative, we have

$$\begin{aligned}
&f^t(tb+(1-t)a)\, g^{(1-t)}(tb+(1-t)a) \\
\leq\ & \left[t^{\alpha_1} f(b) + m_1(1-t^{\alpha_1}) f\left(\frac{a}{m_1}\right)\right]^t \left[t^{\alpha_2} g(b) + m_2(1-t^{\alpha_2}) g\left(\frac{a}{m_2}\right)\right]^{(1-t)}.
\end{aligned} \tag{13}$$

By using the General Cauchy Inequality in (13), we get

$$\begin{aligned}
&f^t(tb+(1-t)a)\, g^{(1-t)}(tb+(1-t)a) \\
\leq\ & t\left[t^{\alpha_1} f(b) + m_1(1-t^{\alpha_1}) f\left(\frac{a}{m_1}\right)\right] + (1-t)\left[t^{\alpha_2} g(b) + m_2(1-t^{\alpha_2}) g\left(\frac{a}{m_2}\right)\right].
\end{aligned}$$

By integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f^t(tb+(1-t)a)\, g^{(1-t)}(tb+(1-t)a)\, dt \\
\leq\ & \frac{1}{\alpha_1+2} f(b) + \frac{m_1}{2(\alpha_1+2)} f\left(\frac{a}{m_1}\right) \\
&+\frac{1}{(\alpha_2+1)(\alpha_2+2)} g(b) + \frac{m_2\left(\alpha_2^2+3\alpha\right)}{2(\alpha_2+1)(\alpha_2+2)} g\left(\frac{a}{m_2}\right).
\end{aligned}$$

Hence, by taking into account the change of the variable $tb+(1-t)a = x$, $(b-a)dt = dx$, we obtain the required result. □

Corollary 3.2. *If we choose $\alpha_1 = \alpha_2 = 1$ in Theorem 7, we have the inequality (5).*

Theorem 3.3. *Suppose that $f, g : [a,b] \to [0,\infty)$, $0 \le a < b < \infty$, are $(\alpha_1, m_1)-$convex and $(\alpha_2, m_2)-$convex functions, respectively, where $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$. If $f, g \in L_1[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\frac{g(b)}{(b-a)^{\alpha_2+1}}\int_a^b (x-a)^{\alpha_2} f(x)\,dx + m_2\frac{g\left(\frac{a}{m_2}\right)}{(b-a)^{\alpha_2+1}}\int_a^b \left[(b-a)^{\alpha_2}-(x-a)^{\alpha_2}\right] f(x)\,dx \\
&+\frac{f(b)}{(b-a)^{\alpha_1+1}}\int_a^b (x-a)^{\alpha_1} g(x)\,dx + m_1\frac{f\left(\frac{a}{m_1}\right)}{(b-a)^{\alpha_1+1}}\int_a^b \left[(b-a)^{\alpha_1}-(x-a)^{\alpha_1}\right] g(x)\,dx \\
\le\; &\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{\alpha_1+\alpha_2+1} f(b)\,g(b) + \frac{m_2\alpha_2}{(\alpha_1+1)(\alpha_1+\alpha_2+1)} g\left(\frac{a}{m_2}\right) f(b) \\
&+\frac{m_1\alpha_1}{(\alpha_1+1)(\alpha_1+\alpha_2+1)} f\left(\frac{a}{m_1}\right) g(b) + \frac{m_1 m_2 \alpha_1\alpha_2(\alpha_1+\alpha_2+2)}{(\alpha_1+1)(\alpha_2+1)(\alpha_1+\alpha_2+1)} f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right).
\end{aligned}$$

Proof. Since f and g are $(\alpha_1, m_1)-$convex and $(\alpha_2, m_2)-$convex functions, respectively, we can write

$$f(tb+(1-t)a) \le t^{\alpha_1} f(b) + m_1(1-t^{\alpha_1}) f\left(\frac{a}{m_1}\right)$$

and

$$g(tb+(1-t)a) \le t^{\alpha_2} g(b) + m_2(1-t^{\alpha_2}) g\left(\frac{a}{m_2}\right).$$

By using the elementary inequality, $e \le f$ and $p \le r$, then $er + fp \le ep + fr$ for $e, f, p, r \in \mathbb{R}$ and by a similar argument to the proof of Theorem 4, we get the required result. □

Corollary 3.4. *If we choose $\alpha_1 = \alpha_2 = 1$ in Theorem 8, we have the inequality (7).*

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